Multi-parameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term

Marius Ghergu and **Vicenţiu Rădulescu**
Department of Mathematics, University of Craiova,
200585 Craiova, Romania (`radulescu@inf.ucv.ro`)

We establish some bifurcation results for the boundary-value problem
$-\Delta u = g(u) + \lambda|\nabla u|^p + \mu f(x,u)$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda, \mu \geqslant 0$, $0 < p \leqslant 2$, f is non-decreasing with respect to the second variable and g is unbounded around the origin. The asymptotic behaviour of the solution around the bifurcation point is also established, provided $g(u)$ behaves like $u^{-\alpha}$ around the origin, for some $0 < \alpha < 1$. Our approach relies on finding explicit sub- and supersolutions combined with various techniques related to the maximum principle for elliptic equations. The analysis we develop in this paper shows the key role played by the convection term $|\nabla u|^p$.

1. Introduction and the main results

In his recent monograph [25], Kielhöfer synthesizes the role of bifurcation problems in applied mathematics:

> Bifurcation theory attempts to explain various phenomena that have been discovered and described in natural sciences over the centuries. The buckling of the Euler rod, the appearance of Taylor vortices, and the onset of oscillations in an electric circuit, for instance, all have a common cause: a specific physical parameter crosses a threshold, and that event forces the system to the organization of a new state that differs considerably from that observed before.

In the present paper we continue the bifurcation analysis developed in our previous works [17, 18] (see also [10]) for a large class of semilinear elliptic equations with singular nonlinearity and Dirichlet boundary condition. Such problems arise in the study of non-Newtonian fluids, boundary-layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically conducting materials (see [7, 12, 28, 30, 37]). The main feature of this paper is the presence of the convection term $|\nabla u|^p$.

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) be a bounded domain with a smooth boundary. We are concerned in this paper with singular elliptic problems of the following type

$$\left.\begin{aligned}
-\Delta u &= g(u) + \lambda|\nabla u|^p + \mu f(x,u) && \text{in } \Omega, \\
u &> 0 && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}\right\} \qquad (1.1)$$

where $0 < p \leqslant 2$ and $\lambda, \mu \geqslant 0$. As remarked in [9, 24, 40], the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

Throughout this paper we suppose that $f : \bar{\Omega} \times [0, \infty) \to [0, \infty)$ is a Hölder continuous function which is non-decreasing with respect to the second variable and is positive on $\bar{\Omega} \times (0, \infty)$. We assume that $g : (0, \infty) \to (0, \infty)$ is a Hölder continuous function which is non-increasing and $\lim_{s \searrow 0} g(s) = +\infty$.

Problems of this type arise in the study of guided modes of an electromagnetic field in a nonlinear medium, satisfying adequate constitutive hypotheses. The following two examples illustrate situations of this type. (i) If $f(u) = u^3 (1 + \gamma u^2)^{-1}$ ($\gamma > 0$), then problem (1.1) describes the variation of the dielectric constant of gas vapours where a laser beam propagates (see [33, 34]). (ii) Nonlinearities of the type $f(u) = (1 - \mathrm{e}^{-\gamma u^2}) u$ arise in the context of laser beams in plasmas (see [35]). If $f(u) = \mathrm{e}^{u/(1+\varepsilon u)}$ ($\varepsilon > 0$), then the corresponding equation describes the temperature dependence of the reaction rate for exothermic reactions obeying the simple Arrhenius rate law in circumstances in which the heat flow is purely conductive (see [4, 38]). In this context, the parameter ε is a dimensionless ambient temperature and the parameter λ is a dimensionless heat evolution rate. The corresponding equation,

$$-\Delta u = g(u) + \lambda |\nabla u|^p + \mu \mathrm{e}^{u/(1+\varepsilon u)} \quad \text{in } \Omega,$$

represents heat balance with reactant consumption ignored, where u is a dimensionless temperature excess. The Dirichlet boundary condition $u = 0$ on $\partial \Omega$ is an isothermal condition and, in this case, it describes the exchange of heat at the surface of the reactant by Newtonian cooling.

Our general setting includes some simple prototype models from boundary-layer theory of viscous fluids (see [39]). If $\lambda = 0$ and $\mu = 0$, then system (1.1) is called the Lane–Emden–Fowler equation (see [15, 16, 20, 26, 31, 36]). Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [14] as a model for long-range Van der Waals interactions in thin films spreading on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [8]), as well as to describe the flow over an impermeable plate (see [5, 6]).

Our aim in this paper is to describe the influence of the gradient term in problem (1.1).

Many papers have been devoted to the case $\lambda = 0$, where problem (1.1) becomes

$$\left.\begin{array}{ll} -\Delta u = g(u) + \mu f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{array}\right\} \tag{1.2}$$

If $\mu = 0$, then (1.2) has a unique solution (see [13, 27]). When $\mu > 0$, the study of (1.2) emphasizes the role played by the nonlinear term $f(x, u)$. For instance, if one of the following assumptions are fulfilled, then problem (1.2) has solutions only if $\mu > 0$ is small enough (see [11]).

(f1) There exists $c > 0$ such that $f(x, s) \geqslant cs$ for all $(x, s) \in \bar{\Omega} \times [0, \infty)$.

(f2) The mapping $(0, \infty) \ni s \mapsto f(x, s)/s$ is non-decreasing for all $x \in \bar{\Omega}$,

In turn, when f satisfies the following assumptions, then problem (1.2) has at least one solution for all $\mu > 0$ (see [10, 11, 17, 32] and the references therein).

(f3) The mapping $(0, \infty) \ni s \mapsto f(x, s)/s$ is non-increasing for all $x \in \bar{\Omega}$.

(f4) $\lim_{s \to \infty} f(x, s)/s = 0$ uniformly for $x \in \bar{\Omega}$.

The same assumptions will be used in the study of (1.1).

If $\lambda > 0$, the following problem was treated in Zhang and Yu [41],

$$\left.\begin{aligned} -\Delta u = \frac{1}{u^{\alpha}} + \lambda |\nabla u|^p + \sigma &\quad \text{in } \Omega, \\ u > 0 &\quad \text{in } \Omega, \\ u = 0 &\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

where $\lambda, \sigma \geqslant 0$, $\alpha > 0$ and $p \in (0, 2]$. By using the change of variable $v = e^{\lambda u} - 1$ in the case $p = 2$, it is proved in [41] that problem (1.3) has classical solutions if $\lambda \sigma < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. This will be used to deduce the existence and nonexistence in the case $0 < p < 2$.

If $f(x, u)$ depends on u, the above change of variable does not preserve the sublinearity condition (f3), (f4) and the monotony of the nonlinear term g in problem (1.1). In turn, if $f(x, u)$ does not depend on u and $p = 2$, this method successfully applies to our study and we will be able to give a complete characterization of (1.1) (see theorem 1.4 below).

Due to the singular term $g(u)$ in (P_λ), we cannot expect to have solutions in $C^2(\bar{\Omega})$. As it was pointed out in [41], if $\alpha > 1$, then the solution of (1.3) is *not* in $C^1(\bar{\Omega})$. We are seeking in this paper classical solutions of (P_λ), that is, solutions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ that verify (1.1).

By the monotony of g, there exists

$$a = \lim_{s \to \infty} g(s) \in [0, \infty).$$

The first result concerns the case $\lambda = 1$ and $1 < p \leqslant 2$. In the statement of the following result we do not need assumptions (f1)–(f4); we just require that f is a Hölder continuous function that is non-decreasing with respect to the second variable and is positive on $\bar{\Omega} \times (0, \infty)$.

THEOREM 1.1. *Assume* $\lambda = 1$ *and* $1 < p \leqslant 2$.

(i) *If* $p = 2$ *and* $a \geqslant \lambda_1$, *then (1.1) has no solutions.*

(ii) *If* $p = 2$ *and* $a < \lambda_1$ *or* $1 < p < 2$, *then there exists* $\mu^* > 0$ *such that (1.1) has at least one classical solution for* $\mu < \mu^*$ *and no solutions exist if* $\mu > \mu^*$.

If $\lambda = 1$ and $0 < p \leqslant 1$, then the study of existence is closely related to the asymptotic behaviour of the nonlinear term $f(x, u)$. In this case, we prove the following result.

THEOREM 1.2. *Assume* $\lambda = 1$ *and* $0 < p \leqslant 1$.

(i) *If f satisfies (f1) or (f2), then there exists $\mu^* > 0$ such that (1.1) has at least one classical solution for $\mu < \mu^*$ and no solutions exist if $\mu > \mu^*$.*

(ii) *If $0 < p < 1$ and f satisfies (f3), (f4), then (1.1) has at least one solution for all $\mu \geqslant 0$.*

Next we are concerned with the case $\mu = 1$. Our result is the following.

THEOREM 1.3. *Assume $\mu = 1$ and f satisfies assumptions (f3) and (f4). Then the following properties hold true.*

(i) *If $0 < p < 1$, then (1.1) has at least one classical solution for all $\lambda \geqslant 0$.*

(ii) *If $1 \leqslant p \leqslant 2$, then there exists $\lambda^* \in (0, \infty]$ such that (1.1) has at least one classical solution for $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$. Moreover, if $1 < p \leqslant 2$, then λ^* is finite.*

Related to the above result, we raise the following *open problem*: if $p = 1$ and $\mu = 1$, is λ^* a finite number?

Theorem 1.3 shows the importance of the convection term $\lambda|\nabla u|^p$ in (1.1). Indeed, according to [17, theorem 1.3] and for any $\mu > 0$, the boundary-value problem

$$\left.\begin{aligned} -\Delta u &= u^{-\alpha} + \lambda|\nabla u|^p + \mu u^\beta && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{1.4}$$

has a unique solution, provided $\lambda = 0$ and $\alpha, \beta \in (0, 1)$. The above theorem shows that if λ is not necessarily 0, then the following situations may occur. (i) Problem (1.4) has solutions if $p \in (0, 1)$ and for all $\lambda \geqslant 0$. (ii) If $p \in (1, 2)$, then there exists $\lambda^* > 0$ such that problem (1.4) has a solution for any $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

To see the dependence between λ and μ in (1.1), we consider the special case $f \equiv 1$ and $p = 2$. In this case, we can say more about the problem (1.1). More precisely, we have the following result.

THEOREM 1.4. *Assume that $p = 2$ and $f \equiv 1$. Then the following properties hold.*

(i) *Problem (1.1) has a solution if and only if $\lambda(a + \mu) < \lambda_1$.*

(ii) *Assume $\mu > 0$ is fixed, g is decreasing and let $\lambda^* = \lambda_1/(a+\mu)$. Then (1.1) has a unique solution u_λ for every $\lambda < \lambda^*$ and the sequence $(u_\lambda)_{\lambda<\lambda^*}$ is increasing with respect to λ.*

Moreover, if $\limsup_{s\searrow 0} s^\alpha g(s) < +\infty$ for some $\alpha \in (0, 1)$, then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ has the following properties.

(ii1) *For all $0 < \lambda < \lambda^*$, there exist two positive constants c_1, c_2 depending on λ such that $c_1 \operatorname{dist}(x, \partial\Omega) \leqslant u_\lambda \leqslant c_2 \operatorname{dist}(x, \partial\Omega)$ in Ω.*

(ii2) *$u_\lambda \in C^{1,1-\alpha}(\bar{\Omega}) \cap C^2(\Omega)$.*

(ii3) *$u_\lambda \to +\infty$ as $\lambda \nearrow \lambda^*$, uniformly on compact subsets of Ω.*
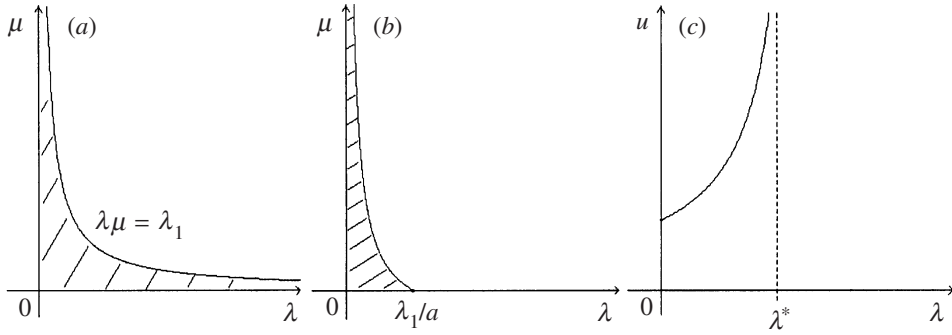
Figure 1. Bifurcation diagrams: (*a*) case 1; (*b*) case 2; (*c*) case 3 ($\lambda^* = \lambda_1/(a + \mu)$).

The assumption $\limsup_{s \searrow 0} s^\alpha g(s) < +\infty$, for some $\alpha \in (0, 1)$, has been used in [17] and it implies the following Keller–Osserman-type growth condition around the origin:

$$\int_0^1 \left(\int_0^t g(s) \, \mathrm{d}s \right)^{-1/2} \mathrm{d}t < +\infty. \tag{1.5}$$

As proved by Bénilan *et al.* in [3], condition (1.5) is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u = g(u) + h \quad \text{a.e. in } \mathbb{R}^N.$$

The situations described in theorem 1.4 are depicted in the bifurcation diagrams of figure 1. Case 1 (respectively, case 2) corresponds to (i) and $a = 0$ (respectively, $a > 0$), while case 3 is related to (ii), $\lambda > 0$ and $\mu = $ fixed.

As regards the uniqueness of the solutions to problem (1.1), we may say that this does not seem to be a feature easy to achieve. Only when $f(x, u)$ is constant in u can we use classical methods in order to prove the uniqueness. It is worth pointing out here that the uniqueness of the solution is a delicate issue even for the simpler problem (1.2). We have shown in [17] that when f fulfils (f3), (f4) and g satisfies the same growth condition as in theorem 1.4, then, if (1.2) has a solution, it follows that this solution is unique. On the other hand, if f satisfies (f2), the uniqueness generally does not occur. In that sense, we refer the interested reader to Haitao [22]. In the case $f(x, u) = u^q$, $g(u) = u^{-\gamma}$, $0 < \gamma < 1/N$ and $1 < q < (N + 2)/(N - 2)$, we learn from [22] that problem (1.2) has at least two classical solutions provided μ belongs to a certain range.

Our approach relies on finding appropriate sub- and supersolutions of (1.1). This allows us to enlarge the study of bifurcation to a class of problems more general to those studied in [41]. However, neither the method used in Zhang and Yu [41] nor our method gives a precise answer if λ^* is finite or not in the case $p = 1$ and $\mu = 1$.

In the next section we state some auxiliary results that will be used in the proofs of the above theorems. This will be done in §§ 3–6.

2. Auxiliary results

Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$-\Delta u = \lambda u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

As it is well known, $\lambda_1 > 0$, $\varphi_1 \in C^2(\bar{\Omega})$ and

$$C_1 \operatorname{dist}(x, \partial\Omega) \leqslant \varphi_1 \leqslant C_2 \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega, \tag{2.1}$$

for some positive constants $C_1, C_2 > 0$. From the characterization of λ_1 and φ_1, we state the following elementary result. For the convenience of the reader, we give a complete proof.

LEMMA 2.1. *Let* $F : \bar{\Omega} \times (0, \infty) \to \mathbb{R}$ *be a continuous function such that* $F(x, s) \geqslant \lambda_1 s + b$ *for some* $b > 0$ *and for all* $(x, s) \in \bar{\Omega} \times (0, \infty)$. *Then the problem*

$$\left.\begin{array}{ll} -\Delta u = F(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{array}\right\} \tag{2.2}$$

has no solutions.

Proof. By contradiction, suppose that (2.2) admits a solution. This will provide a supersolution of the problem

$$\left.\begin{array}{ll} -\Delta u = \lambda_1 u + b & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{2.3}$$

Since 0 is a subsolution, by the sub- and supersolution method and classical regularity theory, it follows that (2.2) has a solution $u \in C^2(\bar{\Omega})$. Multiplying by φ_1 in (2.3) and then integrating over Ω, we get

$$-\int_\Omega \varphi_1 \Delta u = \lambda_1 \int_\Omega \varphi_1 u + b \int_\Omega \varphi_1,$$

that is,

$$\lambda_1 \int_\Omega \varphi_1 u = \lambda_1 \int_\Omega \varphi_1 u + b \int_\Omega \varphi_1,$$

which implies that $\int_\Omega \varphi_1 = 0$. This is clearly a contradiction, since $\varphi_1 > 0$ in Ω. Hence (2.2) has no solutions. $\square$

The growth of φ_1 is prescribed in the following result.

LEMMA 2.2 (see [27]). *We have*

$$\int_\Omega \varphi_1^{-s} \, \mathrm{d}x < +\infty$$

if and only if $s < 1$.

Basic in the study of the existence is the following lemma.

LEMMA 2.3 (see [32]). *Let $F : \bar{\Omega} \times (0, \infty) \to \mathbb{R}$ be a Hölder continuous function on each compact subset of $\bar{\Omega} \times (0, \infty)$ that satisfies the following conditions.*

(F1) $\limsup_{s \to +\infty} (s^{-1} \max_{x \in \bar{\Omega}} F(x, s)) < \lambda_1$.

(F2) *For each $t > 0$, there exists a constant $D(t) > 0$ such that*
$$F(x, r) - F(x, s) \geqslant -D(t)(r - s) \quad \text{for } x \in \bar{\Omega} \text{ and } r \geqslant s \geqslant t.$$

(F3) *There exist a $\eta_0 > 0$ and an open subset $\Omega_0 \subset \Omega$ such that*
$$\min_{x \in \bar{\Omega}} F(x, s) \geqslant 0 \qquad \text{for } s \in (0, \eta_0)$$

and

$$\lim_{s \searrow 0} \frac{F(x, s)}{s} = +\infty \quad \text{uniformly for } x \in \Omega_0.$$

Then the problem
$$\left. \begin{aligned} -\Delta u &= F(x, u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \tag{2.4}$$

has at least one solution.

According to lemma 2.3, there exists $\zeta \in C^2(\Omega) \cap C(\bar{\Omega})$, a solution of the problem
$$\left. \begin{aligned} -\Delta\zeta &= g(\zeta) && \text{in } \Omega, \\ \zeta &> 0 && \text{in } \Omega, \\ \zeta &= 0 && \text{on } \partial\Omega. \end{aligned} \right\} \tag{2.5}$$

Clearly, ζ is a subsolution of (1.1) for all $\lambda \geqslant 0$. It is worth pointing out here that the sub–supersolution method still works for problem (1.1). With the same proof as in Zhang and Yu [41, lemma 2.8], which goes back to the pioneering work of Amann [2], we state the following result.

LEMMA 2.4. *Let $\lambda, \mu \geqslant 0$. If (1.1) has a supersolution $\bar{u} \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $\zeta \leqslant \bar{u}$ in Ω, then (1.1) has at least a solution.*

Another difficulty in the treatment of (1.1) is the lack of the usual maximum principle. The following result, which is due to Shi and Yao [32], gives a comparison principle that applies to singular elliptic equations.

LEMMA 2.5 (see [32]). *Let $F : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0, \infty) \ni s \to F(x, s)/s$ is strictly decreasing at each $x \in \Omega$. Assume $v, w \in C^2(\Omega) \cap C(\bar{\Omega})$ and:*

(a) $\Delta w + F(x, w) \leqslant 0 \leqslant \Delta v + F(x, v)$ *in Ω;*

(b) $v, w > 0$ *in Ω and $v \leqslant w$ on $\partial\Omega$;*

(c) $\Delta v \in L^1(\Omega)$.

Then $v \leqslant w$ in Ω.

LEMMA 2.6 (see [1]). *If $p > 1$, then there exists a real number $\bar{\sigma} > 0$ such that the problem*

$$\left.\begin{aligned} -\Delta u &= |\nabla u|^p + \sigma & in\ \Omega, \\ u &= 0 & on\ \partial\Omega \end{aligned}\right\} \tag{2.6}$$

has no solutions for $\sigma > \bar{\sigma}$.

3. Proof of theorem 1.1

We start with the following more general result.

LEMMA 3.1. *Let $F : \bar{\Omega} \times (0, \infty) \to [0, \infty)$ and $G : (0, \infty) \to (0, \infty)$ be two Hölder continuous functions that verify the following conditions.*

(A1) $F(x, s) > 0$, *for all $(x, s) \in \bar{\Omega} \times (0, \infty)$.*

(A2) *The mapping $[0, \infty) \ni s \mapsto F(x, s)$ is non-decreasing for all $x \in \bar{\Omega}$.*

(A3) *G is non-increasing and $\lim_{s \searrow 0} G(s) = +\infty$.*

  Assume that $\tau > 0$ is a positive real number. Then the following holds.

(i) *If $\tau \lim_{s \to \infty} G(s) \geqslant \lambda_1$, then the problem*

$$\left.\begin{aligned} -\Delta u &= G(u) + \tau|\nabla u|^2 + \mu F(x, u) & in\ \Omega, \\ u &> 0 & in\ \Omega, \\ u &= 0 & on\ \partial\Omega \end{aligned}\right\} \tag{3.1}$$

  has no solutions.

(ii) *If $\tau \lim_{s \to \infty} G(s) < \lambda_1$, then there exists $\bar{\mu} > 0$ such that problem (3.1) has at least one solution for all $0 \leqslant \mu < \bar{\mu}$.*

Proof. (i) With the change of variable $v = e^{\tau u} - 1$, problem (3.1) takes the form

$$\left.\begin{aligned} -\Delta v &= \Psi_\mu(x, u) & in\ \Omega, \\ v &> 0 & in\ \Omega, \\ v &= 0 & on\ \partial\Omega, \end{aligned}\right\} \tag{3.2}$$

where

$$\Psi_\mu(x, s) = \tau(s + 1)G\left(\frac{1}{\tau}\ln(s + 1)\right) + \mu\tau(s + 1)F\left(x, \frac{1}{\tau}\ln(s + 1)\right)$$

for all $(x, s) \in \bar{\Omega} \times (0, \infty)$.

  Taking into account the fact that G is non-increasing and $\tau \lim_{s \to \infty} G(s) \geqslant \lambda_1$, we get

$$\Psi_\mu(x, s) \geqslant \lambda_1(s + 1) \quad in\ \bar{\Omega} \times (0, \infty) \quad for\ all\ \mu \geqslant 0.$$

By lemma 2.1, we conclude that (3.2) has no solutions. Hence (3.1) has no solutions.

(ii) Since

$$\lim_{s \to +\infty} \frac{\tau(s+1)G((1/\tau)\ln(s+1)) + 1}{s} < \lambda_1$$

and

$$\lim_{s \searrow 0} \frac{\tau(s+1)G((1/\tau)\ln(s+1)) + 1}{s} = +\infty,$$

we deduce that the mapping $(0, \infty) \ni s \mapsto \tau(s+1)G((1/\tau)\ln(s+1)) + 1$ fulfils the hypotheses in lemma 2.3. According to this one, there exists $\bar{v} \in C^2(\Omega) \cap C(\bar{\Omega})$, a solution of the problem

$$-\Delta v = \tau(v+1)G\left(\frac{1}{\tau}\ln(v+1)\right) + 1 \quad \text{in } \Omega,$$

$$v > 0 \qquad\qquad \text{in } \Omega,$$

$$v = 0 \qquad\qquad \text{in } \partial\Omega.$$

Define

$$\bar{\mu} := \frac{1}{\tau(\|\bar{v}\|_\infty + 1)} \cdot \frac{1}{\max_{x \in \bar{\Omega}} F(x, (1/\tau)\ln(\|\bar{v}\|_\infty + 1))}.$$

It follows that $\bar{v}$ is a supersolution of (3.2) for all $0 \leqslant \mu < \bar{\mu}$.

Next we provide a subsolution $\underline{v}$ of (3.2) such that $\underline{v} \leqslant \bar{v}$ in Ω. To this end, we apply lemma 2.3 to get that there exists $\underline{v} \in C^2(\Omega) \cap C(\bar{\Omega})$, a solution of the problem

$$-\Delta v = \tau G\left(\frac{1}{\tau}\ln(v+1)\right) \quad \text{in } \Omega,$$

$$v > 0 \qquad\qquad \text{in } \Omega,$$

$$v = 0 \qquad\qquad \text{on } \partial\Omega.$$

Clearly, $\underline{v}$ is a subsolution of (3.2) for all $0 \leqslant \mu < \bar{\mu}$. Let us prove now that $\underline{v} \leqslant \bar{v}$ in Ω. Assuming the contrary, it follows that $\max_{x \in \bar{\Omega}}\{\underline{v} - \bar{v}\} > 0$ is achieved in Ω. At that point, say, x_0, we have

$$0 \leqslant -\Delta(\underline{v} - \bar{v})(x_0)$$

$$\leqslant \tau\left[G\left(\frac{1}{\tau}\ln(\underline{v}(x_0)+1)\right) - G\left(\frac{1}{\tau}\ln(\bar{v}(x_0)+1)\right)\right] - 1$$

$$< 0,$$

which is a contradiction. Thus $\underline{v} \leqslant \bar{v}$ in Ω. We have proved that $(\underline{v}, \bar{v})$ is an ordered pair of sub-super solutions of (3.2), provided $0 \leqslant \mu < \bar{\mu}$. It follows that (3.1) has at least one classical solution for all $0 \leqslant \mu < \bar{\mu}$ and the proof of lemma 3.1 is now complete. $\qquad\square$

Proof of theorem 1.1 completed. According to lemma 3.1 (i), we deduce that (1.1) has no solutions if $p = 2$ and $a \geqslant \lambda_1$. Furthermore, if $p = 2$ and $a < \lambda_1$, in view of

lemma 3.1 (ii), we deduce that (1.1) has at least one classical solution if μ is small enough. Assume now that $1 < p < 2$ and let us fix $C > 0$ such that

$$aC^{p/2-1} + C^{p-1} < \lambda_1. \tag{3.3}$$

Define

$$\psi : [0, \infty) \to [0, \infty), \qquad \psi(s) = \frac{s^p}{s^2 + C}.$$

A careful examination reveals the fact that ψ attains its maximum at

$$\bar{s} = \left(\frac{Cp}{2 - p} \right)^{1/2}.$$

Hence

$$\psi(s) \leqslant \psi(\bar{s}) = \frac{p^{p/2}(2 - p)^{(2-p)/2}}{2C^{1-p/2}} \quad \text{for all } s \geqslant 0.$$

By the classical Young inequality, we deduce that

$$p^{p/2}(2 - p)^{(2-p)/2} \leqslant 2,$$

which yields $\psi(s) \leqslant C^{p/2-1}$ for all $s \geqslant 0$. Thus we have proved

$$s^p \leqslant C^{p/2-1}s^2 + C^{p/2} \quad \text{for all } s \geqslant 0. \tag{3.4}$$

Consider the problem

$$\left.\begin{aligned}
-\Delta u &= g(u) + C^{p/2} + C^{p/2-1}|\nabla u|^2 + \mu f(x, u) && \text{in } \Omega, \\
u &> 0 && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}\right\} \tag{3.5}$$

By virtue of (3.4), any solution of (3.5) is a supersolution of (1.1).

Using (3.3), we get

$$\lim_{s \to \infty} C^{p/2-1}(g(u) + C^{p/2}) < \lambda_1.$$

The above relation enables us to apply lemma 3.1 (ii) with $G(s) = g(s) + C^{p/2}$ and $\tau = C^{p/2-1}$. It follows that there exists $\bar{\mu} > 0$ such that (3.5) has at least a solution u. Using a similar argument to that used in the proof of lemma 3.1, we obtain $\zeta \leqslant u$ in Ω, where ζ is defined in (2.5). By lemma 2.4, we get that (1.1) has at least one solution if $0 \leqslant \mu < \bar{\mu}$.

We have proved that (1.1) has at least one classical solution for both cases $p = 2$ and $a < \lambda_1$ or $1 < p < 2$, provided μ is non-negative and small enough. Next we define

$$A = \{\mu \geqslant 0; \text{ problem (1.1) has at least one solution}\}.$$

The above arguments implies that A is non-empty. Let $\mu^* = \sup A$. We first show that $[0, \mu^*) \subseteq A$. For this purpose, let $\mu_1 \in A$ and $0 \leqslant \mu_2 < \mu_1$. If u_{μ_1} is a solution of (1.1) with $\mu = \mu_1$, then u_{μ_1} is a supersolution of (1.1) with $\mu = \mu_2$. It is easy to prove that $\zeta \leqslant u_{\mu_1}$ in Ω and, by virtue of lemma 2.4, we conclude that problem (1.1) with $\mu = \mu_2$ has at least one solution.

Thus we have proved $[0, \mu^*) \subseteq A$. Next we show $\mu^* < +\infty$.

Since $\lim_{s \searrow 0} g(s) = +\infty$, we can choose $s_0 > 0$ such that $g(s) > \bar{\sigma}$ for all $s \leqslant s_0$, where $\bar{\sigma}$ is defined by lemma 2.6. Let

$$\mu_0 = \frac{\bar{\sigma}}{\min_{x \in \bar{\Omega}} f(x, s_0)}.$$

Using the monotonicity of f with respect to the second argument, the above relations yield

$$g(s) + \mu f(x, s) \geqslant \bar{\sigma} \quad \text{for all } (x, s) \in \bar{\Omega} \times (0, \infty) \text{ and } \mu > \mu_0.$$

If (1.1) has a solution for $\mu > \mu_0$, this would be a supersolution of the problem

$$\left. \begin{array}{ll} -\Delta u = |\nabla u|^p + \bar{\sigma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{3.6}$$

Since 0 is a subsolution, we deduce that (3.6) has at least one solution. According to lemma 2.6, this is a contradiction. Hence $\mu^* \leqslant \mu_0 < +\infty$. This concludes the proof of theorem 1.1. $\qquad\qquad\square$

4. Proof of theorem 1.2

(i) We fix $p \in (0, 1]$ and define

$$q = q(p) = \begin{cases} p + 1 & \text{if } 0 < p < 1, \\ \frac{3}{2} & \text{if } p = 1. \end{cases}$$

Consider the problem

$$\left. \begin{array}{ll} -\Delta u = g(u) + 1 + |\nabla u|^q + \mu f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{4.1}$$

Since $s^p \leqslant s^q + 1$ for all $s \geqslant 0$, we deduce that any solution of (4.1) is a supersolution of (1.1). Furthermore, taking into account the fact that $1 < q < 2$, we can apply theorem 1.1 (ii) in order to get that (4.1) has at least one solution if μ is small enough. Thus, by lemma 2.4, we deduce that (1.1) has at least one classical solution. Following the method used in the proof of theorem 1.1, we set

$$A = \{\mu \geqslant 0; \text{ problem } (1.1) \text{ has at least one solution}\}$$

and let $\mu^* = \sup A$. With the same arguments, we prove that $[0, \mu^*) \subseteq A$. It remains only to show that $\mu^* < +\infty$.

Let us assume first that f satisfies (f1). Since $\lim_{s \searrow 0} g(s) = +\infty$, we can choose $\mu_0 > 2\lambda_1/c$ such that $\frac{1}{2}\mu_0 cs + g(s) \geqslant 1$ for all $s > 0$. Then

$$g(s) + \mu f(x, s) \geqslant \lambda_1 s + 1 \quad \text{for all } (x, s) \in \bar{\Omega} \times (0, \infty) \text{ and } \mu \geqslant \mu_0.$$

By virtue of lemma 2.1, we obtain that (1.1) has no classical solutions if $\mu \geqslant \mu_0$, so μ^* is finite.

Assume now that f satisfies (f2). Since $\lim_{s \searrow 0} g(s) = +\infty$, there exists $s_0 > 0$ such that

$$g(s) \geqslant \lambda_1(s + 1) \quad \text{for all } 0 < s < s_0. \tag{4.2}$$

On the other hand, assumption (f2) and the fact that Ω is bounded implies that the mapping

$$(0, \infty) \ni s \mapsto \frac{\min_{x \in \bar{\Omega}} f(x, s)}{s + 1}$$

is non-decreasing, so we can choose $\tilde{\mu} > 0$ with the property

$$\tilde{\mu} \cdot \frac{\min_{x \in \bar{\Omega}} f(x, s)}{s + 1} \geqslant \lambda_1 \quad \text{for all } s \geqslant s_0. \tag{4.3}$$

Now (4.2) combined with (4.3) yields

$$g(s) + \mu f(x, s) \geqslant \lambda_1(s + 1) \quad \text{for all } (x, s) \in \bar{\Omega} \times (0, \infty) \text{ and } \mu \geqslant \tilde{\mu}.$$

Using lemma 2.1, we deduce that (1.1) has no solutions if $\mu > \tilde{\mu}$, that is, μ^* is finite. The first part of theorem 1.2 is therefore established.

(ii) The strategy is to find a supersolution $\bar{u}_\mu \in C^2(\Omega) \cap C(\bar{\Omega})$ of (1.1) such that $\zeta \leqslant \bar{u}_\mu$ in Ω. To this end, let $h \in C^2(0, \eta] \cap C[0, \eta]$ be such that

$$\left.\begin{array}{ll} h''(t) = -g(h(t)) & \text{for all } 0 < t < \eta, \\ h(0) = 0, \\ h > 0 & \text{in } (0, \eta]. \end{array}\right\} \tag{4.4}$$

The existence of h follows by classical arguments of ordinary differential equations. Since h is concave, there exists $h'(0+) \in (0, +\infty]$. By taking $\eta > 0$ small enough, we can assume that $h' > 0$ in $(0, \eta]$, so h is increasing on $[0, \eta]$.

LEMMA 4.1.

(i) $h \in C^1[0, \eta]$ *if and only if* $\int_0^1 g(s)\,\mathrm{d}s < +\infty$.

(ii) *If* $0 < p \leqslant 2$, *then there exist* $c_1, c_2 > 0$ *such that*

$$(h')^p(t) \leqslant c_1 g(h(t)) + c_2 \quad \text{for all } 0 < t < \eta.$$

Proof. (i) Multiplying by h' in (4.4) and then integrating on $[t, \eta]$, $0 < t < \eta$, we get

$$(h')^2(t) - (h')^2(\eta) = 2 \int_t^\eta g(h(s))h'(s)\,\mathrm{d}s = 2 \int_{h(t)}^{h(\eta)} g(\tau)\,\mathrm{d}\tau. \tag{4.5}$$

This gives

$$(h')^2(t) = 2G(h(t)) + (h')^2(\eta) \quad \text{for all } 0 < t < \eta, \tag{4.6}$$

where

$$G(t) = \int_t^{h(\eta)} g(s)\,\mathrm{d}s.$$

From (4.6), we deduce that $h'(0+)$ is finite if and only if $G(0+)$ is finite, so (i) follows.

(ii) Let $p \in (0, 2]$. Taking into account the fact that g is non-increasing, equality (4.6) leads to

$$(h')^2(t) \leqslant 2h(\eta)g(h(t)) + (h')^2(\eta) \quad \text{for all } 0 < t < \eta. \tag{4.7}$$

Since $s^p \leqslant s^2 + 1$ for all $s \geqslant 0$, from (4.7), we have

$$(h')^p(t) \leqslant c_1 g(h(t)) + c_2 \quad \text{for all } 0 < t < \eta, \tag{4.8}$$

where $c_1 = 2h(\eta)$ and $c_2 = (h')^2(\eta) + 1$. This completes the proof of our lemma. $\square$

Proof of theorem 1.2 completed. Let $p \in (0, 1)$ and $\mu \geqslant 0$ be fixed. We also fix $c > 0$ such that $c\|\varphi_1\|_\infty < \eta$. By Hopf's maximum principle, there exist $\delta > 0$ small enough and $\theta_1 > 0$ such that

$$|\nabla \varphi_1| > \theta_1 \quad \text{in } \Omega_\delta, \tag{4.9}$$

where

$$\Omega_\delta := \{x \in \Omega; \ \text{dist}(x, \partial\Omega) \leqslant \delta\}.$$

Moreover, since $\lim_{s \searrow 0} g(h(s)) = +\infty$, we can pick δ with the property

$$(c\theta_1)^2 g(h(c\varphi_1)) - 3\mu f(x, h(c\varphi_1)) > 0 \quad \text{in } \Omega_\delta. \tag{4.10}$$

Let $\theta_2 := \inf_{\Omega \setminus \Omega_\delta} \varphi_1 > 0$. We choose $M > 1$ with

$$M(c\theta_1)^2 > 3, \tag{4.11}$$

$$Mc\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty) > 3g(h(c\theta_2)). \tag{4.12}$$

Since $p < 1$, we can also assume that

$$(Mc)^{1-p}\lambda_1\theta_2(h')^{1-p}(c\|\varphi_1\|_\infty) \geqslant 3\|\nabla\varphi_1\|_\infty^p. \tag{4.13}$$

On the other hand, by lemma 4.1 (ii), we can choose $M > 1$ such that

$$3(h'(c\varphi_1))^p \leqslant M^{1-p}(c\theta_1)^{2-p}g(h(c\varphi_1)) \quad \text{in } \Omega_\delta. \tag{4.14}$$

Assumption (f4) yields

$$\lim_{s \to \infty} \frac{3\mu f(x, sh(c\|\varphi_1\|_\infty))}{sh(c\|\varphi_1\|_\infty)} = 0 \quad \text{uniformly for } x \in \bar\Omega.$$

So we can choose $M > 1$ large enough such that

$$\frac{3\mu f(x, Mh(c\|\varphi_1\|_\infty))}{Mh(c\|\varphi_1\|_\infty)} < \frac{c\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty)}{h(c\|\varphi_1\|_\infty)} \quad \text{for all } x \in \bar\Omega.$$

This leads us to

$$3\mu f(x, Mh(c\|\varphi_1\|_\infty)) < Mc\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty) \quad \text{for all } x \in \bar\Omega. \tag{4.15}$$

For M satisfying (4.11)–(4.15), we prove that $\bar u_\mu = Mh(c\varphi_1)$ is a supersolution of (1.1). We have

$$-\Delta \bar u_\lambda = Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega. \tag{4.16}$$

First we prove that

$$Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \geqslant g(\bar{u}_\mu) + |\nabla\bar{u}_\mu|^p + \mu f(x, \bar{u}_\mu) \quad \text{in } \Omega_\delta. \qquad (4.17)$$

From (4.9) and (4.11), we get

$$\tfrac{1}{3}Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \geqslant g(h(c\varphi_1)) \geqslant g(Mh(c\varphi_1)) = g(\bar{u}_\mu) \quad \text{in } \Omega_\delta. \qquad (4.18)$$

By (4.9) and (4.14), we also have

$$\tfrac{1}{3}Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \geqslant (Mc)^p (h')^p (c\varphi_1)|\nabla\varphi_1|^p = |\nabla\bar{u}_\mu|^p \quad \text{in } \Omega_\delta. \qquad (4.19)$$

Assumption (f3) and (4.10) produce

$$\tfrac{1}{3}Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \geqslant \mu M f(x, h(c\varphi_1)) \geqslant \mu f(x, Mh(c\varphi_1)) \quad \text{in } \Omega_\delta. \qquad (4.20)$$

Now, by (4.18), (4.19) and (4.20), we conclude that (4.17) is fulfilled.

Next we prove

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant g(\bar{u}_\mu) + |\nabla\bar{u}_\mu|^p + \mu f(x, \bar{u}_\mu) \quad \text{in } \Omega \setminus \Omega_\delta. \qquad (4.21)$$

From (4.12) we obtain

$$\tfrac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant g(h(c\varphi_1)) \geqslant g(Mh(c\varphi_1)) = g(\bar{u}_\mu) \quad \text{in } \Omega \setminus \Omega_\delta. \qquad (4.22)$$

From (4.13) we get

$$\tfrac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant (Mc)^p (h')^p (c\varphi_1)|\nabla\varphi_1|^p = |\nabla\bar{u}_\mu|^p \quad \text{in } \Omega \setminus \Omega_\delta. \qquad (4.23)$$

By (4.15) we deduce that

$$\tfrac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant \mu f(x, Mh(c\varphi_1)) = \mu f(x, \bar{u}_\mu) \quad \text{in } \Omega \setminus \Omega_\delta. \qquad (4.24)$$

Obviously, inequality (4.21) follows now by (4.22), (4.23) and (4.24).

Combining (4.16) with (4.17) and (4.21), we find that $\bar{u}_\mu$ is a supersolution of (1.1). Moreover, $\zeta \leqslant \bar{u}_\mu$ in Ω. Applying lemma 2.4, we deduce that (1.1) has at least one solution for all $\mu \geqslant 0$. This finishes the proof of theorem 1.2. $\qquad \square$

5. Proof of theorem 1.3

The proof case relies on the same arguments used in the proof of theorem 1.2. In fact, the main point is to find a supersolution $\bar{u}_\lambda \in C^2(\Omega) \cap (\bar{\Omega})$ of (1.1), while ζ defined in (2.5) is a subsolution. Since g is non-increasing, the inequality $\zeta \leqslant \bar{u}_\lambda$ in Ω can be proved easily and the existence of solutions to (1.1) follows by lemma 2.4.

Define c, δ and θ_1, θ_2 as in the proof of theorem 1.2. Let M satisfy (4.11) and (4.12). Since $g(h(s)) \to +\infty$ as $s \searrow 0$, we can choose $\delta > 0$ such that

$$(c\theta_1)^2 g(h(c\varphi_1)) - 3f(x, h(c\varphi_1)) > 0 \quad \text{in } \Omega_\delta. \qquad (5.1)$$

Assumption (f4) produces

$$\lim_{s\to\infty} \frac{f(x, sh(c\|\varphi_1\|_\infty))}{sh(c\|\varphi_1\|_\infty)} = 0 \quad \text{uniformly for } x \in \bar{\Omega}.$$

Thus we can take $M > 3$ large enough such that

$$\frac{f(x, Mh(c\|\varphi_1\|_\infty))}{Mh(c\|\varphi_1\|_\infty)} < \frac{c\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty)}{3h(c\|\varphi_1\|_\infty)} \quad \text{for all } x \in \bar{\Omega}.$$

The above relation yields

$$3f(x, Mh(c\|\varphi_1\|_\infty)) < Mc\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty) \quad \text{for all } x \in \bar{\Omega}. \tag{5.2}$$

Using lemma 4.1 (ii), we can take $\lambda > 0$ small enough such that the following inequalities hold:

$$3\lambda M^{p-1}(h')^p(c\varphi_1) \leqslant g(h(c\varphi_1))(c\theta_1)^{2-p} \quad \text{in } \Omega_\delta, \tag{5.3}$$

$$\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty) > 3\lambda(Mc)^{p-1}(h')^p(c\theta_2)\|\nabla\varphi_1\|_\infty^p. \tag{5.4}$$

For M and λ satisfying (4.11), (4.12) and (5.1)–(5.4), we claim that $\bar{u}_\lambda = Mh(c\varphi_1)$ is a supersolution of (1.1). First we have

$$-\Delta\bar{u}_\lambda = Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega. \tag{5.5}$$

Arguing as in the proof of theorem 1.2, from (4.9), (4.11), (5.1), (5.3) and assumption (f3), we obtain

$$Mc^2 g(h(c\varphi_1))|\nabla\varphi_1|^2 \geqslant g(\bar{u}_\lambda) + \lambda|\nabla\bar{u}_\lambda|^p + f(x, \bar{u}_\lambda) \quad \text{in } \Omega_\delta. \tag{5.6}$$

On the other hand, equations (4.12), (5.2) and (5.4) give

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant g(\bar{u}_\lambda) + \lambda|\nabla\bar{u}_\lambda|^p + f(x, \bar{u}_\lambda) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{5.7}$$

Using (5.5) and (5.6), (5.7), we find that $\bar{u}_\lambda$ is a supersolution of (1.1), so our claim follows.

As we have already argued at the beginning of this case, we easily get that $\zeta \leqslant \bar{u}_\lambda$ in Ω and, by lemma 2.4, we deduce that problem (1.1) has at least one solution if $\lambda > 0$ is sufficiently small.

Set

$$A = \{\lambda \geqslant 0; \text{ problem (1.1) has at least one classical solution}\}.$$

From the above arguments, A is non-empty. Let $\lambda^* = \sup A$. First we claim that if $\lambda \in A$, then $[0, \lambda) \subseteq A$. For this purpose, let $\lambda_1 \in A$ and $0 \leqslant \lambda_2 < \lambda_1$. If u_{λ_1} is a solution of (1.1) with $\lambda = \lambda_1$, then u_{λ_1} is a supersolution for (1.1) with $\lambda = \lambda_2$, while ζ defined in (2.5) is a subsolution. Using lemma 2.4 once more, we have that (1.1) with $\lambda = \lambda_2$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $[0, \lambda^*) \subset A$.

Let us assume now that $p \in (1, 2]$. We prove that $\lambda^* < +\infty$. Set

$$m := \inf_{(x,s)\in\bar{\Omega}\times(0,\infty)} (g(s) + f(x, s)).$$

Since $\lim_{s\searrow 0} g(s) = +\infty$ and the mapping $(0, \infty) \ni s \mapsto \min_{x\in\bar{\Omega}} f(x, s)$ is positive and non-decreasing, we deduce that m is a positive real number. Let $\lambda > 0$ be such

that (1.1) has a solution u_λ. If $v = \lambda^{1/(p-1)}u_\lambda$, then v verifies

$$\left.\begin{array}{ll} -\Delta v \geqslant |\nabla v|^p + \lambda^{1/(p-1)}m & \text{in } \Omega, \\ \quad v > 0 & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{5.8}$$

It follows that v is a supersolution of (2.6) for $\sigma = \lambda^{1/(p-1)}m$. Since 0 is a subsolution, we obtain that (2.6) has at least one classical solution for σ defined above. According to lemma 2.6, we have $\sigma \leqslant \bar\sigma$, and so $\lambda \leqslant (\bar\sigma/m)^{p-1}$. This means that λ^* is finite.

Assume now that $p \in (0,1)$ and let us prove that $\lambda^* = +\infty$. Recall that ζ defined in (2.5) is a subsolution. To get a supersolution, we proceed in the same manner. Fix $\lambda > 0$. Since $p < 1$, we can find $M > 1$ large enough such that (4.11), (4.12) and (5.2)–(5.4) hold. From now on, we follow the same steps as above.

The proof of theorem 1.3 is now complete.

6. Proof of theorem 1.4

(i) If $\lambda = 0$, the existence of the solution follows using lemma 2.3. Next we assume that $\lambda > 0$ and fix $\mu \geqslant 0$. With the change of variable $v = \mathrm{e}^{\lambda u} - 1$, problem (1.1) becomes

$$\left.\begin{array}{ll} -\Delta v = \Phi_\lambda(v) & \text{in } \Omega, \\ \quad v > 0 & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{6.1}$$

where

$$\Phi_\lambda(s) = \lambda(s+1)g\left(\frac{1}{\lambda}\ln(s+1)\right) + \lambda\mu(s+1)$$

for all $s \in (0,\infty)$. Obviously, Φ_λ is not monotone, but we still have that the mapping $(0,\infty) \ni s \mapsto \Phi_\lambda(s)/s$ is decreasing for all $\lambda > 0$ and

$$\lim_{s\to+\infty} \frac{\Phi_\lambda(s)}{s} = \lambda(a+\mu) \quad \text{and} \quad \lim_{s\searrow 0} \frac{\Phi_\lambda(s)}{s} = +\infty$$

uniformly for $\lambda > 0$.

We first remark that Φ_λ satisfies the hypotheses in lemma 2.3, provided that $\lambda(a+\mu) < \lambda_1$. Hence (6.1) has at least one solution.

On the other hand, since $g \geqslant a$ on $(0,\infty)$, we get

$$\Phi_\lambda(s) \geqslant \lambda(a+\mu)(s+1) \quad \text{for all } \lambda, s \in (0,\infty). \tag{6.2}$$

Using lemma 2.1, we deduce that (6.1) has no solutions if $\lambda(a+\mu) \geqslant \lambda_1$. The proof of the first part in theorem 1.4 is therefore complete.

(ii) We split the proof into several steps.

STEP 1 (existence of solutions). This follows directly from (i).

STEP 2 (uniqueness of the solution). Fix $\lambda \geqslant 0$. Let u_1 and u_2 be two classical solutions of (1.1) with $\lambda < \lambda^*$. We show that $u_1 \leqslant u_2$ in Ω. Supposing the contrary,

we deduce that $\max_{\bar{\Omega}}\{u_1 - u_2\} > 0$ is achieved in a point $x_0 \in \Omega$. This yields $\nabla(u_1 - u_2)(x_0) = 0$ and

$$0 \leqslant -\Delta(u_1 - u_2)(x_0) = g(u_1(x_0)) - g(u_2(x_0)) < 0,$$

a contradiction. We conclude that $u_1 \leqslant u_2$ in Ω; similarly, $u_2 \leqslant u_1$. Therefore, $u_1 = u_2$ in Ω and the uniqueness is proved.

STEP 3 (dependence on λ). Fix $0 \leqslant \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_1}, u_{λ_2} be the unique solutions of (1.1) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. If $\{x \in \Omega;\ u_{\lambda_1} > u_{\lambda_2}\}$ is non-empty, then $\max_{\bar{\Omega}}\{u_{\lambda_1} - u_{\lambda_2}\} > 0$ is achieved in Ω. At that point, say, $\bar{x}$, we have $\nabla(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = 0$ and

$$0 \leqslant -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = g(u_{\lambda_1}(\bar{x})) - g(u_{\lambda_2}(\bar{x})) + (\lambda_1 - \lambda_2)|\nabla u_{\lambda_1}|^p(\bar{x}) < 0,$$

which is a contradiction.

Hence $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\bar{\Omega}$. The maximum principle also gives $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

STEP 4 (regularity). We fix $0 < \lambda < \lambda^*$, $\mu > 0$ and assume that

$$\limsup_{s \searrow 0} s^\alpha g(s) < +\infty.$$

This means that $g(s) \leqslant cs^{-\alpha}$ in a small positive neighbourhood of the origin. To prove the regularity, we will use again the change of variable $v = e^{\lambda u} - 1$. Thus, if u_λ is the unique solution of (1.1), then $v_\lambda = e^{\lambda u_\lambda} - 1$ is the unique solution of (6.1). Since

$$\lim_{s \searrow 0} \frac{e^{\lambda s} - 1}{s} = \lambda,$$

we conclude that (ii1) and (ii2) of theorem 1.4 are established if we prove the following.

(a) $\tilde{c}_1 \operatorname{dist}(x, \partial\Omega) \leqslant v_\lambda(x) \leqslant \tilde{c}_2 \operatorname{dist}(x, \partial\Omega)$ in Ω for some positive constants $\tilde{c}_1, \tilde{c}_2 > 0$.

(b) $v_\lambda \in C^{1,1-\alpha}(\bar{\Omega})$.

Proof of (a). By the monotonicity of g and the fact that $g(s) \leqslant cs^{-\alpha}$ near the origin, we deduce the existence of $A, B, C > 0$ such that

$$\Phi_\lambda(s) \leqslant As + Bs^{-\alpha} + C \quad \text{for all } 0 < \lambda < \lambda^* \text{ and } s > 0. \tag{6.3}$$

Let us fix $m > 0$ such that $m\lambda_1\|\varphi_1\|_\infty < \lambda\mu$. Combining this with (6.2), we deduce

$$-\Delta(v_\lambda - m\varphi_1) = \Phi_\lambda(v_\lambda) - m\lambda_1\varphi_1 \geqslant \lambda\mu - m\lambda_1\varphi_1 \geqslant 0 \tag{6.4}$$

in Ω. Since $v_\lambda - m\varphi_1 = 0$ on $\partial\Omega$, we conclude that

$$v_\lambda \geqslant m\varphi_1 \quad \text{in } \Omega. \tag{6.5}$$

Now, equations (6.5) and (2.1) imply $v_\lambda \geqslant \tilde{c}_1 \operatorname{dist}(x, \partial\Omega)$ in Ω for some positive constant $\tilde{c}_1 > 0$. The first inequality in the statement of (a) is therefore established. For the second one, we apply an idea found in Gui and Lin [21]. Using (6.5) and

the estimate (6.3), by virtue of lemma 2.2, we deduce $\Phi_\lambda(v_\lambda) \in L^1(\Omega)$, that is, $\Delta v_\lambda \in L^1(\Omega)$.

Using the smoothness of $\partial\Omega$, we can find $\delta \in (0,1)$ such that, for all

$$x_0 \in \Omega_\delta := \{x \in \Omega; \; \mathrm{dist}(x, \partial\Omega) \leqslant \delta\},$$

there exists $y \in \mathbb{R}^N \setminus \bar{\Omega}$ with $\mathrm{dist}(y, \partial\Omega) = \delta$ and $\mathrm{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$.

Let $K > 1$ be such that $\mathrm{diam}(\Omega) < (K-1)\delta$ and let ξ be the unique solution of the Dirichlet problem

$$\left.\begin{array}{rll} -\Delta\xi = \Phi_\lambda(\xi) & \text{in } B_K(0) \setminus B_1(0), \\ \xi > 0 & \text{in } B_K(0) \setminus B_1(0), \\ \xi = 0 & \text{on } \partial(B_K(0) \setminus B_1(0)), \end{array}\right\} \tag{6.6}$$

where $B_r(0)$ denotes the open ball in $\mathbb{R}^N$ of radius r and centred at the origin. By uniqueness, ξ is radially symmetric. Hence $\xi(x) = \tilde{\xi}(|x|)$ and

$$\left.\begin{array}{rll} \tilde{\xi}'' + \dfrac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi}) = 0 & \text{in } (1, K), \\ \tilde{\xi} > 0 & \text{in } (1, K), \\ \tilde{\xi}(1) = \tilde{\xi}(K) = 0. \end{array}\right\} \tag{6.7}$$

Integrating in (6.7), we have

$$\tilde{\xi}'(t) = \tilde{\xi}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}\Phi_\lambda(\tilde{\xi}(r))\,\mathrm{d}r$$

$$= \tilde{\xi}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}\Phi_\lambda(\tilde{\xi}(r))\,\mathrm{d}r,$$

where $1 < a < t < b < K$. With the same arguments as above, we have $\Phi_\lambda(\tilde{\xi}) \in L^1(1, K)$, which implies that both $\tilde{\xi}'(1)$ and $\tilde{\xi}'(K)$ are finite. Hence $\tilde{\xi} \in C^2(1, K) \cap C^1[1, K]$. Furthermore,

$$\xi(x) \leqslant \tilde{C}\min\{K - |x|, |x| - 1\} \quad \text{for any } x \in B_K(0) \setminus B_1(0). \tag{6.8}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \bar{\Omega}$ with $\mathrm{dist}(y_0, \partial\Omega) = \delta$ and $\mathrm{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$. Thus $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define

$$\bar{v}(x) = \xi\left(\frac{x - y_0}{\delta}\right)$$

for all $x \in \bar{\Omega}$. We show that $\bar{v}$ is a supersolution of (6.1). Indeed, for all $x \in \Omega$, we have

$$\Delta\bar{v} + \Phi_\lambda(\bar{v}) = \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}'\right) + \Phi_\lambda(\tilde{\xi})$$

$$\leqslant \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi})\right)$$

$$= 0,$$

where $r = |x - y_0|/\delta$. We have obtained that

$$
\begin{aligned}
\Delta\bar{v} + \Phi_\lambda(\bar{v}) \leqslant 0 \leqslant \Delta v_\lambda + \Phi_\lambda(v_\lambda) & \quad \text{in } \Omega, \\
\bar{v}, v_\lambda > 0 & \quad \text{in } \Omega, \\
\bar{v} = v_\lambda & \quad \text{on } \partial\Omega, \\
\Delta v_\lambda \in L^1(\Omega). &
\end{aligned}
$$

By lemma 2.5, we get $v_\lambda \leqslant \bar{v}$ in Ω. Combining this with (6.8), we obtain

$$
v_\lambda(x_0) \leqslant \bar{v}(x_0) \leqslant \tilde{C} \min\left\{ K - \frac{|x_0 - y_0|}{\delta}, \frac{|x_0 - y_0|}{\delta} - 1 \right\} \leqslant \frac{\tilde{C}}{\delta} \operatorname{dist}(x_0, \partial\Omega).
$$

Hence $v_\lambda \leqslant (\tilde{C}/\delta) \operatorname{dist}(x, \partial\Omega)$ in Ω_δ and the second inequality in the statement of (a) follows. $\qquad\square$

Proof of (b). Let G be Green's function associated with the Laplace operator in Ω. Then, for all $x \in \Omega$, we have

$$
v_\lambda(x) = -\int_\Omega G(x, y)\Phi_\lambda(v_\lambda(y)) \, \mathrm{d}y
$$

and

$$
\nabla v_\lambda(x) = -\int_\Omega G_x(x, y)\Phi_\lambda(v_\lambda(y)) \, \mathrm{d}y.
$$

If $x_1, x_2 \in \Omega$, using (6.3), we obtain

$$
\begin{aligned}
|\nabla v_\lambda(x_1) - \nabla v_\lambda(x_2)| \leqslant \int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot (Av_\lambda + C) \, \mathrm{d}y \\
+ B\int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot v_\lambda^{-\alpha}(y) \, \mathrm{d}y.
\end{aligned}
$$

Now, taking into account that $v_\lambda \in C(\bar{\Omega})$, by the standard regularity theory (see [19]), we get

$$
\int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot (Av_\lambda + C) \, \mathrm{d}y \leqslant \tilde{c}_1 |x_1 - x_2|.
$$

On the other hand, with the same proof as in [21, theorem 1], we deduce

$$
\int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot v_\lambda^{-\alpha}(y) \leqslant \tilde{c}_2 |x_1 - x_2|^{1-\alpha}.
$$

The above inequalities imply that $u_\lambda \in C^2(\Omega) \cap C^{1,1-\alpha}(\bar{\Omega})$. $\qquad\square$

STEP 5 (asymptotic behaviour of the solution). In order to conclude the asymptotic behaviour for u_λ, it is enough to show that $\lim_{\lambda \nearrow \lambda^*} v_\lambda = +\infty$ on compact subsets of Ω. To this end, we use some techniques developed in [29]. Due to the special character of our problem, we will be able to show in what follows that, in certain cases, L^2-boundedness implies H_0^1-boundedness!

We argue by contradiction. Since $(v_\lambda)_{\lambda < \lambda^*}$ is a sequence of non-negative superharmonic functions in Ω, then, by [23, theorem 4.1.9], we can find a subsequence

of $(v_\lambda)_{\lambda<\lambda^*}$ (still denoted by $(v_\lambda)_{\lambda<\lambda^*}$) that converges in $L^1_{\mathrm{loc}}(\Omega)$ to some v^*. The monotony of v_λ yields (up to a subsequence) $v_\lambda \nearrow v^*$ a.e. in Ω.

We first show that $(v_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. Suppose the contrary. Passing eventually at a subsequence, we have $v_\lambda = M(\lambda)w_\lambda$, where

$$M(\lambda) = \|v_\lambda\|_{L^2(\Omega)} \to \infty \quad \text{as } \lambda \nearrow \lambda^* \quad \text{and} \quad w_\lambda \in L^2(\Omega), \quad \|w_\lambda\|_{L^2(\Omega)} = 1. \quad (6.9)$$

Then (6.3) yields

$$\frac{1}{M(\lambda)}\Phi_\lambda(v_\lambda) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \quad \text{as } \lambda \nearrow \lambda^*,$$

that is,

$$-\Delta w_\lambda \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \quad \text{as } \lambda \nearrow \lambda^*. \quad (6.10)$$

By Green's first identity, we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla \phi \, \mathrm{d}x = -\int_\Omega \phi \Delta w_\lambda \, \mathrm{d}x = -\int_{\mathrm{supp}\,\phi} \phi \Delta w_\lambda \, \mathrm{d}x \quad \text{for all } \phi \in C_0^\infty(\Omega). \quad (6.11)$$

Using (6.10), we obtain

$$\left| \int_{\mathrm{supp}\,\phi} \phi \Delta w_\lambda \, \mathrm{d}x \right| \leqslant \int_{\mathrm{supp}\,\phi} |\phi||\Delta w_\lambda| \, \mathrm{d}x \leqslant \|\phi\|_\infty \int_{\mathrm{supp}\,\phi} |\Delta w_\lambda| \, \mathrm{d}x \to 0 \quad \text{as } \lambda \nearrow \lambda^*. \quad (6.12)$$

Now, equations (6.11) and (6.12) yield

$$\int_\Omega \nabla w_\lambda \cdot \nabla \phi \, \mathrm{d}x \to 0 \quad \text{as } \lambda \nearrow \lambda^* \quad \text{for all } \phi \in C_0^\infty(\Omega). \quad (6.13)$$

Recall that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using (6.3) and Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla w_\lambda|^2 &= -\int_\Omega w_\lambda \Delta w_\lambda \\
&= -\frac{1}{M(\lambda)} \int_\Omega w_\lambda \Delta u_\lambda \\
&= \frac{1}{M(\lambda)} \int_\Omega w_\lambda \Phi_\lambda(v_\lambda) \\
&\leqslant \frac{A}{M(\lambda)} \int_\Omega w_\lambda v_\lambda + \frac{B}{M(\lambda)} \int_\Omega w_\lambda v_\lambda^{-\alpha} + \frac{C}{M(\lambda)} \int_\Omega w_\lambda \\
&= A \int_\Omega w_\lambda^2 + \frac{B}{M(\lambda)^{1+\alpha}} \int_\Omega w_\lambda^{1-\alpha} + \frac{C}{M(\lambda)} \int_\Omega w_\lambda \\
&\leqslant A + \frac{B}{M(\lambda)^{1+\alpha}} |\Omega|^{(1+\alpha)/2} + \frac{C}{M(\lambda)} |\Omega|^{1/2}.
\end{aligned}$$

From the above estimates, we can easily conclude that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Thus there exists $w \in H_0^1(\Omega)$ such that

$$w_\lambda \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega) \quad (6.14)$$

and

$$w_\lambda \to w \quad \text{strongly in } L^2(\Omega). \tag{6.15}$$

Combining (6.9) and (6.15), we get $\|w\|_{L^2(\Omega)} = 1$. On the other hand, from (6.13) and (6.14), we obtain

$$\int_\Omega \nabla w \cdot \nabla \phi \, \mathrm{d}x = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Since $w \in H_0^1(\Omega)$, using the above relation and the definition of $H_0^1(\Omega)$, we get $w = 0$, which contradicts the fact that $\|w\|_{L^2(\Omega)} = 1$. Hence $(v_\lambda)_{\lambda < \lambda^*}$ is bounded in $L^2(\Omega)$. As before for w_λ, we can obtain that $(v_\lambda)_{\lambda < \lambda^*}$ is bounded in $H_0^1(\Omega)$. Then, up to a subsequence, we have

$$\left.\begin{aligned}
v_\lambda \rightharpoonup v^* \quad &\text{weakly in } H_0^1(\Omega) \quad &&\text{as } \lambda \nearrow \lambda^*, \\
v_\lambda \to v^* \quad &\text{strongly in } L^2(\Omega) \quad &&\text{as } \lambda \nearrow \lambda^*, \\
v_\lambda \to v^* \quad &\text{a.e. in } \Omega \quad &&\text{as } \lambda \nearrow \lambda^*.
\end{aligned}\right\} \tag{6.16}$$

Now we can proceed to get a contradiction. Multiplying by φ_1 in (6.1) and then integrating over Ω, we have

$$-\int_\Omega \Delta v_\lambda \varphi_1 \, \mathrm{d}x = \int_\Omega \Phi_\lambda(v_\lambda)\varphi_1 \, \mathrm{d}x \quad \text{for all } 0 < \lambda < \lambda^*. \tag{6.17}$$

Using (6.2), we get

$$\lambda_1 \int_\Omega v_\lambda \varphi_1 \geqslant \lambda(a + \mu) \int_\Omega (v_\lambda + 1)\varphi_1 \, \mathrm{d}x \quad \text{for all } 0 < \lambda < \lambda^*. \tag{6.18}$$

By (6.16), we can use Lebesgue's dominated convergence theorem in order to pass to the limit with $\lambda \nearrow \lambda^*$ in (6.18). We obtain

$$\lambda_1 \int_\Omega v^* \varphi_1 \, \mathrm{d}x \geqslant \lambda_1 \int_\Omega (v^* + 1)\varphi_1 \, \mathrm{d}x, \tag{6.19}$$

which is a contradiction since $\varphi_1 > 0$ in Ω. This contradiction shows that

$$\lim_{\lambda \nearrow \lambda^*} v_\lambda = +\infty$$

uniformly on compact subsets of Ω, which implies that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$ uniformly on compact subsets of Ω. The proof of theorem 1.4 is now complete.

Acknowledgments

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