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Competition phenomena for elliptic equations involving a general operator in divergence form

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In this paper, by using variational methods, we study the following elliptic problem

$$\begin{cases} -\mathrm{div} A(x, \nabla u) = \lambda \beta(x) u^q + f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

involving a general operator in divergence form of p-Laplacian type ($p > 1$). In our context, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$, A is a continuous function with potential a, λ is a real parameter, $\beta \in L^\infty(\Omega)$ is allowed to be indefinite in sign, $q > 0$ and $f : [0, +\infty) \to \mathbb{R}$ is a continuous function oscillating near the origin or at infinity. Through variational and topological methods, we show that the number of solutions of the problem is influenced by the competition between the power u^q and the oscillatory term f. To be precise, we prove that, when f oscillates near the origin, the problem admits infinitely many solutions when $q \geq p - 1$ and at least a finite number of solutions when $0 < q < p - 1$. While, when f oscillates at infinity, the converse holds true, that is, there are infinitely many solutions if $0 < q \leq p - 1$, and at least a finite number of solutions if $q > p - 1$. In all these cases, we also give some estimates for the $W^{1,p}$ and L^∞-norm of the solutions. The results presented here extend some recent contributions obtained for equations driven by the Laplace operator, to the case of the p-Laplacian or even to more general differential operators.

Keywords: Dirichlet problems; elliptic operators; p-Laplacian operator; infinitely many solutions; variational methods.

Mathematics Subject Classification 2010: 35J62, 35J92, 35J20, 35J62, 35J15, 47J30

1. Introduction

Competition phenomena in elliptic equations have been widely studied in the literature in different contexts. After the seminal work [4], where Ambrosetti, Brezis and Cerami studied a Laplacian equation involving a concave–convex nonlinearity, a lot of papers appeared on this subject (see, for instance, [5, 6, 10, 15, 16, 18–20, 27] and the references therein). Also when dealing with singular terms, the interactions with different type of nonlinearities were investigated: see, for instance, [8] for supercritical nonlinearities, [21, 28, 29] for equations involving superlinear and subcritical terms, [1] for the concave–convex setting and [12] for the asymptotically linear case, just to name a few.

Equations driven by the p-Laplace operator, or, more generally, by operators in divergence form of p-Laplacian type were widely studied recently in the literature (see, e.g., [1, 10, 17, 14, 24] and the references therein).

In this paper we are interested in problems driven by general operators of p-Laplacian type involving oscillatory terms, in presence of a concave or convex power. Usually, equations involving oscillatory nonlinearities give infinitely many distinct solutions (see, e.g., [25, 26, 30] and references therein for more details), but the presence of an additional term may alter the situation. For instance, in [22] the authors studied a Laplacian equation with an oscillatory term in presence of a power and they showed that the number of solutions depend strongly on this power: when there is an oscillatory term near the origin the equation under consideration admits infinitely many distinct solutions if the power is convex, while it has a finite number of distinct solution when the power is concave. In the case of oscillations at infinity, the converse result holds true.

The aim of the present paper is to extend some of the results obtained in [22] to a general class of quasilinear equations of p-Laplacian type. Precisely, here we deal with the following problem

$$\begin{cases} -\mathrm{div} A(x,\nabla u) = \lambda\beta(x)u^q + f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$, $q > 0$ and $\lambda \in \mathbb{R}$ are parameters, while $\beta \in L^\infty(\Omega)$ and $f : [0, +\infty) \to \mathbb{R}$ is a continuous function.

We also assume that $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that

$$A \text{ is continuous in } \overline{\Omega} \times \mathbb{R}^N; \tag{1.2}$$

there exist $p > 1$ and two positive constants $\Gamma_1 \leq \Gamma_2$ such that

$$A(x,\xi) \cdot \xi \geq \Gamma_1 |\xi|^p \quad \text{and} \quad |A(x,\xi)| \leq \Gamma_2 |\xi|^{p-1} \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}^N; \tag{1.3}$$

$$A = \nabla_\xi a. \tag{1.4}$$

Here, $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is the potential of A (with respect to the second variable) and satisfies the following conditions:

$$a \text{ is continuous in } \overline{\Omega} \times \mathbb{R}^N; \tag{1.5}$$

$$a(x,0) = 0 \quad \text{and} \quad a(x,\xi) = a(x,-\xi) \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}^N; \tag{1.6}$$

$$a(x,\cdot) \text{ is strictly convex in } \mathbb{R}^N \quad \text{for all } x \in \Omega. \tag{1.7}$$

Assumptions (1.2)–(1.7) are natural structural conditions. As a model for A we can take the function

$$A(x,\xi) = |\xi|^{p-2}\xi,$$

(of course, in this case $a(x,\xi) = |\xi|^p/p$) which gives rise to the well-known p-Laplace operator Δ_p, defined as

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

The purpose of this paper is to study the number and the behavior of the solutions of problem (1.1), when f *oscillates near the origin* or *at infinity*. This analysis will be carried on using variational and topological techniques. In the sequel, we state our main results, treating separately the two cases, that is, when the nonlinearity f oscillates near the origin or at infinity, respectively.

Finally, we would like to emphasize that the coefficient $\beta \in L^\infty(\Omega)$ in problem (1.1) is allowed to be *indefinite* in sign, as suggested by several well-known works (see, for instance, [2, 3, 7, 15, 16] and references therein).

The plan of the paper is as follows. In Sec. 2 we will state the main results of the paper in the two different situations when f oscillates near the origin or at infinity. In Sec. 3 we will comment the assumptions on the data of problem (1.1). In Sec. 4 we will consider an auxiliary problem and for it we will prove the existence of solutions by direct minimization. Finally, in Sec. 5 we will study problem (1.1) in presence of an oscillation term near zero, while Sec. 6 is devoted to the case of oscillations at infinity.

We refer to the recent books by Brezis [9] and Ciarlet [11] for related results and complements.

2. Main Results

This section is devoted to the main results of the paper, where we prove the existence of infinitely many solutions for problem (1.1) in these two different contexts:

(i) f oscillating near the origin and $q \geq p-1$;
(ii) f oscillating at infinity and $0 < q \leq p-1$,

while, in the remaining cases, that is, when

(iii) f oscillates near the origin and $0 < q < p-1$;
(iv) f oscillates at infinity and $q > p-1$,

we show the existence of at least a finite number of solutions. Here p is the parameter appearing in (1.3).

In all these cases we assume that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function. Also, we denote by F the function

$$F(s) := \int_0^s f(t)dt \tag{2.1}$$

for any $s > 0$.

As usual, here and in the sequel, $W_0^{1,p}(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}.$$

2.1. *Oscillation near the origin*

In this framework we assume that the following conditions are satisfied:

$$\liminf_{s\to 0^+} \frac{f(s)}{s^{p-1}} =: -\ell_0 \in [-\infty, 0); \tag{2.2}$$

$$-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^p} \leq \limsup_{s\to 0^+} \frac{F(s)}{s^p} = +\infty, \tag{2.3}$$

where p is the parameter given in (1.3) and F is the function defined in (2.1).

As a model for f we can take the function

$$f(s) = \begin{cases} \alpha s^{\alpha-1}(1 - \sin s^{-\sigma}) + \sigma s^{\alpha-\sigma-1}\cos s^{-\sigma} - p\gamma s^{p-1} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \end{cases}$$

where α, σ and γ are such that $1 < \sigma + 1 < \alpha < p$ and $\gamma > 0$. Note that f is continuous in $[0, +\infty)$ and F is the following function

$$F(s) = \int_0^s f(t)dt = s^\alpha(1 - \sin s^{-\sigma}) - \gamma s^p, \quad s > 0.$$

Another prototype for f is given by

$$f(s) = \begin{cases} \alpha s^{\alpha-1}\cos^2 s^{-\sigma} - 2\sigma s^{\alpha-\sigma-1}\cos s^{-\sigma}\sin s^{-\sigma} - p\gamma s^{p-1} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \end{cases}$$

where α, σ and γ are such that $1 < \alpha < p$, $\sigma > 0$, $\alpha - \sigma > 1$ and $\gamma > 0$. Thanks to these choices of the parameters f is continuous in $[0, +\infty)$. Also F is the following function

$$F(s) = \int_0^s f(t)dt = s^\alpha \cos^2 s^{-\sigma} - \gamma s^p, \quad s > 0.$$

In both examples, we deduce by direct calculations that f and F satisfy assumptions (2.2) and (2.3).

In this setting our main result can be stated as follows.

Theorem 2.1. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 3$, *be a bounded domain with smooth boundary,* $\lambda \in \mathbb{R}$ *and let* $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ *and* $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ *be two functions satisfying* (1.2)–(1.4) *and* (1.5)–(1.7), *respectively. Assume that* $\beta \in L^\infty(\Omega)$ *and* $f \in C([0,+\infty);\mathbb{R})$ *satisfies* (2.2) *and* (2.3). *If either*

(a) $q = p-1$, $\ell_0 \in (0,+\infty)$ *and* $\lambda\beta(x) < \lambda_0$ *a.e.* $x \in \Omega$ *for some* $\lambda_0 \in (0, \ell_0)$ *or*
(b) $q = p-1$, $\ell_0 = +\infty$ *and* $\lambda \in \mathbb{R}$ *is arbitrary or*
(c) $q > p-1$ *and* $\lambda \in \mathbb{R}$ *is arbitrary,*

then there exists a sequence $\{u_j\}_j$ *in* $W_0^{1,p}(\Omega)$ *of distinct weak solutions of problem* (1.1) *such that*

$$\lim_{j\to+\infty} \|u_j\|_{W_0^{1,p}(\Omega)} = \lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = 0. \tag{2.4}$$

While, if $0 < q < p-1$, *then for every* $k \in \mathbb{N}$ *there exists* $\Lambda_k > 0$ *such that problem* (1.1) *has at least* k *distinct weak solutions* $u_1, \ldots, u_k \in W_0^{1,p}(\Omega)$ *such that*

$$\|u_j\|_{W_0^{1,p}(\Omega)} \leq 1/j \quad \text{and} \quad \|u_j\|_{L^\infty(\Omega)} \leq 1/j, \;\; j = 1,\ldots,k \tag{2.5}$$

provided $|\lambda| < \Lambda_k$.

Assumption (2.2) yields the existence of solutions for problem (1.1), while (2.3) allows us to deduce some information about the number of the solutions.

We also would like to note that assertion (b) covers also the case when the power q is critical or supercritical, that is the case when $q \geq p^*$, where

$$p^* = Np/(N-p), \quad N > p \tag{2.6}$$

is the Sobolev critical exponent.

2.2. *Oscillation at infinity*

In this framework we assume that the following assumptions hold true:

$$\liminf_{s\to+\infty} \frac{f(s)}{s^{p-1}} =: -\ell_\infty \in [-\infty, 0); \tag{2.7}$$

$$-\infty < \liminf_{s\to+\infty} \frac{F(s)}{s^p} \leq \limsup_{s\to+\infty} \frac{F(s)}{s^p} = +\infty, \tag{2.8}$$

where p is the parameter given in (1.3) and F is as in (2.1).

As in the case of the oscillations near the origin, here we can construct a prototype for f as follows:

$$f(s) = \alpha s^{\alpha-1}(1 - \sin s^\sigma) - \sigma s^{\alpha+\sigma-1} \cos s^\sigma - p\gamma s^{p-1},$$

where α, σ and γ are such that $\alpha > p$, $\sigma > 0$ and $\gamma > 0$. Note that f is continuous in $[0, +\infty)$ and F is the following function

$$F(s) = \int_0^s f(t)dt = s^\alpha(1 - \sin s^\sigma) - \gamma s^p, \quad s > 0.$$

Also in this case, direct calculations show that f and F satisfy assumptions (2.7) and (2.8).

In this setting the counterpart of Theorem 2.1 can be stated as follows.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$ and let $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be two functions satisfying* (1.2)–(1.4) *and* (1.5)–(1.7), *respectively. Assume that $\beta \in L^\infty(\Omega)$ and $f \in C([0, +\infty); \mathbb{R})$ satisfies* (2.7), (2.8) *and $f(0) = 0$. If either*

(a) $q = p - 1$, $\ell_\infty \in (0, +\infty)$ *and $\lambda\beta(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $\lambda_\infty \in (0, \ell_\infty)$ or*
(b) $q = p - 1$, $\ell_\infty = +\infty$ *and $\lambda \in \mathbb{R}$ is arbitrary or*
(c) $0 < q < p - 1$ *and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_j\}_j$ in $W_0^{1,p}(\Omega)$ of distinct weak solutions of problem (1.1) *such that*

$$\lim_{j \to +\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty\,. \tag{2.9}$$

While, if $q > p - 1$, then for every $k \in \mathbb{N}$ there exists $\Lambda_k > 0$ such that problem (1.1) *has at least k distinct weak solutions $u_1, \ldots, u_k \in W_0^{1,p}(\Omega)$ such that*

$$\|u_j\|_{L^\infty(\Omega)} \geq j - 1, \quad j = 1, \ldots, k \tag{2.10}$$

provided $|\lambda| < \Lambda_k$.

As in the case when there is an oscillation near the origin, here assumption (2.7) is used in order to prove the existence of solutions for problem (1.1), while (2.8) guarantees that these solutions are infinitely many, when $0 < q \leq p - 1$, and at least a finite number, if $q > p - 1$.

In all the situations, that is, when there is an oscillation near zero or at infinity and for any value of q, the idea is to prove the existence of solutions for problem (1.1) using variational method. More precisely, we first consider an auxiliary problem and, under suitable assumptions on the data, we prove the existence of solutions for this equation studying the associated energy functional and proving that this functional admits a minimum, using the direct methods of the calculus of variations (see Theorem 4.1). Next, we apply Theorem 4.1 to problem (1.1), in order to get Theorems 2.1 and 2.2.

3. Some Comments on the Assumptions

In this section we comment the assumptions on the data of problem (1.1) and we prove some preliminary results which will be useful in the sequel.

First of all, we would like to emphasize that the technical assumptions (1.2)–(1.7) appeared very recently in [14], where the authors studied elliptic equations involving operators in divergence form and proved the existence of at least three non-trivial weak solutions for their problem.

Now, we derive some relations involving A and a. Since a is the potential of A and (1.6) holds true, it is easily seen that for any $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$

$$a(x,\xi) = \int_0^1 \frac{d(a(x,t\xi))}{dt}\,dt = \int_0^1 A(x,t\xi)\cdot\xi\,dt, \tag{3.1}$$

thanks to (1.6).

Furthermore, assumption (1.3) implies

$$\int_0^1 A(x,t\xi)\cdot\xi\,dt = \int_0^1 \frac{1}{t}A(x,t\xi)\cdot t\xi\,dt \geq \frac{\Gamma_1}{p}|\xi|^p \tag{3.2}$$

and

$$\int_0^1 A(x,t\xi)\cdot\xi\,dt \leq |\xi| \int_0^1 |A(x,t\xi)|\,dt \leq \frac{\Gamma_2}{p}|\xi|^p \tag{3.3}$$

for any $(x,\xi) \in \Omega \times \mathbb{R}^N$.

Hence, as a consequence of (3.1)–(3.3), we easily get that

$$\frac{\Gamma_1}{p}|\xi|^p \leq a(x,\xi) \leq \frac{\Gamma_2}{p}|\xi|^p, \tag{3.4}$$

for every $(x,\xi) \in \Omega \times \mathbb{R}^N$.

Thus, for every $u \in W_0^{1,p}(\Omega)$,

$$\frac{\Gamma_1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} \leq \int_\Omega a(x,\nabla u(x))dx \leq \frac{\Gamma_2}{p}\|u\|^p_{W_0^{1,p}(\Omega)}, \tag{3.5}$$

that is

$$u \mapsto \int_\Omega a(x,\nabla u(x))dx$$

is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the usual one.

In the literature, when dealing with general second-order operators in divergence form, the standard condition required on a is the p-uniformly convexity, that is, that there exists a constant $K > 0$ such that

$$a\left(x,\frac{\xi+\eta}{2}\right) \leq \frac{1}{2}\,a(x,\xi) + \frac{1}{2}\,a(x,\eta) - K|\xi-\eta|^p, \tag{3.6}$$

for every $x \in \Omega$ and $\xi,\eta \in \mathbb{R}^N$ (see, for instance, [17, 24] and references therein). We would like to note that condition (1.7) is weaker than (3.6). Indeed, the function $a(x,\xi) = |\xi|^p/p$ satisfies (1.7) for any $p > 1$, while verifies (3.6) just when $p \geq 2$ (see [17]).

Before ending this section, we would like to discuss a property of the function f, which will be useful in the sequel. As a consequence of assumptions (2.2) and

(2.3) we have that

$$f(0) = 0. \tag{3.7}$$

Indeed, suppose that $f(0) = L \in \mathbb{R}\setminus\{0\}$. Then, by the continuity of f and (2.2) we would get

$$\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = -\infty,$$

so that, by l'Hôpital's rule we would deduce that

$$\lim_{s\to 0^+} \frac{F(s)}{s^p} = \lim_{s\to 0^+} \frac{f(s)}{ps^{p-1}} = -\infty,$$

which contradicts (2.3). Hence, assertion (3.7) holds true.

4. An Auxiliary Problem

In this section we consider the problem

$$\begin{cases} -\mathrm{div} A(x, \nabla u) + K(x)|u|^{p-2}u = h(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_h^K}$$

Here, we assume that $K : \Omega \to \mathbb{R}$ is such that

$$K \in L^\infty(\Omega) \quad \text{with } \operatorname*{ess\,inf}_{x\in\Omega} K(x) > 0, \tag{4.1}$$

while $h : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

$$h(x, 0) = 0 \quad \text{for a.e. } x \in \Omega; \tag{4.2}$$

$$\text{there exists } M > 0 \text{ such that } |h(x, s)| \le M \text{ for a.e. } x \in \Omega \text{ and for any } s \ge 0; \tag{4.3}$$

$$\begin{gathered} \text{there exist } \delta \text{ and } \eta, \text{ with } 0 < \delta < \eta, \text{ such that} \\ h(x, s) \le 0 \text{ for a.e. } x \in \Omega \text{ and for any } s \in [\delta, \eta]. \end{gathered} \tag{4.4}$$

In the sequel we extend the function h on the whole $\Omega \times \mathbb{R}$ by taking $h(x, s) = 0$ for a.e. $x \in \Omega$ and $s < 0$.

The aim of this section is to prove the existence of a non-negative weak solution for problem (P_h^K), that is, a non-negative solution of the following problem:

$$\begin{cases} \displaystyle\int_\Omega A(x, \nabla u(x))\nabla\varphi(x)dx + \int_\Omega K(x)|u(x)|^{p-2}u(x)\varphi(x)dx \\ \quad = \displaystyle\int_\Omega h(x, u(x))\varphi(x)dx \quad \text{for any } \varphi \in W_0^{1,p}(\Omega), \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{4.5}$$

Problem (4.5) has a variational nature and the energy functional $\mathcal{E}_{K,h} : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with it is defined as follows

$$\mathcal{E}_{K,h}(u) = \int_\Omega a(x, \nabla u(x))dx + \frac{1}{p}\int_\Omega K(x)|u(x)|^p\, dx - \int_\Omega H(x, u(x))dx, \tag{4.6}$$

where

$$H(x,s) := \int_0^s h(x,t)dt \quad \text{for any } s \in \mathbb{R}. \tag{4.7}$$

Due to (3.5), hypotheses (4.1)–(4.4) and the embedding properties of the space $W_0^{1,p}(\Omega)$ into the Lebesgue spaces, it is easy to see that $\mathcal{E}_{K,h}$ is well defined. Moreover, standard arguments show that $\mathcal{E}_{K,h}$ is of class C^1 on $W_0^{1,p}(\Omega)$.

Hence, finding non-negative solutions of problem (4.5) means looking for non-negative critical points of the functional $\mathcal{E}_{K,h}$. At this purpose, we introduce the set W^η defined as follows:

$$W^\eta := \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^\infty(\Omega)} \le \eta\},$$

where η is the positive parameter given in (4.4).

The main result of this section is given in the following theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$, be a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$ and let $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be two functions satisfying* (1.2)–(1.4) *and* (1.5)–(1.7)*, respectively. Assume that $K : \Omega \to \mathbb{R}$ is a function verifying* (4.1) *and that $h : \Omega \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function satisfying* (4.2)–(4.4)*. Then,*

(i) *the functional $\mathcal{E}_{K,h}$ is bounded from below on W^η and its infimum is attained at some $u_\eta \in W^\eta$;*
(ii) *$u_\eta \in [0,\delta]$, where δ is the positive parameter given in* (4.4);
(iii) *u_η is a non-negative weak solution of problem (P_h^K).*

Proof. Let us start by proving assertion (i). First of all, it is easy to see that the set W^η is convex. Moreover, W^η is closed in $W_0^{1,p}(\Omega)$. To see this, let $\{u_j\}_j$ be a sequence in W^η such that $u_j \to u$ in $W_0^{1,p}(\Omega)$ as $j \to +\infty$.

We claim that $u \in W^\eta$. Of course $u \in W_0^{1,p}(\Omega)$. Furthermore, by assumption $\{u_j\}_j$ is bounded in $L^\infty(\Omega)$. Since $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$, which is a separable Banach space, then by [9, Corollary III.26] we get that $u_j \to u$ in the weak* topology of $L^\infty(\Omega)$ as $j \to +\infty$. Hence, [9, Proposition III.12] yields that, up to a subsequence, still denoted by $\{u_j\}_j$,

$$\liminf_{j \to +\infty} \|u_j\|_{L^\infty(\Omega)} \ge \|u\|_{L^\infty(\Omega)}.$$

As a consequence of this and taking into account that

$$\|u_j\|_{L^\infty(\Omega)} \le \eta$$

for any $j \in \mathbb{N}$, we get that

$$\|u\|_{L^\infty(\Omega)} \le \eta,$$

that is $u \in W^\eta$, which proves the claim.

Thus, since W^η is convex and closed in $W_0^{1,p}(\Omega)$, then it is weakly closed in $W_0^{1,p}(\Omega)$ by [9, Theorem III.7].

Now, let us consider the functional $\mathcal{E}_{K,h}$. One can easily see that $\mathcal{E}_{K,h}$ is sequentially weakly lower semicontinuous (see [13, Lemma 3.4.3] for details). Moreover, note that (3.5), (4.1), (4.3) and the definition of H yield that the functional $\mathcal{E}_{K,h}$ is bounded from below on W^η. Indeed, for any $u \in W^\eta$

$$\begin{aligned}
\mathcal{E}_{K,h}(u) &= \int_\Omega a(x, \nabla u(x))dx + \frac{1}{p}\int_\Omega K(x)|u(x)|^p\,dx - \int_\Omega H(x,u(x))dx \\
&\geq \frac{\Gamma_1}{p}\|u\|^p_{W^{1,p}_0(\Omega)} - \int_\Omega H(x,u(x))dx \\
&\geq -\int_\Omega H(x,u(x))dx \\
&\geq -M\int_\Omega |u(x)|dx \\
&\geq -\eta M\mathcal{L}(\Omega),
\end{aligned}$$

where $\mathcal{L}(\Omega)$ denotes the Lebesgue measure of Ω.

Let us denote by α_η the infimum of $\mathcal{E}_{K,h}$ on W^η, that is,

$$\alpha_\eta := \inf_{u\in W^\eta} \mathcal{E}_{K,h}(u) > -\infty. \tag{4.8}$$

It is easily seen that for every $k \in \mathbb{N}$, there exists $u_k \in W^\eta$ such that

$$\alpha_\eta \leq \mathcal{E}_{K,h}(u_k) \leq \alpha_\eta + \frac{1}{k}. \tag{4.9}$$

Also, since $u_k \in W^\eta$ and thanks to (4.3), we get

$$\begin{aligned}
\int_\Omega a(x,\nabla u_k(x))dx + \frac{1}{p}\int_\Omega K(x)|u_k(x)|^p dx &= \int_\Omega H(x,u_k(x))dx + \mathcal{E}_{K,h}(u_k) \\
&\leq \eta M\mathcal{L}(\Omega) + \mathcal{E}_{K,h}(u_k) \\
&\leq \eta M\mathcal{L}(\Omega) + \alpha_\eta + \frac{1}{k} \\
&\leq \eta M\mathcal{L}(\Omega) + \alpha_\eta + 1,
\end{aligned}$$

for every $k \in \mathbb{N}$. Thus, by (3.5) and (4.1)

$$\|u_k\|^p_{W^{1,p}_0(\Omega)} \leq \frac{p}{\Gamma_1}(\eta M\mathcal{L}(\Omega) + \alpha_\eta + 1), \tag{4.10}$$

for every $k \in \mathbb{N}$. Then, the sequence $\{u_k\}_k$ is bounded in $W^{1,p}_0(\Omega)$ and so, up to a subsequence, still denoted by $\{u_k\}_k$,

$$u_k \to u_\eta \quad \text{weakly in } W^{1,p}_0(\Omega) \tag{4.11}$$

as $k \to +\infty$ for some $u_\eta \in W^{1,p}_0(\Omega)$.

Now, let us show that u_η is the minimum of $\mathcal{E}_{K,h}$ we are looking for. At this purpose, first of all note that $u_\eta \in W^\eta$, since W^η is weakly closed in $W^{1,p}_0(\Omega)$. Thus, by (4.8)

$$\mathcal{E}_{K,h}(u_\eta) \geq \alpha_\eta. \tag{4.12}$$

On the other hand, thanks to the sequential weak lower semicontinuity of $\mathcal{E}_{K,\,h}$, (4.9) and (4.11), we obtain that

$$\alpha_\eta \geq \liminf_{k\to+\infty} \mathcal{E}_{K,\,h}(u_k) \geq \mathcal{E}_{K,\,h}(u_\eta).$$

By this and (4.12) we obtain that

$$\mathcal{E}_{K,\,h}(u_\eta) = \alpha_\eta.$$

This and (4.8) conclude the proof of statement (i).

Now, let us prove (ii). At this purpose, let δ be as in assumption (4.4) and let B the following set

$$B := \{x \in \Omega : u_\eta(x) \notin [0,\delta]\}.$$

We argue by contradiction and we suppose that $\mathcal{L}(B) > 0$.

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be the function given by

$$\gamma(s) := \min\{s_+, \delta\},$$

where $s_+ = \max\{s, 0\}$. Also, set $w := \gamma \circ u_\eta$, that is

$$w(x) = \begin{cases} \delta & \text{if } u_\eta(x) > \delta, \\ u_\eta(x) & \text{if } 0 \leq u_\eta(x) \leq \delta, \\ 0 & \text{if } u_\eta(x) < 0, \end{cases}$$

for a.e. $x \in \Omega$.

Since γ is a Lipschitz function and $\gamma(0) = 0$, the theorem of Marcus–Mizel (see [9]) shows that $w \in W_0^{1,p}(\Omega)$. Moreover, $0 \leq w(x) \leq \delta$ for a.e. Ω. Consequently, $w \in W^\eta$, being $\delta < \eta$, by assumption (4.4).

We introduce the sets

$$B_1 := \{x \in \Omega : u_\eta(x) < 0\}$$

and

$$B_2 := \{x \in \Omega : u_\eta(x) > \delta\}.$$

Thus, $B = B_1 \cup B_2$, and we have that $w(x) = u_\eta(x)$ for a.e. $x \in \Omega \backslash B, w(x) = 0$ for a.e. $x \in B_1$, and $w(x) = \delta$ for a.e. $x \in B_2$.

As a consequence of this and of (1.6) we get

$$\left(\int_\Omega a(x, \nabla w(x))dx - \int_\Omega a(x, \nabla u_\eta(x))dx\right) = -\int_B a(x, \nabla u_\eta(x))dx,$$

from which it follows that

$$\begin{aligned}\mathcal{E}_{K,\,h}(w) - \mathcal{E}_{K,\,h}(u_\eta) = &\left(\int_\Omega a(x, \nabla w(x))dx - \int_\Omega a(x, \nabla u_\eta(x))dx\right) \\ &+ \frac{1}{p}\int_\Omega K(x)(|w(x)|^p - |u_\eta(x)|^p)dx\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega (H(x, w(x)) - H(x, u_\eta(x)))dx \\
&= - \int_B a(x, \nabla u_\eta(x))dx + \frac{1}{p}\int_B K(x)(|w(x)|^p - |u_\eta(x)|^p)dx \\
&\quad - \int_B (H(x, w(x)) - H(x, u_\eta(x)))dx. \qquad (4.13)
\end{aligned}
$$

Since $\operatorname{ess\,inf}_{x\in\Omega} K(x) > 0$ by (4.1), one has

$$
\begin{aligned}
\int_B K(x)(|w(x)|^p - |u_\eta(x)|^p)dx &= - \int_{B_1} K(x)|u_\eta(x)|^p dx \\
&\quad + \int_{B_2} K(x)(\delta^p - |u_\eta(x)|^p)dx \leq 0. \qquad (4.14)
\end{aligned}
$$

Moreover, due to the fact that $h(x, s) = 0$ for a.e. $x \in \Omega$ and all $s \leq 0$, then

$$\int_{B_1} (H(x, w(x)) - H(x, u_\eta(x)))dx = 0, \qquad (4.15)$$

while, by the mean value theorem, for a.e. $x \in B_2$, there exists $\theta(x) \in [\delta, u_\eta(x)] \subseteq [\delta, \eta]$ such that

$$H(x, w(x)) - H(x, u_\eta(x)) = H(x, \delta) - H(x, u_\eta(x)) = h(x, \theta(x))(\delta - u_\eta(x)).$$

Thus, taking into account (4.4) and the definition of B_2, one has

$$\int_{B_2} (H(x, w(x)) - H(x, u_\eta(x)))dx = \int_{B_2} h(x, \theta(x))(\delta - u_\eta(x))dx \geq 0. \qquad (4.16)$$

Hence, by (4.15) and (4.16), we get that

$$\int_B (H(x, w(x)) - H(x, u_\eta(x)))dx \geq 0. \qquad (4.17)$$

As a consequence of (4.13), (4.14), (4.17) and taking into account (3.5), we get

$$\mathcal{E}_{K,\,h}(w) - \mathcal{E}_{K,\,h}(u_\eta) \leq 0. \qquad (4.18)$$

On the other hand, since $w \in W^\eta$, it is easy to see that $\mathcal{E}_{K,\,h}(w) \geq \mathcal{E}_{K,\,h}(u_\eta)$. By this and (4.18) we get that

$$\mathcal{E}_{K,\,h}(w) = \mathcal{E}_{K,\,h}(u_\eta). \qquad (4.19)$$

Since (4.19) holds true and all the integrals in the right-hand side of (4.13) are non-negative, it is easy to see that every integral term in (4.13) should be zero. In particular,

$$\int_{B_1} K(x)|u_\eta(x)|^p = \int_{B_2} K(x)(|u_\eta(x)|^p - \delta^p)dx = 0.$$

Due to the definition of B_1 and B_2 and to (4.1), we necessarily have $\mathcal{L}(B_1) = \mathcal{L}(B_2) = 0$, that is $\mathcal{L}(B) = 0$, contradicting our assumption. Thus, assertion (ii) is proved.

Finally, let us show (iii). For this let us fix $\varphi \in C_0^\infty(\Omega)$ and let

$$\varepsilon_0 := \frac{\eta - \delta}{\|\varphi\|_{L^\infty(\Omega)} + 1} > 0,$$

where δ and η are given as in (4.4). Moreover, let $E : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ be the function defined as

$$E(\varepsilon) = \mathcal{E}_{K,\,h}(u_\eta + \varepsilon\varphi).$$

First of all, note that, since (ii) holds true, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ we have

$$\begin{aligned} |u_\eta(x) + \varepsilon\varphi(x)| &\le |u_\eta(x)| + \varepsilon|\varphi(x)| \\ &\le u_\eta(x) + \frac{\eta - \delta}{\|\varphi\|_{L^\infty(\Omega)} + 1}\|\varphi\|_{L^\infty(\Omega)} \\ &\le \delta + \eta - \delta = \eta, \end{aligned}$$

for a.e. $x \in \Omega$. So, $u_\eta + \varepsilon\varphi \in W^\eta$.

Consequently, due to (i), one has $E(\varepsilon) \ge E(0)$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, that is 0 is an interior minimum point for E. Then, since E is differentiable at 0, it is easy to see that $E'(0) = 0$ and so also $\langle \mathcal{E}'_{K,\,h}(u_\eta), \varphi\rangle = 0$. Taking into account that $\varphi \in C_0^\infty(\Omega)$ is arbitrary and using the definition of $\mathcal{E}_{K,\,h}$, we obtain that u_η is a weak solution of problem (P_h^K) (that is a solution of (4.5)). Of course, u_η is non-negative in Ω thanks to (ii) and this ends the proof of (iii). $\square$

We would like to note that, as a consequence of (1.3), it is easily seen that $A(x, 0) = 0$ for any $x \in \Omega$. Hence, since also $h(x, 0) = 0$ a.e. $x \in \Omega$ by (4.2), the function $u \equiv 0$ is a weak solution of problem (P_h^K). Theorem 4.1 does not guarantee that the solution u_η of problem (P_h^K) is not the trivial one. In spite of this, by Theorem 4.1 we will derive the existence of non-trivial solutions for the original problem (1.1), provided the nonlinear term f is chosen appropriately.

We conclude this section by constructing a special function which will be useful in the proof of our main theorems. In the sequel, let $x_0 \in \Omega$ and $r > 0$ be such that $B(x_0, r) \subset \Omega$. For any $s > 0$ we define the function z_s as follows:

$$z_s(x) := \begin{cases} 0 & \text{if } x \in \Omega \backslash B(x_0, r), \\ \dfrac{2s}{r}(r - |x - x_0|) & \text{if } x \in B(x_0, r) \backslash B(x_0, r/2), \\ s & \text{if } x \in B(x_0, r/2). \end{cases} \tag{4.20}$$

It is clear that $z_s \ge 0$ in Ω and $z_s \in W_0^{1,p}(\Omega)$. Moreover, $\|z_s\|_{L^\infty(\Omega)} = s$ and

$$\|z_s\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla z_s(x)|^p\, dx \le \frac{2^p s^p \omega_N r^N}{r^p} \equiv C(r, p, N)s^p, \tag{4.21}$$

where $C(r, p, N)$ is a positive constant and ω_N is the volume of the unit ball in $\mathbb{R}^N$.

We also introduce the truncation function $\tau_\eta : [0, +\infty) \to \mathbb{R}$ defined as

$$\tau_\eta(s) := \min\{\eta, s\} \tag{4.22}$$

for any $s \geq 0$, where η is the positive constant given in assumption (4.4). Note that τ_η is a continuous function in $[0, +\infty)$.

5. Oscillation Near the Origin

In this section we study problem (1.1) in the case when the nonlinear term f oscillates near the origin.

In order to prove Theorem 2.1 we first give an auxiliary result obtained as a consequence of Theorem 4.1. Precisely, we prove the existence of infinitely many solutions for problem (P_h^K) under the following assumptions on the function h:

$$\text{there exists } \bar{s} > 0 \text{ such that } \sup_{s\in[0,\bar{s}]} |h(\cdot, s)| \in L^\infty(\Omega); \tag{5.1}$$

$$\begin{gathered}\text{there exist two sequences } \{\delta_j\}_j \text{ and } \{\eta_j\}_j \text{ with } 0 < \eta_{j+1} < \delta_j < \eta_j \text{ and} \\ \lim_{j\to+\infty} \eta_j = 0 \text{ such that } h(x, s) \leq 0 \text{ for a.e. } x \in \Omega \\ \text{and for every } s \in [\delta_j, \eta_j],\ j \in \mathbb{N};\end{gathered} \tag{5.2}$$

$$-\infty < \liminf_{s\to0^+} \frac{H(x, s)}{s^p} \leq \limsup_{s\to0^+} \frac{H(x, s)}{s^p} = +\infty \text{ uniformly for a.e. } x \in \Omega, \tag{5.3}$$

where H is the function given in (4.7).

In this setting our result for problem (P_h^K) is the following theorem.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$ and let $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be two functions satisfying* (1.2)–(1.4) *and* (1.5)–(1.7), *respectively. Moreover, assume that $K : \Omega \to \mathbb{R}$ satisfies* (4.1) *and $h : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function verifying* (4.2) *and* (5.1)–(5.3).

Then, there exists a sequence $\{u_j\}_j \subset W_0^{1,p}(\Omega)$ of distinct non-trivial non-negative weak solutions of problem (P_h^K) such that

$$\lim_{j\to+\infty} \|u_j\|_{W_0^{1,p}(\Omega)} = \lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = 0. \tag{5.4}$$

Proof. Since $\eta_j \to 0$ as $j \to +\infty$, by (5.2), without loss of generality, we may assume that

$$\delta_j < \eta_j < \bar{s} \tag{5.5}$$

for j sufficiently large, where $\bar{s} > 0$ comes from (5.1).

For every $j \in \mathbb{N}$, let $h_j : \Omega \times [0, +\infty) \to \mathbb{R}$ be the function defined by

$$h_j(x, s) = h(x, \tau_{\eta_j}(s)), \tag{5.6}$$

and

$$H_j(x,s) := \int_0^s h_j(x,t)dt$$

for a.e. $x \in \Omega$ and $s \geq 0$, where τ_{η_j} is the function defined in (4.22) with $\eta = \eta_j$. Also, in what follows for any $j \in \mathbb{N}$ we denote by

$$\mathcal{E}_j := \mathcal{E}_{K,\,h_j}, \tag{5.7}$$

where $\mathcal{E}_{K,\,h_j}$ is the functional given in (4.6), with $h = h_j$. Note that $\mathcal{E}_j$ is the energy functional associated with problem $(P^K_{h_j})$, that is (P^K_h) with $h = h_j$.

The function h_j verifies all the assumptions of Theorem 4.1 for $j \in \mathbb{N}$ large enough. Indeed, due to the regularity of h, the continuity of τ_η and (4.2), the function h_j is Carathéodory and such that $h_j(x,0) = 0$ a.e. $x \in \Omega$. Moreover, by (5.1), (5.5) and (5.6), h_j satisfies (4.3). Finally, condition (4.4) comes from (5.2).

Hence, as a consequence of Theorem 4.1, for j sufficiently large there exists $u_j \in W^{\eta_j}$ such that

$$\min_{u \in W^{\eta_j}} \mathcal{E}_j(u) = \mathcal{E}_j(u_j) \tag{5.8}$$

$$u_j(x) \in [0, \delta_j] \quad \text{for a.e. } x \in \Omega, \tag{5.9}$$

and

$$u_j \text{ is a non-negative weak solution of } (P^K_{h_j}). \tag{5.10}$$

By the definition of τ_η, (5.6) and the fact that $u_j(x) \leq \delta_j < \eta_j$ a.e. $x \in \Omega$, then

$$h_j(x, u_j(x)) = h(x, \tau_{\eta_j}(u_j(x)) = h(x, u_j(x))$$

a.e. $x \in \Omega$. Thus, by this and (5.10), u_j is a non-negative weak solution not only for $(P^K_{h_j})$ but also for problem (P^K_h).

To conclude the proof of Theorem 5.1, we have to prove that there are infinitely many distinct elements in the sequence $\{u_j\}_j$. In order to see this, we first claim that

$$\mathcal{E}_j(u_j) < 0 \quad \text{for } j \in \mathbb{N} \text{ large enough.} \tag{5.11}$$

Assumption (5.3) implies the existence of some $\ell > 0$ and $\zeta \in (0, \eta_1)$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x,s) \geq -\ell s^p \quad \text{for all } s \in (0,\zeta) \tag{5.12}$$

and that there is a sequence $\{s_j\}_j$ such that $0 < s_j \to 0$ as $j \to +\infty$ (here we use the definition of H, to take $s_j > 0$) such that

$$\lim_{j\to+\infty} \frac{\operatorname*{ess\,inf}_{x\in\Omega} H(x,s_j)}{s_j^p} = +\infty, \tag{5.13}$$

namely, for any $L > 0$

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x,s_j) > L s_j^p \tag{5.14}$$

for $j \in \mathbb{N}$ large enough.

Since $\delta_j \searrow 0$ as $j \to +\infty$, we can choose a subsequence of $\{\delta_j\}_j$, still denoted by $\{\delta_j\}_j$, such that

$$s_j \le \delta_j \tag{5.15}$$

for all $j \in \mathbb{N}$.

Now, let us fix $j \in \mathbb{N}$ sufficiently large and let

$$z_j := z_{s_j} \in W_0^{1,p}(\Omega)$$

be the function defined as in (4.20) with $s = s_j$. Then, $z_j \in W_0^{1,p}(\Omega)$ and $\|z_j\|_{L^\infty(\Omega)} = s_j \le \delta_j < \eta_j$ by (5.15) and (5.2). Hence, $z_j \in W^{\eta_j}$ and $0 \le z_j(x) \le s_j \le \delta_j < \eta_j$ a.e. $x \in \Omega$. This yields that for a.e. $x \in \Omega$

$$\int_0^{z_j(x)} h_j(x,t)dt = \int_0^{z_j(x)} h(x, \tau_{\eta_j}(t))dt = \int_0^{z_j(x)} h(x,t)dt.$$

By this and taking into account (3.5), (4.1) and (4.21), for j sufficiently large one has:

$$\begin{aligned}
\mathcal{E}_j(z_j) &= \int_\Omega a(x, \nabla z_j(x))dx + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_\Omega H_j(x, z_j(x))dx \\
&= \int_\Omega a(x, \nabla z_j(x))dx + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_\Omega H(x, z_j(x))dx \\
&\le C(r,p,N)\frac{\Gamma_2}{p}s_j^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx \\
&\quad - \int_{B(x_0,r/2)} H(x,s_j)dx - \int_{B(x_0,r)\setminus B(x_0,r/2)} H(x, z_j(x))dx \\
&\le \left(C(r,p,N)\frac{\Gamma_2}{p} + \|K\|_{L^\infty(\Omega)}\frac{\mathcal{L}(\Omega)}{p} - L(r/2)^N\omega_N + \ell\mathcal{L}(\Omega)\right) s_j^p,
\end{aligned} \tag{5.16}$$

thanks to (5.12), (5.14) and using the fact that $z_j(x) < \eta_j < \eta_1$ (being $\{\eta_j\}_j$ decreasing by (5.2)). Here ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Choosing $L > 0$ large enough so that

$$L(r/2)^N\omega_N > C(r,p,N)\frac{\Gamma_2}{p} + \|K\|_{L^\infty(\Omega)}\frac{\mathcal{L}(\Omega)}{p} + \ell\mathcal{L}(\Omega),$$

we get that, for j large enough

$$\mathcal{E}_j(z_j) < 0.$$

Consequently, using also (5.8), we obtain that, if j is sufficiently large

$$\mathcal{E}_j(u_j) = \min_{u \in W^{\eta_j}(u)} \mathcal{E}_j \le \mathcal{E}_j(z_j) < 0, \tag{5.17}$$

which proves (5.11). Also, this guarantees that $u_j \not\equiv 0$ in Ω, being $\mathcal{E}_j(0) = 0$.

Now, we claim that

$$\lim_{j \to +\infty} \mathcal{E}_j(u_j) = 0. \tag{5.18}$$

At this purpose, note that for $j \in \mathbb{N}$ sufficiently large, by using the definition of H_j, (5.1), (5.2), (5.5), (5.6) and (5.9), we have

$$\begin{aligned}\mathcal{E}_j(u_j) &\geq -\int_\Omega H_j(x, u_j(x))dx \\ &= -\int_\Omega \int_0^{u_j(x)} h(x,s)ds \\ &\geq -\int_\Omega \sup_{s\in[0,\bar{s}]} |h(x,s)|\, u_j(x)dx \\ &\geq -\mathcal{L}(\Omega) \left\| \sup_{s\in[0,\bar{s}]} |h(\cdot,s)| \right\|_{L^\infty(\Omega)} \delta_j. \end{aligned} \tag{5.19}$$

Since $\lim_{j\to+\infty} \delta_j = 0$ by (5.2), the above inequality and (5.17) leads to (5.18) and so the claim is proved.

Combining (5.11) and (5.18), we deduce that the sequence $\{u_j\}_j$ contains infinitely many distinct elements, that is problem (P_h^K) has infinitely many distinct weak solutions.

Finally, it remains to prove relation (5.4). Since $\|u_j\|_{L^\infty(\Omega)} \leq \delta_j$ for $j \in \mathbb{N}$ sufficiently large by (5.9), and $\lim_{j\to+\infty} \delta_j = 0$ (see (5.2)), we easily get that $\|u_j\|_{L^\infty(\Omega)} \to 0$ as $j \to +\infty$.

For the latter limit, observe that by (4.1), we have

$$\begin{aligned}\int_\Omega a(x, \nabla u_j(x))dx &\leq \int_\Omega a(x, \nabla u_j(x))dx + \frac{1}{p}\int_\Omega K(x)|u_j(x)|^p dx \\ &< \int_\Omega H_j(x, u_j(x))dx \\ &= \int_\Omega H(x, u_j(x))dx \\ &\leq \mathcal{L}(\Omega) \left\| \sup_{s\in[0,\bar{s}]} |h(\cdot,s)| \right\|_{L^\infty(\Omega)} \delta_j, \end{aligned}$$

thanks to (5.1), (5.9) and (5.11).

Thus, by (3.5) and (5.2) it is easy to see that

$$\lim_{j\to+\infty} \|u_j\|^p_{W_0^1(\Omega)} \leq \frac{p}{\Gamma_1} \lim_{j\to+\infty} \int_\Omega a(x, \nabla u_j(x))dx = 0,$$

which concludes the proof of the theorem. □

Now, we are ready to prove Theorem 2.1. The strategy will consists in applying Theorems 4.1 and 5.1 with a suitable choice of the functions K and h.

5.1. *Proof of Theorem* 2.1

First of all, we show that, under suitable assumptions, problem (1.1) has infinitely many distinct weak solutions, provided $q \geq p - 1$. Let us consider separately the case when $q = p-1$ and the one when $q > p-1$: in both the situations the strategy will consist in using Theorem 5.1.

Let us start proving assertion (a). In this setting we suppose that $q = p-1$, $\ell_0 \in (0, +\infty)$ and $\lambda \in \mathbb{R}$ is such that $\lambda\beta(x) < \lambda_0$ a.e. $x \in \Omega$ for some $\lambda_0 \in (0, \ell_0)$. Let us choose $\tilde{\lambda}_0 \in (\lambda_0, \ell_0)$ and let

$$K(x) := \tilde{\lambda}_0 - \lambda\beta(x) \quad \text{and} \quad h(x,s) := \tilde{\lambda}_0 s^{p-1} + f(s), \tag{5.20}$$

a.e. $x \in \Omega$ and $s \geq 0$.

Now, we show that the functions K and h given in (5.20) satisfy all the assumptions of Theorem 5.1. First of all, note that $K \in L^\infty(\Omega)$ and

$$\operatorname*{ess\,inf}_{x \in \Omega} K(x) \geq \tilde{\lambda}_0 - \lambda_0 > 0,$$

thanks to the fact that $\beta \in L^\infty(\Omega)$. Hence, (4.1) is satisfied.

Moreover, using the regularity of f, it is easy to see that h is a continuous function in $\Omega \times [0, +\infty)$ and $h(x, 0) = 0$ for any $x \in \Omega$, since $f(0) = 0$, due to (3.7). Thus, h verifies assumption (4.2). Also, the continuity of $s \mapsto h(\cdot, s)$ and the Weierstrass Theorem yield (5.1). Furthermore, since for any $x \in \Omega$ and $s > 0$

$$\frac{H(x,s)}{s^p} = \frac{\tilde{\lambda}_0}{p} + \frac{F(s)}{s^p},$$

hypothesis (2.3) immediately implies (5.3).

It remains to show that h satisfies (5.2). At this purpose, note that, by (2.2), we get that there exists a sequence $\{s_j\}_j$ converging to 0 such that

$$\frac{f(s_j)}{s_j^{p-1}} \to -\ell_0 \tag{5.21}$$

as $j \to +\infty$. Now, since $\tilde{\lambda}_0 < \ell_0$ by assumption, there exists $\overline{\varepsilon} > 0$ such that $\tilde{\lambda}_0 + \overline{\varepsilon} < \ell_0$. By this and (5.21) we get that, for j large enough, say $j \geq j^* \in \mathbb{N}$,

$$\frac{f(s_j)}{s_j^{p-1}} < -\tilde{\lambda}_0. \tag{5.22}$$

Consequently, by using the continuity of f, there exists a neighborhood of s_j, say (δ_j, η_j) such that

$$h(x,s) = \tilde{\lambda}_0 s^{p-1} + f(s) \leq 0,$$

for any $x \in \Omega$ and all $s \in [\delta_j, \eta_j]$ and $j \geq j^*$. Therefore, (5.2) holds too.

Now, we can apply Theorem 5.1 to problem (P_h^K) with K and h given in (5.20). As a consequence of this, we get the existence of infinitely many distinct non-trivial non-negative solutions $\{u_j\}_j$ for problem (P_h^K), satisfying condition (2.4). Due to the choice of K and h in (5.20) and taking into account that $q = p-1$, it is easy to see that u_j is a weak solution of problem (1.1) and this ends the proof of Theorem 2.1 in the case $q = p-1$.

Now, let us consider assertion (b). At this purpose, let $q = p - 1$, $\ell_0 = +\infty$ and $\lambda \in \mathbb{R}$. In this case we choose $\tilde{\lambda}_0 \in (\lambda_0, \ell_0)$ and

$$K(x) := \tilde{\lambda}_0 \quad \text{and} \quad h(x,s) := (\lambda\beta(x) + \tilde{\lambda}_0)s^{p-1} + f(s), \tag{5.23}$$

a.e. $x \in \Omega$ and $s \geq 0$. In this setting we can argue exactly as in the proof of assertion (a), just replacing formula (5.22) with the following one

$$\frac{f(s_j)}{s_j^{p-1}} < -(|\lambda|\|\beta\|_{L^\infty(\Omega)} + \tilde{\lambda}_0) \tag{5.24}$$

for j large enough, and taking into account that

$$h(x,s) = (\lambda\beta(x) + \tilde{\lambda}_0)s^{p-1} + f(s) \leq (|\lambda|\|\beta\|_{L^\infty(\Omega)} + \tilde{\lambda}_0)s^{p-1} + f(s).$$

Now, let us prove assertion (c). At this purpose, let $q > p - 1$ and $\lambda \in \mathbb{R}$. Let $\tilde{\lambda}_0 \in (0, \ell_0)$ and

$$K(x) := \tilde{\lambda}_0 \quad \text{and} \quad h(x,s) := \lambda\beta(x)s^q + \tilde{\lambda}_0 s^{p-1} + f(s), \tag{5.25}$$

for a.e. $x \in \Omega$ and $s \geq 0$. Also in this setting our aim is to prove that K and h given in (5.25) satisfy the conditions required by Theorem 5.1.

Clearly, (4.1) and (4.2) are trivially satisfied, also thanks to (2.2). Moreover, since $\beta \in L^\infty(\Omega)$, the continuity of $s \mapsto h(\cdot, s)$ and the Weierstrass Theorem yield that (5.1) holds true. Moreover, for a.e. $x \in \Omega$ and $s > 0$ we have

$$\frac{H(x,s)}{s^p} = \lambda\frac{\beta(x)}{q+1}s^{q-p+1} + \frac{\tilde{\lambda}_0}{p} + \frac{F(s)}{s^p},$$

so that hypothesis (2.3) and the fact that $q > p - 1$ imply (5.3).

Finally, note that for a.e $x \in \Omega$ and any $s \geq 0$, we have

$$h(x,s) \leq |\lambda|\|\beta\|_{L^\infty(\Omega)}s^q + \lambda_0 s^{p-1} + f(s). \tag{5.26}$$

As a consequence of this and of (2.2) we get

$$\liminf_{s\to 0^+} \frac{h(x,s)}{s^{p-1}} \leq \liminf_{s\to 0^+} \left(\lambda|\,\|\beta\|_{L^\infty(\Omega)}s^{q-p+1} + \tilde{\lambda}_0 + \frac{f(s)}{s^{p-1}}\right) = \tilde{\lambda}_0 - \ell_0 < 0 \tag{5.27}$$

uniformly a.e. $x \in \Omega$, thanks to the choice of q. Thus, there exists a sequence $\{s_j\}_j$ converging to 0 as $j \to +\infty$ such that $h(x, s_j) < 0$ for $j \in \mathbb{N}$ large enough and uniformly a.e. $x \in \Omega$. Thus, by using the continuity of $s \mapsto h(\cdot, s)$, there exist two sequences $\{\delta_j\}_j, \{\eta_j\}_j$ such that $0 < \eta_{j+1} < \delta_j < s_j < \eta_j, \lim_{j\to+\infty} \eta_j = 0$, and

$$h(x,s) \leq 0,$$

for a.e. $x \in \Omega$ and all $s \in [\delta_j, \eta_j]$ and j large enough. Therefore, hypothesis (5.2) holds. Arguing as in the proof of assertion (a) and applying Theorem 5.1 we get (c).

Finally, let us consider the case when $0 < q < p - 1$. In this setting the strategy will consist in applying Theorem 4.1 to problem (P_h^K) with a suitable choices of K

and h. At this purpose, let $\tilde{\lambda}_0 \in (0, \ell_0)$, where $\ell_0 > 0$ is given in assumption (2.2), and let

$$K(x) := \tilde{\lambda}_0 \quad \text{and} \quad h(x, s, \lambda) := \lambda\beta(x)s^q + \tilde{\lambda}_0 s^{p-1} + f(s), \tag{5.28}$$

a.e. $x \in \Omega$, $s \geq 0$ and $\lambda \in \mathbb{R}$.

Using the fact that

$$h(x, s, 0) = \tilde{\lambda}_0 s^{p-1} + f(s),$$

and arguing as in (5.26)–(5.27), we get that there exist the sequences $\{\delta_j\}_j, \{\eta_j\}_j, \{s_j\}_j$ and $\{\lambda_j\}_j$ such that $\lambda_j > 0$,

$$0 < \eta_{j+1} < \delta_j < s_j < \eta_j < 1, \lim_{j\to+\infty} \eta_j = 0, \tag{5.29}$$

and

$$h(x, s, \lambda) \leq 0 \tag{5.30}$$

a.e. $x \in \Omega$, for all $s \in [\delta_j, \eta_j]$, $\lambda \in [-\lambda_j, \lambda_j]$ and $j \in \mathbb{N}$ large enough.

For any $j \in \mathbb{N}$, let $h_j : \Omega \times [0, +\infty) \times [-\lambda_j, \lambda_j] \to \mathbb{R}$ be the function defined by

$$h_j(x, s, \lambda) = h(x, \tau_{\eta_j}(s), \lambda), \tag{5.31}$$

and

$$H_j(x, s, \lambda) := \int_0^s h_j(x, t, \lambda)dt$$

for a.e. $x \in \Omega$, $s \geq 0$ and $\lambda \in [-\lambda_j, \lambda_j]$.

Let us prove that K given in (5.28) and h_j satisfy all the assumptions of Theorem 4.1. Of course, (4.1) and (4.2) are trivially verified, also thanks to (3.7). Moreover, the regularity of h and the continuity of τ_η show that h_j is a Carathéodory function. Also, thanks to (5.31), (4.22), the continuity of $s \mapsto h(\cdot, s, \cdot)$ and the Weierstrass Theorem give that h_j satisfies (4.3). Finally, (5.30) and (5.31) yield (4.4) for j large enough. Hence, h_j satisfies all the assumptions of Theorem 4.1 for j large.

Now, for any $j \in \mathbb{N}$ let $\mathcal{E}_{j,\lambda}$ be the energy functional

$$\mathcal{E}_{j,\lambda} := \mathcal{E}_{K,h_j(\cdot,\cdot,\lambda)}, \tag{5.32}$$

where $\mathcal{E}_{K,\, h_j(\cdot,\cdot,\lambda)}$ is the functional given in (4.6), with $h = h_j(\cdot, \cdot, \lambda)$. By Theorem 4.1 we get that, for j sufficiently large and provided $|\lambda| \leq \lambda_j$, there exists $u_{j,\lambda} \in W^{\eta_j}$ such that

$$\min_{u\in W^{\eta_j}} \mathcal{E}_{j,\lambda}(u) = \mathcal{E}_{j,\lambda}(u_{j,\lambda}) \tag{5.33}$$

$$u_{j,\lambda}(x) \in [0, \delta_j] \quad \text{for a.e. } x \in \Omega, \tag{5.34}$$

and

$$u_{j,\lambda} \text{ is a non-negative weak solution of } (P^K_{h_j(\cdot,\cdot,\lambda)}). \tag{5.35}$$

Since for j sufficiently large

$$0 \leq u_{j,\lambda}(x) \leq \delta_j < \eta_j \tag{5.36}$$

a.e. $x \in \Omega$ by (5.29) and (5.34), we get

$$h_j(x, u_{j,\lambda}(x), \lambda) = h(x, u_{j,\lambda}(x), \lambda),$$

so that by (5.28) it is easily seen that $u_{j,\lambda}$ is a non-negative weak solution of problem (1.1), provided j is large and $|\lambda| \le \lambda_j$.

It remains to prove that for any $k \in \mathbb{N}$ problem (1.1) admits at least k distinct solutions, for suitable values of λ. At this purpose, first of all note that, thanks to the choices of K and h_j and (5.36), the functional $\mathcal{E}_{j,\lambda}$ is given by

$$\begin{aligned}\mathcal{E}_{j,\lambda}(u) &= \int_\Omega a(x, \nabla u(x))dx - \frac{\lambda}{q+1}\int_\Omega \beta(x)|u(x)|^{q+1}\,dx - \int_\Omega F(u(x))dx \\ &= \mathcal{E}_{j,0}(u) - \frac{\lambda}{q+1}\int_\Omega \beta(x)|u(x)|^{q+1}\,dx \end{aligned} \tag{5.37}$$

for any $u \in W_0^{1,p}(\Omega)$.

We claim that there exists an increasing sequence $\{\theta_j\}_j$ such that $\theta_j < 0$, $\lim_{j\to+\infty}\theta_j = 0$ and

$$\theta_{j-1} < \mathcal{E}_{j,0}(u_{j,0}) < \theta_j \tag{5.38}$$

for $j \ge j^*$, with $j^* \in \mathbb{N}$.

First, note that the function

$$(x, s) \mapsto h(x, s, 0) = \tilde{\lambda}_0 s^{p-1} + f(s)$$

verifies all the assumptions of Theorem 5.1, in particular thanks to (2.3). Hence, arguing as in the proof of Theorem 5.1 (see, in particular, formulas (5.12)–(5.15)) we get that there exist $\ell > 0$ and $\zeta \in (0, \eta_1)$ such that

$$F(s) \ge -\ell s^p \quad \text{for all } s \in (0, \zeta) \tag{5.39}$$

and that there is a sequence $\{\tilde{s}_j\}_j$ such that $0 < \tilde{s}_j \to 0$ as $j \to +\infty$ such that for any $L > 0$

$$F(s_j) > L s_j^p \tag{5.40}$$

for $j \in \mathbb{N}$ large enough. Also, since $\delta_j \searrow 0$ as $j \to +\infty$, we can choose a subsequence of $\{\delta_j\}_j$, still denoted by $\{\delta_j\}_j$, such that

$$\tilde{s}_j \le \delta_j \tag{5.41}$$

for all $j \in \mathbb{N}$.

Now, let us fix $j \in \mathbb{N}$ sufficiently large and let

$$\tilde{z}_j := z_{\tilde{s}_j} \in W_0^{1,p}(\Omega)$$

be the function defined as in (4.20) with $s = \tilde{s}_j$. Arguing as in (5.16), by (5.39) and (5.40) we get that for j large enough

$$\mathcal{E}_{j,0}(u_{j,0}) \le \mathcal{E}_{j,0}(\tilde{z}_j) < -C_1\tilde{s}_j^p =: c_j < 0, \tag{5.42}$$

for a suitable positive constant C_1. Also, as in (5.19) and taking into account the continuity of f and (5.36), we have that, for j sufficiently large

$$\begin{aligned}\mathcal{E}_{j,0}(u_{j,0}) &\geq -\int_\Omega F(u_{j,0}(x))dx\\ &\geq -\int_\Omega \int_0^{u_{j,0}(x)} |f(s)|\,ds\,dx\\ &\geq -\int_\Omega \int_0^{\delta_j} |f(s)|\,ds\,dx\\ &\geq -C_2\delta_j =: d_j < 0,\end{aligned} \tag{5.43}$$

where C_2 is a positive constant.

Note that $\{c_j\}_j$ and $\{d_j\}_j$ are such that $d_j < c_j < 0$ for any $j \in \mathbb{N}$ and $\lim_{j\to+\infty} c_j = \lim_{j\to+\infty} d_j = 0$, since $\{\delta_j\}_j$ does, thanks to (5.29). Thus, we can extract two subsequences, still denoted by $\{c_j\}_j$ and $\{d_j\}_j$, such that the above properties hold true and the sequences $\{c_j\}_j$ and $\{d_j\}_j$ are non-decreasing. Now, we define

$$\theta_j := \begin{cases} c_j & \text{if } j \in \mathbb{N} \text{ is even,}\\ d_j & \text{if } j \in \mathbb{N} \text{ is odd.}\end{cases}$$

As a consequence of this definition and of (5.42) and (5.43) we have that for i large enough

$$\theta_{2i-1} = d_{2i-1} \leq d_{2i} < \mathcal{E}_{2i,0}(u_{2i,0}) < c_{2i} = \theta_{2i},$$

which, putting $2i = j$ gives (5.38). Hence, the claim is proved. Note that θ_j is independent of λ, since c_j and d_j do.

Now, for any $j \geq j^*$ let

$$\lambda'_j := \frac{(q+1)(\mathcal{E}_{j,0}(u_{j,0}) - \theta_{j-1})}{(\|\beta\|_{L^\infty(\Omega)}+1)\mathcal{L}(\Omega)}, \quad \lambda''_j := \frac{(q+1)(\theta_j - \mathcal{E}_{j,0}(u_{j,0}))}{\|\beta\|_{L^1(\Omega)}+1}. \tag{5.44}$$

Note that λ'_j and λ''_j are strictly positive, thanks to (5.38), and they are independent of λ.

Now, for any fixed $k \in \mathbb{N}$, let

$$\Lambda_k := \min\{\lambda_{j^*+1}, \dots, \lambda_{j^*+k}, \lambda'_{j^*+1}, \dots, \lambda'_{j^*+k}, \lambda''_{j^*+1}, \dots, \lambda''_{j^*+k}\}.$$

Of course, $\Lambda_k > 0$ is independent of λ. Also, if $|\lambda| \leq \Lambda_k$, then $|\lambda| \leq \lambda_j$ for any $j = j^*+1, \dots, j^*+k$. As a consequence of this, for any $\lambda \in \mathbb{R}$ with $|\lambda| \leq \Lambda_k$

$$u_{j,\lambda} \text{ is a non-negative weak solution of problem (1.1)}$$

for any $j = j^*+1, \dots, j^*+k$.

Let us show that these solutions are distinct. At this purpose, note that $u_{j,\lambda} \in W^{\eta_j}$ by (5.36) and so

$$\mathcal{E}_{j,0}(u_{j,0}) = \min_{u\in W^{\eta_j}} \mathcal{E}_{j,0}(u) \leq \mathcal{E}_{j,0}(u_{j,\lambda}). \tag{5.45}$$

By (5.37) and (5.45), for any λ such that $|\lambda| \leq \Lambda_k$ we get

$$\begin{aligned}
\mathcal{E}_{j,\lambda}(u_{j,\lambda}) &= \mathcal{E}_{j,0}(u_{j,\lambda}) - \frac{\lambda}{q+1} \int_\Omega \beta(x)|u_{j,\lambda}(x)|^{q+1}\, dx \\
&\geq \mathcal{E}_{j,0}(u_{j,0}) - \frac{|\lambda|}{q+1} \|\beta\|_{L^\infty(\Omega)} \eta_j^{q+1} \mathcal{L}(\Omega) \\
&\geq \mathcal{E}_{j,0}(u_{j,0}) - \frac{\Lambda_k}{q+1} \|\beta\|_{L^\infty(\Omega)} \mathcal{L}(\Omega) \\
&\geq \mathcal{E}_{j,0}(u_{j,0}) - \frac{\lambda'_j}{q+1} \|\beta\|_{L^\infty(\Omega)} \mathcal{L}(\Omega) \\
&= \theta_{j-1},
\end{aligned} \tag{5.46}$$

for any $j = j^* + 1, \ldots, j^* + k$, thanks to (5.29), (5.36), the choice of Λ_k and the definition of λ'_j.

On the other hand, by (5.42), (5.37) and using the fact that $\|\tilde{z}_j\|_{L^\infty(\Omega)} = \tilde{s}_j \leq \delta_j < \eta_j < 1$ (see (5.29) and (5.41)), for any λ with $|\lambda| \leq \Lambda_k$ we deduce that

$$\begin{aligned}
\mathcal{E}_{j,\lambda}(u_{j,\lambda}) &= \min_{u \in W^{\eta_j}} \mathcal{E}_{j,\lambda}(u) \\
&\leq \mathcal{E}_{j,\lambda}(\tilde{z}_j) \\
&= \mathcal{E}_{j,0}(\tilde{z}_j) - \frac{\lambda}{q+1} \int_\Omega \beta(x)|\tilde{z}_j(x)|^{q+1}\, dx \\
&\leq \mathcal{E}_{j,0}(\tilde{z}_j) - \frac{\lambda}{q+1} \int_{\{x\in\Omega:\lambda\beta(x)<0\}} \beta(x)|\tilde{z}_j(x)|^{q+1}\, dx \\
&\leq \mathcal{E}_{j,0}(\tilde{z}_j) - \frac{\lambda}{q+1} \int_{\{x\in\Omega:\lambda\beta(x)<0\}} \beta(x) dx \\
&\leq \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{|\lambda|}{q+1} \int_{\{x\in\Omega:\lambda\beta(x)<0\}} |\beta(x)| dx \\
&\leq \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{\lambda}{q+1} \int_\Omega |\beta(x)| dx \\
&\leq \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{\Lambda_k}{q+1} \|\beta\|_{L^1(\Omega)} \\
&\leq \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{\lambda''_j}{q+1} \|\beta\|_{L^1(\Omega)} \\
&= \theta_j,
\end{aligned} \tag{5.47}$$

for any $j = j^* + 1, \ldots, j^* + k$, again thanks to the choice of Λ_k and the definition of λ''_j.

Hence, by (5.46), (5.47) and the properties of $\{\theta_j\}_j$ we deduce that for any $j = j^* + 1, \ldots, j^* + k$

$$\theta_{j-1} < \mathcal{E}_{j,\lambda}(u_{j,\lambda}) < \theta_j < 0, \tag{5.48}$$

which yields that

$$\mathcal{E}_{1,\lambda}(u_{1,\lambda}) < \ldots < \mathcal{E}_{k,\lambda}(u_{k,\lambda}) < 0,$$

that is the solutions $\{u_{1,\lambda}, \ldots, u_{k,\lambda}\}$ are all distinct and non-trivial, provided $|\lambda| \leq \Lambda_k$.

Finally, let us estimate the $W_0^{1,p}$-norm of $u_{j,\lambda}$. For this, by (3.5), (5.29), (5.36), (5.37) and (5.48) we have that for any $j = j^* + 1, \ldots, j^* + k$ and $|\lambda| \leq \Lambda_k$

$$\begin{aligned} \frac{\Gamma_1}{p} \|u_{j,\lambda}\|^p_{W_0^{1,p}(\Omega)} &\leq \mathcal{E}_{j,\lambda}(u_{j,\lambda}) + \frac{\lambda}{q+1} \int_\Omega \beta(x)|u_{j,\lambda}(x)|^{q+1}\, dx + \int_\Omega F(u_{j,\lambda}(x))dx \\ &< \theta_j + \frac{|\lambda|}{q+1} \|\beta\|_{L^\infty(\Omega)} \delta_j^{q+1} + \int_\Omega \int_0^{\delta_j} |f(s)|\, ds\, dx \\ &< \frac{\Lambda_k}{q+1} \|\beta\|_{L^\infty(\Omega)} \delta_j + \bar{C}\delta_j, \end{aligned}$$

for a suitable positive constant $\bar{C}$, that is

$$\|u_{j,\lambda}\|_{W_0^{1,p}(\Omega)} \leq \tilde{C}\delta_j^{1/p},$$

where $\tilde{C} > 0$. Since $\delta_j \to 0$ as $j \to +\infty$, without loss of generality, we may assume that

$$\delta_j \leq \min\{\tilde{C}^{-p}, 1\} 1/j^p \tag{5.49}$$

and this gives

$$\|u_{j,\lambda}\|_{W_0^{1,p}(\Omega)} \leq 1/j$$

for any $j = j^* + 1, \ldots, j^* + k$, provided $|\lambda| \leq \Lambda_k$, which is the desired assertion. Finally, by (5.36) and (5.49) it is easily seen that

$$\|u_{j,\lambda}\|_{L^\infty(\Omega)} \leq 1/j^p < 1/j$$

for any $j = j^* + 1, \ldots, j^* + k$, provided $|\lambda| \leq \Lambda_k$ This completes the proof of Theorem 2.1.

6. Oscillation at Infinity

This section is devoted to the study of problem (1.1) in the case when f oscillates at infinity.

In order to prove Theorem 2.2 we apply some techniques used in the previous section. However, for completeness, we give all the details.

Here, we consider again problem (P_h^K), under the following assumptions on the function h:

$$\text{for any } s \geq 0 \ \sup_{t \in [0,s]} |h(\cdot, t)| \in L^\infty(\Omega); \tag{6.1}$$

there exist two sequences $\{\delta_j\}_j$ and $\{\eta_j\}_j$ with $0 < \delta_j < \eta_j < \delta_{j+1}$ and

$$\lim_{j\to+\infty} \delta_j = +\infty \text{ such that } h(x,s) \le 0 \text{ for a.e. } x \in \Omega$$
$$\text{and for all } s \in [\delta_j, \eta_j], j \in \mathbb{N}; \tag{6.2}$$
$$-\infty < \liminf_{s\to+\infty} \frac{H(x,s)}{s^p} \le \limsup_{s\to+\infty} \frac{H(x,s)}{s^p} = +\infty \text{ uniformly for a.e. } x \in \Omega, \tag{6.3}$$

where H is the function given in (4.7).

In this context our existence result for problem (P_h^K) is given by the following theorem.

Theorem 6.1. *Let* $\Omega \subset \mathbb{R}^N$, $N \ge 3$, *be a bounded domain with smooth boundary,* $\lambda \in \mathbb{R}$ *and let* $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ *and* $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ *be two functions satisfying* (1.2)–(1.4) *and* (1.5)–(1.7), *respectively. Moreover, assume that* $K : \Omega \to \mathbb{R}$ *satisfies* (4.1) *and* $h : \Omega \times [0,+\infty) \to \mathbb{R}$ *is a Carathéodory function verifying* (4.2) *and* (6.1)–(6.3).

Then, there exists a sequence $\{u_j\}_j \subset W_0^{1,p}(\Omega)$ *of distinct non-negative weak solutions of problem* (P_h^K) *such that*

$$\lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty. \tag{6.4}$$

Proof. Here we consider again the function h_j and the functional $\mathcal{E}_j$ defined in (5.6) and (5.7), respectively. Taking into account hypotheses (6.1) and (6.2), it is easily seen that h_j fulfills the assumptions of Theorem 4.1 for any $j \in \mathbb{N}$. Thus, for every $j \in \mathbb{N}$, there is an element $u_j \in W^{\eta_j}$ such that

$$u_j \text{ is the minimum point of the functional } \mathcal{E}_j \text{ on } W^{\eta_j}, \tag{6.5}$$
$$u_j(x) \in [0, \delta_j] \quad \text{for a.e. } x \in \Omega, \tag{6.6}$$

and

$$u_j \text{ is a non-negative weak solution of } (P_{h_j}^K). \tag{6.7}$$

Arguing as in the proof of Theorem 5.1 and taking into account the definition of h_j, (6.2) and (6.6), it is easily seen that

$$h_j(x, u_j(x)) = h(x, \tau_{\eta_j}(u_j(x))) = h(x, u_j(x)),$$

so that, by (6.7), u_j is also a non-negative weak solution of problem (P_h^K).

In order to get the assertion of Theorem 6.1, we need to show that there are infinitely many distinct elements in the sequence $\{u_j\}_j$. To this end, first of all we claim that, up to a subsequence,

$$\lim_{j\to+\infty} \mathcal{E}_j(u_j) = -\infty. \tag{6.8}$$

For this, note that, by (6.3), there exist $\ell > 0$ and $\zeta > 0$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x,s) \ge -\ell s^p \quad \text{for all } s > \zeta \tag{6.9}$$

and there exists a sequence $\{s_j\}_j$ such that $\lim_{j\to+\infty} s_j = +\infty$ and

$$\limsup_{j\to+\infty} \frac{H(x,s_j)}{s_j^p} = +\infty,$$

that is, for any $L > 0$

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x,s_j) > Ls_j^p \tag{6.10}$$

for $j \in \mathbb{N}$ sufficiently large.

Since $\delta_j \nearrow +\infty$ by (6.2), we can choose a subsequence of $\{\delta_j\}_j$, still denoted by $\{\delta_j\}_j$, such that

$$s_j \le \delta_j \tag{6.11}$$

for all $j \in \mathbb{N}$. Let us fix $j \in \mathbb{N}$ and let

$$z_j := z_{s_j} \in W_0^{1,p}(\Omega)$$

be the function from (4.20) with $s = s_j$. Then, $z_j \in W_0^{1,p}(\Omega)$ and $\|z_j\|_{L^\infty(\Omega)} = s_j$, so that, by (6.2) and (6.11), $0 \le z_j(x) \le \delta_j < \eta_j$ a.e. $x \in \Omega$. Taking into account (3.4), (4.21), (6.9) and (6.10), we have

$$\begin{aligned}
\mathcal{E}_j(z_j) &= \int_\Omega a(x,\nabla z_j(x))dx + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_\Omega H_j(x,z_j(x))dx \\
&\le C(r,p,N)\frac{\Gamma_2}{p}s_j^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_{B(x_0,r/2)} H(x,s_j)dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_j>\zeta\}} H(x,z_j(x))dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_j\le\zeta\}} H(x,z_j(x))dx \\
&\le \left(C(r,p,N)\frac{\Gamma_2}{p} + \frac{\|K\|_{L^\infty(\Omega)}\mathcal{L}(\Omega)}{p} - L(r/2)^N\omega_N + \ell\mathcal{L}(\Omega)\right)s_j^p \\
&\quad + \left\|\sup_{s\in[0,\,\zeta]}|h(\cdot,s)|\right\|_{L^\infty(\Omega)} \mathcal{L}(\Omega)\zeta.
\end{aligned} \tag{6.12}$$

Choosing $L > 0$ sufficiently large, so that

$$L(r/2)^N\omega_N > C(r,p,N)\frac{\Gamma_2}{p} + \frac{\|K\|_{L^\infty(\Omega)}\mathcal{L}(\Omega)}{p} + \ell\mathcal{L}(\Omega),$$

and taking into account that $\lim_{j\to+\infty} s_j = +\infty$, by (6.12) we get

$$\lim_{j\to+\infty} \mathcal{E}_j(z_j) = -\infty. \tag{6.13}$$

On the other hand, using (6.5), we have

$$\mathcal{E}_j(u_j) = \min_{u\in W^{\eta_j}} \mathcal{E}_j(u) \le \mathcal{E}_j(z_j),$$

so that, by this and (6.13) it easily follows claim (6.8).

As a consequence of (6.8) we get that the sequence $\{u_j\}_j$ has infinitely many distinct elements (and, in particular, $u_j \not\equiv 0$ in Ω, being $\mathcal{E}_j(0) = 0$). Indeed, let us assume that in the sequence $\{u_j\}_j$ there is only a finite number of elements, say $\{u_1, \dots, u_k\}$ for some $k \in \mathbb{N}$. Consequently, the sequence $\{\mathcal{E}_j(u_j)\}_j$ reduces to at most the finite set $\{\mathcal{E}_1(u_1), \dots, \mathcal{E}_k(u_k)\}$ and this fact contradicts relation (6.8). Hence, problem (P_h^K) admits infinitely many distinct weak solutions.

Now, we prove (6.4), arguing by contradiction. We assume that, up to a subsequence, still denoted by $\{u_j\}_j$, there exists $L > 0$ such that

$$\|u_j\|_{L^\infty(\Omega)} \leq L \tag{6.14}$$

for all $j \in \mathbb{N}$.

Since $\eta_j \to +\infty$ as $j \to +\infty$, for j large enough, say $j \geq j^*$, with $j^* \in \mathbb{N}$, we have that $\eta_j \geq L$. As a consequence of this and (6.14), we deduce that

$$u_j \in W^{\eta_{j^*}} \quad \text{for any } j \geq j^*. \tag{6.15}$$

Here we used also the fact that the sequence $\{\eta_j\}_j$ is increasing by (6.2).

Also, note that, as a consequence of the monotonicity of $\{\eta_j\}_j$, it is easy to see that when $u \in W^{\eta_j}$ and $j < k$, then $u \in W^{\eta_k}$, that is

$$W^{\eta_j} \subseteq W^{\eta_k} \tag{6.16}$$

and this implies that a.e. $x \in \Omega$

$$\begin{aligned} H_j(x, u(x)) &= \int_0^{u(x)} h(x, \tau_{\eta_j}(t))dt \\ &= \int_0^{u(x)} h(x, t)dt \\ &= \int_0^{u(x)} h(x, \tau_{\eta_k}(t))dt \\ &= H_k(x, u(x)), \end{aligned} \tag{6.17}$$

provided $u \in W^{\eta_j}$.

Furthermore, the sequence $\{\mathcal{E}_j(u_j)\}_j$ is non-increasing. Indeed, for $j < k$, by (6.16) and (6.17) we have

$$\begin{aligned} \mathcal{E}_j(u_j) &= \min_{u \in W^{\eta_j}} \mathcal{E}_j(u) \\ &= \min_{u \in W^{\eta_j}} \mathcal{E}_k(u) \\ &\geq \min_{u \in W^{\eta_k}} \mathcal{E}_k(u) \\ &= \mathcal{E}_k(u_k). \end{aligned} \tag{6.18}$$

Then, by (6.15)–(6.18), for any $j \geq j^*$ we get

$$\begin{aligned} \mathcal{E}_{j^*}(u_{j^*}) &\geq \mathcal{E}_j(u_j) \\ &\geq \min_{u \in W^{\eta_{j^*}}} \mathcal{E}_j(u) \end{aligned}$$

$$= \min_{u \in W^{\eta_{j^*}}} \mathcal{E}_{j^*}(u)$$
$$= \mathcal{E}_{j^*}(u_{j^*}).$$

As a consequence,

$$\mathcal{E}_j(u_j) = \mathcal{E}_{j^*}(u_{j^*}) \quad \text{for all } j \geq j^*.$$

This fact contradicts (6.8). Hence, $\|u_j\|_{L^\infty(\Omega)} \to +\infty$ as $j \to +\infty$ and this concludes the proof of the theorem. $\square$

Requiring the following extra condition on the function h

$$\sup_{s\in[0,+\infty)} \frac{|h(x,s)|}{1+s^{p^*-1}} < +\infty \text{ uniformly a.e. } x \in \Omega, \tag{6.19}$$

where p^* is the critical Sobolev exponent given in (2.6), we have the next result.

Corollary 6.2. *Let all the assumptions of Theorem* 6.1 *be satisfied. In addition assume that* (6.19) *holds true.*

Then,

$$\lim_{j\to+\infty} \|u_j\|_{W_0^{1,p}(\Omega)} = +\infty,$$

where $\{u_j\}_j$ *is the sequence of distinct weak solutions of problem* (P_h^K)*, given by Theorem* 6.1*.*

Proof. We argue by contradiction and we assume that, up to a subsequence, still denoted by $\{u_j\}_j$, there exists $L > 0$ such that for any $j \in \mathbb{N}$

$$\|u_j\|_{W_0^{1,p}(\Omega)} \leq L. \tag{6.20}$$

Hence, for any $j \in \mathbb{N}$ by (3.5) and (6.19) we have

$$\begin{aligned} |\mathcal{E}_j(u_j)| &\leq \frac{\Gamma_2}{p}\|u_j\|^p_{W_0^{1,p}(\Omega)} + \|K\|_{L^\infty(\Omega)}\|u_j\|^p_{L^p(\Omega)} + C_1 \int_\Omega \int_0^{u_j(x)} (1+|s|^{p^*-1})\, ds\, dx \\ &\leq \frac{\Gamma_2}{p}\|u_j\|^p_{W_0^{1,p}(\Omega)} + \|K\|_{L^\infty(\Omega)}\|u_j\|^p_{L^p(\Omega)} + C_2\|u_j\|_{L^1(\Omega)} + C_3\|u_j\|^{p*}_{L^{p^*}(\Omega)} \end{aligned} \tag{6.21}$$

for suitable positive constants C_1, C_2 and C_3. Since the Sobolev space $W_0^{1,p}(\Omega)$ is continuously embedded into $L^\gamma(\Omega)$ for any $\gamma \in [1, p^*]$, by (6.20) and (6.21) we easily obtain that the sequence $\{\mathcal{E}_j(u_j)\}_j$ is bounded in $\mathbb{R}$. This contradicts (6.8) and this ends the proof of the corollary. $\square$

6.1. *Proof of Theorem* 2.2

We use some techniques developed in the proof of Theorem 2.1. The main idea consists in applying Theorems 4.1 and 6.1 to problem (P_h^K) with a suitable choice for the functions K and h appearing in the equation.

In the case when $q = p-1$ and $\ell_\infty \in (0, +\infty)$ we fix $\lambda \in \mathbb{R}$ such that $\lambda\beta(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $\lambda_\infty \in (0, \ell_\infty)$. In this setting we take $\tilde{\lambda}_\infty \in (\lambda_\infty, \ell_\infty)$ and

$$K(x) := \tilde{\lambda}_\infty - \lambda\beta(x) \quad \text{and} \quad h(x,s) := \tilde{\lambda}_\infty s^{p-1} + f(s), \tag{6.22}$$

for a.e. $x \in \Omega$ and $s \geq 0$, in Theorem 6.1. With the same arguments used in the proof of Theorem 2.1, it is easily seen that K and h satisfy the assumptions of Theorem 6.1 (here we use also the fact that $f(0) = 0$ by assumption) and this gives the assertion of Theorem 2.2.

In the case when $q = p-1$ and $\ell_\infty = +\infty$ we take $\lambda \in \mathbb{R}$ and we use Theorem 6.1 with

$$K(x) := \tilde{\lambda}_\infty \quad \text{and} \quad h(x,s) := (\lambda\beta(x) + \tilde{\lambda}_\infty)s^{p-1} + f(s), \tag{6.23}$$

for a.e. $x \in \Omega$ and $s \geq 0$. The arguments are the same of the ones used in the previous case.

In the case when $0 < q < p-1$ we choose K and h in Theorem 6.1 as follows:

$$K(x) := \tilde{\lambda}_\infty \quad \text{and} \quad h(x,s) := \lambda\beta(x)s^q + \tilde{\lambda}_\infty s^{p-1} + f(s), \tag{6.24}$$

for a.e. $x \in \Omega$ and $s \geq 0$, where $\tilde{\lambda}_\infty \in (0, \ell_\infty)$, and we argue as above.

In the case when $q > p-1$ the strategy will consist in applying Theorem 4.1 to problem (P_h^K) with a suitable choices of K and h. At this purpose, let $\tilde{\lambda}_\infty \in (\lambda_\infty, \ell_\infty)$, where $\ell_\infty > 0$ is given in assumption (2.7), and let

$$K(x) := \tilde{\lambda}_\infty \quad \text{and} \quad h(x,s,\lambda) := \lambda\beta(x)s^q + \tilde{\lambda}_\infty s^{p-1} + f(s)$$

a.e. $x \in \Omega$, $s \geq 0$ and $\lambda \in \mathbb{R}$. Arguing as in the proof of Theorem 2.1 we get the assertion.

Finally, when $q > p-1$ we can adapt the arguments used in the proof of Theorem 2.1. This ends the proof of Theorem 2.2.

As a consequence of Corollary 6.2, we have the following result.

Corollary 6.3. *Let* $q \leq p-1$ *and let all the assumptions of Theorem* 2.2 *be satisfied. In addition, assume that*

$$\sup_{s\in[0,+\infty)} \frac{|f(s)|}{1 + s^{p^*-1}} < +\infty, \tag{6.25}$$

where p^* *is given in* (2.6).

Then,

$$\lim_{j\to+\infty} \|u_j\|_{W_0^{1,p}(\Omega)} = +\infty,$$

where $\{u_j\}_j$ *is the sequence of distinct weak solutions of problem* (1.1), *given by Theorem* 2.2.

Proof. It is enough to apply Corollary 6.2 with K and h given in (6.22) when $q = p-1$, and in (6.24) when $0 < q < p-1$.

Let us consider the case when $q = p - 1$. Note that $q > 0$, being $p > 1$. Also, by (6.25), for $s \in [0, 1]$

$$\begin{aligned}\frac{|h(x,s)|}{1+s^{p^*-1}} &= \frac{\tilde{\lambda}_\infty s^{p-1}}{1+s^{p^*-1}} + \frac{|f(s)|}{1+s^{p^*-1}} \\ &\leq \tilde{\lambda}_\infty s^{p-1} + \frac{|f(s)|}{1+s^{p^*-1}} \\ &\leq \tilde{\lambda}_\infty + \frac{|f(s)|}{1+s^{p^*-1}} \\ &\leq \tilde{\lambda}_\infty + \sup_{s\in[0,+\infty)} \frac{|f(s)|}{1+s^{p^*-1}} < +\infty,\end{aligned}$$

while for $s > 1$ we have

$$\begin{aligned}\frac{|h(x,s)|}{1+s^{p^*-1}} &= \frac{\tilde{\lambda}_\infty s^{p-1}}{1+s^{p^*-1}} + \frac{|f(s)|}{1+s^{p^*-1}} \\ &\leq \tilde{\lambda}_\infty s^{p-p^*} + \frac{|f(s)|}{1+s^{p^*-1}} \\ &\leq \tilde{\lambda}_\infty s^{p-p^*} + \sup_{s\in[0,+\infty)} \frac{|f(s)|}{1+s^{p^*-1}} < +\infty,\end{aligned}$$

since $p < p^*$ and $s^{p-p^*} \to 0$ as $s \to +\infty$. Thus, in any case (6.19) is satisfied.

In the case when $0 < q < p - 1$ we argue in a similar way, taking into account that $q < p - 1 < p^* - 1$. This concludes the proof of the corollary. □

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