# THE INFLUENCE OF THE DISTANCE FUNCTION IN SOME SINGULAR ELLIPTIC PROBLEMS

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*This paper is dedicated to the memory of Aurel Cornea*

Abstract. We present some existence and nonexistence results for classical solutions to singular elliptic problems of the form

$$-\Delta u \pm \delta(x)^{-\alpha} u^{-\beta} = \lambda f(x,u) \quad \text{in } \Omega,$$

subject to homogeneous Dirichlet boundary condition. Here $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a smooth and bounded domain, $\delta(x)$ =dist$(x, \partial\Omega)$, $\alpha$, $\beta$ and $\lambda$ are positive real numbers, while $f$ has either a linear or a sublinear growth with respect to the second variable.

## 1. Introduction and the main results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a smooth boundary. We are concerned in this paper with singular elliptic problems of the following type

$$\begin{cases} -\Delta u \pm \delta(x)^{-\alpha} u^{-\beta} = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)^{\pm}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$, $\alpha, \beta, \lambda > 0$. Throughout this paper we suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and such that $f$ is positive on $\overline{\Omega} \times (0,\infty)$. The analysis we develop in this paper concerns the cases where $f$ is either linear or $f$ is sublinear with respect to the second variable. This last case means that $f$ fulfills the hypotheses

$$(f1) \qquad \text{the mapping } (0,\infty) \ni t \longmapsto \frac{f(x,t)}{t} \quad \text{is nonincreasing for all } x \in \overline{\Omega};$$

$$(f2) \qquad \lim_{t\to 0^+} \frac{f(x,t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to +\infty} \frac{f(x,t)}{t} = 0, \ \text{ uniformly for } x \in \overline{\Omega}.$$

Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics (such as the Langmuir–Hinshelwool model), in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids (we refer for more details to [5, 6, 8, 9, 11, 12] and the more recent papers [7, 14, 15, 17, 20, 21, 23, 24]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive

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solutions are relevant in most cases. For instance, problems of this type characterize some reaction-diffusion processes where $u \geq 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see Aris [3] for the study of a single, irreversible steady-state reaction). Nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see Callegari and Nachman [6] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence).

To the best of our knowledge, there does not exist a qualitative theory for the study of singular boundary value problems with nonlinearities in the Kato class $K_N^{\rm loc}(\mathbb{R}^N)$. This theory was introduced by Aizenman and Simon in [2] to describe wide classes of functions arising in Potential Theory. We refer to the recent paper [22] for existence and bifurcation results on Dirichlet boundary value problems with indefinite nonlinearities.

In this paper we aim to study the influence of the distance function $\delta(x)$ in such singular elliptic problems. We first establish the following result related to problem $(P_\lambda)^+$.

**Theorem 1.1.** *Assume that $f$ satisfies $(f1), (f2)$.*

(i) *If $\alpha+\beta \leq 1$ then $(P_\lambda)^+$ has no classical solutions;*

(ii) *If $\alpha+\beta > 1$, then there exists $\lambda^* > 0$ such that $(P_\lambda)^+$ has at least one classical solution for all $\lambda > \lambda^*$ and no solution exists if $0 < \lambda < \lambda^*$.*

In contrast to the results in Theorem 1.1, the study of $(P_\lambda)^-$ is quite different. The one dimensional case, namely the problem

$$\begin{cases} -H''(t) = \delta(t)^{-\alpha}H(t)^{-\beta}, H > 0 & \text{in } (0,1), \\ H(0) = H(1) = 0, \end{cases} \tag{1.1}$$

was discussed in [25] and [1, Section 2]. It has been shown that (1.1) has solutions if and only if $\alpha < 2$. We first obtain that condition $\alpha < 2$ is also necessary and sufficient to obtain a solution in higher dimension provided $f$ has a sublinear growth. More precisely we have:

**Theorem 1.2.** *Assume that $f$ satisfies $(f1), (f2)$.*

(i) *If $\alpha \geq 2$ then $(P_\lambda)^-$ has no classical solutions;*

(ii) *If $0 < \alpha < 2$, then for all $\lambda > 0$ problem $(P_\lambda)^-$ has al least one solution. Moreover, there exist $0 < \eta < 1$ and $C_1, C_2 > 0$ such that $u_\lambda$ satisfies*

(ii1) *If $\alpha+\beta > 1$, then*

$$C_1\delta(x)^{\frac{2-\alpha}{1+\beta}} \leq u_\lambda(x) \leq C_2\delta(x)^{\frac{2-\alpha}{1+\beta}}, \quad \textit{for all } x \in \Omega; \tag{1.2}$$

(ii2) *If $\alpha+\beta = 1$, then*

$$C_1 d(x)(-\ln\delta(x))^{\frac{1}{2-\alpha}} \leq u_\lambda(x) \leq C_2 d(x)(-\ln\delta(x))^{\frac{1}{2-\alpha}}, \tag{1.3}$$

*for all $x \in \Omega$ with $\delta(x) < \eta$;*

(ii3) *If $\alpha+\beta < 1$, then*

$$C_1\delta(x) \leq u_\lambda(x) \leq C_2\delta(x), \quad \textit{for all } x \in \Omega. \tag{1.4}$$

If $\alpha+\beta < 1$, we are able to obtain the uniqueness and to provide the regularity of solution to $(P_\lambda)^-$ by means of the associated Green function.

**Theorem 1.3.** *Assume that $f$ satisfies $(f1),(f2)$ and $\alpha+\beta<1$. Then, for all $\lambda>0$ problem $(P_\lambda)^-$ has a unique solution $u_\lambda$ which in addition satisfies*

(i) *$(0,\infty)\ni\lambda\longmapsto u_\lambda$ is increasing in $\Omega$;*

(ii) *$u_\lambda\in C^2(\Omega)\cap C^{1,1-\alpha-\beta}(\overline{\Omega})$.*

We now consider the case where $f$ is asymptotically linear. More precisely, we assume that $f$ satisfies $(f1)$ and

$$(f3)\qquad m:=\lim_{t\to 0^+}\frac{f(x,t)}{t}\in(0,\infty).$$

In case $\alpha+\beta<1$ we have the following result.

**Theorem 1.4.** *Assume that $\alpha+\beta<1$ and $f$ satisfies $(f1),(f3)$. Then, problem $(P_\lambda)^-$ has solutions if and only if $0<\lambda<\lambda_1/m$. Moreover, for all $0<\lambda<\lambda_1/m$, there exists a unique solution $u_\lambda$ of $(P_\lambda)^-$ such that*

(i) *$u_\lambda\in C^2(\Omega)\cap C^{1,1-\alpha-\beta}(\overline{\Omega})$;*

(ii) *$\lim_{\lambda\nearrow\lambda_1/m}u_\lambda=\infty$ uniformly on compact subsets of $\Omega$.*

The next sections contains the proofs of the above results.

## 2. Proof of Theorem 1.1

Several times in this paper we apply the following comparison result (we refer to [16, Lemma 2.1] for a complete proof).

*Lemma* 2.1. Let $\Phi:\Omega\times(0,\infty)\to\mathbb{R}$ be a Hölder continuous function such that the mapping $(0,\infty)\ni t\longmapsto\Phi(x,t)/t$ is strictly decreasing for each $x\in\Omega$. Assume that there exist $v$, $w\in C^2(\Omega)\cap C(\overline{\Omega})$ such that

(*a*) $\Delta w+\Phi(x,w)\le 0\le\Delta v+\Phi(x,v)$ in $\Omega$;

(*b*) $v,w>0$ in $\Omega$ and $v\le w$ on $\partial\Omega$;

(*c*) $\Delta v\in L^1(\Omega)$ or $\Delta w\in L^1(\Omega)$.

Then $v\le w$ in $\Omega$.

**Proof of Theorem 1.1.** (i) Assume that $\alpha+\beta\ge 1$ and there exists $\lambda>0$ such that $(P_\lambda)^+$ has a classical solution $u$ and let $C=\max_{\overline{\Omega}}\lambda f(u)>0$. Let also $v\in C^2(\overline{\Omega})$ be the unique solution of

$$(2.1)\qquad \begin{cases}-\Delta v=C & \text{in }\Omega,\\ v>0 & \text{in }\Omega,\\ v=0 & \text{on }\partial\Omega.\end{cases}$$

Moreover, there exist $c_1,c_2>0$ such that

$$(2.2)\qquad c_1d(x)\le v\le c_2d(x),\quad \text{for all } x\in\Omega.$$

By maximum principle, it follows that $u\le v$ in $\Omega$. Next we consider the perturbed problem

$$(2.3)\qquad \begin{cases}-\Delta u+(\delta(x)+\varepsilon)^{-\alpha}(u+\varepsilon)^{-\beta}=C & \text{in }\Omega,\\ u>0 & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{cases}$$

Then, $u$ and $v$ are, respectively, sub- and super-solution of (2.3). By standard arguments and elliptic regularity (see [18]), there exists $u_\varepsilon\in C^2(\overline{\Omega})$ a solution of

(2.3) such that $u \le u_\varepsilon \le v$ in $\Omega$. Integrating in (2.3) we obtain

$$-\int_\Omega \Delta u_\varepsilon dx + \int_\Omega (\delta(x)+\varepsilon)^{-\alpha}(u_\varepsilon+\varepsilon)^{-\beta}dx = C|\Omega|.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} ds + \int_\Omega (\delta(x)+\varepsilon)^{-\alpha}(u_\varepsilon+\varepsilon)^{-\beta}dx \le M, \tag{2.4}$$

where $M$ is a positive constant. Taking into account that $\partial u_\varepsilon/\partial n \le 0$ on $\partial\Omega$ and $v \le u_\varepsilon$ in $\Omega$ we have $\displaystyle\int_\Omega (\delta(x)+\varepsilon)^{-\alpha}(u_\varepsilon+\varepsilon)^{-\beta}dx \le M$. Thus, for any compact subset $\omega \subset\subset \Omega$ we have

$$\int_\omega (\delta(x)+\varepsilon)^{-\alpha}(u_\varepsilon+\varepsilon)^{-\beta}dx \le M.$$

Passing to the limit with $\varepsilon \to 0^+$ we obtain $\displaystyle\int_\omega \delta(x)^{-\alpha}u_\varepsilon^{-\beta}dx \le M$, for all $\omega \subset\subset \Omega$. Therefore

$$\int_\Omega \delta(x)^{-\alpha}u_\varepsilon^{-\beta}dx \le M. \tag{2.5}$$

On the other hand, using (2.2) and the hypothesis $\alpha+\beta \ge 1$, it follows that

$$M \ge \int_\Omega \delta(x)^{-\alpha}u_\varepsilon(x)^{-\beta}dx \ge c\int_\Omega \delta(x)^{-\alpha-\beta}dx = +\infty,$$

which is a contradiction. Hence, problem $(P_\lambda)^+$ has no classical solutions and the proof of Theorem 1.1 is complete.

(ii) We first establish the existence of solutions to $(P_\lambda)^+$ for $\lambda > 0$ large. By virtue of [23, Lemma 2.4] (see also ([24, Theorem 2.2]), the problem

$$\begin{cases} -\Delta U = \lambda f(x,U) & \text{in } \Omega, \\ \;\; U > 0 & \text{in } \Omega, \\ \;\; U = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

has at least one classical solution $U_\lambda$, for all $\lambda > 0$ and $U_\lambda$ is a super-solution of $(P_\lambda)^+$. The main point is to find a sub-solution $\underline{u}_\lambda$ of $(P_\lambda)^+$ such that $\underline{u}_\lambda \le U_\lambda$ in $\Omega$. To this aim, let

$$\Psi : [0,\infty) \to [0,\infty), \quad \Psi(t) = \int_0^t \left[2\left(1 - \frac{1}{1-\alpha-\beta}s^{1-\alpha-\beta}\right)\right]^{-1/2} ds.$$

Since $\alpha+\beta<1$, $\Psi$ is well defined. Moreover, $\Psi$ is a bijective map. Let $h : [0,\infty) \to [0,\infty)$ be the inverse of $\Psi$. Then $h \in C^2(0,\infty) \cap C^1[0,\infty)$ and $h$ satisfies

$$\begin{cases} h''(t) = h(t)^{-\alpha-\beta}, h > 0 & \text{in } \; (0,\infty), \\ h(0) = h'(0) = 0. \end{cases} \tag{2.7}$$

The following technical result provides a suitable sub-solution for our problem.

*Lemma* 2.2. (see [13]) There exist two positive constants $c > 0$ and $M > 0$ such that $\underline{u}_\lambda := Mh(c\varphi_1)$ is a sub-solution of $(P_\lambda)^+$ provided $\lambda > 0$ is large enough.

Using Lemma 2.1, it follows that $\underline{u}_\lambda \le U_\lambda$ in $\Omega$ and by standard elliptic arguments (see [18]) we obtain a classical solution $u_\lambda$ of $(P_\lambda)^+$ such that $\underline{u}_\lambda \le u_\lambda \le U_\lambda$ in $\Omega$.

Next we prove that $(P_\lambda)^+$ has no solutions for $\lambda > 0$ small. since $f$ satisfies $(f1)-(f2)$, we can find $m > 0$ such that

$$f(x,t) - p(d(x))g(t) < mt, \qquad \text{for all } (x,t) \in \Omega \times (0,+\infty). \tag{2.8}$$

Set $\lambda_0 = \min\{1, \lambda_1/2m\}$. We claim that problem $(P_\lambda)^+$ has no classical solution for $0 < \lambda \le \lambda_0$. Indeed, assume by contradiction that $u_0$ is a classical solution of $(P_\lambda)^+$ with $\lambda \in (0,\lambda_0]$. Then, according to (2.8), $u_0$ is a sub-solution of

$$\begin{cases} -\Delta v = \dfrac{\lambda_1}{2} v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

By Lemma 2.1 we have $u_0 \le U_\lambda$ in $\Omega$ which yields $cu_0 \le \varphi_1$ in $\Omega$ for some positive constant $c > 0$. Note that $cu_0$ is still a sub-solution of (2.9) while $\varphi_1$ is a super-solution of (2.9). By standard elliptic arguments, it follows that problem (2.9) has a solution $v \in C^2(\overline{\Omega})$. Multiplying by $\varphi_1$ in (2.9) and integrating on $\Omega$ we have

$$-\int_\Omega \varphi_1 \Delta v dx = \frac{\lambda_1}{2} \int_\Omega v\varphi_1 dx,$$

that is,

$$\lambda_1 \int_\Omega v\varphi_1 dx = -\int_\Omega v\Delta\varphi_1 dx = \frac{\lambda_1}{2}\int_\Omega v\varphi_1 dx.$$

The above equality yields $\int_\Omega v\varphi_1 dx = 0$, but this is clearly a contradiction, since $v$ and $\varphi_1$ are both positive in $\Omega$. It follows that $(P_\lambda)^+$ has no classical solutions for $0 < \lambda \le \lambda_0$.

Next we define

$$A = \left\{\lambda > 0;\ \text{problem } (P_\lambda)^+ \text{ has at least one classical solution}\right\}.$$

From the above arguments we deduce that $A$ is nonempty and $\lambda^* := \inf A$ is positive. We only need to show that if $\lambda \in A$, then $(\lambda,\infty) \subseteq A$ but this follows by the sub and super-solution method. Hence $(\lambda^*,\infty) \subseteq A \subseteq [\lambda^*,\infty)$ and the proof is now complete. □

## 3. Proof of Theorem 1.2

(i) We proceed similarly as in the proof of Theorem 1.1.

(ii) Remark first that for all $\lambda > 0$, the solution $U_\lambda$ of (2.6) is a sub-solution for $(P_\lambda)^-$. However, we need a suitable sub-solution for $(P_\lambda)^-$ that will provide the asymptotic behavior for $u_\lambda$ as describet in (1.2)-(1.4).

*Lemma* 3.1. (see [13]) Assume $0 < \alpha < 2$ and let $H$ be the solution of (1.1). Then, for all $\lambda > 0$ there exist positive constants $C, c > 0$ and $M, m > 0$ (depending on $\lambda$) such that $\underline{u}_\lambda := mH(c\varphi_1)$, $\overline{u}_\lambda := MH(C\varphi_1)$ is a sub-solution and a super-solution of $(P_\lambda)^-$ such that $\underline{u}_\lambda \le \overline{u}_\lambda$ in $\Omega$..

Hence, for all $\lambda > 0$ there exists an ordered pair of sub- and super-solution of $(P_\lambda)^-$. By standard elliptic arguments, problem $(P_\lambda)^-$ has at least one solution $u_\lambda$ such that

$$mH(c\varphi_1) \le u_\lambda \le MH(C\varphi_1) \quad \text{in } \Omega. \tag{3.1}$$

(ii1) Remark that

$$H(t) = \left(\frac{(1+\beta)^2}{(2-\alpha)(\alpha+\beta-1)}\right)^{1/(1+\beta)} t^{\frac{2-\alpha}{1+\beta}}, \qquad t > 0,$$

is a solution of (1.1) provided $\alpha + \beta > 1$. The conclusion in this case follows now from (3.1).

(ii2) Since $H$ is concave, there exists $H'(0^+) \in (0, \infty]$. Taking $0 < b < 1$ small enough, we can assume that $H' > 0$ on $(0, b]$. Hence $H$ satisfies

$$\begin{cases} H''(t) = -t^{-\alpha} H^{\alpha-1}(t), & \text{for all } 0 < t \le b < 1, \\ H(0) = 0, \\ H, H' > 0 \quad \text{in } (0, b]. \end{cases} \tag{3.2}$$

Since $H$ is concave, it follows that

$$H(t) > tH'(t), \quad \text{for all } \ 0 < t \le b. \tag{3.3}$$

Relations (3.2) and (3.3) yield

$$-H''(t) < t^{-1}(H'(t))^{\alpha-1}, \quad \text{for all } \ 0 < t \le b.$$

Hence

$$-H''(t)(H'(t))^{1-\alpha} \le \frac{1}{t}, \quad \text{for all } \ 0 < t \le b. \tag{3.4}$$

Integrating in (3.4) over $[t, b]$ we get

$$(H')^{2-\alpha}(t) - (H')^{2-\alpha}(b) \le (2-\alpha)(\ln b - \ln t), \quad \text{for all } \ 0 < t \le b.$$

Hence, there exist $c_1 > 0$ and $\delta_1 \in (0, b)$ such that

$$H'(t) \le c_1(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } \ 0 < t \le \delta_1. \tag{3.5}$$

Fix $t \in (0, \delta_1]$. Integrating over $[\varepsilon, t]$, $0 < \varepsilon < t$, in (3.5) we have

$$H(t) - H(\varepsilon) \le c_1 t(-\ln t)^{\frac{1}{2-\alpha}} + \frac{c_1}{2-\alpha}\int_\varepsilon^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}} ds. \tag{3.6}$$

Note that

$$\int_0^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}} ds < +\infty \qquad \text{and} \qquad \lim_{t\to 0^+} \frac{\int_0^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}} ds}{t(-\ln t)^{\frac{1}{2-\alpha}}} = 0. \tag{3.7}$$

Therefore, taking $\varepsilon \to 0^+$ in (3.6) we deduce that there exist $c_2 > 0$ and $\delta_2 \in (0, \delta_1)$ such that

$$H(t) \le c_2 t(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } \ 0 < t \le \delta_2. \tag{3.8}$$

From (3.2) and (3.8) we obtain

$$-H''(t) \ge c_2^{\alpha-1} t^{-1}(-\ln t)^{\frac{\alpha-1}{2-\alpha}}, \quad \text{for all } \ 0 < t \le \delta_2.$$

Integrating over $[t, \delta_2]$ in the above inequality we get

$$H'(t) \ge (2-\alpha)c_2^{\alpha-1}\left[(-\ln t)^{\frac{1}{2-\alpha}} - (-\ln \delta_2)^{\frac{1}{2-\alpha}}\right], \quad \text{for all } \ 0 < t \le \delta_2.$$

Therefore, there exist $c_3 > 0$ and $\delta_3 \in (0, \delta_2)$ such that

$$H'(t) \geq c_3(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } 0 < t \leq \delta_3.$$

With the same arguments as in (3.5)-(3.8) we obtain $c_4 > 0$ and $\delta_4 \in (0, \delta_3)$ such that

$$H(t) \geq c_4 t(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } 0 < t \leq \delta_4. \tag{3.9}$$

The conclusion of (ii2) in Theorem 1.2 follows now from (3.8) and (3.9).

(ii3) Using the fact that $H'(0+) \in (0, \infty]$ and the inequality (3.3), we get the existence of $c > 0$ such that

$$H(t) > ct, \quad \text{for all } 0 < t \leq b.$$

This yields

$$-H''(t) \leq c^{-\beta} t^{-(\alpha+\beta)}, \quad \text{for all } 0 < t \leq b.$$

Since $\alpha + \beta < 1$, it follows that $H'(0+) < +\infty$, that is, $H \in C^1[0, b]$. Thus, there exists $c_1, c_2 > 0$ such that

$$c_1 t \leq H(t) \leq c_2 t, \quad \text{for all } 0 < t \leq b. \tag{3.10}$$

The conclusion in Theorem 1.2 (ii3) is now immediately by (3.10). This finishes the proof of Theorem 1.2. □

## 4. Proof of Theorem 1.3

(i) From Theorem 1.2, for all $\lambda > 0$ there exists a solution $u_\lambda$ of $(P_\lambda)^-$ such that (1.4) holds. Since $\alpha + \beta < 1$ this implies

$$0 \leq -\Delta u_\lambda = \delta(x)^{-\alpha} u_\lambda^{-\alpha+\beta} + \lambda f(x, u_\lambda) \leq C\delta(x)^{-\alpha-\beta} \in L^1(\Omega).$$

Therefore, by Lemma 2.1 we derive that $u_\lambda$ is the unique solution of $(P_\lambda)^-$.

(ii) Let $0 < \lambda_1 < \lambda_2$ and $u_{\lambda_1}$, $u_{\lambda_2}$ be the corresponding solutions to $(P_{\lambda_1})^-$ and $(P_{\lambda_2})^-$ respectively. From the above arguments we have that $\Delta u_{\lambda_i} \in L^1(\Omega)$, $i = 1, 2$. We only have to apply Lemma 2.1 for the mapping $\Phi(x, t) = \delta(x)^{-\alpha} t^{-\beta} + \lambda_1 f(x, t)$, $(x, t) \in \Omega \times (0, \infty)$ in order to deduce $u_{\lambda_1} \leq u_{\lambda_2}$ in $\Omega$. By maximum principle in follows that $u_{\lambda_1} < u_{\lambda_2}$ in $\Omega$.

(iii) For the proof of regularity we use a similar method to that developed by Gui and Lin [19]. Let $\mathcal{G}$ be the Green function associated with the Laplace operator in $\Omega$. We shall use the following estimates, which are due to Widman [26].

*Lemma* 4.1. There exists a positive constant $c > 0$ such that for all $x, y \in \Omega$, $x \neq y$ we have

(i) $|\mathcal{G}_x(x, y)| \leq c \dfrac{\min\{|x-y|, d(y)\}}{|x-y|^N}$;

(ii) $|\mathcal{G}_{xx}(x, y)| \leq c \dfrac{\min\{|x-y|, d(y)\}}{|x-y|^{N+1}}$.

Then

$$u_\lambda(x) = -\int_\Omega \mathcal{G}(x, y)\Phi(y)dy \quad \text{for all } x \in \Omega,$$

where $\Phi(y) = \delta(y)^{-\alpha} u_\lambda(y)^{-\beta} + \lambda f(y, u_\lambda(y))$. Hence

$$\nabla u_\lambda(x) = -\int_\Omega \mathcal{G}_x(x, y)\Phi(y)dy \quad \text{for all } x \in \Omega. \tag{4.1}$$

A very useful tool in our approach is the following technical result.

*Lemma* 4.2. There exists $c_0 = c_0(\Omega) > 0$ and $\delta_0 = \delta_0(\Omega) > 0$ such that for all $x_1, x_2 \in \Omega$, $0 < |x_1 - x_2| < \delta_0$ there exists a $C^1$ path $\xi : [0,1] \to \Omega$ with the properties

(i) $\xi(0) = x_1$ and $\xi(1) = x_2$;
(ii) $|\xi'(t)| \le c_0|x_1 - x_2|$, for all $0 \le t \le 1$.

Fix $x_1, x_2 \in \Omega$, $0 < |x_1 - x_2| < \delta_0$ and let $\xi : [0,1] \to \Omega$ be the corresponding path in Lemma 4.2.

By (4.1) we have

$$\begin{aligned} |\nabla u_\lambda(x_1) - \nabla u_\lambda(x_2)| &\le \int_\Omega |\mathcal{G}_x(x_1,y) - \mathcal{G}_x(x_2,y)|\Phi(y)dy \\ &\le \underbrace{\int_{B_r(x_1)} |\mathcal{G}_x(x_1,y) - \mathcal{G}_x(x_2,y)|\Phi(y)dy}_{I} \\ &\quad + \underbrace{\int_{\Omega\setminus B_r(x_1)} |\mathcal{G}_x(x_1,y) - \mathcal{G}_x(x_2,y)|\Phi(y)dy}_{II}\ , \end{aligned} \tag{4.2}$$

where $r = (c_0+1)|x_1 - x_2|$ and $c_0$ is the constant appearing in Lemma 4.2.

Before evaluating $I$ and $II$, let us remark that there exists $c_1 > 0$ such that $\Phi(y) \le c_1\delta(y)^{-\alpha-\beta}$ for all $y \in \Omega$. Then

$$\begin{aligned} I &\le c_1 \int_{B_r(x_1)} |\mathcal{G}_x(x_1,y) - \mathcal{G}_x(x_2,y)|\delta(y)^{-\alpha-\beta}dy \\ &\le c_1 \int_{B_r(x_1)} |\mathcal{G}_x(x_1,y)|\delta(y)^{-\alpha-\beta}dy + c_1 \int_{B_R(x_2)} |\mathcal{G}_x(x_2,y)|\delta(y)^{-\alpha-\beta}dy, \end{aligned}$$

where $R = r + |x_1 - x_2|$. Let $y \in B_r(x_1)$.
If $d(y) \ge |x_1 - y|$ then, by Lemma 4.1 (i), we have

$$|\mathcal{G}_x(x_1,y)|\delta(y)^{-\alpha-\beta} \le c|x_1 - y|^{-N+1}\delta(y)^{-\alpha-\beta} \le c|x_1 - y|^{-N+1-\alpha-\beta}.$$

If $\delta(y) < |x_1 - y|$ then, by Lemma 4.1 (i), we obtain

$$|\mathcal{G}_x(x_1,y)|\delta(y)^{-\alpha-\beta} \le c|x_1 - y|^{-N}\delta(y)^{1-\alpha-\beta} \le c|x_1 - y|^{-N+1-\alpha-\beta}.$$

Therefore, for all $y \in B_r(x_1)$ we have

$$|\mathcal{G}_x(x_1,y)|\delta^{-lpha-\beta}(y) \le c|x_1 - y|^{-N+1-\alpha-\beta},$$

and similarly

$$|\mathcal{G}_x(x_2,y)|\delta(y)^{-\alpha-\beta} \le c|x_2 - y|^{-N+1-\alpha-\beta} \quad \text{for all } y \in B_R(x_2).$$

Hence,

$$\begin{aligned} I &\le c_2 \int_{B_r(x_1)} |x_1 - y|^{-N+1-\alpha-\beta}dy + c_2 \int_{B_R(x_2)} |x_2 - y|^{-N+1-\alpha-\beta}dy \\ &\le c_3 \int_0^R t^{-\alpha-\beta}dt \le c_4|x_1 - x_2|^{1-\alpha-\beta}. \end{aligned} \tag{4.3}$$

To evaluate $II$, we first apply the mean value theorem. We have

$$\begin{aligned} II &\le c_1 \int_{\Omega\setminus B_r(x_1)} |\mathcal{G}_x(\xi(0),y) - \mathcal{G}_x(\xi(1),y)|\delta(y)^{-\alpha-\beta}dy \\ &\le c_1 \int_{\Omega\setminus B_r(x_1)} \int_0^1 |\mathcal{G}_{xx}(\xi(t),y)||\xi'(t)|\delta(y)^{-\alpha-\beta}dtdy \\ &\le c_5|x_1-x_2| \int_{\Omega\setminus B_r(x_1)} \int_0^1 |\mathcal{G}_{xx}(\xi(t),y)|\delta(y)^{-\alpha-\beta}dtdy. \end{aligned}$$

As shown earlier, by Lemma 4.1 (ii) we obtain

$$|\mathcal{G}_x(\xi(t),y)|\delta(y)^{-\alpha-\beta} \le c|\xi(t)-y|^{-N-\alpha-\beta} \quad \text{for all } y \in B_r(x_1) \text{ and } 0 \le t \le 1.$$

Let $c_6 = 1/(1+c_0)$, where $c_0$ is the constant from Lemma 4.2 that depends only on $\Omega$ and not on $x_1, x_2$. Then for all $y \in \Omega \setminus B_r(x_1)$,

$$\begin{aligned} |\xi(t)-y| &\ge |x_1-y| - |\xi(t)-x_1| = |x_1-y| - |\xi(t)-\xi(0)| \\ &\ge |x_1-y| - t|\xi'(c_t)| \ge |x_1-y| - c_0|x_1-x_2| \\ &\ge c_6|x_1-y|. \end{aligned}$$

Combining the last two estimates we obtain

$$|\mathcal{G}_x(\xi(t),y)|d^{-\alpha-\beta}(y) \le c_7|x_1-y|^{-N-\alpha-\beta} \quad \text{for all } y \in B_r(x_1) \text{ and } 0 \le t \le 1.$$

Hence, we may write

$$\begin{aligned} II &\le c_7|x_1-x_2| \int_{\Omega\setminus B_r(x_1)} |x_1-y|^{-N-\alpha-\beta}dy \\ &\le c_7|x_1-x_2| \int_r^\infty t^{-1-\alpha-\beta}dt \le c_8|x_1-x_2|r^{-\alpha-\beta} \\ &\le c_9|x_1-x_2|^{1-\alpha-\beta}. \end{aligned} \tag{4.4}$$

The conclusion follows now from (4.2), (4.3), and (4.4). The proof of Theorem 1.3 is now complete. □

## 5. Proof of Theorem 1.4

Let $\lambda^* = \lambda_1/m$. We first prove that $(P_\lambda)^-$ has no classical solutions for $\lambda \ge \lambda^*$. Indeed, assume that there exists $\lambda \ge \lambda^*$ such that $(P_\lambda)^-$ has a solution $u$ and let $c = \max_{\overline{\Omega}} \delta(x)^{-\alpha}u(x)^{-\beta}$. Since the mapping $t \longmapsto f(x,t)/t$ is nonincreasing, it follows that $\lambda f(x,u) \ge \lambda_1 u$ in $\Omega$. Then, $u$ satisfies $-\Delta u \ge c + \lambda_1 u$ in $\Omega$. Multiplying by $\varphi_1$ in the last inequality and integrating by parts, we have

$$\lambda_1 \int_\Omega u\varphi_1 dx \ge \int_\Omega (c+\lambda_1 u)\varphi_1 dx,$$

which is clearly a contradiction.

Let $0 < \lambda < \lambda^*$. Since $\alpha+\beta < 1$, by [7, Theorem 2] there exists $v \in C^2(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v = v^{-\alpha-\beta} + \lambda f(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

and $c_1\delta(x) \le v \le c_2\delta(x)$ in $\Omega$, for some $c_1, c_2 > 0$. Therefore, $v$ satisfies

$$-\Delta v \ge c\delta(x)^{-\alpha}v^{-\beta} + \lambda f(v) \quad \text{in } \Omega, \tag{5.2}$$

where $c > 0$. Let now $M > 1$ large such that $M^{1+\beta}c > 1$. Since $Mf(v) \ge f(Mv)$ in $\Omega$, from (5.2) we deduce that $\overline{u}_\lambda := Mv$ satisfies

$$-\Delta\overline{u}_\lambda \ge \delta(x)^{-\alpha}\overline{u}_\lambda^{-\beta} + \lambda f(\overline{u}_\lambda) \quad \text{in } \Omega.$$

Hence $\overline{u}_\lambda$ is a super-solution of $(P_\lambda)^-$ and obviously $\overline{u}_\lambda \ge \underline{u}_\lambda$ in $\Omega$. Thus, $(P_\lambda)^-$ has at least one solution $u_\lambda$. Furthermore, from the last inequality we also have $u_\lambda \ge c\delta(x)$ in $\Omega$, for some $c > 0$. This implies that $\Delta u_\lambda \in L^1(\Omega)$ and by Lemma 2.1 we obtain the uniqueness. The regularity in (i) follows exactly in the same way as in Theorem 1.3. We now prove (ii).

Remark that $\{u_\lambda\}_{0<\lambda<\lambda^*}$ is a sequence of positive increasing super-harmonic functions. By [4, Theorem 3.7.3] it follows that

$$u^*(x) := \lim_{\lambda\nearrow\lambda^*} u_\lambda(x) \quad x \in \Omega,$$

is either identically $\infty$ or a superharmonic function. Assume by contradiction that $u^* \not\equiv \infty$, so that $u^*$ is superharmonic. Hence, $u^* \in L^1_{\rm loc}(\Omega)$ (see, e.g., Theorem 3.1.3 in [4]).

We claim that $\{u_\lambda\}_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We argue by contradiction. Thus, passing eventually at a subsequence we have $u_\lambda = k(\lambda)w_\lambda$, where

$$k(\lambda) = \|u_\lambda\|_2 \to \infty \text{ as } \lambda \nearrow \lambda^* \text{ and } w_\lambda \in L^2(\Omega),\ \ \|w_\lambda\|_2 = 1. \tag{5.3}$$

Notice that $f(x,t) \le at + b$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$, where $a, b > 0$. This implies

$$\frac{1}{k(\lambda)}(\delta(x)^{-\alpha}u_\lambda^{-\beta} + \lambda f(x,u_\lambda)) \to 0 \quad \text{in } L^1_{\rm loc}(\Omega) \text{ as } \lambda \nearrow \lambda^*.$$

That is,

$$-\Delta w_\lambda \to 0 \quad \text{in } L^1_{\rm loc}(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{5.4}$$

By Green's first identity, for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla\phi\, dx = -\int_\Omega \phi\, \Delta w_\lambda dx = -\int_{\mathrm{Supp}\,\phi} \phi\, \Delta w_\lambda dx. \tag{5.5}$$

Using (5.4) we obtain

$$\begin{aligned}\left|\int_{\mathrm{Supp}\,\phi} \phi\, \Delta w_\lambda\, dx\right| &\le \int_{\mathrm{Supp}\,\phi} |\phi||\Delta w_\lambda|\, dx \\ &\le \|\phi\|_\infty \int_{\mathrm{Supp}\,\phi} |\Delta w_\lambda|\, dx \to 0 \quad \text{as } \lambda \nearrow \lambda^*.\end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) we derive that for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla\phi\, dx \to 0 \quad \text{as } \lambda \nearrow \lambda^*. \tag{5.7}$$

By definition, the sequence $(w_\lambda)_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We claim that $(w_\lambda)_{0<\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using Sobolev-Hardy and Hölder inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla w_\lambda|^2 dx &= -\int_\Omega w_\lambda \Delta w_\lambda dx \\
&= \frac{1}{k(\lambda)} \int_\Omega w_\lambda(\delta(x)^{-\alpha} u_\lambda^{-\beta} + \lambda f(x, u_\lambda))dx \\
&= \frac{1}{k(\lambda)^{1+\beta}} \int_\Omega \frac{w_\lambda^{1-\beta}}{\delta(x)^\alpha} dx + a\lambda \int_\Omega w_\lambda^2 dx + \frac{\lambda}{k(\lambda)} \int_\Omega w_\lambda dx \\
&\le \frac{1}{k(\lambda)^{1+\beta}} \left( \int_\Omega |\nabla w_\lambda|^2 dx \right)^{(1-\beta)/2} + a\lambda^* + \frac{\lambda^*}{k(\lambda)} |\Omega|^{1/2}.
\end{aligned}$$

From the above estimates it follows that $\{w_\lambda\}_{0<\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$, so the claim follows. Thus, there exists $w \in H_0^1(\Omega)$ such that up to a subsequence and as $\lambda \nearrow \lambda^*$ we have

$$\begin{aligned}
&w_\lambda \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \\
&w_\lambda \to w \quad \text{strongly in } L^2(\Omega).
\end{aligned} \tag{5.8}$$

On the one hand, by (5.3) and (5.8), we derive that $\|w\|_2 = 1$. Furthermore, using (5.7) and (5.8), we infer that

$$\int_\Omega \nabla w \cdot \nabla \phi \, dx = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Because $w \in H_0^1(\Omega)$, using the previous relation and the definition of $H_0^1(\Omega)$, we find $w = 0$. This contradiction shows that $\{u_\lambda\}_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. As noted earlier for $\{w_\lambda\}_{0<\lambda<\lambda^*}$, we derive that $\{u_\lambda\}_{0<\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Hence, there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence and as $\lambda \nearrow \lambda^*$, there holds

$$\begin{aligned}
&u_\lambda \rightharpoonup u^* \quad \text{weakly in } H_0^1(\Omega), \\
&u_\lambda \to u^* \quad \text{strongly in } L^2(\Omega), \\
&u_\lambda \to u^* \quad \text{almost everywhere in } \Omega.
\end{aligned} \tag{5.9}$$

Now we obtain the desired contradiction in the same manner as in the proof of (i)$-$(ii). By $(f1)$ we have $f(x, u_\lambda) \ge m u_\lambda$ in $\Omega$. We next multiply by $\varphi_1$ in $(P_\lambda)^-$ and then we integrate over $\Omega$. We obtain

$$\lambda_1 \int_\Omega u_\lambda \varphi_1 dx = \int_\Omega (\delta(x)^{-\alpha} u_\lambda^{-\beta} + \lambda f(u_\lambda))\varphi_1 dx \ge \int_\Omega (\delta(x)^{-\alpha} u_\lambda^{-\beta} + \lambda m u_\lambda)\varphi_1 dx, \tag{5.10}$$

for all $0 < \lambda < \lambda^*$. Passing to the limit in (5.10) with $\lambda \nearrow \lambda^*$, by virtue of Lebesgue's theorem on dominated convergence we find

$$\lambda_1 \int_\Omega u^* \varphi_1 = \int_\Omega (\delta(x)^{-\alpha} u^{*\,-\beta} + \lambda_1 u^*)\varphi_1 dx,$$

which is a contradiction. Hence $u^* \equiv \infty$. It is easy now to see, using the monotonicity of $\{u_\lambda\}_{0<\lambda<\lambda^*}$ that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = \infty$ uniformly on compact subsets of $\Omega$, and the proof is now complete. □

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