GAZETA MATEMATICĂ

SERIA A

ANUL XXVIII(CVII) **Nr. 3 – 4/ 2010**

ARTICOLE

Picard and Krasnoselski sequences: applications to fixed point problems[1)]

Teodora–Liliana T. Rădulescu[2)] and Vicenţiu D. Rădulescu[3)]

Abstract. We are concerned with the role of *Picard* and *Krasnoselski* sequences in the approximation of fixed points in various classes of nonlinear equations. We also give a connection with the cobweb method that describes equilibrium phenomena in mathematical economics.

Keywords: fixed point; Brouwer fixed point theorem; Knaster fixed point theorem; successive approximation; Picard sequence; Krasnoselski sequence; asymptotic regularity; cobweb method.

MSC : 26A15; 26A18; 47H10.

1. Introduction

The development of fixed point theory is closely related with the study of various problems arising in the theory of ordinary differential equations. One of the first contributions to this field is due to the French mathematician *Henri Poincaré*[4)] (1854-1912) in his famous paper [14] of 1890 on the three-body problem crowned by *King Oscar Prize*. This problem concerns the free

---

1)This paper contains the main topics of a plenary conference delivered at the XIVth Conference of the Romanian Mathematical Society, Alba Iulia, 15–17 October 2010.

2)Department of Mathematics, „Fraţii Buzeşti“ National College, 200352 Craiova, Romania, teodoraradulescu@yahoo.com

3)Institute of Mathematics „Simion Stoilow“ of the Romanian Academy, Bucharest, Romania, Department of Mathematics, University of Craiova, 200585 Craiova, Romania, vicentiu.radulescu@imar.ro

4)One of the most influential mathematician of modern times, with crucial contributions to the development of applied mathematics, mathematical physics, and celestial mechanics. According to *H. Brezis* and *F. Browder* [3], "Poincaré emphasized that a wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance – un “air de famille” in Poincaré’s words – and should be treated by common methods”.

motion of multiple orbiting bodies and *Poincaré* reduced the study to the qualitative analysis of the T–periodic solutions of a differential system in $\mathbb{R}^n$

$$x' = f(t, x) \tag{1}$$

to the study of the fixed points of the operator $P_T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$P_T(y) = p(T; 0, y),$$

where $p(t; s, y)$ denotes the solution of equation (1) verifying the initial condition $x(s) = y$. We refer to the survey paper by *Mawhin* [11] for more details and related results.

As early as 1883, *Poincaré* stated in [13] a theorem shown much later to be equivalent with a fixed point theorem for continuous functions on a closed ball into itself, published by *L.E.J. Brouwer*[1)] in 1912 (see [4]). In its simplest one-dimensional case (see Figure), the *Brouwer* fixed point theorem asserts the following property: *any continuous function* $f : [a, b] \to [a, b]$ *has at least a fixed point.* The proof combines very simple arguments, which strongly rely

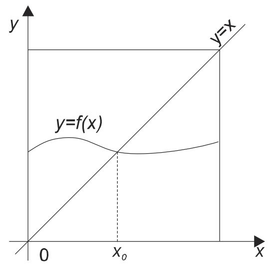

Fig. 1. *Brouwer*'s fixed point theorem (one dimensional case)

on the main continuity assumption combined with the order structure of the set of real numbers. In the general form, the *Brouwer* fixed point theorem asserts that any continuous function with domain the closed unit ball B in $\mathbb{R}^N$ and range contained in B must have at least one fixed point. This result was first applied in 1943 to some forced *Liénard* equations by *Lefschetz* [9] and *Levinson* [10]. If $N \geq 2$, the proof of the the *Brouwer* theorem is much more complicated. However, simpler proofs have been found by means of powerful topological tools, such as the topological degree.

In this paper we are concerned with the following natural related questions:

– what happens if the continuity hypothesis in the *Brouwer* fixed point theorem is replaced with a monotonicity assumption;

– what about the approximation of the fixed point by means of two classical successive approximations:

$$x_{n+1} = f(x_n) \qquad (\textit{Picard} \text{ sequence})$$

[1)] Dutch mathematician (1881–1966).

$$x_{n+1} = \frac{x_n + f(x_n)}{2} \qquad (\textit{Krasnoselski} \text{ sequence}).$$

If f is continuous, these sequences provide fixed points, provided that they *are convergent*. Indeed, taking the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 - x$, let us consider $x_0 \in [0,1]$ and $x_{n+1} = f(x_n)$, for all $n \geq 0$. Then the *Picard* sequence (x_n) converges if and only if $x_0 = 1/2$. However, if we construct the *Krasnoselski* sequence $x_{n+1} = [x_n + f(x_n)]/2$, then (x_n) converges for any initial value $x_0 \in [0,1]$.

We also establish in the present paper related fixed point properties. A particular interest is given to the cobweb method arising in mathematical economics, in strong relationship with successive approximations. We refer to the recent problem books [15], [16] for further results and relevant applications.

2. Knaster fixed point theorem

In this section we argue that the fixed point property stated in the *Brouwer* theorem remains true if the continuity assumption is replaced with the hypothesis that the function $f : [a,b] \to [a,b]$ is non-decreasing. The same property does not hold provided that if f is *decreasing*. In the non-decreasing case, the proof strongly relies on the order structure of the real axis.

Theorem 1. *Let $f : [a,b] \to [a,b]$ be a non-decreasing function.*
(i) *Then f has at least one fixed point.*
(ii) *There are decreasing function $f : [a,b] \to [a,b]$ with no fixed points.*

Proof. (i) Set

$$A = \{a \leq x \leq b;\ f(x) \geq x\}$$

and $x_0 = \sup A$. The following situations may occur.

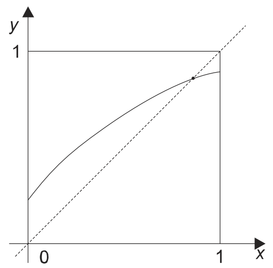

Fig. 2. Knaster's fixed point theorem

Case 1: $x_0 \in A$. By the definition of x_0 it follows that $f(x_0) \geq x_0$. If $f(x_0) = x_0$, then the proof is concluded. If not, we argue by contradiction and assume that $f(x_0) > x_0$. By the definition of x_0 we obtain $f(x) < x$, $\forall x > x_0$.

On the other hand, for any $x_0 < x < f(x_0)$ we have $x > f(x) \geq f(x_0)$, contradiction since $x \in (x_0, f(x_0))$, that is, $x < f(x_0)$. It follows that the assumption $f(x_0) > x_0$ is false, so f has a fixed point.

Case 2: $x_0 \notin A$. We prove that, in fact, it is impossible to have $x_0 \notin A$, so $x_0 \in A$, which reduces the problem to Case 1. If $x_0 \notin A$ then there exists a sequence $x_n \to x_0$, $x_n < x_0$, such that $x_n \in A$. Since f is increasing, it follows that $\lim_{n\to\infty} f(x_n) = x_0$. On the other hand, from $f(x_0) < x_0$ we deduce that there exists $x_n < x_0$ such that $f(x_n) > f(x_0)$, contradiction with the fact that f is increasing.

(ii) Consider the function

$$f(x) = \begin{cases} 1 - x & \text{if } \ 0 \leq x < \dfrac{1}{2} \\ \dfrac{1}{2} - \dfrac{x}{2} & \text{if } \ \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Then $f : [0,1] \to [0,1]$ is decreasing but does not have any fixed point. $\square$

A related counterexample is depicted in Figure 3.

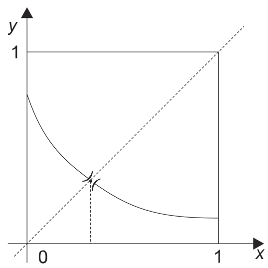

Fig. 3. *Knaster*'s fixed point theorem fails for decreasing functions

3. Further fixed point properties

We start with some simple facts regarding the hypotheses of the *Brouwer* fixed point theorem on the real axis.

(i) While a fixed point in $[a, b]$ exists for a continuous function $f : [a, b] \to [a, b]$, it need **not** be unique. Indeed, any point $x \in [a, b]$ is a fixed point of the function $f : [a, b] \to [a, b]$ defined by $f(x) = x$.

(ii) The condition that f is defined on a closed subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [0, 1) \to \mathbb{R}$ is defined by $f(x) = (1+x)/2$, then f maps $[0, 1)$ into itself, and f is continuous. However, f has **no** fixed point in $[0, 1)$.

(iii) The condition that f be defined on a bounded subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [1, \infty) \to \mathbb{R}$ is defined by $f(x) = x + x^{-1}$, then f maps $[1, \infty)$ into itself, f is continuous, but f has **no** fixed point in $[1, \infty)$.

(iv) The condition that f be defined on an interval in $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $D = [-2, -1] \cup [1, 2]$ and $f : D \to \mathbb{R}$ is defined by $f(x) = -x$, then f maps D into itself, f is continuous, but f has **no** fixed point in D.

We prove in what follows some elementary fixed point properties of real-valued functions.

Proposition 1. *Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$, $f(1) = 1$. Denote $f^n := f \circ f \circ \cdots \circ f$ (n times) and assume that there exists a positive integer m such that $f^m(x) = x$ for all $x \in [0, 1]$. Then $f(x) = x$ for any $x \in [0, 1]$.*

Proof. Our hypothesis implies that f is one–to–one, so increasing (since f is continuous). Assume, by contradiction, that there exists $x \in (0, 1)$ such that $f(x) > x$. Then, for any $n \in \mathbb{N}$, we have $f^n(x) > f^{n-1}(x) > \cdots > f(x) > x$. For $n = m$ we find a contradiction. A similar argument shows that the case $f(x) < x$ (for some x) is not possible. □

Proposition 2. *Let a, b be real numbers, $a < b$ and consider a continuous function $f : [a, b] \to \mathbb{R}$.*

(i) If $[a, b] \subset f([a, b])$ then f has a fixed point.

(ii) Assume that there exists a closed interval $I' \subset f([a, b])$. Then $I' = f(J)$, where J is a closed interval contained in $[a, b]$.

(iii) Assume that there exists n closed intervals $I_0, \ldots, I_{n-1}$ contained in $[a, b]$ such that for all $0 \le k \le n-2$, $I_{k+1} \subset f(I_k)$ and $I_0 \subset f(I_{n-1})$. Then f^n has a fixed point ($f^n = f \circ \ldots \circ f$).

Proof. (i) Denote $f([a, b]) = [m, M]$ and let x_m, $x_M \in [a, b]$ be such that $f(x_m) = m$ and $f(x_M) = M$. Since $f(x_m) - x_m \le 0$ and $f(x_M) - x_m \ge 0$, it follows by the intermediate value property that f has at least a fixed point.

(ii) Set $I' = [c, d]$ and consider $u, v \in I$ such that $f(u) = c$ and $f(v) = d$. Assume, without loss of generality, that $u \le v$.

The set $A = \{x \in [u, v];\ f(x) = c\}$ is compact and non-empty, so there exists $\alpha = \max\{x;\ x \in A\}$ and, moreover, $\alpha \in A$. Similarly, the set $B = \{x \in [\alpha, v];\ f(x) = d\}$ has a minimum point β. Then $f(\alpha) = c$, $f(\beta) = d$ and for all $x \in (\alpha, \beta)$ we have $f(x) \ne c$ and $f(x) \ne d$. So, by the intermediate value property, $[c, d] \subset f((\alpha, \beta))$ and $f((\alpha, \beta))$ is an interval which contains neither c nor d. It follows that $I' = f(J)$, where $J = [\alpha, \beta]$.

(iii) Since $I_0 \subset f(I_{n-1})$, it follows by b) that there exists a closed interval $J_{n-1} \subset I_{n-1}$ such that $I_0 = f(J_{n-1})$. But $J_{n-1} \subset I_{n-1} \subset f(I_{n-2})$. So, by (ii), there exists a closed interval $J_{n-2} \subset I_{n-2}$ such that $J_{n-1} = f(J_{n-2})$. Thus, we obtain n closed intervals $J_0, \ldots, J_{n-1}$ such that

$$J_k \subset I_k, \qquad \text{for all } 0 \le k \le n-1$$

and

$$J_{k+1} = f(J_k), \qquad \text{for all } 0 \le k \le n-2 \text{ and } I_0 = f\,(J_{n-1}).$$

Consequently, J_0 is included in the domain of the nth iterate f^n and $J_0 \subset I_0 = f^n(J_0)$. By a) we deduce that f^n has a unique fixed point in J_0. □

Proposition 3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$. Then there is some ξ in $[-\infty, +\infty]$ such that, for any real x, $f^n(x) \to \xi$ as $n \to \infty$.*

Proof. We suppose first that f has a fixed point, say ξ, in $\mathbb{R}$. Then, from the contracting property of f, ξ is the only fixed point of f. We may assume that $\xi = 0$, and this implies that $|f(x)| < |x|$ for all nonzero x. Thus, for any x, the sequence $|f^n(x)|$ is decreasing, so converges to some nonnegative number $\mu(x)$. We want to show that $\mu(x) = 0$ for every x, so suppose now that x is such that $\mu(x) > 0$. Then f maps $\mu(x)$ and $-\mu(x)$ to points y_1 and y_2, say, where $y_j < |\mu(x)|$ for each j. Thus, as f is continuous, there are open neighborhoods of $\pm\mu(x)$ that are mapped by f into the open interval $I = (-\mu(x), \mu(x))$ that contains y_1 and y_2. This implies that, for sufficiently large n, $f^n(x)$ lies in I, which contradicts the fact that $|f^n(x)| \ge \mu(x)$ for all n. Thus, for all x, $\mu(x) = 0$ and $f^n(x) \to 0$.

Now suppose that f has no fixed point in $\mathbb{R}$. Then the function $f(x) - x$ is continuous and nonzero in $\mathbb{R}$. By the intermediate value theorem, $f(x) > x$ for al x, or $f(x) < x$ for all x. We may assume that $f(x) > x$ for all x, as similar argument holds in the other case. Now the sequence $f^n(x)$ is strictly increasing, hence converges to some ξ in $\mathbb{R} \cup \{+\infty\}$. Moreover, $\xi \notin \mathbb{R}$, else ξ would be a fixed point of f. Thus $f^n(x) \to +\infty$ for all x. □

We conclude this paper with the following elementary property, which is due to *M.W. Botsko* [2].

Proposition 4. *Let $f : [0,1] \to [0,1]$ be a function such that $|f(x) - f(y)| \le |x - y|$ for all $x, y \in [0,1]$. Then the set of all fixed points of f is either a single point or an interval.*

Proof. Let $F = \{x \in [0,1];\ f(x) = x\}$. Since F is continuous, it follows that F is compact. Let a be the smallest number in F and b the largest number in F. It follows that $F \subset [a,b]$. Fix arbitrarily $x_0 \in [a,b]$. Since a is a fixed point of f, we have

$$f(x_0) - a \le |f(x_0) - a| = |f(x_0) - f(a)| \le x_0 - a\,.$$

Therefore, $f(x_0) \le x_0$. Similarly,

$$b - f(x_0) \le |b - f(x_0)| = |f(b) - f(x_0)| \le b - x_0\,,$$

which shows that $f(x_0) \ge x_0$. It follows that $f(x_0) = x_0$, hence x_0 is a fixed point of f. Thus, $F = [a,b]$. □

4. Approximation of fixed points

We have observed that if $f : [a, b] \to [a, b]$ is a continuous function then f must have at least one fixed point, that is, a point $x \in [a, b]$ such that $f(x) = x$. A natural question in applications is to provide an algorithm for finding (or approximating) this point. One method of finding such a fixed point is by successive approximation. This technique is due to the French mathematician *Émile Picard* (1856–1941) and was introduced in his classical textbook on analysis [12]. More precisely, if $x_1 \in [a, b]$ is chosen arbitrarily, define $x_{n+1} = f(x_n)$ and the resulting sequence $(x_n)_{n\geq 1}$ is called the *sequence of successive approximations* of f (or a *Picard sequence* for the function f). If the sequence $(x_n)_{n\geq 1}$ converges to some x, then a direct argument based on the continuity of f shows that x is a fixed point of f. Indeed,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = f\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} x_n = x\,.$$

The usual method of showing that the sequence $(x_n)_{n\geq 1}$ of successive approximations converges is to show that it satisfies the *Cauchy* convergence criterion: for every $\varepsilon > 0$ there is an integer N, such that for all integers j, $k \geq N$, we have $|x_j - x_k| < \varepsilon$. The next exercise asserts that it is enough to set $j = k + 1$ in the *Cauchy* criterion.

Proposition 5. *Let $f : [a, b] \to [a, b]$ be a continuous function. Let x_1 be a point in $[a, b]$ and let $(x_n)_{n\geq 1}$ denote the resulting sequence of successive approximations. Then the sequence $(x_n)_{n\geq 1}$ converges to a fixed point of f if and only if $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.*

Proof. Clearly $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$ if $(x_n)_{n\geq 1}$ converges to a fixed point. Suppose $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$ and the sequence $(x_n)_{n\geq 1}$ does not converge. Since $[a, b]$ is compact, there exist two subsequences of $(x_n)_{n\geq 1}$ that converge to ξ_1 and ξ_2 respectively. We may assume $\xi_1 < \xi_2$. It suffices to show that $f(x) = x$ for all $x \in (\xi_1, \xi_2)$. Suppose this is not the case, hence there is some $x^* \in (\xi_1, \xi_2)$ such that $f(x^*) \neq x^*$. Then a $\delta > 0$ could be found such that $[x^* - \delta, x^* + \delta] \subset (\xi_1, \xi_2)$ and $f(\tilde{x}) \neq \tilde{x}$ whenever $\tilde{x} \in (x^* - \delta, x^* + \delta)$. Assume $\tilde{x} - f(\tilde{x}) > 0$ (the proof in the other case being analogous) and choose N so that $|f^n(x) - f^{n+1}(x)| < \delta$ for $n > N$. Since ξ_2 is a cluster point, there exists a positive integer $n > N$ such that $f^n(x) > x^*$. Let n_0 be the smallest such integer. Then, clearly,

$$f^{n_0-1}(x) < x^* < f^{n_0}(x)$$

and since $f^{n_0}(x) - f^{n_0-1}(x) < \delta$ we must have

$$f^{n_0-1}(x) - f^{n_0}(x) > 0 \text{ so that } f^{n_0}(x) < f^{n_0-1}(x) < x^*,$$

a contradiction. $\square$

The usual method of showing that the sequence $(x_n)_{n\geq 0}$ of successive approximations converges is to show that it satisfies the *Cauchy* convergence criterion. The next result establishes that this happens if and only if the difference of two consecutive terms in this iteration converges to zero. The American mathematician *Felix Browder* has called this condition *asymptotic regularity.*

The next result is due to *H.G. Barone* [1] and was established in 1939.

Theorem 2. *Let $(x_n)_{n\geq 0}$ be a sequence of real numbers such that the sequence $(x_{n+1} - x_n)$ converges to zero. Then the set of cluster points of $(x_n)_{n\geq 0}$ is a closed interval in $\overline{\mathbb{R}}$, eventually degenerated.*

Proof. Set $\ell_- := \liminf\limits_{n\to\infty} x_n$, $\ell_+ := \limsup\limits_{n\to\infty} x_n$ and choose $a \in (\ell_-, \ell_+)$. By the definition of ℓ_-, there exists $x_{n_1} < a$. Let n_2 be the least integer greater than n_1 such that $x_{n_2} > a$ (the existence of n_2 follows by the definition of ℓ_+). Thus, $x_{n_2-1} \leq a < x_{n_2}$. Since $\ell_- < a$, there exists a positive integer $n_3 > n_2$ such that $x_{n_3} < a$. Next, by the definition of ℓ_+, there exists an integer $N_4 > n_3$ such that $x_{N_4} > a$. If n_4 denotes the least integer with these properties, then $x_{n_4-1} \leq a < x_{n_4}$. In this manner we construct an increasing sequence of positive numbers $(n_{2k})_{k\geq 1}$ such that, for all $k \geq 1$, $x_{n_{2k}-1} \leq a < x_{n_{2k}}$. Using the hypothesis we deduce that the sequences $(x_{n_{2k}-1})_{k\geq 1}$ and $(x_{n_{2k}})_{k\geq 1}$ converge to a, so a is a cluster point. $\square$

The following convergence result was established by *B.P. Hillam* [7] in 1976.

Theorem 3. *Let $f : [a,b] \to [a,b]$ be a continuous function. Consider the sequence $(x_n)_{n\geq 0}$ defined by $x_0 \in [a,b]$ and, for any positive integer n, $x_n = f(x_{n-1})$. Then the sequence $(x_n)_{n\geq 0}$ converges if and only if $(x_{n+1} - x_n)$ converges to zero.*

Proof. Assume that the sequence of successive approximations $(x_n)_{n\geq 0}$ satisfies $x_{n+1} - x_n \to 0$, as $n \to \infty$. With the same notations as above, assume that $\ell_- < \ell_+$. The proof of (i) combined with the continuity of f imply $a = f(a)$, for all $a \in (\ell_-, \ell_+)$. But this contradicts our assumption $\ell_- < \ell_+$. Indeed, choose $\ell_- < c < d < \ell_+$ and $0 < \varepsilon < (d-c)/3$. Since $x_{n+1} - x_n \to 0$, there exists N_ε such that for all $n \geq N_\varepsilon$, $-\varepsilon < x_{n+1} - x_n < \varepsilon$. Let $N_2 > N_1 > N_\varepsilon$ be such that $x_{N_1} < c < d < x_{N_2}$. Our choice of ε implies that there exists an integer $n \in (N_1, N_2)$ such that $a := x_n \in (c,d)$. Hence $x_{n+1} = f(a) = a$, $x_{n+2} = a$, and so on. Therefore $x_{N_2} = a$, contradiction.

The reversed assertion is obvious. $\square$

The following result is a particular case of a fixed point theorem due to *Krasnoselski* (see [8]). We refer to [6] for the general framework corresponding to functions defined on a closed convex subset of strictly convex *Banach* spaces.

Theorem 4. *Let* $f : [a,b] \to [a,b]$ *a function satisfying* $|f(x) - f(y)| \le |x - y|$, *for all* $x,\ y \in [a,b]$. *Define the sequence* $(x_n)_{n\ge1}$ *by* $x_1 \in [a,b]$ *and, for all* $n \ge 1$, $x_{n+1} = [x_n + f(x_n)]/2$. *Then the sequence* $(x_n)_{n\ge1}$ *converges to some fixed point of* f.

Proof. We observe that it is enough to show that $(x_n)_{n\ge1}$ converges. In this case, by the recurrence relation and the continuity of f, it follows that the limit of $(x_n)_{n\ge1}$ is a fixed point of f. We argue by contradiction and denote by A the set of all limit points of $(x_n)_{n\ge1}$, that is,

$$A := \{\ell \in [a,b];\ \text{there exists a subsequence } (x_{n_k})_{k\ge1} \text{ of } (x_n)_{n\ge1} \ \text{ such that } x_{n_k} \to \ell\}\,.$$

By our hypothesis and the compactness of $[a,b]$, we deduce that A contains at least two elements and is a closed set.

We split the proof into several steps.

(i) For any $\ell \in A$ we have $f(\ell) \ne \ell$. Indeed, assume that $\ell \in A$ and fix $\varepsilon > 0$ and $n_k \in \mathbb{N}$ such that $|x_{n_k} - \ell| \le \varepsilon$. Then

$$|\ell - x_{n_k+1}| = \left|\frac{\ell + f(\ell)}{2} - \frac{x_{n_k} + f(x_{n_k})}{2}\right| \le$$

$$\le \frac{|\ell - f(x_{n_k})|}{2} + \frac{|f(\ell) - f(x_{n_k})|}{2} \le |\ell - x_{n_k}| \le \varepsilon$$

and so on. This shows that $|x_n - \ell| \le \varepsilon$, for all $n \ge n_k$. Hence $(x_n)_{n\ge1}$ converges to ℓ, contradiction.

(ii) There exists $\ell_0 \in A$ such that $f(\ell_0) > \ell_0$. Indeed, arguing by contradiction, set $\ell_- = \min\limits_{\ell\in A} \ell$. Then $\ell_- \in A$ and $f(\ell_-) \le \ell_-$. The variant $f(\ell_-) = \ell_-$ is excluded, by (i). But $f(\ell_-) < \ell_-$ implies that $\frac{[\ell_- + f(\ell_-)]}{2} \in A$ and $\frac{[\ell_- + f(\ell_-)]}{2} < \ell_-$, which contradicts the definition of ℓ_-.

(iii) There exists $\varepsilon > 0$ such that $|f(\ell) - \ell| \ge \varepsilon$, for all $\ell \in A$. For if not, let $\ell_n \in A$ such that $|f(\ell_n) - \ell_n| < \frac{1}{n}$, for all $n \ge 1$. This implies that any limit point of $(\ell_n)_{n\ge1}$ (which lies in A, too) is a fixed point of f. This contradicts (i).

(iv) Conclusion. By (ii) and (iii), there exists a largest $\ell_+ \in A$ such that $f(\ell_+) > \ell_+$. Let $\ell' = \frac{[\ell_+ + f(\ell_+)]}{2}$ and observe that $f(\ell_+) > \ell' > \ell_+$ and $f(\ell') < \ell'$. By (iii), there exists a smallest $\ell'' \in A$ such that $\ell'' > \ell_+$ and $f(\ell'') < \ell''$. It follows that $\ell_+ < \ell'' < f(\ell_+)$. Next note that $f(\ell'') < \ell_+$; for, if not, $\ell''' := \frac{[\ell'' + f(\ell'')]}{2}$ satisfies $\ell_+ < \ell''' < \ell''$ and, by definitions of ℓ_+ and ℓ'', it follows that $f(\ell''') = \ell'''$, contrary to (i). Thus $f(\ell'') < \ell_+ < \ell'' < f(\ell_+)$. It then follows that $|f(\ell'') - f(\ell_+)| > |\ell'' - \ell_+|$. This contradicts the hypothesis and concludes the proof. $\square$

Remark. The iteration scheme described in the above *Krasnoselski*'s property does not apply to arbitrary continuous mappings of a closed interval into itself. Indeed, consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} \dfrac{3}{4}, & \text{if} \quad 0 \le x \le \dfrac{1}{4} \\ -3x + \dfrac{3}{2}, & \text{if} \quad \dfrac{1}{4} < x < \dfrac{1}{2} \\ 0, & \text{if} \quad \dfrac{1}{2} \le x \le 1. \end{cases}$$

Then the sequence defined in the above statement is defined by $x_{2n} = \frac{1}{2}$ and $x_{2n+1} = \frac{1}{4}$, for any $n \ge 1$. So, $(x_n)_{n\ge1}$ is a divergent sequence.

The contraction mapping theorem states that if $f : \mathbb{R} \to \mathbb{R}$ (or $f : [a, \infty) \to [a, \infty)$) is a map such that for some k in $(0, 1)$ and all x and y in $\mathbb{R}$, $|f(x) - f(y)| \le k\,|x - y|$, then the iterates $f^n = f \circ \ldots \circ f$ (n terms) of f converge to a (unique) fixed point ξ of f. This theorem can be accompanied by an example to show that the inequality cannot be replaced by the weaker condition $|f(x) - f(y)| < |x - y|$. The most common example of this type is $f(x) = x + 1/x$ acting on $[1, \infty)$. Then $f(x) > x$, so that f has no fixed points. Also, for every x, the sequence x, $f(x)$, $f^2(x)$, $\ldots$ is strictly increasing and so must converge in the space $[-\infty, +\infty]$. In fact, $f^n(x) \to +\infty$, for otherwise $f^n(x) \to a$ for some real a, and then $f(f^n(x)) \to f(a)$ (because f is continuous) so that $f(a) = a$, which is not so. Thus we define $f(+\infty)$ to be $+\infty$ and deduce that this example is no longer a counterexample. The following property clarifies these ideas and provides an elementary, but interesting, adjunct to the contraction mapping theorem. We just point out that a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| < |x - y|$ for all $x \ne y$ is called a *contractive* function.

We say that a function $f : [a, b] \to \mathbb{R}$ satisfies the Lipschitz condition with constant $L > 0$ if for all x and y in $[a, b]$, $|f(x) - f(y)| \le L\,|x - y|$. A function that satisfies a *Lipschitz* condition is clearly continuous. Geometrically, if $f : [a, b] \to \mathbb{R}$ satisfies the Lipschitz condition

$$|f(x) - f(y)| \le L\,|x - y| \qquad \text{for all } x,\ y \in [a, b],$$

then for any x, $y \in [a, b]$, $x \ne y$, the inequality

$$\left| \frac{f(x) - f(y)}{x - y} \right| \le L$$

indicates that the *slope* of the chord joining the points $(x, f(x))$ and $(y, f(y))$ on the graph of f is bounded by L.

Using the fact that the real line is totally ordered, the following more general theorem with much more elementary proof is possible.

Proposition 6. *Let $f : [a, b] \to [a, b]$ be a function that satisfies a Lipschitz condition with constant L. Let x_1 in $[a, b]$ be arbitrary and define $x_{n+1} = (1 - \lambda)x_n + \lambda f(x_n)$ where $\lambda = 1/(L + 1)$. If $(x_n)_{n\ge1}$ denotes the resulting* *sequence then* $(x_n)_{n\geq 1}$ *converges monotonically to a point* z *in* $[a,b]$ *where* $f(z)=z$.

Proof. Without loss of generality we can assume $f(x_n)\neq x_n$ for all n. Suppose $f(x_1)>x_1$ and let p be the first point greater than x_1 such that $f(p)=p$. Since $f(x_1)>x_1$ and $f(b)\leq b$, the continuity of f implies there is such a point.

Next, we prove the following claim. If $x_1<x_2<\ldots<x_n<p$ and $f(x_i)>x_i$ for $i=1,2,\ldots,n$, then $f(x_{n+1})>x_{n+1}$ and $x_{n+1}<p$. Indeed, suppose $p<x_{n+1}$, then $x_n<p<x_{n+1}$, hence $0<p-x_n<x_{n+1}-x_n=\lambda(f(x_n)-x_n)$. Therefore

$$\begin{aligned} 0 &< \frac{1}{\lambda}\,|x_n-p| = (L+1)\,|x_n-p| < |f(x_n)-x_n| \leq \\ &\leq |f(x_n)-f(p)|+|p-x_n|\,. \end{aligned}$$

It follows that

$$L|x_n-p|<|f(x_n)-f(p)|\,,$$

which contradicts the fact that f is a Lipschitz function. Thus $x_{n+1}<p$ and $f(x_{n+1})>x_{n+1}$ by the choice of p, and the claim is proved.

Using the induction hypothesis it follows that $x_n<x_{n+1}<p$ for all integers n. Since a bounded monotonic sequence converges, $(x_n)_{n\geq 1}$ converges to some point z. By the triangle inequality it follows that

$$\begin{aligned} |z-f(z)| &\leq |z-x_n|+|x_n-f(x_n)|+|f(x_n)-f(z)| = \\ &= |z-x_n|+\frac{1}{\lambda}\,|x_{n+1}-x_n|+|f(x_n)-f(z)|\,. \end{aligned}$$

Since the right-hand side tends to 0 as $n\to\infty$, we conclude that $f(z)=z$. If $f(x_1)<x_1$ a similar argument holds. □

Applying a somewhat more sophisticated argument, one can allow λ to be any number less than $2/(L+1)$ but the resulting sequence $(x_n)_{n\geq 1}$ need not converge monotonically. The following example shows this last result is best possible.

Let $f:[0,1]\to[0,1]$ be defined by

$$f(x)=\begin{cases} 1, & 0\leq x<\dfrac{L-1}{2L} \\ -Lx+\dfrac{1}{2}(L+1), & \dfrac{L-1}{2L}\leq x\leq\dfrac{L+1}{2L} \\ 0, & \dfrac{L+1}{2L}<x\leq 1, \end{cases}$$

where $L>1$ is arbitrary. Note that f satisfies a *Lipschitz* condition with constant L. Let $\lambda=2/(L+1)$ and let $x_1=(L-1)/2L$. Then $x_2= =(1-\lambda)x_1+\lambda f(x_1)=(L+1)/2L$, $x_3=(1-\lambda)x_2+\lambda f(x_2)=(L-1)/2L$, etc.

5. Picard sequences versus the cobweb model and qualitative analysis of markets

We start with the following elementary geometric interpretation of the *Picard* method. First, take a point $A(x_1, f(x_1))$ on the curve $y = f(x)$. Next, consider the point $B_1(f(x_1), f(x_1))$ on the diagonal line $y = x$ and then, project the point B_1 vertically onto the curve $y = f(x)$ to obtain a point $A_2(x_2, f(x_2))$. Again, project A_2 horizontally to B_2 on $y = x$ and then, project B_2 vertically onto $y = f(x)$ to obtain $A_3(x_3, f(x_3))$. This process can be repeated a number of times. Often, it will weave a cobweb in which the fixed point of f, that is, the point of intersection of the curve $y = f(x)$ and the diagonal line $y = x$, gets trapped. In fact, such trapping occurs if the slopes of tangents to the curve $y = f(x)$ are smaller (in absolute value) than the slope of the diagonal line $y = x$. The situation described above is illustrated in Figure 4.

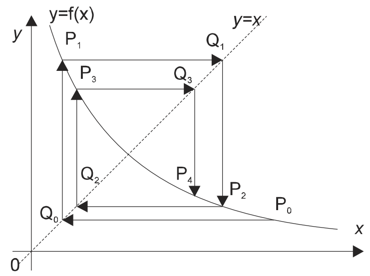

Fig. 4. *Picard* sequence converging to a fixed point

When the slope condition is not met, then the points A_1, $A_2, \ldots$ may move away from a fixed point. This case is depicted in Figure 5.

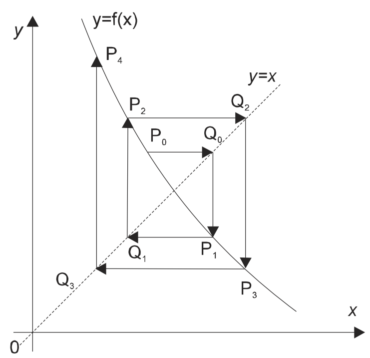

Fig. 5. *Picard* sequence diverging away from a fixed point

In mathematical economics, the behaviour described in Figure 4 corresponds to the convergence to an equilibrium point, while the framework described in Figure 5 describes the divergence from equilibrium.

A sufficient condition for the convergence of a *Picard* sequence, which is a formal analogue of the geometric condition of slopes mentioned above, is stated in the following result, which is also referred as the *Picard convergence theorem*.

Theorem 5. *Let $f : [a, b] \to [a, b]$ be a continuous function which is differentiable on (a, b), with $|f'(x)| < 1$ for all $x \in (a, b)$.*

Then f has a unique fixed point. Moreover, any Picard sequence for f is convergent and converges to the unique fixed point of f.

Proof. We first observe that the *Brouwer* fixed point theorem implies that f has at least one fixed point. Next, assuming that f has two fixed points x_* and x^*, the *Lagrange* mean value theorem implies that there exists $\xi \in (a, b)$ such that

$$|x_* - x^*| = |f(x_*) - f(x^*)| = |f'(\xi)| \cdot |x_* - x^*| < |x_* - x^*| ,$$

a contradiction. Thus, f has a unique fixed point.

We point out that the condition $|f'(x)| < 1$ for all $x \in (a, b)$ is essential for the uniqueness of a fixed point. For example, if $f : [a, b] \to [a, b]$ is defined by $f(x) = x$, then $f'(x) = 1$ for all $x \in [a, b]$ and every point of $[a, b]$ is a fixed point of f.

We prove in what follows that the corresponding *Picard* sequence converges. Let x^* denote the unique fixed point of f. Consider arbitrarily $x_1 \in [a, b]$ and let $(x_n)_{n\geq 1} \subset [a, b]$ be the *Picard* sequence for f with its initial point x_1. This means that $x_n = f(x_{n-1})$ for all $n \geq 2$. Fix an integer $n \geq 1$. Thus, by the *Lagrange* mean value theorem, there exists ξ_n between x_n and x^* such that

$$x_{n+1} - x^* = f(x_n) - f(x^*) = f'(\xi_n)(x_n - x^*) .$$

This implies that $|x_{n+1} - x^*| < |x_n - x^*|$. Next, we prove that $x_n \to x^*$ as $n \to \infty$. Since $(x_n)_{n\geq 1}$ is bounded, it suffices to show that every convergent subsequence of $(x_n)_{n\geq 1}$ converges to x^*. Let $x \in \mathbb{R}$ and $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$ converging to x. Then

$$|x_{n_{k+1}} - x^*| \leq |x_{n_k+1} - x^*| \leq |x_{n_k} - x^*| .$$

But $|x_{n_{k+1}} - x^*| \to |x - x^*|$ and

$$|x_{n_{k+1}} - x^*| = |f(x_{n_k}) - f(x^*)| \to |f(x) - f(x^*)|, \qquad \text{as } k \to \infty .$$

It follows that $|f(x) - f(x^*)| = |x - x^*|$. Now, if $x \neq x^*$, then by the *Lagrange* mean value theorem, there exists $\xi \in (a, b)$ such that

$$|x - x^*| = |f(x) - f(x^*)| = |f'(\xi)| \cdot |x - x^*| < |x - x^*| ,$$

which is a contradiction. This proves that $x \neq x^*$. $\square$

Remark. If the condition $|f'(x)| < 1$ for all $x \in (a,b)$ is not satisfied, then f can still have a unique fixed point x^* but the *Picard* sequence $(x_n)_{n\geq 1}$ with initial point $x_1 \neq x^*$ may not converge to x^*.

Example. Let $f : [-1,1] \to [-1,1]$, $f(x) = -x$. Then f is differentiable, $|f'(x)| = 1$ for all $x \in [-1,1]$, and $x^* = 0$ is the unique fixed point of f. If $x_1 \neq 0$ then the corresponding *Picard* sequence is $x_1, -x_1, x_1, -x_1, \ldots$, which oscillates between x_1 and $-x_1$ and never reaches the fixed point. In geometric terms, the cobweb that we hope to weave just traces out a square over and over again.

When the hypotheses of the *Picard* convergence theorem are satisfied, a *Picard* sequence for $f : [a,b] \to [a,b]$ with arbitrary $x_1 \in [a,b]$ as its initial point, converges to a fixed point of f. It is natural to expect that if x_1 is closer to the fixed point, then the convergence rate will be better. A fixed point of f lies not only in the range of f but also in the ranges of the iterates $f \circ f$, $f \circ f \circ f$, and so on. Thus, if $\mathcal{R}_n$ is the range of the n–fold composite $f \circ \cdots \circ f$ (n times), then a fixed point is in each $\mathcal{R}_n$. If only a single point belongs to $\bigcap\limits_{n=1}^{\infty} \mathcal{R}_n$, then we have found our fixed point. In fact, the *Picard* method amounts to starting with any $x_1 \in [a,b]$ and considering the image of x_1 under the n–fold composite $f \circ \cdots \circ f$.

Example. If $f : [0,1] \to [0,1]$ is defined by $f(x) = \frac{x+1}{4}$, then the nth iterate of f is given by

$$f \circ \cdots \circ f(n \text{ times})(x) = \frac{3x + 4^n - 1}{3 \cdot 4^n}$$

and

$$\mathcal{R}_n = \left[\frac{1}{3}\left(1 - \frac{1}{4^n}\right), \frac{1}{3}\left(1 + \frac{2}{4^n}\right)\right].$$

Thus, $\bigcap\limits_{n=1}^{\infty} \mathcal{R}_n = \{\frac{1}{3}\}$, hence $\frac{1}{3}$ is the unique fixed point of f. In general, it is not convenient to determine the ranges $\mathcal{R}_n$ for all n. So, it is simpler to use the *Picard* method, but this tool will be more effective if the above observations are used to some extent in choosing the initial point.

The *Picard* convergence theorem was extended in [5] to a framework arising frequently in mathematical economics. This corresponds to the *cobweb model* that concerns a qualitative analysis of markets in which supply adjustments have a time lag and demand adjustments occur with no delay. We briefly describe in what follows the cobweb model and we conclude with the *cobweb theorem*, which is a generalization of Theorem 5. Let $s(p)$ denote the total quantity of the product that sellers are willing to supply at a given price level $p > 0$. Assume the demand function $d(p)$ represent the total quantity of the product that buyers are willing to purchase at a given price level p. The situation described in the next result corresponds to price converging to an equilibrium price and it is described by economists as a "stable equilibrium". This means that if a small extraneous disturbance occurs in the market, eventually price will again converges to some equilibrium price. The same result shows that disturbance should be be large enough to remove the equilibrium. Such a disturbance might be a depression, drought, or large recession.

Theorem 6. *(Cobweb Theorem) Let s and d be real-valued functions of the real variable $p > 0$, and suppose that the graphs of s and d intersect at the point (p^*, q^*) where $q^* > 0$. Let I be a closed interval centered at p^* on which functions s and d have nonvanishing continuous derivatives. Define sequences (p_n) and (q_n) by letting p_0 be any element of I, $q_n = s(p_{n-1})$ and $p_n = d^{-1}(q_n)$ for all $n \geq 1$. Assume that $|s'(p)| < |d'(p)|$ for all p in I. Then $\lim_{n\to\infty} p_n = p^*$ and $\lim_{n\to\infty} q_n = q^*$.*

The proof of Theorem 6 relies on the *Cauchy* mean value theorem; we refer to [5] for details and related properties.

Acknowledgments. The authors are greatly indebted to Professor *Jean Mawhin* for his interest in this work and for his numerous remarks, which have considerably improved the initial version of this paper.

References

[1] H.G. Barone, Limit points of sequences and their transforms by methods of summability, *Duke Math. J.* **5** (1939). 740–752.

[2] M.W. Botsko, Problem Q982, *Mathematics Magazine* **81** (2008), p. 304.

[3] H. Brezis, F. Browder, Partial differential equations in the 20th century, *Adv. Math.* **135** (1998), 76–144.

[4] L.E.J. Brouwer, Ueber Abbildungen von Mannigfaltigkeiten, *Math. Ann.* **71** (1912), 97–115.

[5] E.H. Davis, Cauchy and economic cobwebs, *Mathematics Magazine* **53** (1980), 171–173.

[6] M. Edelstein, A remark on a theorem of M. A. Krasnoselski, *Amer. Math. Monthly* **73** (1966), 509–510.

[7] B.P. Hillam, A characterization of the convergence of successive approximations, *Amer. Math. Monthly* **83** (1976), p. 273.

[8] M.A. Krasnoselski, Two remarks on the method of successive approximations, *Math. Nauk (N. S.)* **10** (1955), 123–127.

[9] S. Lefschetz, Existence of periodic solutions for certain differential equations, *Proc. Nat. Acad. Sci.* **29** (1943), 1–4.

[10] N. Levinson, On the existence of periodic solutions for second order differential equations with a forcing term, *J. Math. Phys.* **22** (1943), 41–48.

[11] J. Mawhin, Topological fixed point theory and nonlinear differential equations, in *Handbook of Topological Fixed Point Theory* (R.F. Brown, M. Furi, L. Górniewicz, B. Jiang, Eds.), pp. 867–904, Springer, Dordrecht, 2005.

[12] É. Picard, *Traité d'Analyse*, Gauthier-Villars, Paris, 1891.

[13] H. Poincaré, Sur certaines solutions particulières du problème de trois corps, *C.R. Acad. Sci. Paris Sér. I* **97** (1883), 251–252.

[14] H. Poincaré, Sur le problème de trois corps et les équations de la dynamique, *Acta Math.* **13** (1890), 1–270.

[15] M. Rădulescu, S. Rădulescu, V. Rădulescu: *Selected Problems in Mathematical Analysis*, Problem Books in Mathematics, Springer-Verlag, Heidelberg, in preparation.

[16] T.-L. Rădulescu, V. Rădulescu, T. Andreescu, *Problems in Real Analysis: Advanced Calculus on the Real Axis*, Springer, New York, 2009.