Research Article

Nikolaos S. Papageorgiou and Vicenţiu D. Rădulescu

Multiplicity theorems for semilinear Robin problems

Abstract: We consider a semilinear Robin problem driven by the Laplacian with a reaction which does not satisfy a global growth condition, only a local one. Using variational methods coupled with truncation and perturbation techniques and Morse theory, we prove two multiplicity theorems producing four and three nontrivial solutions respectively, all with precise sign. Also, we show that our results incorporate as a special case a semilinear parametric problem which has been considered primarily in the context of Dirichlet problems.

Keywords: Robin boundary conditions, Green's identity, nodal solutions, maximum principle, local minimizers, critical groups

MSC 2010: Primary 35J20, 35J60; secondary 58E05

Nikolaos S. Papageorgiou: National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece, e-mail: npapg@math.ntua.gr
Vicenţiu D. Rădulescu: Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia, e-mail: vicentiu.radulescu@math.cnrs.fr

Communicated by: Frank Duzaar

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem:

$$-\Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}(z) + \beta(z)u(z) = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

In this problem, $n(\cdot)$ is the outward unit normal on $\partial\Omega$ and the reaction $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). The interesting feature of our analysis of problem (1.1) is that on the nonlinearity $x \mapsto f(z,x)$ we do not impose any global growth condition. Instead, we assume that $f(z,\cdot)$ admits z-dependent zeros of constant sign. Using variational methods coupled with suitable truncation and perturbation techniques and Morse theory (critical groups), we prove a multiplicity theorem producing four nontrivial solutions, all with sign information (two of constant sign and two nodal (sign changing)). A second multiplicity theorem establishing three nontrivial solutions is also proved.

Recently a similar problem was investigated by Zhang, Li and Xue [20], who deal with a semilinear Robin problem driven by the differential operator $u \mapsto -\Delta u + \alpha u$ with $\alpha > 0$. Therefore their differentiable operator is coercive and this makes easier the use of the direct method. Also their reaction f is z-independent (autonomous) and the zeros are constant functions. Their main multiplicity theorem (see [20, Theorem 1.1]) produces four nontrivial solutions, but without providing sign information for all of them. On the other hand, we should mention that in [20] it is assumed that $f \in C^1(\mathbb{R} \setminus \{0\})$ and so $f'(\cdot)$ can have jump discontinuities at $x = 0$. We should also mention the recent work of the authors [14] who studied a parametric Robin problem driven by the p-Laplacian. They proved that if $\hat{\lambda}_2 > 0$ is the second eigenvalue of the negative Robin p-Laplacian and $\lambda > \hat{\lambda}_2$ ($\lambda > 0$ being the parameter), then the problem has at least three nontrivial solutions and in the semilinear case ($p = 2$) four nontrivial solutions, all with sign information. Our framework here is more general and recovers as a special case the setting of [14] (see Section 5).

In the next section, for easy reference, we present the main mathematical tools which will be used in this paper.

2 Mathematical background

We start with critical point theory. Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the Palais–Smale condition (PS-condition for short) if the following holds:

Palais–Smale Condition. Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(x_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This compactness-type condition on the functional φ compensates for the fact that the ambient space is not in general locally compact. It leads to a deformation theorem, from which we can derive the minimax theory for the critical values of φ. One result in this direction is the so-called "Mountain Pass Theorem".

Theorem 2.1. *Assume that X is a Banach space. Let $\varphi \in C^1(X)$ satisfy the PS-condition, and let $x_0, x_1 \in X$ with $\|x_1 - x_0\| > r > 0$ and $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = m_r$. Set*

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)) \quad \text{where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}.$$

Then $c \geqslant m_r$ and c is a critical value of φ.

In the analysis of problem (1.1), in addition to the Sobolev space $H^1(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$, which is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In what follows, by $\|\cdot\|$ we denote the norm of $H^1(\Omega)$. We have

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

Let $\sigma(\cdot)$ denote the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$. Then we can consider the Lebesgue space $L^2(\partial\Omega)$ and the Sobolev space of fractional order $H^{\frac{1}{2},2}(\partial\Omega)$. From the trace theory, we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the trace map, such that

$$\operatorname{range} \gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker \gamma_0 = H_0^1(\Omega).$$

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$|f_0(z,x)| \leqslant a(z)(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}$$

with $a \in L^\infty(\Omega)_+$ and

$$1 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geqslant 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\psi_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_0(u) = \frac{1}{2}\|Du\|_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u(z)^2 d\sigma - \int_\Omega F_0(z, u(z))dz \quad \text{for all } u \in H^1(\Omega).$$

Hereafter, we assume the following for the boundary weight function $\beta(\cdot)$:

Hypothesis $H(\beta)$. We have $\beta \in C^{0,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$ and $\beta(z) \geqslant 0$ for all $z \in \overline{\Omega}$, $\beta \neq 0$.

The next result is a special case of a result in Papageorgiou and Rădulescu [14]. It is an outgrowth of the regularity theory for Robin problems (see Lieberman [12]).

Proposition 2.2. *Assume that hypotheses $H(\beta)$ holds and $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of ψ_0, that is, there exists $\varrho_0 > 0$ such that*

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leqslant \varrho_0.$$

Then $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$ and u_0 is a local $H^1(\Omega)$-minimizer of ψ_0, that is, there exists $\varrho_1 > 0$ such that

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h) \quad \text{for all } h \in H^1(\Omega) \text{ with } \|h\| \leqslant \varrho_1.$$

Remark 1. We should point out that the first such result relating local minimizers was proved for the space $H_0^1(\Omega)$ by Brezis and Nirenberg [6].

Next we recall some basic facts about the spectrum of the negative Robin Laplacian. So, we consider the following linear eigenvalue problem:

$$-\Delta u(z) = \hat{\lambda} u(z) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}(z) + \beta(z)u(z) = 0 \quad \text{on } \partial\Omega.$$

From the spectral theory for compact self-adjoint operators, we know that this eigenvalue problem has a sequence of eigenvalues $\{\hat{\lambda}_k\}_{k\geqslant 1}$ such that $\hat{\lambda}_k \to +\infty$ as $k \to \infty$ and this sequence exhausts the spectrum of the negative Robin Laplacian.

We know that $\hat{\lambda}_1 > 0$ and it is simple (that is, the corresponding eigenspace is one-dimensional),

$$\hat{\lambda}_1 = \inf\left\{\frac{\|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2 d\sigma}{\|u\|_2^2} : u \in H^1(\Omega),\ u \neq 0\right\}. \tag{2.1}$$

The infimum in (2.1) is realized on the one-dimensional eigenspace $E(\hat{\lambda}_1)$ corresponding to $\hat{\lambda}_1 > 0$. It is clear from (2.1) that the elements of this eigenspace do not change sign. By $\hat{u}_1$ we denote the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) nonnegative eigenfunction corresponding to $\hat{\lambda}_1 > 0$. The regularity theory of Lieberman [12] and the maximum principle of Vazquez [18] imply $\hat{u}_1 \in \operatorname{int} C_+$. We mention that $\hat{\lambda}_1 > 0$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal eigenfunctions.

For every integer $k \geqslant 1$, by $E(\hat{\lambda}_k)$ we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. From the regularity theory (see Lieberman [12]), we have

$$E(\hat{\lambda}_k) \subseteq C^1(\overline{\Omega}) \quad \text{for all } k \geqslant 1.$$

Moreover, these spaces have the so-called unique continuation property (UCP for short), namely, if $u \in E(\hat{\lambda}_k)$ vanishes on a set of positive Lebesgue measure, then $u(z) = 0$ for all $z \in \overline{\Omega}$. We set

$$\overline{H}_m = \bigoplus_{k=1}^{m} E(\hat{\lambda}_k) \quad \text{and} \quad \hat{H}_m = \overline{H}_m^{\perp} = \overline{\bigoplus_{k\geqslant m+1} E(\hat{\lambda}_k)}.$$

Using these spaces we have the following variational characterizations of the higher eigenvalues ($m \geqslant 2$):

$$\hat{\lambda}_m = \max\left\{\frac{\|D\overline{u}\|_2^2 + \int_{\partial\Omega} \beta(z)\overline{u}^2 d\sigma}{\|\overline{u}\|_2^2} : \overline{u} \in \overline{H}_m,\ \overline{u} \neq 0\right\} = \min\left\{\frac{\|D\hat{u}\|_2^2 + \int_{\partial\Omega} \beta(z)\hat{u}^2 d\sigma}{\|\hat{u}\|_2^2} : \hat{u} \in \hat{H}_{m-1},\ \hat{u} \neq 0\right\}. \tag{2.2}$$

Both the maximum and the minimum in (2.2) are realized on $E(\hat{\lambda}_m)$.

The next lemma is an easy consequence of the UCP (see also Gasinski and Papageorgiou [10]). In what follows, we set $\xi(u) = \|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u^2 d\sigma$ for all $u \in H^1(\Omega)$.

Lemma 2.3. *The following statements hold.*

(a) *If $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \leqslant \hat{\lambda}_{m+1}$ for a.a. $z \in \Omega$, $\vartheta \neq \hat{\lambda}_{m+1}$, $m \geqslant 0$, then there exists $\xi_0 > 0$ such that*

$$\xi(u) - \int_\Omega \vartheta(z)u^2 dz \geqslant \xi_0\|u\|^2 \quad \text{for all } u \in \hat{H}_m \ \ (\hat{H}_0 = H^1(\Omega)).$$

(b) *If $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \leqslant \hat{\lambda}_m$ for a.a. $z \in \Omega$, $\vartheta \neq \hat{\lambda}_m$, $m \geqslant 1$, then there exists $\xi_1 > 0$ such that*

$$\xi(u) - \int_\Omega \vartheta(z)u^2 dz \leqslant -\xi_1\|u\|^2 \quad \text{for all } u \in \overline{H}_m.$$

Let X be a Banach space and let $\varphi \in C^1(X)$, $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leqslant c\}, \quad K_\varphi = \{x \in X : \varphi'(x) = 0\} \quad \text{and} \quad K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$ by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Recall that $H_k(Y_1, Y_2) = 0$ for $k < 0$. The critical groups of φ at an isolated $x \in K_\varphi^c$ are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x\}) \quad \text{for all } k \geqslant 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all integers } k \geqslant 0.$$

The Second Deformation Theorem (see, for example, Gasinski and Papageorgiou [10, p. 628]) implies that the above definition of critical groups at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define

$$M(t, x) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, x) t^k \qquad \text{for all } t \in \mathbb{R} \text{ and all } x \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

Then the Morse relation says

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t), \tag{2.3}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Suppose that $X = H$ is a Hilbert space, $x \in H$, U is a neighborhood of x and $\varphi \in C^2(U)$. If $x \in K_\varphi$, then the Morse index of φ at x, denoted by $\mu = \mu(x)$, is defined as the supremum of the dimensions of the vector subspaces of H for which $\varphi''(x)$ is negative definite. The nullity of φ at x, denoted by $\nu = \nu(x)$, is defined to be the dimension of the subspace $\ker \varphi''(x)$.

Suppose that $H = V \oplus Y$ with $\dim V < +\infty$. We say that $\varphi \in C^1(X)$ has a "local linking at 0" with respect to the decomposition (V, Y) if we can find $\varrho > 0$ such that

$$\varphi(v) \leqslant 0 \quad \text{for all } v \in V \text{ with } \|v\| \leqslant \varrho,$$
$$\varphi(y) \geqslant 0 \quad \text{for all } y \in Y \text{ with } \|y\| \leqslant \varrho.$$

From [17, Proposition 2.3], we have the following result.

Proposition 2.4. *If H is a Hilbert space, U is a neighborhood of $u = 0$, $\varphi \in C^2(U)$, $u = 0$ is an isolated critical point of φ, $\varphi''(0)$ is a Fredholm operator and φ has a local linking at 0 with respect to the decomposition (V, Y) with $H = V \oplus Y$, $d = \dim V < +\infty$, then $C_k(\varphi, 0) = \delta_{k,\mu}\mathbb{Z}$ for all $k \geqslant 0$ if $d = \mu$ with μ the Morse index of φ at 0, and $C_k(\varphi, 0) = \delta_{k,\mu+\nu}\mathbb{Z}$ for all $k \geqslant 0$ if $d = \mu + \nu$ with ν the nullity of φ at 0.*

In the sequel, $A : H^1(\Omega) \to H^1(\Omega)^*$ denotes the bounded linear operator defined by

$$\langle A(u), y \rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N} \, dz \quad \text{for all } u, y \in H^1(\Omega).$$

Moreover, if $x \in \mathbb{R}$, then we set $x^\pm = \max\{0, \pm x\}$. For $u \in H^1(\Omega)$, we define

$$u^\pm(\cdot) = u(\cdot)^\pm.$$

We know that

$$u^\pm \in H^1(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also, given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in H^1(\Omega)$$

(the Nemytskii operator corresponding to $h(\cdot,\cdot)$). In what follows, by $\|\cdot\|_p$ we denote the norm of the Lebesgue space $L^p(\Omega)$ or of $L^p(\Omega, \mathbb{R}^N)$. Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3 Constant sign solutions

In this section we establish the existence of solutions constant sign for problem. To this end, we introduce the following conditions on the reaction $f(z, x)$:

Hypothesis H_1. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\varrho(z) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in [-\varrho, \varrho],$$

(ii) there exist functions $w_\pm \in H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{aligned} w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad & \text{for a.a. } z \in \Omega, \\ f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \quad & \text{a.e. in } \Omega, \\ A(w_-) + \beta(\cdot)u_- \leqslant 0 \leqslant A(w_+) + \beta(\cdot)u_+ \quad & \text{in } H^1(\Omega)^*, \end{aligned}$$

(iii) there exist an integer $m \geqslant 2$ and $\delta_0 > 0$ such that

$$\hat{\lambda}_m x^2 \leqslant f(z, x)x \quad \text{for a.a. } z \in \Omega \text{ and all } x \in [-\delta_0, \delta_0].$$

Remark 2. Note that the above hypotheses do not impose any growth restriction on $f(z,\cdot)$ near $\pm\infty$. Hypotheses H_1 (ii)–(iii) imply that near zero, $f(z,\cdot)$ exhibits an oscillatory behavior. Evidently, hypothesis H_1 (ii) is satisfied if we can find $\xi_- < 0 < \xi_+$ such that

$$f(z, \xi_+) \leqslant 0 \leqslant f(z, \xi_-) \quad \text{a.e. in } \Omega.$$

Hypothesis H_1 (iii) permits for resonance to occur asymptotically at zero but it also incorporates reactions which are concave near zero. Clearly, in H_1 (iii) we can always assume that $\delta_0 \in (0, \min\{c_+, -c_-, 1\})$. As we will see in the proof of the next proposition, hypotheses H_1 (i) and (iii) imply that given $\varrho > 0$, $r > 2$, we can find $\hat{\xi} > 0$ such that

$$f(z, x)x + \hat{\xi}x^r \geqslant 0 \quad \text{for a.a. } z \in \Omega \text{ and all } x \in [-\varrho, \varrho].$$

For the boundary weight function $\beta(\cdot)$, we keep the hypothesis $H(\beta)$ introduced in Section 2.

Proposition 3.1. *Assume that hypotheses $H(\beta)$ and H_1 hold. Then problem* (1.1) *admits at least two nontrivial constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof. First we produce a positive solution. To this end, we introduce the following truncation-perturbation of $f(z,\cdot)$:

$$g_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z, x) + x & \text{if } 0 \leqslant x \leqslant w_+(z), \\ f(z, w_+(z)) + w_+(z) & \text{if } w_+(z) < x. \end{cases} \tag{3.1}$$

This is a Carathéodory function. We set

$$G_+(z, x) = \int_0^x g_+(z, s)\,ds$$

and consider the C^1-functional $\varphi_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \int_\Omega G_+(z, u(z))dz \quad \text{for all } u \in H^1(\Omega),$$

where

$$\xi(u) = \|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u(z)^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

As before, for the sake of notational simplicity, we drop the use of the trace map γ_0 to denote the restriction of a Sobolev function on $\partial\Omega$.

From (3.1) it is clear that φ_+ is coercive. Also, using the Sobolev Embedding Theorem, we show easily that φ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass Theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$\varphi_+(u_0) = \inf\{\varphi_+(u) : u \in H^1(\Omega)\}. \tag{3.2}$$

Let $t \in (0, 1)$ be small such that $t\hat{u}_1(z) \in (0, \delta_0]$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_1 \in \operatorname{int} C_+$). We have

$$\begin{aligned} \varphi_+(t\hat{u}_1) &= \frac{t^2}{2}\xi(\hat{u}_1) - \int_\Omega F(z, t\hat{u}_1)dz && \text{(see (3.1))} \\ &\leqslant \frac{t^2}{2}[\hat{\lambda}_1 - \hat{\lambda}_m] && \text{(see } H_1 \text{ (iii) and recall } \|\hat{u}_1\|_2 = 1) \\ &< 0 && \text{(since } m \geqslant 2), \end{aligned}$$

which implies $\varphi_+(u_0) < 0 = \varphi_+(0)$ (see (3.2)), hence $u_0 \neq 0$. From (3.2) we have $\varphi_+'(u_0) = 0$, which shows

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z)u_0 h d\sigma + \int_\Omega u_0 h dz = \int_\Omega g_+(z, u_0)h dz \quad \text{for all } h \in H^1(\Omega). \tag{3.3}$$

In (3.3), first we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\|Du_0^-\|_2^2 + \|u_0^-\|_2^2 \leqslant 0 \quad \text{(see } H(\beta) \text{ and (3.1))},$$

hence $u_0 \geqslant 0$, $u_0 \neq 0$. Next, in (3.3) we choose $(u_0 - w_+)^+ \in H^1(\Omega)$. Then

$$\begin{aligned} &\langle A(u_0), (u_0 - w_+)^+\rangle + \int_{\partial\Omega} \beta(z)u_0(u_0 - w_+)^+ d\sigma + \int_\Omega u_0(u_0 - w_+)^+ dz \\ &\quad = \int_\Omega [f(z, w_+) + w_+](u_0 - w_+)^+ dz && \text{(see (3.1))} \\ &\quad \leqslant \langle A(w_+), (u_0 - w_+)^+\rangle + \int_{\partial\Omega} \beta(z)w_+(u_0 - w_+)^+ d\sigma + \int_\Omega w_+(u_0 - w_+)^+ dz && \text{(see } H_1 \text{ (ii))} \end{aligned}$$

and thus

$$\langle A(u_0 - w_+), (u_0 - w_+)^+\rangle + \int_\Omega [u_0 - w_+](u_0 - w_+)^+ dz \leqslant 0$$

showing $\|(u_0 - w_+)^+\|^2 \leqslant 0$, hence $u_0 \leqslant w_+$. So, we have proved that

$$u_0 \in [0, w_+] = \{u \in H^1(\Omega) : 0 \leqslant u(z) \leqslant w_+(z) \text{ for a.a. } z \in \Omega\}, \quad u_0 \neq 0. \tag{3.4}$$

Because of (3.1) and (3.4), (3.3) becomes

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z)u_0 h d\sigma = \int_\Omega f(z, u_0)h dz \quad \text{for all } h \in H^1(\Omega). \tag{3.5}$$

From Green's identity (see, for example, Gasinski and Papageorgiou [10, p. 211]), we have

$$\langle A(u_0), h\rangle = \langle -\Delta u_0, h\rangle + \left\langle \frac{\partial u_0}{\partial n}, h\right\rangle_{\partial\Omega} \quad \text{for all } h \in H^1(\Omega), \tag{3.6}$$

with $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denoting the duality brackets for the pair $(H^{-\frac{1}{2},2}(\partial\Omega), H^{\frac{1}{2},2}(\partial\Omega))$. We know that

$$\Delta u_0 = \operatorname{div} Du_0 \in H^{-1}(\Omega) = H_0^1(\Omega)^*$$

(see, for example, Gasinski and Papageorgiou [10, p. 212]). Hence from (3.5) and (3.6) it follows that

$$\langle -\Delta u_0, h\rangle = \int_\Omega f(z, u_0) h dz \quad \text{for all } h \in H_0^1(\Omega) \subseteq H^1(\Omega)$$

and thus

$$-\Delta u_0(z) = f(z, u_0(z)) \quad \text{a.e. in } \Omega.$$

Then from (3.5) and (3.6) it follows that

$$\left\langle \frac{\partial u_0}{\partial n} + \beta(z)u, h\right\rangle_{\partial\Omega} = 0 \quad \text{for all } h \in H^1(\Omega).$$

Recall that the trace map is surjective on $H^{\frac{1}{2},2}(\partial\Omega)$. Hence

$$\frac{\partial u_0}{\partial n} + \beta(z)u_0 = 0 \quad \text{on } \partial\Omega.$$

Therefore u_0 is a nontrivial positive solution of (1.1) such that $0 \leqslant u_0 \leqslant w_+$. From the regularity result of Lieberman [12], we have $u_0 \in C_+ \setminus \{0\}$. Hypotheses H_1 (i) and (iii) imply that we can find $c_1 > 0$ such that

$$f(z,x)x \geqslant \hat{\lambda}_m x^2 - c_1|x|^r \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \leqslant \varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\},$$

with $r > 2$. Then we have

$$-\Delta_p u_0(z) + c_1 u_0(z)^{r-1} = f(z, u_0(z)) + c_1 u_0(z)^{r-1} \geqslant 0 \quad \text{a.e. in } \Omega,$$

which implies

$$\Delta u_0(z) \leqslant c_1\|u_0\|^{r-2} u_0(z) \leqslant c_1 \varrho_0^{r-2} u_0(z) \quad \text{a.e. in } \Omega,$$

hence (see Vazquez [18])

$$u_0 \in \operatorname{int} C_+.$$

To produce a negative solution, we introduce the following Carathéodory function:

$$g_-(z,x) = \begin{cases} f(z, w_-(z)) + w_-(z) & \text{if } x < w_-(z), \\ f(z,x) + x & \text{if } w_-(z) \leqslant x \leqslant w_+(z), \\ 0 & \text{if } 0 < x. \end{cases}$$

We set

$$G_-(z,x) = \int_0^x g_-(z,s) ds$$

and consider the C^1-functional $\varphi_- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \int_\Omega G_-(z, u(z)) dz \quad \text{for all } u \in H^1(\Omega).$$

Reasoning as above, using this time the functional φ_-, we produce a negative solution

$$v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+). \qquad \square$$

Next we show that in fact we can produce extremal solutions of constant sign namely the smallest positive solution and the biggest negative solution. To this end, we introduce the following solution sets:

$$S_+ = \{u \in H^1(\Omega) : u \neq 0,\ u \in [0, w_+],\ u \text{ is a solution of (1.1)}\},$$
$$S_- = \{v \in H^1(\Omega) : v \neq 0,\ v \in [w_-, 0],\ v \text{ is a solution of (1.1)}\}.$$

We introduce the following Carathéodory function:

$$g(z,x) = \begin{cases} f(z, w_-(z)) + w_-(z) & \text{if } x < w_-(z), \\ f(z,x) + x & \text{if } w_-(z) \leqslant x \leqslant w_+(z), \\ f(z, w_+(z)) + w_+(z) & \text{if } w_+(z) < x. \end{cases} \tag{3.7}$$

From (3.7) and hypotheses H_1 (i) and (iii), we see that given $r \in (2, 2^*)$ we can find $c_2 > 0$ big such that

$$g(z,x)x \geqslant (\hat{\lambda}_m + 1)x^2 - c_2|x|^r \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.8}$$

This unilateral growth estimate for $g(z,\cdot)$ leads to the following auxiliary Robin problem:

$$-\Delta u(z) = \hat{\lambda}_m u(z) - c_2|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n}(z) + \beta(z)u = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

For this auxiliary problem, we have the following existence and uniqueness result.

Proposition 3.2. *Assume that hypothesis $H(\beta)$ holds. Then problem* (3.9) *has a unique positive solution $\tilde{u} \in \operatorname{int} C_+$ and since* (3.9) *is odd, $\tilde{v} = -\tilde{u} \in -\operatorname{int} C_+$ is the unique negative solution of* (3.9).

Proof. First we establish the existence of a positive solution for problem (3.9). To this end, let $\tau_+ : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\tau_+(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \frac{\hat{\lambda}_m}{2}\|u^+\|_2^2 - \frac{1}{2}\|u^+\|_2^2 + \frac{c_2}{r}\|u^+\|_r^r \quad \text{for all } u \in H^1(\Omega).$$

Since $r > 2$, it is clear that τ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in H^1(\Omega)$ such that

$$\tau_+(\tilde{u}) = \inf\{\tau_+(u) : u \in H^1(\Omega)\}. \tag{3.10}$$

For $t \in (0,1)$ we have

$$\tau_+(t\hat{u}_1) = \frac{t^2}{2}\xi(\hat{u}_1) + \frac{c_2 t^r}{r}\|\hat{u}_1\|_r^r - \frac{\hat{\lambda}_m t^2}{2}\|\hat{u}_1\|_2^2 = \frac{t^2}{2}[\hat{\lambda}_1 - \hat{\lambda}_m] + \frac{c_2 t^r}{2}\|\hat{u}_1\|_r^r \quad (\text{recall that } \|\hat{u}_1\|_2 = 1).$$

Since $\hat{\lambda}_m > \hat{\lambda}_1$ (recall $m \geqslant 2$) and $r > 2$, choosing $t \in (0,1)$ small we have $\tau_+(t\hat{u}_1) < 0$ and so $\tau_+(\tilde{u}) < 0 = \tau_+(0)$ (see (3.10)), hence $\tilde{u} \neq 0$. From (3.10), we have $\tau_+'(\tilde{u}) = 0$, which implies

$$\langle A(\tilde{u}), h\rangle + \int_{\partial\Omega} \beta(z)\tilde{u}h\,d\sigma + \int_\Omega \tilde{u}h\,dz = \int_\Omega (\hat{\lambda}_m + 1)\tilde{u}^+ h\,dz - c_2 \int_\Omega (\tilde{u}^+)^{r-1}h\,dz \quad \text{for all } h \in H^1(\Omega). \tag{3.11}$$

If in (3.11) we choose $h = -\tilde{u}^- \in H^1(\Omega)$, then we obtain $\tilde{u} \geqslant 0$, $\tilde{u} \neq 0$. Hence

$$\langle A(\tilde{u}), h\rangle + \int_{\partial\Omega} \beta(z)\tilde{u}h\,d\sigma = \hat{\lambda}_m \int_\Omega \tilde{u}h\,dz - c_2 \int_\Omega \tilde{u}^{r-1}h\,dz \quad \text{for all } h \in H^1(\Omega). \tag{3.12}$$

From (3.12) as in the proof of Proposition 3.1, via Green's identity, we obtain

$$-\Delta\tilde{u}(z) = \hat{\lambda}_m\tilde{u}(z) - c_2\tilde{u}(z)^{r-1} \quad \text{a.e. in } \Omega, \quad \frac{\partial\tilde{u}}{\partial n} + \beta(z)\tilde{u} = 0 \quad \text{on } \partial\Omega.$$

So $\tilde{u}$ is a positive solution for the auxiliary problem (3.9). From Winkert [19] we know that $\tilde{u} \in L^\infty(\Omega)$. Then applying the regularity result of Lieberman [12, p. 320], we have that $\tilde{u} \in C_+ \setminus \{0\}$. Also, we have

$$\Delta\tilde{u}(z) \leqslant c_2\|\tilde{u}\|_\infty^{r-2}\tilde{u}(z) \quad \text{a.e. in } \Omega$$

and thus $\tilde{u} \in \operatorname{int} C_+$ (see Vazquez [18]).

Next we prove the uniqueness of this positive solution. To this end, we introduce the integral functional $\gamma_+ : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\gamma_+(u) = \begin{cases} \frac{1}{2}\|Du^{1/2}\|_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u(z)d\sigma & \text{if } u \geqslant 0,\ u^{1/2} \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Benguria, Brezis and Lieb [5], we have that γ_+ is convex. Also, via Fatou's lemma, we check that γ_+ is lower semicontinuous.

Let u be a positive solution of the auxiliary problem (3.9). From the first part of the proof, we have that $u \in \operatorname{int} C_+$. Hence $u^2 \in \operatorname{dom} \gamma_+$ and if $h \in C^1(\overline{\Omega})$, then for $t \in (-1, 1)$ with $|t|$ small, we have $u^2 + th \in \operatorname{dom} \gamma_+$. So the Gâteaux derivative of γ_+ at u in the direction h exists. Moreover, via the chain rule, we have

$$\begin{aligned} \gamma_+'(u^2)(h) &= \left\langle A(u), \frac{h}{u} \right\rangle + \int_{\partial\Omega} \beta(z)hd\sigma \\ &= \left\langle -\Delta u, \frac{h}{u} \right\rangle + \int_{\partial\Omega} \left[\frac{\partial u}{\partial n} + \beta(z)u\right]hd\sigma \quad \text{(by Green's identity)} \\ &= \int_\Omega [\hat{\lambda}_m - c_2 u^{r-2}]hdz \quad \text{(since } u \text{ is a solution of (3.9)).} \end{aligned}$$

Similarly, if $v \in H^1(\Omega)$ is another positive solution of (3.9), then $v \in \operatorname{int} C_+$ and we have

$$\gamma_+'(v^2)(h) = \int_\Omega [\hat{\lambda}_m - c_1 v^{r-2}]hdz.$$

The convexity of γ_+ implies the monotonicity of γ_+'. Therefore

$$0 \leqslant c_1 \int_\Omega [v^{r-2} - u^{r-2}](u^2 - v^2)dz \leqslant 0,$$

hence $u = v$ and this proves the uniqueness of $\tilde{u} \in \operatorname{int} C_+$.

Since problem (3.9) is odd, $\tilde{v} = -\tilde{u} \in -\operatorname{int} C_+$ is the unique negative solution of (3.9). □

Proposition 3.3. *Assume that hypotheses $H(\beta)$ and H_1 hold. Then $\tilde{u} \leqslant u$ for all $u \in S_+$ and $v \leqslant \tilde{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+$ and consider the following Carathéodory function:

$$k_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ (\hat{\lambda}_m + 1)x - c_2 x^{r-1} & \text{if } 0 \leqslant u(z) \leqslant x, \\ (\hat{\lambda}_m + 1)u(z) - c_2 u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{3.13}$$

We set

$$K_+(z, x) = \int_0^x k_+(z, s)ds$$

and consider the C^1-functional $\mu_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\mu_+(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \int_\Omega K_+(z, u(z))dz \quad \text{for all } u \in H^1(\Omega).$$

It is clear from (3.13) that μ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_* \in H^1(\Omega)$ such that

$$\mu_+(\tilde{u}_*) = \inf\{\mu_+(u) : u \in H^1(\Omega)\}. \tag{3.14}$$

As before (see the proof of Proposition 3.1), using hypothesis H_1 (iii), for $t \in (0, 1)$ small we have $\mu_+(t\hat{u}_1) < 0$, which implies $\mu_+(\tilde{u}_*) < 0 = \mu_+(0)$ (see (3.14)), hence $\tilde{u}_* \neq 0$. From (3.14) we have $\mu_+'(\tilde{u}_*) = 0$ and so

$$\langle A(\tilde{u}_*), h\rangle + \int_{\partial\Omega} \beta(z)\tilde{u}_* hd\sigma + \int_\Omega \tilde{u}_* hdz = \int_\Omega k_+(z, \tilde{u}_*)hdz \quad \text{for all } h \in H^1(\Omega). \tag{3.15}$$

In (3.15), first we choose $h = -\tilde{u}_*^- \in H^1(\Omega)$. Then $\tilde{u}_* \geqslant 0$, $\tilde{u}_* \neq 0$. Also, in (3.15) we choose $h = (\tilde{u}_* - u)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
\langle A(\tilde{u}_*), (\tilde{u}_* - u)^+\rangle &+ \int_{\partial\Omega} \beta(z)\tilde{u}_*(\tilde{u}_* - u)^+ d\sigma + \int_\Omega \tilde{u}_*(\tilde{u}_* - u)^+ dz \\
&= \int_\Omega [(\hat{\lambda}_m + 1)u - c_2 u^{r-1}](\tilde{u}_* - u)^+ dz && \text{(see (3.13))} \\
&\leqslant \int_\Omega g(z,u)(\tilde{u}_* - u)^+ dz + \int_{\partial\Omega} \beta(z)\tilde{u}_*(\tilde{u}_* - u)^+ d\sigma && \text{(see (3.8) and } H(\beta)) \\
&= \int_\Omega f(z,u)(\tilde{u}_* - u)^+ dz + \int_{\partial\Omega} \beta(z)\tilde{u}_*(\tilde{u}_* - u)^+ d\sigma + \int_\Omega u(\tilde{u}_* - u)^+ dz && \text{(see (3.7) and recall } u \leqslant w_+) \\
&= \langle A(u), (\tilde{u}_* - u)^+\rangle + \int_{\partial\Omega} \beta(z)\tilde{u}_*(\tilde{u}_* - u)^+ d\sigma + \int_\Omega u(\tilde{u}_* - u)^+ dz && \text{(since } u \in S_+),
\end{aligned}$$

which shows

$$\langle A(\tilde{u}_*) - A(u), (\tilde{u}_* - u)^+\rangle + \int_\Omega (\tilde{u}_* - u)(\tilde{u}_* - u)^+ dz \leqslant 0$$

implying $\|(\tilde{u}_* - u)^+\|^2 \leqslant 0$, hence $\tilde{u}_* \leqslant u$. So we have proved that

$$\tilde{u}_* \in [0, u] = \{y \in H^1(\Omega) : 0 \leqslant y(z) \leqslant u(z) \text{ a.e. in } \Omega\}, \quad \tilde{u}_* \neq 0. \tag{3.16}$$

From (3.13) and (3.16), we see that (3.15) becomes

$$\langle A(\tilde{u}_*), h\rangle + \int_{\partial\Omega} \beta(z)\tilde{u}_* h d\sigma = \int_\Omega [\hat{\lambda}_m \tilde{u}_* - c_2 \tilde{u}_*^{r-1}] h dz \quad \text{for all } h \in H^1(\Omega). \tag{3.17}$$

From (3.17), as in the proof of Proposition 3.1, via Green's identity, we obtain

$$-\Delta \tilde{u}_*(z) = \hat{\lambda}_m \tilde{u}_*(z) - c_2 \tilde{u}_*(z)^{r-1} \quad \text{a.e. in } \Omega, \quad \frac{\partial \tilde{u}_*}{\partial n} + \beta(z)\tilde{u}_* = 0 \quad \text{on } \partial\Omega.$$

It follows that $\tilde{u}_* = \tilde{u} \in \operatorname{int} C_+$ (see Proposition 3.2), thus $\tilde{u} \leqslant u$ (see (3.16)). Similarly we show that $v \leqslant \tilde{v}$ for all $v \in S_-$. □

Now we are ready to produce extremal nontrivial constant sign solutions for problem (1.1).

Proposition 3.4. *Assume that hypotheses $H(\beta)$ and H_1 hold. Then problem* (1.1) *admits a smallest positive solution $u_* \in \operatorname{int} C_+$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$*

Proof. From Dunford and Schwartz [9, p. 336], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq C$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n.$$

We have

$$-\Delta u_n(z) = f(z, u_n(z)) \quad \text{a.e. in } \Omega, \quad \frac{\partial u_n}{\partial n} + \beta(z)u_n = 0 \quad \text{on } \partial\Omega, \quad u_n \in \operatorname{int} C_+,\ n \geqslant 1. \tag{3.18}$$

From Lieberman [12, p. 320], we know that we can find $\eta \in (0,1)$ and $M_1 > 0$ such that

$$u_n \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\eta}(\overline{\Omega})} \leqslant M_1 \quad \text{for all } n \geqslant 1.$$

By virtue of the compact embedding of $C^{1,\eta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and by passing to a subsequence if necessary, we may assume that

$$u_n \to u_* \quad \text{in } C^1(\overline{\Omega}). \tag{3.19}$$

From (3.18) and Green's identity, we have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z)u_n h d\sigma = \int_\Omega f(z, u_n) h dz,$$

which implies

$$\langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_* h d\sigma = \int_{\Omega} f(z, u_*) h dz \quad \text{(see (3.19))},$$

hence

$$\langle -\Delta u_*, h\rangle + \left\langle \frac{\partial u_*}{\partial n} + \beta(z) u_*, h \right\rangle_{\partial\Omega} = \int_{\Omega} f(z, u_*) h dz \quad \text{(using again Green's identity).} \tag{3.20}$$

From (3.20) as in the proof of Proposition 3.1, we obtain

$$-\Delta u_*(z) = f(z, u_*(z)) \quad \text{a.e. in } \Omega, \quad \frac{\partial u_*}{\partial n} + \beta(z) u_* = 0 \quad \text{on } \partial\Omega. \tag{3.21}$$

Moreover, from Proposition 3.3, we have $\tilde{u} \leqslant u_n$ for all $n \geqslant 1$, which implies $\tilde{u} \leqslant u_*$ (see (3.19)), hence $u_* \neq 0$ and so $u_* \in S_+$ (see (3.21)). Evidently $u_* = \inf S_+$.

Similarly, working with the set S_- we produce $v_* \in -\operatorname{int} C_+$ ($v_* \leqslant \tilde{v}$) the biggest negative solution of problem (1.1). □

Remark 3. Suppose that instead of H_1 (iii) we assume that

$$\hat{\lambda}_m x^2 \leqslant f(z, x)x \leqslant \hat{\eta}(z) x^2 \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \leqslant \delta_0, \tag{3.22}$$

with $\hat{\eta} \in L^\infty(\Omega)_+$. Evidently (3.22), in contrast to H_1 (iii), excludes from consideration reactions which are concave near zero. Using (3.22) the proof of the nontriviality of u_* is much easier and we do not need to go through Propositions 3.2 and 3.3. Indeed, as before we argue by contradiction and assume that $u = 0$. We set $y_n = \frac{u_n}{\|u_n\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \geqslant 1$ and so we may assume that

$$\begin{cases} y_n \xrightarrow{w} y & \text{in } H^1(\Omega), \\ y_n \to y & \text{in } L^2(\Omega), \end{cases} \quad \text{as } n \to \infty \text{ with } y \geqslant 0. \tag{3.23}$$

From the proof of Proposition 3.4, we have

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma = \int_{\Omega} \frac{f(z, u_n)}{\|u_n\|} h dz \quad \text{for all } n \geqslant 1. \tag{3.24}$$

By virtue of (3.22) and H_1 (i), we see that $\{\frac{f(\cdot, u_n(\cdot))}{\|u_n\|}\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded. So, we may assume that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|} \xrightarrow{w} \eta_0 y \quad \text{in } L^2(\Omega) \quad \text{with } \hat{\lambda}_m \leqslant \eta_0(z) \leqslant \hat{\eta}(z) \text{ a.e. in } \Omega \quad \text{(see (3.22))}. \tag{3.25}$$

In (3.24) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.23) and (3.25). Then we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

which implies $\|Dy_n\|_2 \to \|Dy\|_2$ and so

$$y_n \to y \text{ in } H^1(\Omega) \quad \text{(by the Kadec–Klee property, see (3.23))}, \quad \text{hence } \|y\| = 1. \tag{3.26}$$

So, if in (3.24) we pass to the limit as $n \to \infty$ and use (3.25) and (3.26), then

$$\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_{\Omega} \eta_0 y dz \quad \text{for all } h \in H^1(\Omega) \quad \text{(recall that the trace map is continuous).} \tag{3.27}$$

From (3.27) it follows that

$$-\Delta y(z) = \eta_0(z) y(z) \quad \text{a.e. in } \Omega, \quad \frac{\partial y}{\partial n} + \beta(z) y = 0 \quad \text{on } \partial\Omega, \quad y \geqslant 0, \ y \neq 0,$$

a contradiction to the fact that $\eta_0(z) \geqslant \hat{\lambda}_m$ a.e. in Ω, with $m \geqslant 2$ (see Section 2).

4 Nodal solutions

In this section, we produce nodal (that is, sign changing) solutions for problem (1.1). To do this, we need to strengthen our hypotheses on the reaction $f(z,x)$. So, we assume the following:

Hypothesis H_2. Assume that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is measurable function such that for a.a. $z\in\Omega$, $f(z,0)=0$, $f(z,\cdot)\in C^1(\mathbb{R})$ and

(i) for every $\varrho>0$, there exists $a_\varrho\in L^\infty(\Omega)_+$ such that $|f'_x(z,x)|\leqslant a_\varrho(z)$ for a.a. $z\in\Omega$ and all $|x|\leqslant\varrho$,

(ii) there exist functions $w_\pm\in H^1(\Omega)\cap L^\infty(\Omega)$ such that

$$\begin{aligned} w_-(z)\leqslant c_-<0<c_+\leqslant w_+(z) &\quad \text{for a.a. } z\in\Omega,\\ f(z,w_+(z))\leqslant 0\leqslant f(z,w_-(z)) &\quad \text{for a.a. } z\in\Omega,\\ A(w_-)+\beta(\cdot)w_-\leqslant 0\leqslant A(w_+)+\beta(\cdot)w_+ &\quad \text{in } H^1(\Omega)^*,\end{aligned}$$

(iii) there exist an integer $m\geqslant 2$ and $\delta_0>0$ such that

$$\hat\lambda_m x^2\leqslant f(z,x)x\leqslant\hat\eta(z)x^2 \quad \text{for a.a. } z\in\Omega \text{ and all } |x|\leqslant\delta_0$$

with $\hat\eta\in L^\infty(\Omega)_+$, $\hat\eta(z)\leqslant\hat\lambda_{m+1}$ a.e. in Ω, $\hat\eta\neq\hat\lambda_{m+1}$ and

$$f'_x(z,0)=\lim_{x\to 0}\frac{f(z,x)}{x} \quad \text{uniformly for a.a. } z\in\Omega.$$

Remark 4. It is clear from hypothesis H_2 (iii) that

$$\hat\lambda_m\leqslant f'_x(z,0)\leqslant\hat\eta(z) \quad \text{for a.a. } z\in\Omega. \tag{4.1}$$

We observe, by hypotheses H_2 (i) and (iii) and the differentiability of $f(z,\cdot)$, that if $\varrho=\max\{\|w_+\|_\infty,\|w_-\|_\infty\}$, then we can find $\xi_0>0$ such that for a.a. $z\in\Omega$, $x\mapsto f(z,x)+\xi_0 x$ is nondecreasing on $[-\varrho_0,\varrho_0]$.

Let $g(z,x)$ be the Carathéodory function introduced in (3.7). Let

$$G(z,x)=\int_0^x g(z,s)ds$$

and consider the functional $\varphi:H^1(\Omega)\to\mathbb{R}$ defined by

$$\varphi(u)=\frac12\xi(u)+\frac12\|u\|_2^2-\int_\Omega G(z,u(z))dz \quad \text{for all } u\in H^1(\Omega).$$

Recall that

$$\xi(u)=\|Du\|_2^2+\int_{\partial\Omega}\beta(z)u(z)^2d\sigma \quad \text{for all } u\in H^1(\Omega).$$

We have that $\varphi\in C^{2-0}(H^1(\Omega))$.

Proposition 4.1. *Assume that hypotheses $H(\beta)$ and H_2 hold. Then we have $C_k(\varphi,0)=\delta_{k,d_m}\mathbb{Z}$ for all $k\geqslant 0$ with $d_m=\dim\bigoplus_{i=1}^m E(\hat\lambda_i)$.*

Proof. Let $\psi:H^1(\Omega)\to\mathbb{R}$ be the C^2-functional defined by

$$\psi(u)=\frac12\xi(u)-\frac12\int_\Omega f'_x(z,0)u(z)^2dz \quad \text{for all } u\in H^1(\Omega).$$

Note that by virtue of hypothesis H_2 (iii), given $\varepsilon>0$, we can find $\delta=\delta(\varepsilon)>0$ such that

$$-\varepsilon x\leqslant f(z,x)-f'_x(z,0)x\leqslant\varepsilon x \quad \text{for a.a. } z\in\Omega \text{ and all } |x|\leqslant\delta,$$

that is,

$$-\frac{\varepsilon}{2}x^2\leqslant F(z,x)-\frac12 f'_x(z,0)x^2\leqslant\frac{\varepsilon}{2}x^2 \quad \text{for a.a. } z\in\Omega \text{ and all } |x|\leqslant\delta.$$

So, if $\overline{B}_\delta^c = \{u \in C^1(\overline{\Omega}) : \|u\|_{C^1(\overline{\Omega})} \leqslant \delta\}$, then

$$\|\varphi - \psi\|_{C^1(\overline{B}_\delta^c)} \leqslant \varepsilon.$$

Choosing $\varepsilon > 0$ sufficiently small and using the continuity of the critical groups with respect to the C^1-topology (see Chang [7, p. 336]), we have

$$C_k(\varphi|_{C^1(\overline{\Omega})}, 0) = C_k(\psi|_{C^1(\overline{\Omega})}, 0) \quad \text{for all } k \geqslant 0. \tag{4.2}$$

But from Palais [13] and Bartsch [4], we know that

$$C_k(\varphi|_{C^1(\overline{\Omega})}, 0) = C_k(\varphi, 0) \quad \text{and} \quad C_k(\psi|_{C^1(\overline{\Omega})}, 0) = C_k(\psi, 0) \quad \text{for all } k \geqslant 0. \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$C_k(\varphi, 0) = C_k(\psi, 0) \quad \text{for all } k \geqslant 0. \tag{4.4}$$

Let $\overline{H} = \bigoplus_{k=1}^m E(\hat{\lambda}_k)$ and $\hat{H} = \overline{\bigoplus_{k\geqslant m+1} E(\hat{\lambda}_k)} = \overline{H}^\perp$. We have the orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{H} \oplus \hat{H}.$$

Then for $\overline{u} \in \overline{H}$ we have

$$\psi(u) = \frac{1}{2}\xi(u) - \frac{1}{2}\int_\Omega f_x'(z,0)u^2 dz \overset{(4.1)}{\leqslant} \frac{1}{2}\xi(u) - \frac{\hat{\lambda}_m}{2}\|u\|_2^2 \overset{(2.2)}{\leqslant} 0.$$

Also, if $\hat{u} \in \hat{H}$, then

$$\begin{aligned} \psi(u) &= \frac{1}{2}\xi(u) - \frac{1}{2}\int_\Omega f_x'(z,0)u^2 dz \\ &\geqslant \frac{1}{2}\xi(u) - \frac{1}{2}\int_\Omega \hat{\eta}(z)u^2 dz \qquad \text{(see (4.1))} \\ &\geqslant \frac{\xi_0}{2}\|u\|^2 \quad \text{for some } \xi_0 > 0 \quad \text{(see Lemma 2.3).} \end{aligned}$$

So, we can apply Proposition 2.4 and infer that $C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geqslant 0$ with $d_m = \dim \bigoplus_{k=1}^m E(\hat{\lambda}_k)$, which implies $C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geqslant 0$ (see (4.4)). □

Using this result, we can produce nodal solutions. In what follows, $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are the two extremal constant sign solutions produced in Proposition 3.4.

Proposition 4.2. *Assume that hypotheses $H(\beta)$ and H_2 hold. Then problem* (1.1) *has at least two nodal solutions*

$$y_0, \hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*].$$

Proof. Using the extremal constant sign solutions $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ produced in Proposition 3.4, we introduce the following Carathéodory function

$$h(z,x) = \begin{cases} f(z, v_*(z)) + v_*(z) & \text{if } x < v_*(z), \\ f(z,x) + x & \text{if } v_*(z) \leqslant x \leqslant u_*(z), \\ f(z, u_*(z)) + u_*(z) & \text{if } u_*(z) < x. \end{cases} \tag{4.5}$$

Let $H(z,x) = \int_0^x h(z,s)ds$ and consider a C^1-functional $\tau : H^1(\Omega) \to \mathbb{R}$ defined by

$$\tau(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \int_\Omega H(z,u(z))dz \quad \text{for all } u \in H^1(\Omega),$$

where

$$\xi(u) = \|Du\|_2^2 + \int_{\partial\Omega} \beta(z)u(z)^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

Also, we consider the positive and negative truncations of $h(z,\cdot)$, namely the Carathéodory function

$$h_\pm(z,x) = h(z,\pm x^\pm).$$

Let $H_\pm(z,x) = \int_0^x h_\pm(z,x)ds$ and introduce the C^1-functionals $\tau_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\tau_\pm(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \int_\Omega H_\pm(z,u(z))dz \quad \text{for all } u \in H^1(\Omega).$$

As in the proof of Proposition 3.1, we can check that

$$K_\tau \subseteq [v_*, u_*], \quad K_{\tau_+} \subseteq [0, u_*], \quad K_{\tau_-} \subseteq [v_*, 0].$$

The extremality of u_* and v_* implies that

$$K_\tau \subseteq [v_*, u_*], \quad K_{\tau_+} = \{0, u_*\}, \quad K_{\tau_-} = \{0, v_*\}. \tag{4.6}$$

Claim 1. *The functions $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of τ.*

From (4.5) it is clear that τ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_* \in H^1(\Omega)$ such that

$$\tau_+(\overline{u}_*) = \inf\{\tau_+(u) : u \in H^1(\Omega)\}. \tag{4.7}$$

As in the proof of Proposition 3.1, using hypothesis H_2 (iii) and the fact that $\hat{u}_1 \in \operatorname{int} C_+$, for $t \in (0,1)$ small, we have $\tau_+(t\hat{u}_1) < 0$ and so

$$\tau_+(\overline{u}_*) < 0 = \tau_+(0) \quad \text{(see (4.7)),} \quad \text{hence } \overline{u}_* \neq 0. \tag{4.8}$$

From (4.7) we have $\overline{u}_* \in K_{\tau_+} = \{0, u_*\}$ (see (4.6)), hence $\overline{u}_* = u_*$ (see (4.8)). Since $\tau|_{C_+} = \tau_+|_{C_+}$ and $u_* \in \operatorname{int} C_+$, it follows that u_* is a local $C^1(\overline{\Omega})$-minimizer of τ, hence by Proposition 2.2 it is also a local $H^1(\Omega)$-minimizer of τ. Similarly, for $v_* \in -\operatorname{int} C_+$ using this time the functional τ_-. This proves the claim.

Without any loss of generality, we may assume that $\tau(v_*) \leqslant \tau(u_*)$ (the analysis is similar if the opposite inequality holds). By virtue of the claim, we can find $\varrho \in (0,1)$ small such that

$$\tau(v_*) \leqslant \tau(u_*) < \inf\{\tau(u) : \|u - u_*\| = \varrho\} = m_\varrho, \quad \|v_* - u_*\| > \varrho \tag{4.9}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 29]). The functional τ is coercive (see (4.5)) and so it satisfies the PS-condition. This fact and (4.9) permit the use of Theorem 2.1 (the Mountain Pass Theorem). So, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \in K_\tau \subseteq [v_*, u_*] \quad \text{(see (4.6))} \quad \text{and} \quad m_\varrho \leqslant \tau(y_0). \tag{4.10}$$

From (4.9) and (4.10) it follows that $y_0 \notin \{v_*, u_*\}$ and $\tau'(y_0) = 0$. We have

$$\langle A(y_0), v\rangle + \int_{\partial\Omega} \beta(z) y_0 v d\sigma + \int_\Omega y_0 v dz = \int_\Omega h(z, y_0) v dz \quad \text{for all } v \in H^1(\Omega). \tag{4.11}$$

From (4.5), (4.10), (4.11) as before, we infer that $y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega})$ is a solution of (1.1). Since y_0 is a critical point of τ of mountain pass-type, we have

$$C_1(\tau, y_0) \neq 0. \tag{4.12}$$

On the other hand, since $\tau|_{[v_*,u_*]} = \varphi|_{[v_*,u_*]}$ (see (4.5)) and $v_* \in -\operatorname{int} C_+$, $u_* \in \operatorname{int} C_+$, we have

$$C_k(\tau, 0) = C_k(\varphi, 0) \quad \text{for all } k \geqslant 0 \quad \text{(see the proof of Proposition 4.1),}$$

hence

$$C_k(\tau, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geqslant 0 \quad \text{(see Proposition 4.1).} \tag{4.13}$$

Comparing (4.12) and (4.13) and since $d_m \geqslant 2$ (recall $m \geqslant 2$), we infer $y_0 \neq 0$. Because $y_0 \in [v_*, u_*] \setminus \{v_*, u_*\}$, by virtue of the extremality of v_* and u_*, we conclude that $y_0 \in C^1(\overline{\Omega})$ is nodal.

To produce a second nodal solution, we will use Morse theory. This requires that we improve (4.12) and compute the critical groups of τ exactly. We assume that $K_\tau = \{0, u_*, v_*, y_0\}$ (otherwise we already have a second nodal solution of (1.1) and so we are done, see (4.6)). Note that because of the truncations at $u_*(z)$ and $v_*(z)$, the functional τ is not C^2, it belongs to $C^{2-0}(H^1(\Omega))$. So, we cannot apply directly the results of Morse theory for critical points of mountain pass-type. For this reason, we proceed as follows. We have

$$-\Delta y_0(z) = f(z, y_0(z)) \quad \text{a.e. in } \Omega, \quad \frac{\partial y_0}{\partial n} + \beta(z) y_0 = 0 \quad \text{on } \partial\Omega.$$

Recall that $u_*, v_* \in [w_-, w_+]$ and let $\xi_0 > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z,x) + \xi_0 x$ is nondecreasing on $[-\varrho_0, \varrho_0]$ with $\varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$ (see the remark after hypotheses H_2). Then we have

$$\begin{aligned} -\Delta y_0(z) + \xi_0 y_0(z) &= f(z, y_0(z)) + \xi_0 y_0(z) \\ &\leqslant f(z, u_*(z)) + \xi_0 u_*(z) \quad (\text{since } y_0 \leqslant u_*) \\ &= -\Delta u_*(z) + \xi_0 u_*(z) \qquad \text{a.e. in } \Omega, \end{aligned}$$

which shows $\Delta(u_* - y_0)(z) \leqslant \xi_0 (u_* - y_0)(z)$ a.e. in Ω, hence $u_* - y_0 \in \operatorname{int} C_+$ (see Vazquez [18]). Similarly, we show that $y_0 - v_* \in \operatorname{int} C_+$, hence $y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]$.

Let $0 < \varepsilon < \min\{\min_{\overline{\Omega}}(u_* - y_0), \min_{\overline{\Omega}}(y_0 - v_*)\}$ and consider the multifunction $L_\varepsilon : \Omega \to 2^{C^1(\mathbb{R})}$ defined by

$$L_\varepsilon(z) = \{\vartheta \in C^1(\mathbb{R}) : \|h(z,\cdot) - \vartheta\|_{C(\mathbb{R})} \leqslant \varepsilon,\ h(z,x) = \vartheta(x) \text{ for } |x - u_*(z)| \geqslant \varepsilon,\ |x - v_*(z)| \geqslant \varepsilon\}.$$

Since $h(z,\cdot)$ need not be differentiable at $v_*(z)$ and $u_*(z)$, by a smooth modification of $h(z,\cdot)$ near $v_*(z)$ and $u_*(z)$, we see that $L_\varepsilon(z) \neq \emptyset$ for all $z \in \Omega$. Clearly $\operatorname{Gr} L_\varepsilon \in \mathcal{L}(\Omega) \times \mathcal{B}(C^1(\mathbb{R}))$, with $\mathcal{L}(\Omega)$ being the Lebesgue σ-field of Ω and $\mathcal{B}(C^1(\mathbb{R}))$ the Borel σ-field of $C^1(\mathbb{R})$. Since $C^1(\mathbb{R})$ is a separable Fréchet space, we can apply the Yankov–von Neumann–Aumann Selection Theorem (see Hu and Papageorgiou [11, pp. 158–159]) and obtain a Borel measurable map $\hat{\vartheta} : \Omega \to C^1(\mathbb{R})$ such that $\hat{\vartheta}(z) \in L_\varepsilon(z)$ for a.a. $z \in \Omega$. We set $\hat{h}(z,x) = \hat{\vartheta}(z)(x)$ for all $x \in \mathbb{R}$. Then $\hat{h}(\cdot,\cdot)$ is jointly measurable and for a.a. $z \in \Omega$, $\hat{h}(z,\cdot) \in C^1(\mathbb{R})$. We set

$$\hat{H}(z,x) = \int_0^x \hat{h}(z,x)\,ds$$

and consider the functional $\hat{\tau} : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\tau}(u) = \frac{1}{2}\xi(u) + \frac{1}{2}\|u\|_2^2 - \int_\Omega \hat{H}(z, u(z))\,dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently $\hat{\tau} \in C^2(H^1(\Omega))$ and $y_0 \in K_{\hat{\tau}}$. Choosing $\varepsilon > 0$ small, by virtue of the continuity of the critical groups in the C^1-topology, we have

$$C_k(\tau, y_0) = C_k(\hat{\tau}, y_0) \quad \text{for all } k \geqslant 0, \tag{4.14}$$

which implies $C_1(\hat{\tau}, y_0) \neq 0$ (see (4.12)), hence $C_k(\hat{\tau}, y_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geqslant 0$ (see Bartsch [4]), and so

$$C_k(\tau, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0 \quad (\text{see } (4.14)). \tag{4.15}$$

By virtue of the claim, we have

$$C_k(\tau, u_*) = C_k(\tau, v_*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{4.16}$$

Moreover, since τ is coercive (see (4.5)), we have

$$C_k(\tau, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{4.17}$$

Recall that $K_\tau = \{0, u_*, v_*, y_0\}$. From (4.13), (4.15), (4.16), (4.17) and the Morse relation (see (2.3)) with $t = -1$, we have $(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0$ and so $(-1)^{d_m} = 0$, a contradiction. So, we can find $\hat{y} \in K_\tau \setminus \{0, u_*, v_*, y_0\}$. Then $\hat{y} \in [v_*, u_*] \cap C^1(\overline{\Omega})$ (see (4.6)) and so it is the second nodal solution of (1.1). As we did for y_0, we show that $\hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]$. ☐

Summarizing the situation for problem (1.1), we can state the following multiplicity theorem.

Theorem 4.3. *Assume that hypotheses $H(\beta)$ and H_2 hold. Then problem* (1.1) *has at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0, \hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal}.$$

A careful reading of the above proof leads to another multiplicity theorem producing three nontrivial solutions all with sign information and without imposing any differentiability condition on the reaction $f(z,\cdot)$.

So, the new hypotheses on the reaction $f(z,x)$ are the following:

Hypothesis H_3. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$ such that $|f(z,x)| \leqslant a_\varrho(z)$ for a.a. $z \in \Omega$, all $|x| \leqslant \varrho$,

(ii) there exist functions $w_\pm \in H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{aligned} w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) &\quad \text{for a.e. } z \in \Omega, \\ f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) &\quad \text{for a.e. } z \in \Omega, \\ A(w_-) + \beta(\cdot)w_- \leqslant 0 \leqslant A(w_+) + \beta(\cdot)w_+ &\quad \text{in } H^1(\Omega)^*, \end{aligned}$$

(iii) there exist an integer $m \geqslant 2$ and functions $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_m \leqslant \eta(z) \leqslant \hat{\eta}(z) \leqslant \hat{\lambda}_{m+1} \quad \text{a.e. in } \Omega, \quad \eta \neq \hat{\lambda}_m,\ \hat{\eta} \neq \hat{\lambda}_{m+1}$$

and

$$\eta(z) \leqslant \liminf_{x \to 0} \frac{f(z,x)}{x} \leqslant \limsup_{x \to 0} \frac{f(z,x)}{x} \leqslant \hat{\eta}(z) \quad \text{uniformly for a.a. } z \in \Omega,$$

(iv) if $\varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$, then there exists $\xi_0 > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z,x) + \xi_0 x$ is nondecreasing on $[-\varrho_0, \varrho_0]$.

Remark 5. Now, in contrast to the previous setting, we do not allow resonance at zero, only nonuniform nonresonance.

In this case, hypothesis H_3 (iii), the homotopy invariance of critical groups and the results of Dancer [8] imply that $C_{d_m}(\tau, 0) \neq 0$ and $C_1(\tau, 0) = 0$, with τ as in the proof of Proposition 4.2 and with $d_m = \dim \bigoplus_{k=1}^m E(\hat{\lambda}_k) \geqslant 2$. Then the argument in the proof of Proposition 4.2 remains valid and we have.

Theorem 4.4. *Assume that hypotheses $H(\beta)$ and H_1 hold. Then problem* (1.1) *has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal}.$$

5 Parametric equations

We consider the following parametric semilinear Robin problem:

$$-\Delta u(z) = \lambda u(z) - f(z, u(z)) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n}(z) + \beta(z)u(z) = 0 \quad \text{on } \partial\Omega. \tag{P_λ}$$

Here $\lambda > 0$ is a parameter and $f(z,x)$ is a Carathéodory perturbation. The precise conditions on the nonlinearity $f(z,x)$ are the following:

Hypothesis H_4. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$ such that $|f'_x(z,x)| \leqslant a_\varrho(z)$ for a.a. $z \in \Omega$ and all $|x| \leqslant \varrho$,

(ii) $\lim_{x \to \pm\infty} \frac{f(z,x)}{x} = +\infty$ uniformly for a.a. $z \in \Omega$,

(iii) $\lim_{x \to 0} \frac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$,

(iv) there exists $\hat{\delta} > 0$ such that $f(z,x)x \geqslant 0$ for a.a. $z \in \Omega$ and all $|x| \leqslant \hat{\delta}$.

Remark 6. The above hypotheses imply that given $\varrho > 0$, we can find $\xi_\varrho > 0$ such that for a.a. $z \in \Omega$ the function $x \mapsto (\lambda + \xi_\varrho)x - f(z, x)$ is nondecreasing on $[-\varrho, \varrho]$.

Theorem 5.1. *Assume that hypotheses $H(\beta)$ and H_4 hold and $\lambda > \hat{\lambda}_2$. Then problem (P_λ) admits at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0, \hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal.}$$

Proof. Hypothesis H_4 (ii) implies that given $\vartheta > 0$, we can find $M = M(\vartheta) > 0$ such that

$$f(z, x) \geqslant \vartheta x \quad \text{for a.a. } z \in \Omega \text{ and all } x \geqslant M.$$

Since $\hat{u}_1 \in \operatorname{int} C_+$, for $t > 0$ large we will have $t\hat{u}_1(z) \geqslant M$ for all $z \in \overline{\Omega}$. Then

$$f(z, t\hat{u}_1(z)) \geqslant \vartheta(t\hat{u}_1(z)) \quad \text{for a.a. } z \in \Omega. \tag{5.1}$$

We have

$$-\Delta(t\hat{u}_1)(z) = \hat{\lambda}_1(t\hat{u}_1)(z) \quad \text{a.e. in } \Omega, \quad \frac{\partial(t\hat{u}_1)}{\partial n} + \beta(z)(t\hat{u}_1) = 0 \quad \text{on } \partial\Omega. \tag{5.2}$$

For every $h \in H^1(\Omega)$, $h \geqslant 0$, we have

$$\langle -\Delta(t\hat{u}_1), h\rangle = \int_\Omega \hat{\lambda}_1(t\hat{u}_1)h\,dz$$

and so

$$\langle A(t\hat{u}_1), h\rangle - \left\langle \frac{\partial(t\hat{u}_1)}{\partial n}, h\right\rangle_{\partial\Omega} = \int_\Omega \hat{\lambda}_1(t\hat{u}_1)h\,dz \quad \text{(by Green's identity),}$$

hence

$$\langle A(t\hat{u}_1), h\rangle + \int_{\partial\Omega} \beta(z)(t\hat{u}_1)\,d\sigma = \int_\Omega \hat{\lambda}_1(t\hat{u}_1)h\,dz \quad \text{(see (5.2)).} \tag{5.3}$$

Choosing $\vartheta = \lambda - \hat{\lambda}_1 > 0$, from (5.1) and (5.3), we have

$$\int_\Omega [\lambda(t\hat{u}_1) - f(z, t\hat{u}_1)]h\,dz \leqslant \int_\Omega \hat{\lambda}_1(t\hat{u}_1)h\,dz = \langle A(t\hat{u}_1), h\rangle + \int_{\partial\Omega} \beta(z)(t\hat{u}_1)h\,d\sigma \quad \text{for all } h \in H^1(\Omega) \text{ with } h \geqslant 0.$$

Then $w_+ = t\hat{u}_1 \in \operatorname{int} C_+$ satisfies hypothesis H_2 (ii). Similarly, we produce $w_- = -t\hat{u}_1 \in -\operatorname{int} C_+$ which also satisfies hypothesis H_2 (ii).

Finally note that by virtue of hypothesis H_4 (iii) given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in (0, \hat{\delta}]$ (see H_4 (iv)) such that

$$|f(z, x)| \leqslant \varepsilon|x| \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \leqslant \delta_0. \tag{5.4}$$

Then for a.a. $z \in \Omega$ and all $|x| \leqslant \delta_0$, we have

$$\lambda x^2 - f(z, x)x \geqslant (\lambda - \varepsilon)x^2.$$

Choosing $\varepsilon \in (0, \lambda - \hat{\lambda}_2]$, we see that

$$\lambda x^2 - f(z, x)x \geqslant \hat{\lambda}_2 x^2 \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \leqslant \delta_0.$$

Also by hypothesis H_4 (iv) we have

$$\lambda x^2 - f(z, x)x \leqslant \lambda x^2 \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \leqslant \delta_0.$$

So, we have satisfied hypothesis H_2 (iii). Therefore, we are within the framework of Theorem 4.4. Applying that theorem, we produce four nontrivial solutions

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \text{and} \quad y_0, \hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \text{ nodal.} \qquad \square$$

Remark 7. Theorem 5.1 above extends Papageorgiou and Rădulescu [14, Theorem 14], since here we do not impose a global growth condition on $f'_x(z, \cdot)$ (see H_4 (i)). Moreover, it extends analogous results for the semilinear Dirichlet problem by Ambrosetti and Lupo [2], Ambrosetti and Mancini [3] and Struwe [15, 16], where the authors produce only three nontrivial solutions, without obtaining nodal solutions.

References

[1] S. Aizicovici, N. S. Papageorgiu and V. Staicu, Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints, *Mem. Amer. Math. Soc.* **196** (2008), bo. 915.

[2] A. Ambrosetti and D. Lupo, On a class of nonlinear Dirichlet problems with multiple solutions, *Nonlinear Anal.* **8** (1984), 1145–1150.

[3] A. Ambrosetti and G. Mancini, Sharp nonuniqueness results for some nonlinear problems, *Nonlinear Anal.* **3** (1979), 635–645.

[4] T. Bartsch, Critical point theory on partially ordered Hilbert spaces, *J. Funct. Anal.* **186** (2001), 117–152.

[5] R. Benguria, H. Brezis and E. H. Lieb, The Thomas–Fermi–von Weizsäcker theory of atoms and molecules, *Comm. Math. Phys.* **79** (1981), 167–180.

[6] H. Brezis and L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Sér. I* **317** (1993), 465–472.

[7] K.-C. Chang, *Methods in Nonlinear Analysis*, Springer-Verlag, Berlin, 2005.

[8] E. N. Dancer, Remark on jumping nonlinearity, in: *Topics in Nonlinear Analysis*, Birkhäuser-Verlag, Basel (1998), 101–116.

[9] N. Dunford and J. Schwartz, *Linear Operators I*, Wiley-Interscience, New York, 1958.

[10] L. Gasinski and N. S. Papageorgiou, *Nonlinear Analysis*, Chapman–Hall/CRC, Boca Raton, 2006.

[11] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Volume I: Theory*, Kluwer, Dordrecht, 1997.

[12] G. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva, *Comm. Partial Differential Equations* **16** (1991), 311–361.

[13] R. Palais, Homotopy theory of infinite dimensional manifolds, *Topology* **5** (1966), 1–16.

[14] N. S. Papageorgiou and V. Rădulescu, Multiple solutions with precise sign for nonlinear parametric Robin problems, *J. Differential Equations* **256** (2014), 2449–2479.

[15] M. Struwe, A note on a result of Ambrosetti and Mancini, *Ann. Mat. Pura Appl. (4)* **131** (1982), 107–115.

[16] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin, 1990.

[17] J. Su, Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues, *Nonlinear Anal.* **48** (2002), 881–895.

[18] J. Vazquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (1984), 191–202.

[19] P. Winkert, L^∞-estimates for nonlinear elliptic Neumann boundary value problems, *NoDEA Nonlinear Differential Equations Appl.* **17** (2010), 289–302.

[20] J. Zhang, S. Li and X. Xue, Multiple solutions for a class of semilinear elliptic problems with Robin boundary condition, *J. Math. Anal. Appl.* **388** (2012), 435–442.

Received December 6, 2013; accepted March 27, 2014.