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Variational analysis of anisotropic Schrödinger equations without Ambrosetti–Rabinowitz-type condition

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Abstract. This article is concerned with the qualitative analysis of weak solutions to nonlinear stationary Schrödinger-type equations of the form

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ -2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

without the Ambrosetti–Rabinowitz growth condition. Our arguments rely on the existence of a Cerami sequence by using a variant of the mountain-pass theorem due to Schechter.

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1. Introduction

In quantum mechanics, the Schrödinger equation is a partial differential equation that describes how the quantum state of a quantum system changes with time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Schrödinger [31]. In classical mechanics, Newton's second law ($F = ma$) is used to make a mathematical prediction as to what path a given system will take following a set of known initial conditions. In quantum mechanics, the analogue of Newton's law is Schrödinger's equation for a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localized). It is not a simple algebraic equation, but in general a linear partial differential equation, describing the time-evolution of the system's wave function (also called a "state function"). The nonlinear Schrödinger equation also describes various phenomena arising in the theory of Heisenberg ferromagnets and magnons, self-channeling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, electromagnetic fields, plasma physics (e.g., the Kurihara superfluid film equation). We also refer to the pioneering paper by Gamow [13] who was particularly interested in the *tunneling effect*, which lead to the construction of the electronic microscope and the correct study of the alpha radioactivity. The notion of "solution" used by him was not explicitly mentioned in the paper, but it is coherent with the notion of weak solution introduced several years later by other authors such as Leray, Sobolev and Schwartz. We refer to Ablowitz et al. [1], Cazenave [9], Sulem [32] for a modern overview and relevant applications. Recent contributions to the analysis of nonlinear Schrödinger equations may be found in [14,16,22].

Our main purpose is to consider the nonlinear Schrödinger equation in a new setting corresponding to anisotropic spaces of Sobolev-type. More precisely, the standard linear Laplace operator Δ is replaced

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with the non-homogeneous differential operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$. This is an anisotropic operator with a complicated structure, in which different space directions have different roles.

2. Abstract setting

In the present paper, we are interested in the study of the anisotropic nonlinear problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ - 2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. We assume that λ is a positive parameter, $b \in L^\infty(\Omega)$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, the functions $a_i(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are of the type $|t|^{p_i(x)-2}t$ with $p_i(x) \in C(\overline{\Omega})$, $p_i(x) > 1$ and $\min p_i(x) > 1$ for all $i = 1, \dots, N$. Let $P_+^+ = \max_{i \in \{1,\dots,N\}} \sup_{x \in \Omega} p_i(x)$ and assume that $P_+^+ \geq 2$ (see also the basic assumption (3.2)). We denote by $a_i(x,\eta)$ the continuous derivative with respect to η of the mapping $A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $A_i = A_i(x,\eta)$, that is, $a_i(x,\eta) = \frac{\partial}{\partial \eta} A_i(x,\eta)$ for all $i \in \{1, \dots, N\}$.

The study of this kind of nonlinear problems described by non-homogeneous differential operators has been an interesting topic in relationship with several relevant applications, such as electro-rheological fluids (see Ružička [29]). The first major discovery in electro-rheological fluids was due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance, lithium polymethacrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. We refer to the monograph [27] for more details.

Throughout this paper, we assume that the following hypotheses are fulfilled for all $i = 1, \dots, N$:

($\mathbf{A_0}$) $A_i(x, 0) = 0$ for a.e. $x \in \Omega$.

($\mathbf{A_1}$) The mapping a_i is continuous and satisfies the growth condition

$$|a_i(x,\eta)| \leq 1 + |\eta|^{p_i(x)-1},$$

for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}$.

($\mathbf{A_2}$) The inequalities

$$|\eta|^{p_i(x)} \leq a_i(x,\eta)\eta \leq p_i(x) A_i(x,\eta)$$

hold for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}$.

($\mathbf{A_3}$) There exists $k_i > 0$ such that

$$A_i\left(x, \frac{\eta+\xi}{2}\right) \leq \frac{1}{2} A_i(x,\eta) + \frac{1}{2} A_i(x,\xi) - k_i |\eta - \xi|^{p_i(x)},$$

for all $x \in \overline{\Omega}$ and $\eta, \xi \in \mathbb{R}$, with equality if and only if $\eta = \xi$.

($\mathbf{B}$) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

($\mathbf{f_1}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

$$\lim_{t \to 0} \frac{f(x,t)}{|t|^{P_+^+ - 1}} = 0$$

uniformly in $x \in \Omega$.

($\mathbf{f_2}$) f satisfies the subcritical growth condition

$$|f(x,t)| \le a + b|t|^{q(x)-1} \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where a and b are positive constants and $q(x)$ is a continuous function such that $P_+^+ < q^- \le q^+ < P_-^*$, ($P_-^*$ is defined in (3.3) while q^- and q^+ are defined as in (2.2)).

($\mathbf{f_3}$) $\lim_{|t|\to+\infty} \dfrac{F(x,t)}{|t|^{P_+^+}} = +\infty$, uniformly in $x \in \Omega$, where $F(x,t) := \int\limits_0^t f(x,s)\,\mathrm{d}s$.

($\mathbf{f_4}$) There exists a constant $C_* > 0$ such that

$$G(x,t) \le G(x,s) + C_*,$$

for each $x \in \Omega$, $0 < t < s$ or $s < t < 0$, where $G(x,t) := tf(x,t) - F(x,t)$.

The differential operator in problem (2.1) is the anisotropic $\overrightarrow{p}(x)$–Laplace type operator (where $\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x))$ because, if we take

$$a_i(x,\eta) = |\eta|^{p_i(x)-2}\eta,$$

for all $i \in \{1,\ldots,N\}$, then $A_i(x,\eta) = \frac{1}{p_i(x)}|\eta|^{p_i(x)}$ for all $i \in \{1,\ldots,N\}$, that is,

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^N \partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big).$$

Obviously, there are many other operators deriving from $\sum_{i=1}^N \partial_{x_i} a_i(x, \partial_{x_i}u)$. Indeed, if

$$a_i(x,\eta) = (1+|\eta|^2)^{\frac{p_i(x)-2}{2}}\eta, \quad \text{for all } i \in \{1,\ldots,N\},$$

then we have $A_i(x,\eta) = \frac{1}{p_i(x)}[(1+|\eta|^2)^{\frac{p_i(x)}{2}} - 1]$ for all $i \in \{1,\ldots,N\}$ and we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^N \partial_{x_i}\Big[(1+|\partial_{x_i}u|^2)^{\frac{p_i(x)-2}{2}}\partial_{x_i}u\Big].$$

Anisotropic type problems have received specific attention in recent decades. We refer to [3–8] and [12,17,21,24,26,28]. In a recent paper [10], the authors have studied the following anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum\limits_{i=1}^N \partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big) = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with non-standard growth conditions. By using variational methods, they obtained existence and multiplicity results. Our results in the present paper extend to a general abstract setting the existence property obtained in [10].

Now, we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We will also introduce an adequate functional space where problems of type (2.1) can be studied. Such a space will be called an anisotropic variable exponent Sobolev space, and it can be characterized as a functional space of Sobolev's type in which different space directions have different roles.

For any $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\overline{\Omega}) = \{h(x) \in C(\overline{\Omega}) : \ 1 < \min_{x\in\overline{\Omega}} h(x) < \max_{x\in\overline{\Omega}} h(x) < \infty\},$$

and we define

$$h^+ = \sup\{h(x) : \ x \in \overline{\Omega}\}, \qquad h^- = \inf\{h(x) : \ x \in \overline{\Omega}\}. \tag{2.2}$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \Big\{ u : \ u \text{ is a measurable real-valued function and } \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x < \infty \Big\},$$

endowed with the *Luxemburg norm*

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \Big\{ \mu > 0 : \ \int_\Omega \Big| \frac{u(x)}{\mu} \Big|^{p(x)} \, \mathrm{d}x \leq 1 \Big\}.$$

As established by Kováčik and Rákosník [19], $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space.

Proposition 2.1. (see Edmunds and Rákosník [11]) *For all $p(x) \in C_+(\overline{\Omega})$, we have the following properties:* **(i)** *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\Big| \int_\Omega uv \, \mathrm{d}x \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) |u|_{p(x)} |v|_{q(x)} \leq 2 |u|_{p(x)} |v|_{q(x)}.$$

(ii) *If $p_1(x), \ p_2(x) \in C_+(\overline{\Omega}), \ p_1(x) \leq p_2(x), \ \forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. (see Rădulescu and Repovš [27]) *For all $u \in L^{p(x)}(\Omega), \ (u_n) \subset L^{p(x)}(\Omega)$, we have*

(1) $|u|_{p(x)} < 1$ $(\textit{respectively} = 1; \ > 1) \Longleftrightarrow \int_\Omega |u|^{p(x)} \, \mathrm{d}x < 1$ $(\textit{respectively} = 1; \ > 1)$;

(2) *for* $u \neq 0, \ |u|_{p(x)} = \lambda \Longleftrightarrow \int_\Omega \Big| \frac{u}{\lambda} \Big|^{p(x)} \, \mathrm{d}x = 1$;

(3) *if* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \leq \int_\Omega |u|^{p(x)} \, \mathrm{d}x \leq |u|_{p(x)}^{p^+}$;

(4) *if* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \leq \int_\Omega |u|^{p(x)} \, \mathrm{d}x \leq |u|_{p(x)}^{p^-}$;

(5) $|u_n|_{p(x)} \to 0$ $(\textit{respectively} \to \infty) \Longleftrightarrow \int_\Omega |u_n|^{p(x)} \, \mathrm{d}x \to 0$ $(\textit{respectively} \to \infty)$.

The Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : \ \partial_{x_i} u \in L^{p(x)}(\Omega), \ i \in \{1, \ldots, N\} \}.$$

Then, $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{p(x)} = |u(x)|_{p(x)} + |\nabla u(x)|_{p(x)}.$$

As shown by Zhikov [33], smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$. This property is related to the *Lavrentiev phenomenon*, which asserts that there are variational problems for which the infimum over the smooth functions is strictly greater than the infimum over all functions that satisfy the same boundary conditions. An equivalent formulation asserts that a Lagrangian L exhibits the Lavrentiev phenomenon if the infimum taken over the set of absolutely continuous trajectories $\mathbf{AC}[0, 1]$ is strictly lower than the infimum taken over the set of Lipschitzian trajectories $\mathbf{Lip}[0, 1]$, with fixed boundary conditions. We refer to [27, pp. 12–13] for more details. Zhikov [33] also established that if the exponent variable p in $C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is, there exists $M>0$ such that

$$|p(x)-p(y)| \leq \frac{-M}{\log(|x-y|)} \quad \text{for all} \quad x,y\in\Omega \quad \text{such that} \quad |x-y|\leq \frac{1}{2},$$

then smooth functions are dense in $W^{1,p(x)}(\Omega)$. The Sobolev space with zero boundary values $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$. Of course, also the norms $\|u\|_{p(x)} = |\nabla u|_{p(x)}$ and $\|u\|_{p(x)} = \sum_{i=1}^N |\partial_{x_i} u|_{p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$. Note that when $s\in C_+(\overline{\Omega})$ and $s(x)<p^*(x)$ for all $x\in\overline{\Omega}$, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ if $p(x)<N$ and $p^*(x)=\infty$ if $p(x)\geq N$, then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega)$ is compact. Details, extensions and further references can be found in [15,19].

Finally, we introduce a natural generalization of the function space $W_0^{1,p(x)}(\Omega)$, which will enable us to study with sufficient accuracy problem (2.1). For this purpose, let us denote by $\overrightarrow{p}:\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\overrightarrow{p}(x)=(p_1(x),p_2(x),\dots,p_N(x))$ with $p_i\in C_+(\overline{\Omega}),\ i\in\{1,\dots,N\}$. We define $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, the *anisotropic variable exponent Sobolev space*, as the closure of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\| = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}. \tag{2.3}$$

As it was pointed out in [25], $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a reflexive Banach space.

The above definition shows that the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a function space of Sobolev's type in which different space directions have different roles.

As pointed out in [27], the function spaces with variable exponent have some striking properties, such as:

(i) If $1<p^-\leq p^+<\infty$ and $p:\overline{\Omega}\to[1,\infty)$ is smooth, then the formula

$$\int\limits_\Omega |u(x)|^p \mathrm{d}x = p\int\limits_0^\infty t^{p-1}\,|\{x\in\Omega;\ |u(x)|>t\}|\,\mathrm{d}t$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the mean continuity property. More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u\in L^{p(x)}(B)$ such that $u(x+h)\notin L^{p(x)}(B)$ for all $h\in\mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are *never* translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f*g|_{p(x)} \leq C\,|f|_{p(x)}\,\|g\|_{L^1}$$

holds if and only if p is constant.

We refer to the books [2,24,27] for related results and complements.

3. Notations and auxiliary results

A central role in our analysis will be played by the vectors $\overrightarrow{P}_+,\ \overrightarrow{P}_-\in\mathbb{R}^N$ and by the positive numbers $P_+^+,\ P_-^+,\ P_+^-,\ P_-^-$ defined as follows (see (2.2) for notations):

$$\begin{array}{ll} \overrightarrow{P}_+ = (p_1^+,p_2^+,\dots,p_N^+), & \overrightarrow{P}_- = (p_1^-,p_2^-,\dots,p_N^-), \\ P_+^+ = \max\{p_1^+,p_2^+,\dots,p_N^+\}, & P_-^+ = \max\{p_1^-,p_2^-,\dots,p_N^-\}, \\ P_+^- = \min\{p_1^+,p_2^+,\dots,p_N^+\}, & P_-^- = \min\{p_1^-,p_2^-,\dots,p_N^-\}. \end{array} \tag{3.1}$$

Throughout this paper, we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{3.2}$$

By [27, Theorem 1, p. 13], this condition ensures that the anisotropic space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is embedded into some Lebesgue space $L^r(\Omega)$. If hypothesis (3.2) is no longer fulfilled, then one has embeddings into Orlicz or Hölder spaces.

Define $P_-^* \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \qquad P_{-,\infty} = \max\{P_-^+, P_-^*\}. \tag{3.3}$$

It should be noticed that if $p_i^- = p$ then P_-^* gives the usual Sobolev conjugate p^*.

Now we recall a theorem concerning the embedding of the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ into the Lebesgue space with variable exponent $L^{q(x)}(\Omega)$.

Proposition 3.1. (see [20]) *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary. Assume that relation (3.2) is satisfied and that $q \in C(\overline{\Omega})$ verifies*

$$1 < q(x) < P_{-,\infty}, \qquad \text{for all} \quad x \in \overline{\Omega}.$$

Then, the embedding

$$W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact.

It should be noticed that, due to the condition $P_+^+ < P_-^*$ imposed by $(\mathbf{f_2})$, we have

$$P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^*. \tag{3.4}$$

We give some definitions and results stated in a general Banach space X. Of course, in our setting, $X = W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

Definition 3.2. Let $(X, \|\cdot\|_X)$ be a real Banach space with its dual space $(X^*, \|\cdot\|_{X^*})$ and $I \in C^1(X, \mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the Cerami condition at level c if for any sequence $(u_n) \subset X$ with

$$I(u_n) \to c, \quad \|I'(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.5}$$

there is a subsequence (u_n) such that (u_n) converges strongly in X. Any sequence $(u_n) \subset X$ for which (3.5) holds true is called a Cerami sequence at level c.

The following version of the mountain-pass theorem, which can be found in [30], gives us the existence of a Cerami sequence at the mountain-pass level.

Lemma 3.3. *Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the Cerami condition at level c for any $c \in \mathbb{R}$, $I(0) = 0$ and*

(i) *there are constants $\rho, \ \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$;*
(ii) *there exists $e \in X \setminus B_\rho$ such that $I(e) \leq 0$.*

Then,

$$c_0 = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \geq \alpha \tag{3.6}$$

is a critical value of I where

$$\Gamma = \left\{\gamma \in C^0([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = e\right\}.$$

We refer to the survey paper by Pucci and Rădulescu [23] for various applications of the mountain-pass theorem.

In the present paper, we choose $X = W_0^{1,\overrightarrow{p}(x)}(\Omega)$. In order to associate a variational framework with problem (2.1) and to associate an energy functional, we introduce $\Lambda_i : X \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, defined on $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ by setting

$$\Lambda_i(u) = \int_\Omega A_i(x, \partial_{x_i} u)\, \mathrm{d}x.$$

for all $u \in X$.

Proposition 3.4. (see [18]) *For $i \in \{1, \ldots, N\}$,*

(i) Λ_i is well defined on X,

(ii) $\Lambda_i \in C^1(X, \mathbb{R})$ and

$$\langle \Lambda_i'(u), \varphi \rangle = \int_\Omega a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi\, \mathrm{d}x,$$

for all $u, \varphi \in X$. In addition, Λ_i' is a continuous, bounded and strictly monotone operator.

(iii) Λ_i is weakly lower semi-continuous.

Denote by $\Lambda : X \to \mathbb{R}$ the functional

$$\Lambda(u) = \sum_{i=1}^N \Lambda_i(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\, \mathrm{d}x.$$

Note that, by $(\mathbf{A_2})$ and since $p_i(x) \leq P_+^+$, we have

$$\Lambda(u) \geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} \mathrm{d}x. \tag{3.7}$$

We recall the following result concerning the functional Λ and which establishes that Λ fulfills the Kadec-Klee property.

Lemma 3.5. (see [21]) *Assume that hypotheses $(\mathbf{A_1})$ and $(\mathbf{A_3})$ hold. Let (u_n) be a sequence that weakly converges to u in X and*

$$\limsup_{n\to\infty} \langle \Lambda'(u_n), u_n \rangle = \limsup_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u)\, \mathrm{d}x \leq 0.$$

Then, (u_n) strongly converges to u in X.

In the sequel, we use c_i, to denote a generic nonnegative or positive constant (the exact value may change from line to line).

4. Main result

This section is devoted to the statement and to the proof of our main results. Given $\lambda > 0$, let us define the energy functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+} |u|^{P_+^+} - \lambda F(x, u) \Big\} \mathrm{d}x.$$

We recall that $F(x,t)=\int_0^t f(x,s)\mathrm{d}s$ and $A(x,\eta)=\int_0^\eta a_i(x,z)\mathrm{d}z$.

By hypotheses $(\mathbf{A_0})$, $(\mathbf{A_1})$, $(\mathbf{f_1})$ and $(\mathbf{f_2})$, the functional I_λ associated with problem (2.1) is well defined and of C^1 class on X. Moreover, by Proposition 3.4 (ii) we have

$$\langle I'_\lambda(u),\varphi\rangle=\int_\Omega\Big\{\sum_{i=1}^N a_i(x,\partial_{x_i}u)\partial_{x_i}\varphi+b(x)|u|^{P_+^+-2}u\varphi-\lambda f(x,u)\varphi\Big\}\mathrm{d}x,$$

for all $u,\varphi\in X$. Thus, weak solutions of problem (2.1) are exactly the critical points of the functional I_λ, so by means of Lemma 3.3, we intend to establish the existence of critical points in order to deduce the existence of weak solutions.

Our main result is the following existence property.

Theorem 4.1. *Assume that hypotheses* $(\mathbf{A_0})$*–*$(\mathbf{A_3})$*,* $(\mathbf{f_1})$*–*$(\mathbf{f_4})$ *and* $(\mathbf{B})$ *are fulfilled. Then, problem* (2.1) *has a weak solution.*

For the proof of Theorem 4.1, we show that the energy functional I_λ satisfies hypotheses of Lemma 3.3. To this end, we proceed with the following auxiliary result, which establishes that I_λ satisfies the geometrical configuration required by Lemma 3.3.

Lemma 4.2. *If the conditions* $(\mathbf{A_0})$*,* $(\mathbf{A_2})$*,* $(\mathbf{B})$ *and* $(\mathbf{f_1})$*–*$(\mathbf{f_3})$ *hold, then*

(**a**) *there exist* $\rho,\alpha>0$ *such that* $I_\lambda(u)\ge\alpha>0$ *for any* $u\in X$ *with* $\|u\|=\rho$.
(**b**) *there exists* $e\in X$ *with* $\|e\|>\rho$ *such that* $I_\lambda(e)<0$.

Proof. (**a**) By hypothesis $(\mathbf{B})$, we get

$$\frac{1}{P_+^+}\int_\Omega b(x)|u|^{P_+^+}\,\mathrm{d}x\ge\frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)}\ge 0 \tag{4.1}$$

for all $u\in X$. Since $P_+^+<q^-\le q^+<P_-^*$, it follows that the embeddings $X\hookrightarrow L^{P_+^+}(\Omega)$ and $X\hookrightarrow L^{q(x)}(\Omega)$ are continuous. Thus, there exist positive constants c_1 and c_2 such that

$$\|u\|_{L^{P_+^+}(\Omega)}\le c_1\|u\|,\quad \|u\|_{L^{q(x)}(\Omega)}\le c_2\|u\|. \tag{4.2}$$

In view of $(\mathbf{f_1})$ and $(\mathbf{f_2})$, we see that for any $\epsilon>0$, there exists $c_3>0$ such that

$$|F(x,t)|\le\epsilon|t|^{P_+^+}+c_3|t|^{q(x)} \tag{4.3}$$

for any $(x,t)\in\overline{\Omega}\times\mathbb{R}$. Fix $\epsilon>0$ small enough to get

$$\frac{1}{P_+^+N^{P_+^+-1}}-\lambda\epsilon c_1>0.$$

Next, we focus our attention on the case when $u\in X$ and $\|u\|<1$. Then, by taking into account (2.3), we have $|\partial_{x_i}u|_{p_i(x)}<1,\ i\in\{1,\dots,N\}$ and by Proposition 2.2, we obtain

$$\begin{aligned}\sum_{i=1}^N\int_\Omega|\partial_{x_i}u|^{p_i(x)}\mathrm{d}x&\ge\sum_{i=1}^N|\partial_{x_i}u|^{p_i^+}_{p_i(x)}\ge\sum_{i=1}^N|\partial_{x_i}u|^{P_+^+}_{p_i(x)}\\&\ge N\Big(\frac{\sum_{i=1}^N|\partial_{x_i}u|_{p_i(x)}}{N}\Big)^{P_+^+}=\frac{\|u\|^{P_+^+}}{N^{P_+^+-1}}.\end{aligned} \tag{4.4}$$

Considering (4.1)–(4.4), thanks to $(\mathbf{A_2})$ we can write

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \lambda F(x,u)\Big\} \mathrm{d}x \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} \mathrm{d}x + \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} - \lambda \int_\Omega F(x,u)\,\mathrm{d}x \\
&\geq \frac{1}{P_+^+ N^{P_+^+-1}} \|u\|^{P_+^+} - \lambda\epsilon \int_\Omega |u|^{P_+^+}\,\mathrm{d}x - \lambda c_3 \int_\Omega |u|^{q(x)}\,\mathrm{d}x \\
&\geq \frac{1}{P_+^+ N^{P_+^+-1}} \|u\|^{P_+^+} - \lambda\epsilon c_1^{P_+^+} \|u\|^{P_+^+} - \lambda c_3 c_2^{q^-} \|u\|^{q^-} \\
&\geq \left(\frac{1}{P_+^+ N^{P_+^+-1}} - \lambda\epsilon c_1 \right) \|u\|^{P_+^+} - \lambda c_4 \|u\|^{q^-}.
\end{aligned}$$

Since $q^- > P_+^+$, we can choose $\alpha > 0$ and $\rho > 0$ small enough such that $I_\lambda(u) \geq \alpha > 0$ for all $u \in X$ with $\|u\| = \rho$. This proves $(\mathbf{a})$.

$(\mathbf{b})$ By $(\mathbf{f_3})$, for all $M > 0$ there exists $c_5 = c(M)$ depending on M, such that

$$|F(x,t)| \geq M|t|^{P_+^+} - c_5 \tag{4.5}$$

for any $x \in \Omega$ and $t \in \mathbb{R}$. From $(\mathbf{A_0})$ and $(\mathbf{A_1})$, we have

$$A_i(x,\eta) = \int_0^1 a_i(x, t\eta)\,\eta\,\mathrm{d}t \leq |\eta| + \frac{1}{p_i(x)}|\eta|^{p_i(x)}, \tag{4.6}$$

for all $x \in \overline{\Omega}$ and $\eta \in \mathbb{R}$.

Take $\varphi \in X$ with $\varphi > 0$ on Ω. Using (4.5) and (4.6) for $t > 1$, we have

$$\begin{aligned}
I_\lambda(t\varphi) &= \int_\Omega \Big\{ \sum_{i=1}^N A_i(x, \partial_{x_i} t\varphi) + \frac{b(x)}{P_+^+}|t\varphi|^{P_+^+} - \lambda F(x,t\varphi)\Big\} \mathrm{d}x \\
&\leq \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} t\varphi| + \frac{|\partial_{x_i} t\varphi|^{p_i(x)}}{p_i(x)} \right) \mathrm{d}x + \frac{1}{P_+^+} \int_\Omega b(x)|t\varphi|^{P_+^+}\,\mathrm{d}x - \lambda M \int_\Omega |t\varphi|^{P_+^+}\,\mathrm{d}x \\
&\quad + c_5|\Omega| \\
&\leq t^{P_+^+} \left[\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\varphi| + \frac{|\partial_{x_i}\varphi|^{p_i(x)}}{P_-^-} \right) \mathrm{d}x + \frac{1}{P_+^+} \int_\Omega b(x)|\varphi|^{P_+^+}\,\mathrm{d}x - \lambda M \int_\Omega |\varphi|^{P_+^+}\,\mathrm{d}x \right] \\
&\quad + c_5|\Omega|.
\end{aligned}$$

If M is chosen large enough to have

$$\sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\varphi| + \frac{|\partial_{x_i}\varphi|^{p_i(x)}}{P_-^-} \right) \mathrm{d}x + \frac{1}{P_+^+} \int_\Omega b(x)|\varphi|^{P_+^+}\,\mathrm{d}x - \lambda M \int_\Omega |\varphi|^{P_+^+}\,\mathrm{d}x < 0,$$

then

$$\lim_{t\to+\infty} I_\lambda(t\varphi) = -\infty.$$

Take $e = t_0\varphi$ with t_0 large enough that $I_\lambda(t_0\varphi) < 0$. This proves $(\mathbf{b})$. $\square$

Lemma 4.3. *Let (u_n) be a Cerami sequence for the functional I_λ. If $(\mathbf{A_0})$–$(\mathbf{A_2})$ and $(\mathbf{f_1})$–$(\mathbf{f_3})$ hold, then (u_n) is bounded.*

Proof. Fix $c > 0$ and assume that $(u_n) \subset X$ is a Cerami sequence at level c for $I_\lambda(u)$ (see Definition 3.2). Then, (3.5) shows that

$$c = I_\lambda(u_n) + o(1), \quad \langle I'_\lambda(u_n), u_n \rangle = o(\|u_n\|) \tag{4.7}$$

where $o(1) \to 0$ as $n \to \infty$.

Suppose by the contrary, that

$$\|u_n\| \to +\infty \text{ as } \quad n \to +\infty. \tag{4.8}$$

Let $w_n(x) = \dfrac{u_n(x)}{\|u_n\|}$, then $(w_n) \subset X$ with $\|w_n\| = 1$. Therefore, up to a subsequence, we have $w_n \rightharpoonup w$ weakly in X. Since Ω is bounded, using the Sobolev embedding theorem (Proposition 3.1), see also assumption on $q(x)$ given in $(\mathbf{f_2})$ and equality (3.4), we get

$$\begin{cases} w_n(x) \to w(x) & \text{a.e. in } \Omega, \\ w_n \to w & \text{strongly in } L^{q(x)}(\Omega), \\ w_n \to w & \text{strongly in } L^{P_+^+}(\Omega). \end{cases} \tag{4.9}$$

We claim that $w(x) = 0$ a.e. in Ω. For this goal, set $\Omega_{\neq} = \{x \in \Omega : \ w(x) \neq 0\}$ and show that $|\Omega_{\neq}| = 0$. Considering (4.9), we have $|u_n(x)| = |w_n(x)|\|u_n\| \to +\infty$, as $n \to +\infty$ for all $x \in \Omega_{\neq}$. Taking into account $(\mathbf{f_3})$, we see that

$$\lim_{n \to +\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{P_+^+}} = +\infty \quad \text{a.e. in} \quad \Omega_{\neq}.$$

In particular, since $w_n \to w \neq 0$ a.e. in $\Omega_{\neq}$, we deduce

$$\lim_{n \to +\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{P_+^+}} |w_n(x)|^{P_+^+} = +\infty \quad \text{a.e. in} \quad \Omega_{\neq}. \tag{4.10}$$

Now, we prove that $F(x,t)$ is bounded from below $\forall (x,t) \in \overline{\Omega} \times \mathbb{R}$. Indeed, by $(\mathbf{f_3})$, there exists $n_0 > 0$ such that

$$\frac{F(x,t)}{|t|^{P_+^+}} > 1 \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R} \text{ with } |t| \geq n_0.$$

On the other hand, $F(x,t)$ is continuous on $\overline{\Omega} \times [-n_0, n_0]$, then there exists $c_6 > 0$ such that

$$|F(x,t)| \leq c_6$$

for any $(x,t) \in \overline{\Omega} \times [-n_0, n_0]$. Thus, we see that there exists a constant c_7 (not necessarily positive) such that

$$F(x,t) \geq c_7 \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Therefore, by (4.8) for n large enough

$$\frac{F(x, u_n(x)) - c_7}{\|u_n\|^{P_+^+}} \geq 0 \quad \forall x \in \Omega.$$

This means that

$$\frac{F(x, u_n(x))}{|u_n(x)|^{P_+^+}} |w_n(x)|^{P_+^+} - \frac{c_7}{\|u_n\|^{P_+^+}} \geq 0. \tag{4.11}$$

For each $i \in \{1, 2, \ldots, N\}$ and n, we define

$$\alpha_{i,n} = \begin{cases} P_+^+ & \text{if } |\partial_{x_i} u_n|_{p_i(x)} < 1, \\ P_-^- & \text{if } |\partial_{x_i} u_n|_{p_i(x)} > 1. \end{cases}$$

Using Proposition 2.2 and Jensen's inequality (applied to the convex function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(t) = |t|^{P_-^-}$, $P_-^- > 1$) or the generalized mean inequality, for n large enough we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)} \mathrm{d}x &\geq \sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(x)}^{\alpha_{i,n}} \\
&= \sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(x)}^{P_-^-} - \sum_{\{i:\ \alpha_{i,n}=P_+^+\}} \left(|\partial_{x_i} u_n|_{p_i(x)}^{P_-^-} - |\partial_{x_i} u_n|_{p_i(x)}^{P_+^+} \right) \\
&\geq N \left(\frac{\sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(x)}}{N} \right)^{P_-^-} - N \\
&= \frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} - N.
\end{aligned} \tag{4.12}$$

We point out that in the second inequality we used the fact that if $\alpha_{i,n} = P_+^+$, then $|\partial_{x_i} u_n| \leq 1$ and so

$$0 \leq |\partial_{x_i} u_n|^{P_-^-} - |\partial_{x_i} u_n|^{P_+^+} \leq 1.$$

Using (4.1), (4.7), (4.12), $(\mathbf{A_2})$ and $(\mathbf{B})$ for sufficiently large n,

$$\begin{aligned}
c &= I_\lambda(u_n) + o(1) \\
&= \int_{\Omega} \Big\{ \sum_{i=1}^{N} A_i(x, \partial_{x_i} u_n) + \frac{b(x)}{P_+^+} |u_n|^{P_+^+} - \lambda F(x, u_n) \Big\} \, \mathrm{d}x + o(1) \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)} \, \mathrm{d}x + \frac{b_0}{P_+^+} |u_n|_{L^{P_+^+}(\Omega)}^{P_+^+} \, \mathrm{d}x - \lambda \int_{\Omega} F(x, u_n) \, \mathrm{d}x + o(1) \\
&\geq \frac{1}{P_+^+ N^{P_-^- - 1}} \|u_n\|^{P_-^-} - \frac{N}{P_+^+} - \lambda \int_{\Omega} F(x, u_n) \, \mathrm{d}x + o(1).
\end{aligned}$$

So

$$\lambda \int_{\Omega} F(x, u_n) \, \mathrm{d}x \geq \frac{1}{P_+^+ N^{P_-^- - 1}} \|u_n\|^{P_-^-} - c - \frac{N}{P_+^+} + o(1) \to +\infty,$$

thus

$$\int_{\Omega} F(x, u_n) \, \mathrm{d}x \to +\infty. \tag{4.13}$$

Similarly, for each $i \in \{1, 2, \ldots, N\}$ and $n \in \mathbb{N}$ we define

$$\beta_{i,n} = \begin{cases} P_-^- & \text{if } |\partial_{x_i} u_n|_{p_i(x)} < 1, \\ P_+^+ & \text{if } |\partial_{x_i} u_n|_{p_i(x)} > 1. \end{cases}$$

Using Proposition 2.2, for n large enough we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}u_n|^{p_i(x)}\,\mathrm{d}x &\leq \sum_{i=1}^{N}|\partial_{x_i}u_n|_{p_i(x)}^{\beta_{i,n}}\\ &=\sum_{i=1}^{N}|\partial_{x_i}u_n|_{p_i(x)}^{P_+^+}-\sum_{\{i:\ \beta_{i,n}=P_-^-\}}\left(|\partial_{x_i}u_n|_{p_i(x)}^{P_+^+}-|\partial_{x_i}u_n|_{p_i(x)}^{P_-^-}\right)\\ &\leq\left(\sum_{i=1}^{N}|\partial_{x_i}u_n|_{p_i(x)}\right)^{P_+^+}+N\\ &=\|u_n\|^{P_+^+}+N.\end{aligned} \tag{4.14}$$

Also, from (4.6), (4.7) and (4.14), $b\in L^\infty$ and since $\partial_{x_i}u_n\in L^{p_i(x)}\hookrightarrow L^1$ we deduce that $\int_\Omega|\partial_{x_i}u_n|\,\mathrm{d}x\leq\int_\Omega|\partial_{x_i}u_n|^{p_i(x)}\,\mathrm{d}x$ and therefore, by (4.14) we have

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}u_n|\leq(\|u_n\|^{P_+^+}+N).$$

We deduce that

$$\begin{aligned}c&=I_\lambda(u_n)+o(1)\\ &=\int_\Omega\Big\{\sum_{i=1}^{N}A_i(x,\partial_{x_i}u_n)+\frac{b(x)}{P_+^+}|u_n|^{P_+^+}-\lambda F(x,u_n)\Big\}\,\mathrm{d}x+o(1)\\ &\leq\sum_{i=1}^{N}\int_\Omega\left(|\partial_{x_i}u_n|+\frac{|\partial_{x_i}u_n|^{p_i(x)}}{p_i(x)}\right)\mathrm{d}x+\frac{1}{P_+^+}\int_\Omega b(x)|u_n|^{P_+^+}\,\mathrm{d}x-\lambda\int_\Omega F(x,u_n)\,\mathrm{d}x+o(1)\\ &\leq\left(c_8+\frac{1}{P_-^-}\right)\left(\|u_n\|^{P_+^+}+N\right)+\frac{c_9}{P_+^+}\|u_n\|^{P_+^+}-\lambda\int_\Omega F(x,u_n)\,\mathrm{d}x+o(1)\\ &\leq c_{10}\|u_n\|^{P_+^+}+c_{11}N-\lambda\int_\Omega F(x,u_n)\,\mathrm{d}x+o(1),\end{aligned}$$

or

$$\|u_n\|^{P_+^+}\geq\frac{\lambda}{c_{10}}\int_\Omega F(x,u_n)\,\mathrm{d}x+\frac{c}{c_{10}}-\frac{c_{11}N}{c_{10}}-o(1) \tag{4.15}$$

for n large enough.

If $|\Omega_{\neq}|\neq 0$, using relations (4.10), (4.11), (4.15) and Fatou's lemma, by (4.8) we have

$$\begin{aligned}+\infty=(+\infty)\,|\Omega_{\neq}|&=\int_{\Omega_{\neq}}\liminf_{n\to+\infty}\left(\frac{F(x,u_n)}{|u_n|^{P_+^+}}|w_n|^{P_+^+}-\frac{c_7}{\|u_n\|^{P_+^+}}\right)\mathrm{d}x\\ &\leq\liminf_{n\to+\infty}\int_{\Omega_{\neq}}\left(\frac{F(x,u_n)}{|u_n|^{P_+^+}}|w_n|^{P_+^+}-\frac{c_7}{\|u_n\|^{P_+^+}}\right)\mathrm{d}x\\ &\leq\liminf_{n\to+\infty}\int_{\Omega}\left(\frac{F(x,u_n)}{|u_n|^{P_+^+}}|w_n|^{P_+^+}-\frac{c_7}{\|u_n\|^{P_+^+}}\right)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{n\to+\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{P_+^+}}\,\mathrm{d}x - \limsup_{n\to+\infty} \int_\Omega \frac{c_7}{\|u_n\|^{P_+^+}}\,\mathrm{d}x \\
&= \liminf_{n\to+\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{P_+^+}}\,\mathrm{d}x - \limsup_{n\to+\infty} \frac{c_7|\Omega|}{\|u_n\|^{P_+^+}} \\
&= \liminf_{n\to+\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{P_+^+}}\,\mathrm{d}x \\
&\le \liminf_{n\to+\infty} \frac{\int_\Omega F(x,u_n)\,\mathrm{d}x}{\frac{\lambda}{c_{10}}\int_\Omega F(x,u_n) + \frac{c}{c_{10}} - \frac{c_{11}N}{c_{10}} - o(1)}.
\end{aligned}
\tag{4.16}
$$

Due to (4.13), the right-hand side of the above inequality goes to $\frac{c_{10}}{\lambda}$, and hence we deduce that $\dfrac{c_{10}}{\lambda} \geq +\infty$, which is a contradiction. This shows that, as claimed, $|\Omega_{\neq}| = 0$. Hence $w(x) = 0$ a.e. in Ω.

Since $I_\lambda(tu_n)$ is continuous in $t \in [0,1]$, there exists $t_n \in [0,1]$ such that

$$
I_\lambda(t_n u_n) = \max_{t\in[0,1]} I_\lambda(tu_n). \tag{4.17}
$$

Clearly, $t_n > 0$ and we can write $I_\lambda(t_n u_n) \geq c_{12} > 0 = I_\lambda(0)$, by Lemma 4.2 (**a**). When $t_n < 1$, we have $\dfrac{\mathrm{d}}{\mathrm{d}t} I_\lambda(tu_n)|_{t=t_n} = 0$, which gives $\langle I'_\lambda(t_n u_n), t_n u_n\rangle = 0$. If $t_n = 1$, by (4.7) we get $\langle I'_\lambda(u_n), u_n\rangle = o(1)$. So, if t_n makes (4.17) hold true, then

$$
\langle I'_\lambda(t_n u_n), t_n u_n\rangle = o(1). \tag{4.18}
$$

Let (r_k) be a sequence of real positive numbers such that $r_k > 1$ for any k and $\lim_{n\to+\infty} r_k = +\infty$. Since $\|u_n\| \to +\infty$ as $n \to +\infty$, we can choose (r_k) such that $0 < \frac{r_k}{\|u_n\|} < 1$ for large n. For fixed k, using ($\mathbf{f_2}$) we deduce that

$$
F(x, r_k w_n) \leq a|r_k w_n| + \frac{b}{q}|r_k w_n|^{q(x)}.
$$

Next, by (4.9) (since we have proved that $w \equiv 0$) we obtain

$$
\lim_{n\to+\infty} \int_\Omega F(x, r_k w_n)\,\mathrm{d}x = 0. \tag{4.19}
$$

Therefore, considering (3.7) and (**B**), we have

$$
\begin{aligned}
I_\lambda(t_n u_n) &\geq I_\lambda\Big(\frac{r_k}{\|u_n\|}u_n\Big) \\
&= I_\lambda(r_k w_n) \\
&= \int_\Omega \Big\{\sum_{i=1}^N A_i(x, \partial_{x_i} r_k w_n) + \frac{b(x)}{P_+^+}|r_k w_n|^{P_+^+} - \lambda F(x, r_k w_n)\Big\}\mathrm{d}x \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i} r_k w_n|^{p_i(x)}\,\mathrm{d}x + \frac{b_0}{P_+^+}|r_k w_n|^{P_+^+}_{L^{P_+^+}(\Omega)} - \lambda\int_\Omega F(x, r_k w_n)\,\mathrm{d}x \\
&\geq \frac{r_k^{P_-^-}}{P_+^+ N^{P_-^- -1}} - \frac{N}{P_+^+} - \lambda\int_\Omega F(x, r_k w_n)\,\mathrm{d}x,
\end{aligned}
$$

where the last inequality follows by replacing u_n with $r_k w_n$ in (4.12) since $\|r_k w_n\| = r_k\|w_n\| = r_k$. So, by (4.19), we get

$$
\lim_{n\to+\infty} I_\lambda(t_n u_n) = +\infty. \tag{4.20}
$$

Putting together (4.6) and $(\mathbf{f_4})$, for $t_n \in (0,1]$ such that $(\mathbf{H})$ holds, we get by (4.18)

$$\begin{aligned}
I_\lambda(t_n u_n) &= I_\lambda(t_n u_n) - \langle I'_\lambda(t_n u_n), t_n u_n\rangle + o(1)\\
&= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} t_n u_n) + \frac{b(x)}{P_+^+}|t_n u_n|^{P_+^+} - \lambda F(x, t_n u_n)\right\} \mathrm{d}x\\
&\quad - \int_\Omega \left[\sum_{i=1}^N a_i(x, \partial_{x_i} t_n u_n)\partial_{x_i}(t_n u_n) + b(x)|t_n u_n|^{P_+^+} - \lambda f(x, t_n u_n) t_n u_n \right] \mathrm{d}x + o(1)\\
&\le \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} t_n u_n| + \frac{|\partial_{x_i} t_n u_n|^{p_i(x)}}{p_i(x)} \right) \mathrm{d}x + \left(\frac{1}{P_+^+} - 1\right) \int_\Omega b(x)|t_n u_n|^{P_+^+}\,\mathrm{d}x\\
&\quad - \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} t_n u_n)\partial_{x_i}(t_n u_n)\,\mathrm{d}x + \lambda \int_\Omega G(x, t_n u_n)\,\mathrm{d}x + o(1)\\
&\le \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} u_n| + \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} \right) \mathrm{d}x + \left(\frac{1}{P_+^+} - 1\right) \int_\Omega b(x)|t_n u_n|^{P_+^+}\,\mathrm{d}x\\
&\quad - \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} t_n u_n)\partial_{x_i}(t_n u_n)\,\mathrm{d}x + \lambda \int_\Omega \Big(G(x, u_n) + C_*\Big)\,\mathrm{d}x + o(1).
\end{aligned}$$

We deduce that

$$\begin{aligned}
I_\lambda(t_n u_n) &\le I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n\rangle + O(1) + o(1)\\
&= O(1) \quad \text{as} \quad n \to +\infty, \quad \text{by} \quad (4.7).
\end{aligned} \tag{4.21}$$

From (4.20) and (4.21), we get a contradiction. Therefore, we have proved that (u_n) is bounded. □

Proposition 4.4. *If* $(\mathbf{A_0})$–$(\mathbf{A_2})$ *and* $(\mathbf{f_1})$–$(\mathbf{f_3})$ *hold true, then the functional* I_λ *satisfies the Cerami condition at any level* c.

Proof. By taking into account Lemma 4.3 and the fact that X is reflexive, any Cerami sequence (u_n) yields to the existence of $u_0 \in X$ such that, up to a subsequence, still denote by (u_n),

$$\begin{cases} u_n \rightharpoonup u_0 & \text{weakly in } X,\\ u_n \to u_0 & \text{strongly in } L^{q(x)}(\Omega),\\ u_n \to u_0 & \text{strongly in } L^{P_+^+}(\Omega). \end{cases} \tag{4.22}$$

We deduce in what follows that the convergence is strong in X. To this aim, we check the assumptions in Lemma 3.5. Let us first remark that, by (4.7) we have

$$\lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u_0\rangle = 0.$$

More precisely,

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \Big[&\sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0)\\
&+ b(x)|u_n|^{P_+^+ - 2} u_n (u_n - u_0) - \lambda f(x, u_n)(u_n - u_0)\Big]\,\mathrm{d}x = 0.
\end{aligned} \tag{4.23}$$

By using condition $(\mathbf{f_2})$, (4.22) and the Hölder-type inequality stated in Proposition 2.1, we can write

$$\begin{aligned}
\left| \int_\Omega f(x, u_n)(u_n - u_0)\,\mathrm{d}x \right| &\le \int_\Omega |f(x, u_n)||u_n - u_0|\,\mathrm{d}x\\
&\le c_{12}\Big(a + b\Big\| |u_n|^{q(x)-1}\Big\|_{L^{\frac{q(x)}{q(x)-1}}(\Omega)}\Big) \|u_n - u_0\|_{L^{q(x)}(\Omega)}\\
&\to 0 \quad \text{as} \quad n \to +\infty,
\end{aligned}$$

so we deduce

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u_0)\,\mathrm{d}x=0. \tag{4.24}$$

Next, using (**B**) and Hölder's inequality we have

$$\Big|\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)\,\mathrm{d}x\Big|\le\|b\|_{L^\infty(\Omega)}\Big\||u_n|^{P_+^+-2}u_n\Big\|_{L^{\frac{P_+^+}{P_+^+-1}}(\Omega)}\|u_n-u_0\|_{L^{P_+^+}(\Omega)},$$

and, so by (4.22) we get

$$\lim_{n\to+\infty}\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)\,\mathrm{d}x=0. \tag{4.25}$$

Combining relations (4.23)–(4.25), we obtain

$$\lim_{n\to\infty}\int_\Omega\sum_{i=1}^N a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)\mathrm{d}x=0,$$

and, so by Lemma 3.5, we deduce that (u_n) converges strongly to u_0 in X. Thus, the functional I_λ satisfies the Cerami condition at level c. □

4.1. Proof of Theorem 4.1

Lemma 4.2 and Proposition 4.4 guarantee that Lemma 3.3 applies to the functional I_λ. Therefore, the real number c_0 given in (3.6) is a critical level for I_λ to which corresponds at least a nontrivial weak solution to problem (2.1).

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