

ASYMPTOTICS FOR THE MINIMIZERS OF THE GINZBURG-LANDAU ENERGY WITH VANISHING WEIGHT

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Abstract. We study the asymptotic behavior of the minimizers for the Ginzburg-Landau energy with a weight which vanishes. We find the link between the growth rate of the weight near its zeroes and the number of singularities of the limiting configuration, as well as their degrees. We give the expression of the corresponding renormalized energy which governs the location of singularities at the limit.

Introduction

F. Bethuel, H. Brezis and F. Hélein have studied in [BBH4] the asymptotic behavior as $\varepsilon \rightarrow 0$ of minimizers of the Ginzburg-Landau energy

$$E_\varepsilon(u, G) = E_\varepsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2$$

in the class

$$H_g^1 = H_g^1(G) = \{u \in H^1(G; \mathbb{R}^2); u = g \text{ on } \partial G\},$$

where $G \subset \mathbb{R}^2$ is a smooth bounded domain and $g : \partial G \rightarrow S^1$ is a smooth data with the topological degree $d > 0$.

For each sequence $\varepsilon_n \rightarrow 0$, they have proved the existence of a subsequence, also denoted (ε_n) and of a finite configuration $\{a_1, \dots, a_d\}$ in G such that (u_{ε_n}) converges in certain topologies to u_\star , which is the canonical harmonic map with values in S^1 associated to $\{a_1, \dots, a_d\}$ with degrees $+1$ and to the boundary data g . This means that

$$u_\star(z) = \frac{z - a_1}{|z - a_1|} \cdots \frac{z - a_d}{|z - a_d|} e^{i\varphi(z)} \quad \text{in } G \setminus \{a_1, \dots, a_d\}$$

with

$$(1) \quad \begin{cases} \Delta\varphi = 0 & \text{in } G \\ u_\star = g & \text{on } \partial G. \end{cases}$$

Moreover, the configuration $a = (a_1, \dots, a_d)$ minimizes the renormalized energy $W(a, g)$. The renormalized energy $W(a, \bar{d}, g)$ associated to a given configuration $a = (a_1, \dots, a_k)$ with corresponding degrees $\bar{d} = (d_1, \dots, d_k)$ and to the boundary data g with $\deg(g, \partial G) = d$, $d = d_1 + \dots + d_k$ was introduced in [BBH2], [BBH4]. If all d_j equal $+1$ (that is $k = d$) then $W(a, g)$ denotes $W(a, \bar{d}, g)$.

In [LR1] we have studied the Ginzburg-Landau energy with weight

$$E_\varepsilon^w(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2 w,$$

where $w \in C^1(\bar{G})$, $w > 0$ in \bar{G} . We proved a similar behavior of minimizers, but the limiting configuration minimizes the modified renormalized energy. More precisely, u_{ε_n} converges to u_\star in certain topologies but now the limiting configuration $a = (a_1, \dots, a_d)$ is a minimum point of

$$\widetilde{W}(b, g) = W(b, g) + \frac{\pi}{2} \sum_{j=1}^d \log w(b_j), \quad b \in G^d.$$

A natural question is to see what happens if w vanishes. We first study the case when $w \geq 0$ and it has a unique zero $x_0 \in G$ and suppose that $w(x) \sim |x - x_0|^p$ around x_0 , where $p > 1$. This means that $w(x) = |x - x_0|^p + f(x)|x|^{p+1}$ in a neighbourhood of x_0 , where f is a C^1 function. We show that, up to a subsequence, u_ε converges to a harmonic map u_\star associated to singularities x_0, a_1, \dots, a_k with $d_0 = \deg(u_\star, x_0) > 0$ and $\deg(u_\star, a_j) = +1$ for $j = 1, \dots, k$. More precisely, we have (see Theorems 1 and 7)

$$u_\star(z) = \left(\frac{z - x_0}{|z - x_0|} \right)^{d_0} \frac{z - a_1}{|z - a_1|} \dots \frac{z - a_k}{|z - a_k|} e^{i\varphi}$$

with $d_0 + k = d$. Here φ is such that (1) holds. Remark that in some situations the set $a = (a_1, \dots, a_k)$ is empty. We next complete this result by finding:

- a) the exact value of k as a function of p and d ;
 - b) the position of a_1, \dots, a_k through the corresponding renormalized energy.
- Our main results are the following:

Theorem A. Assume that $d < \frac{p}{4} + 1$. Then $d_0 = d$ and x_0 is the only singularity of u_* .

Theorem B. Assume that $d \geq \frac{p}{4} + 1$ and that p is not an integer multiple of 4. Then $d_0 = \left\lceil \frac{p}{4} \right\rceil + 1$ (here $[x]$ denotes the integer part of the real number x).

Theorem C. Assume that $d \geq \frac{p}{4} + 1$ and that p is an integer multiple of 4. Then either $d_0 = \frac{p}{4}$ or $d_0 = \frac{p}{4} + 1$.

Theorem D. Assume that $d \geq \frac{p}{4} + 1$ and u_ε converges to the canonical harmonic map associated to the configuration $a = (x_0, a_1, \dots, a_k)$ with degrees $\bar{d} = (d_0, +1, \dots, +1)$ and to the boundary data g . Then the limiting configuration a minimizes the renormalized energy

$$\widehat{W}(b) = W(b, \bar{d}, g) + \frac{\pi}{2} \sum_{j=1}^k \log w(b_j)$$

among all configurations $b = (x_0, b_1, \dots, b_k)$.

We show, by considering two examples, that in Theorem C both cases actually occur (see Examples 1 and 3).

The proofs of Theorems A-D follow immediately from Theorems 6, 7, 8 and 9.

1 Estimates of the energy in the case of a ball

We start with a preliminary result.

Theorem 1. For each sequence $\varepsilon_n \rightarrow 0$, there exist a subsequence (also denoted by ε_n), k points a_1, \dots, a_k in G and positive integers d_0, d_1, \dots, d_k with $d_0 + d_1 + \dots + d_k = d$ such that (u_{ε_n}) converges in $H_{\text{loc}}^1(\overline{G} \setminus \{x_0, a_1, \dots, a_k\}; \mathbb{R}^2)$ to u_* , which is the canonical harmonic map with values in S^1 associated to the points x_0, a_1, \dots, a_k with corresponding degrees d_0, d_1, \dots, d_k and to the boundary data g . Moreover, $d_0 \geq 0$ and $d_1 = \dots = d_k = \pm 1$.

Proof. As in [BBH4], the estimate

$$(2) \quad \frac{1}{\varepsilon^2} \int_{G \setminus U} (1 - |u_\varepsilon|^2)^2 w \leq C$$

is fundamental to prove the convergence of (u_ε) , where U is an arbitrary neighbourhood of x_0 and $C = C(U)$. The estimate (2) may be obtained with the techniques of Struwe (see [S2]) used by Hong in the case $w > 0$ (see [H]).

Let V be a closed neighbourhood of x_0 . With the methods developed in [BBH4], Chapters III-VI, one obtains a finite number of “bad” discs in $G \setminus V$. By this way we find a finite configuration $\{a_1, \dots, a_k\}$ (k depending on V) in $G \setminus V$ such that, up to a subsequence, (u_{ε_n}) converges in $H_{\text{loc}}^1(\overline{G} \setminus (V \cup \{a_1, \dots, a_k\}); \mathbb{R}^2)$ to some u_\star . The limit u_\star is a harmonic map with values in S^1 and singularities a_1, \dots, a_k , such that the degree of u_\star around each a_j ($j \geq 1$) is some non-zero integer d_j . The fact that all the singularities lie in G follows as in [BBH4], Theorem VI.2.

Taking arbitrary small neighbourhoods V of x_0 and passing to a further subsequence, we obtain by a diagonal argument a sequence (a_k) of points in G without cluster point in $G \setminus \{x_0\}$ and a sequence (d_k) of non-zero integers such that (u_{ε_n}) converges in

$$H_{\text{loc}}^1(\overline{G} \setminus (\{x_0\} \cup \{a_k; k \geq 1\}); \mathbb{R}^2)$$

to u_\star , which is a harmonic map from $\overline{G} \setminus (\{x_0\} \cup \{a_k; k \geq 1\})$ with values in S^1 and singularities a_k of degrees d_k .

As in [BBH4], Theorem III.1,

$$(3) \quad E_\varepsilon(u_\varepsilon) \leq \pi d \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \rightarrow 0.$$

Taking into account the energy estimates in [BBH4] (see also [LR1]) we obtain that

$$(4) \quad \sum_{j \geq 1} d_j^2 \leq d.$$

This means that there is a finite number of singularities a_j , say k .

Denote $d_0 = \deg(u_\star, x_0)$, which is well defined, since x_0 is an isolated singularity. By adapting the proof of Lemma V.2 from [BBH4] in our case and on $G \setminus V$ we obtain that all degrees d_j , $j = 1, \dots, k$ have the same sign. Moreover, as in Theorem VI.2 from [BBH4], $|d_j| = +1$, for all $j \geq 1$.

We now prove that $d_0 \geq 0$. Indeed, if not, there would be at least $d + 1$ singularities different from 0. This would contradict (4). ■

We shall see later that $d_0 > 0$ and $d_j = +1$, for all $j = 1, \dots, k$. This will be done after obtaining stronger energy estimates.

At this stage we are in position to point out the following estimate, which will be used in what follows: for each compact $K \subset \overline{G} \setminus \{x_0, a_1, \dots, a_k\}$,

$$(5) \quad \|\nabla(u_\varepsilon - u_\star)\|_{L^\infty(K)} \leq C_K \varepsilon .$$

This follows with the techniques from [BBH3] in the case of a null degree (see also [M]).

We shall next establish, when G is a ball and $w(x) = |x|^p$, upper and lower bounds for the energy E_ε . These will be accomplished by using the techniques developed in [BBH4], Chapter I. We shall also take into account some results from [LR1] (see Theorem 1).

For fixed $p > 0$, $\varepsilon, R > 0$ and $g(x) = \left(\frac{x}{|x|}\right)^d$, set

$$J_d(\varepsilon, R) = J_d^p(\varepsilon, R) = \min_{H_g^1(B_R)} \left\{ \frac{1}{2} \int_{B_R} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{B_R} (1 - |u|^2)^2 |x|^p \right\} .$$

By scaling, it is easy to see that

$$(6) \quad J_d(\varepsilon, R) = J_d\left(\frac{\varepsilon}{R^{1+\frac{p}{2}}}, 1\right) .$$

Hence, in order to obtain an asymptotic formula for J_d^p , it suffices to study the functional $J_d(\varepsilon) := J_d(\varepsilon, 1)$. If $p = 0$, denote $I_d(\varepsilon, R) = J_d^0(\varepsilon, R)$. Throughout, u_ε will denote a point where $J_d(\varepsilon)$ is achieved.

We first establish an upper bound for $J_d(\varepsilon)$.

Theorem 2. *The following estimate holds*

$$(7) \quad J_d(\varepsilon) \leq \frac{2d^2}{p+2} \pi \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \rightarrow 0 .$$

Proof. For $\alpha > 0$ and $0 < \varepsilon < 1$, let w_ε be a minimizer of E_ε on $H_g^1(B(0, \varepsilon^\alpha))$. In order to obtain (7), we choose the following comparison function:

$$v_\varepsilon(x) = \begin{cases} \left(\frac{x}{|x|}\right)^d & \text{for } \varepsilon^\alpha \leq |x| \leq 1 \\ w_\varepsilon(x) & \text{for } 0 < |x| < \varepsilon^\alpha . \end{cases}$$

A straightforward computation shows that

$$(8) \quad E_\varepsilon(v_\varepsilon; \{x; \varepsilon^\alpha < |x| < 1\}) = \frac{1}{2} \int_{\varepsilon^\alpha < |x| < 1} |\nabla v_\varepsilon|^2 = \pi d^2 \alpha \log \frac{1}{\varepsilon} .$$

On the other hand, using Lemma III.1 in [BBH4] and the fact that $|x|^p \leq \varepsilon^{p\alpha}$ on $B(0, \varepsilon^\alpha)$, we obtain

$$(9) \quad E_\varepsilon(v_\varepsilon; B(0, \varepsilon^\alpha)) \leq I_d(\varepsilon^{1-\frac{p\alpha}{2}}, \varepsilon^\alpha) = I_d(\varepsilon^{1-\frac{p+2}{2}\alpha}, 1) \leq \pi d \left| \log \frac{1}{\varepsilon^{1-\frac{p+2}{2}\alpha}} \right| + O(1).$$

Now, choosing $\alpha = \frac{2}{p+2}$ and taking into account (8) and (9) we obtain (7). \blacksquare

We next establish a lower bound for the energy.

Theorem 3. *Assume that the only limit point of u_ε obtained in Theorem 1 is $\left(\frac{x}{|x|}\right)^d$, that is 0 is the unique singularity of the limit. Then*

$$(10) \quad J_d(\varepsilon) \geq \frac{2d^2}{p+2} \pi \log \frac{1}{\varepsilon} - O(1) \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. We first estimate $\frac{d}{d\varepsilon} E_\varepsilon(u_\varepsilon)$ using an idea from [S1]. Let $\varepsilon_1 < \varepsilon_2$. Then

$$E_{\varepsilon_1}(u_{\varepsilon_2}) \geq E_{\varepsilon_1}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_1}) \geq E_{\varepsilon_2}(u_{\varepsilon_2}).$$

Therefore, if $\nu(\varepsilon) := E_\varepsilon(u_\varepsilon)$ then

$$|\nu(\varepsilon_1) - \nu(\varepsilon_2)| \leq |\varepsilon_1 - \varepsilon_2| \cdot \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1^2 \varepsilon_2^2} \int_{B_1} (1 - |u_{\varepsilon_2}|^2)^2 w(x) dx.$$

This implies that ν is locally Lipschitz on $(0, +\infty)$, that is locally absolutely continuous on $(0, +\infty)$ and ν equals to the integral of its derivative. On the other hand

$$\frac{E_{\varepsilon_1}(u_{\varepsilon_2}) - E_{\varepsilon_2}(u_{\varepsilon_2})}{\varepsilon_1 - \varepsilon_2} \leq \frac{E_{\varepsilon_1}(u_{\varepsilon_1}) - E_{\varepsilon_2}(u_{\varepsilon_2})}{\varepsilon_1 - \varepsilon_2} \leq \frac{E_{\varepsilon_1}(u_{\varepsilon_1}) - E_{\varepsilon_2}(u_{\varepsilon_1})}{\varepsilon_1 - \varepsilon_2}.$$

Letting $\varepsilon_1 \nearrow \varepsilon_2$ and $\varepsilon_2 \searrow \varepsilon_1$ we have

$$(11) \quad \nu'(\varepsilon) = \frac{d}{d\varepsilon} E_\varepsilon(u_\varepsilon) = -\frac{1}{2\varepsilon^3} \int_{B_1} (1 - |u_\varepsilon|^2)^2 |x|^p \quad \text{a.e. on } (0, +\infty).$$

Recall that u_ε satisfies the equation

$$(12) \quad \begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) |x|^p & \text{in } B_1 \\ u_\varepsilon = x^d & \text{on } \partial B_1. \end{cases}$$

As in the proof of the Pohozaev identity, multiplying (12) by $(x \cdot \nabla u_\varepsilon)$ and integrating by parts we obtain

$$\int_{\partial B_1} \frac{\partial u_\varepsilon}{\partial \nu} (x \cdot \nabla u_\varepsilon) + \int_{B_1} \sum_{i,j} \frac{\partial u_\varepsilon}{\partial x_j} \left(\delta_{ij} \frac{\partial u_\varepsilon}{\partial x_i} + x_i \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} \right) = \frac{p+2}{4\varepsilon^2} \int_{B_1} (1 - |u_\varepsilon|^2)^2 |x|^p.$$

Therefore

$$(13) \quad \frac{p+2}{2\varepsilon^2} \int_{B_1} (1 - |u_\varepsilon|^2)^2 |x|^p = \int_{\partial B_1} \left(\left| \frac{\partial u_\varepsilon}{\partial \tau} \right|^2 - \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 \right).$$

Thus

$$(14) \quad \frac{1}{2\varepsilon^2} \int_{B_1} (1 - |u_\varepsilon|^2)^2 |x|^p = \frac{2d^2}{p+2} \pi - \frac{1}{p+2} \int_{\partial B_1} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2.$$

Taking into account the estimate (5) we obtain from (14) that

$$(15) \quad \frac{1}{2\varepsilon^2} \int_{B_1} (1 - |u_\varepsilon|^2)^2 |x|^p = \frac{2d^2}{p+2} \pi + O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0.$$

Integrating (11) from ε to 1 we find together with (15) that

$$(16) \quad E_\varepsilon(u_\varepsilon) = \frac{2d^2}{p+2} \pi \log \frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \rightarrow 0.$$

■

Theorem 4. Suppose, in the case of the ball B_1 and $w(x) = |x|^p$, that u_{ε_n} converges as in Theorem 1 to u_\star which has singularities 0 with degree d_0 and a_1, \dots, a_k such that

$$\deg(u_\star, a_1) = \dots = \deg(u_\star, a_k) = \pm 1.$$

Then

$$(17) \quad \frac{1}{4\varepsilon_n^2} \int_{B_1} (1 - |u_{\varepsilon_n}|^2)^2 |x|^p = \frac{d_0^2}{p+2} \pi + \frac{k\pi}{2} + O(\varepsilon) \quad \text{as } n \rightarrow \infty.$$

Proof. We follow the strategy of the proof of Theorem VII.2 from [BBH4]. From (13) we have that

$$W_n = \frac{1}{4\varepsilon_n^2} (1 - |u_{\varepsilon_n}|^2)^2 |x|^p$$

is bounded in $L^1(B_1)$ as $n \rightarrow \infty$. We also remark at this stage that there exists $C > 0$ such that, for all $\varepsilon > 0$ (and not only for a subsequence),

$$\frac{1}{4\varepsilon^2} \int_{B_1} (1 - |u_\varepsilon|^2)^2 |x|^p \leq C.$$

Indeed, if not, passing to a subsequence ε_n such that (u_{ε_n}) converges, we would contradict the previous result.

By the boundedness of (W_n) it follows its convergence weak \star in $C(\overline{B_1})^\star$ to a measure W_\star supported by $0, a_1, \dots, a_k$. Hence

$$W_\star = m_0 \delta_0 + \sum_{j=1}^k m_j \delta_{a_j} \quad \text{with } m_j \in \mathbb{R}.$$

We now determine m_0 .

Consider $B_R = B(0, R)$ for R small enough so that B_R contains no other point a_i ($i \neq 0$). Multiplying (12) by $x \cdot \nabla u_\varepsilon$ and integrating on B_R we obtain

$$\begin{aligned} (18) \quad & \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 + \frac{p+2}{4\varepsilon^2} \int_{B_R} (1 - |u_\varepsilon|^2)^2 |x|^p = \\ & = \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\varepsilon}{\partial \tau} \right|^2 + \frac{R}{4\varepsilon^2} \int_{\partial B_R} (1 - |u_\varepsilon|^2)^2 |x|^p. \end{aligned}$$

Passing to the limit in (18) as $\varepsilon \rightarrow 0$ and using the convergence of W_n we find

$$(19) \quad \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\star}{\partial \nu} \right|^2 + (p+2)m_0 = \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\star}{\partial \tau} \right|^2.$$

The fact that u_\star is canonical implies that

$$u_\star(x) = \left(\frac{x}{|x|} \right)^{d_0} e^{iH_0(x)} \quad \text{on } B_R$$

with

$$\Delta H_0 = 0.$$

Therefore, on ∂B_R ,

$$(20) \quad \left| \frac{\partial u_\star}{\partial \nu} \right|^2 = \left| d_0 \frac{\partial \theta}{\partial \nu} + \frac{\partial H_0}{\partial \nu} \right|^2 = \left| \frac{\partial H_0}{\partial \nu} \right|^2.$$

$$(21) \quad \left| \frac{\partial u_\star}{\partial \tau} \right|^2 = \left| d_0 \frac{\partial \theta}{\partial \tau} + \frac{\partial H_0}{\partial \tau} \right|^2 = \frac{d_0^2}{R^2} + 2 \frac{d_0}{R} \frac{\partial H_0}{\partial \tau} + \left| \frac{\partial H_0}{\partial \tau} \right|^2.$$

Inserting (20) and (21) into (19) we obtain

$$(22) \quad \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial H_0}{\partial \nu} \right|^2 + (p+2)m_0 = d_0^2 \pi + \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial H_0}{\partial \tau} \right|^2 .$$

On the other hand, by multiplying $\Delta H_0 = 0$ with $x \cdot \nabla H_0$ and integrating on B_R we find

$$(23) \quad \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial H_0}{\partial \nu} \right|^2 = \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial H_0}{\partial \tau} \right|^2 .$$

Thus, from (22) and (23) we obtain

$$m_0 = \frac{\pi}{p+2} d_0^2 .$$

A similar computation for a_j , $j \neq 0$ gives $m_j = \frac{\pi}{2}$ (see [BBH4], Theorem VII.2). ■

Remark 1. By analyzing the proofs of Theorems 3 and 4 we observe that we may replace the weight $|x|^p$ by a weight which, in a neighbourhood of 0 is of the form $w(x) = |x|^p + f(x)|x|^{p+1}$, with $f \in C^1$.

Remark 2. The conclusion of Theorem 4 remains valid for a general domain G and a weight $w(x) = |x|^p$ around 0. In this case, the boundedness of

$$\frac{1}{4\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 w$$

follows by the same computation as in the proof of Theorem 4.

Until now we have obtained a lower bound for the energy under the supplementary hypotheses that $G = B_1$, $g = e^{id\theta}$ and $w(x) = |x|^p$. We now establish a general lower bound for $E_\varepsilon(u_\varepsilon)$ when w is like in Remark 1; this will be useful to deduce the exact value of d_0 .

Theorem 5. *Let*

$$(24) \quad C = \liminf_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 w .$$

Then

i) $C > 0$.

ii) *The following hold:*

$$(25) \quad \frac{1}{4\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 w \geq C - O(\varepsilon) .$$

and

$$(25') \quad E_\varepsilon(u_\varepsilon) \geq 2C \pi \log \frac{1}{\varepsilon} - O(\varepsilon).$$

iii) We have

$$(26) \quad C \geq \min \left\{ \frac{(d-\ell)^2}{p+2} + \frac{\ell}{2}; 0 \leq \ell \leq d \right\}.$$

Proof. ii) Suppose (25) does not hold. Then there are $\varepsilon_n \rightarrow 0$ and $C_n \rightarrow +\infty$ such that

$$\frac{1}{4\varepsilon_n^2} \int_G (1 - |u_{\varepsilon_n}|^2)^2 w \leq C - C_n \varepsilon_n.$$

We may suppose that u_{ε_n} converges as in Theorem 1. Taking into account (18) and the rate of convergence of u_ε away from singularities (see [BBH4], Theorem VI.1) we easily observe that

$$\frac{1}{4\varepsilon_n^2} \int_G (1 - |u_{\varepsilon_n}|^2)^2 w = C + O(\varepsilon_n),$$

which gives a contradiction.

The inequality (25') follows by integrating (11) for small ε .

i),iii) By Theorem 4, any limit point as $\varepsilon \rightarrow 0$ of

$$\frac{1}{4\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 w$$

is of the form

$$\frac{(d-\ell)^2}{p+2} \pi + \frac{|\ell| \pi}{2} \quad \text{with } -d \leq \ell \leq d$$

and i), iii) follow immediately. ■

Theorem 1'. *Under the assumptions of Theorem 1, we have $d_0 > 0$.*

Proof. We already know that $d_0 \geq 0$. Suppose $d_0 = 0$. Then, as in [LR1], Theorem 1,

$$E_\varepsilon(u_\varepsilon) \geq \pi d \log \frac{1}{\varepsilon} - C.$$

On the other hand, by Theorem 2 and choosing an appropriate test function,

$$E_\varepsilon(u_\varepsilon) \leq \left(\frac{2}{p+2} + (d-1) \right) \pi \log \frac{1}{\varepsilon} + C.$$

This gives a contradiction. ■

Theorem 6. *Let $G = B_1$, $g(\theta) = e^{id\theta}$ and $w(x) = |x|^p$. If $d < \frac{p}{4} + 1$ then, for the corresponding minimizers u_ε of E_ε , we have*

$$u_\varepsilon(x) \rightarrow \left(\frac{x}{|x|} \right)^d \quad \text{as } \varepsilon \rightarrow 0.$$

If p is not an integer multiple of 4 and $d > \frac{p}{4} + 1$, then u_\star has singularities $0, a_1, \dots, a_k$ with degrees $d_0, +1, \dots, +1$, where $d_0 = \left\lceil \frac{p}{4} \right\rceil + 1$.

Proof. We prove the assertion of the theorem by induction. Let $d = 1$ and let k be the number of singularities different from 0. On the one hand, it follows from Theorem 2 that

$$E_\varepsilon(u_\varepsilon) \leq \frac{2\pi}{p+2} \log \frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \rightarrow 0.$$

On the other hand, it follows as in [LR1], Theorem 1 that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \pi k \log \frac{1}{\varepsilon_n} + O(1) \quad \text{as } \varepsilon_n \rightarrow 0.$$

We thus obtain $k \leq \frac{2}{p+2} < 1$, that is $k = 0$.

Suppose now the assertion true for any $0 \leq k \leq d-1$ with $d < \frac{p}{4} + 1$. If the conclusion of the theorem does not hold, there is a sequence $\varepsilon_n \rightarrow 0$ and there are $k \geq 1$ points a_1, \dots, a_k in $G \setminus \{0\}$ such that (u_{ε_n}) has at the limit the singularities a_1, \dots, a_k . These singularities have equal degrees $d' = +1$ or $d' = -1$. We shall examine the two cases:

i) If $d' = +1$ then $d_0 < d$. Taking into account the induction hypotheses and Theorem 5 we obtain, for $R > 0$ sufficiently small,

$$E_\varepsilon(u_\varepsilon; B_R) \geq \frac{2d_0^2}{p+2} \pi \log \frac{1}{\varepsilon} - C, \quad \text{as } \varepsilon \rightarrow 0.$$

Thus

$$(27) \quad E_\varepsilon(u_\varepsilon) \geq \left(\frac{2d_0^2}{p+2} + k \right) \pi \log \frac{1}{\varepsilon} - C, \quad \text{as } \varepsilon \rightarrow 0.$$

But Theorem 2 implies

$$(28) \quad E_\varepsilon(u_\varepsilon) \leq \frac{2d^2}{p+2} \pi \log \frac{1}{\varepsilon} + C, \quad \text{as } \varepsilon \rightarrow 0.$$

If we compare (27) and (28) we find that

$$\frac{2d^2}{p+2} \geq \frac{2d_0^2}{p+2} + k.$$

This inequality is clearly false if $k > 0$ and $d_0 > 0$, contradiction.

ii) Let $d' = -1$. There are two cases:

Case 1: $d+k \leq \frac{p}{4} + 1$. In this case, the corresponding minimum in (26) for d replaced by $d+k$ is achieved for $\ell = 0$ and we obtain from Theorem 5 that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \left(\frac{2(d+k)^2}{p+2} - \delta + k \right) \pi \log \frac{1}{\varepsilon_n} - C \quad \text{as } \varepsilon_n \rightarrow 0.$$

This contradicts the upper bound (7).

Case 2: $d+k > \frac{p}{4} + 1$. In this case, the minimum in (26) (for d replaced by $d+k$) is $> \frac{d^2}{p+2}$. This yields again a contradiction. ■

Theorem 7. *Under the assumptions of Theorem 1, we have $d_i = +1$, for $i = 1, \dots, k$.*

If p is an integer multiple of 4 and $d \geq \frac{p}{4} + 1$ then $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4} + 1 \right\}$.

Proof. The fact that $d_i = +1$ follows as in Theorem 6. The statement that $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4} + 1 \right\}$ for $d \geq \frac{p}{4} + 1$ is a consequence of Theorem 5 and of the fact that the quantity

$$\frac{2d_0^2}{p+2} + (d - d_0)$$

attains its minimum in the set $d_0 \in \{1, \dots, d\}$ for $d_0 = \frac{p}{4}$ or $d_0 = \frac{p}{4} + 1$. ■

2 The renormalized energy

In [BBH4], F. Bethuel, H. Brezis and F. Hélein have introduced the concept of renormalized energy associated to a given configuration of points with prescribed degrees and to a boundary data. They observed that the limiting configuration of singularities is a minimum point of this functional. We shall find the renormalized energy in the case of a ball, say B_1 , when the weight is $w(x) = |x|^p$. In the case of a vanishing weight the introduction of a concept

of renormalized energy is useful only for $d \geq \frac{p}{4} + 1$. Indeed, for $d < \frac{p}{4} + 1$ there is only one singularity at the limit, namely the zero of w .

Theorem 8. *Let $g : \partial B_1 \rightarrow S^1$, $\deg(g, \partial B_1) = d > \frac{p}{4} + 1$, $w(x) = |x|^p$. If u_{ε_n} converges to the canonical harmonic map u_\star associated to $a = (0, a_1, \dots, a_k)$ with corresponding degrees $\bar{d} = (d_0, +1, \dots, +1)$, then the configuration a minimizes the functional*

$$\widehat{W}(a, g) = W(a, \bar{d}, g) + \frac{\pi}{2} \sum_{j=1}^k \log w(a_j) .$$

The proof follows the same lines as of the proof of Theorem 1 in [LR1]. ■

It has been observed in the preceding Section that if p is an integer multiple of 4, then $d_0 \in \left\{ \frac{p}{4}, \frac{p}{4} + 1 \right\}$. In what follows we show that both cases may occur.

Example 1. If p is an integer multiple of 4, $G = B_1$, $w(x) = |x|^p$, $g(\theta) = e^{di\theta}$ and $d = \frac{p}{4} + 1$ then $d_0 = \frac{p}{4} + 1$. Assume, by contradiction, that $d_0 \neq \frac{p}{4}$. As observed in Theorem 7, the only possibility in this case is $d_0 = \frac{p}{4}$. By Theorem 8, the limiting configuration $a = (0, a_1)$ with degrees $\bar{d} = (\frac{p}{4}, 1)$ minimizes the functional \widehat{W} . We may now make use of the explicit form of the renormalized energy W found in [LR2], Proposition 2:

$$\begin{aligned} W(a, \bar{d}, g) &= -\frac{\pi}{2} p \log |a_1| - \pi \log(1 - |a_1|^2) - \frac{\pi}{2} p \log(|a_1|^2 + 1 - |a_1|^2) \\ &= -\frac{\pi}{2} p \log |a_1| - \pi \log(1 - |a_1|^2) . \end{aligned}$$

Hence

$$\widehat{W}(a, g) = -\pi \log(1 - |a_1|^2) .$$

But this functional does not achieve its infimum on $B_1 \setminus \{0\}$. So, this case is impossible, that is $d_0 = \frac{p}{4} + 1$.

Example 2. If p is an integer multiple of 4, $G = B_1$, $w(x) = |x|^p$, $g(\theta) = e^{di\theta}$ and $d = \frac{p}{4} + 2$ then $d_0 = \frac{p}{4}$. Indeed, with the explicit form of the renormalized energy (see [LR2]) we compute \widehat{W} when $d_0 = \frac{p}{4} + 1$ (that is $k = 1$) and $d_0 = \frac{p}{4}$ (that is $k = 2$).

If $d_0 = \frac{p}{4} + 1$ then

$$\widehat{W}(0, a_1) = -\pi \log \left(|a_1|^2 (1 - |a_1|^2) \right)$$

which achieves its infimum on $\overline{B}_1 \setminus \{0\}$ and

$$\inf \widehat{W}(0, a_1) = \pi \log 4.$$

If $d_0 = \frac{p}{4}$ then

$$\begin{aligned} \widehat{W}(0, a_1, a_2) = & -\pi \log |a_1 - a_2|^2 - \pi \log(1 - |a_1|^2) - \pi \log(1 - |a_2|^2) - \\ & -\pi \log \left(|a_1 - a_2|^2 + (1 - |a_1|^2)(1 - |a_2|^2) \right). \end{aligned}$$

In this case, with an argument from [LR2], the infimum of $\widehat{W}(0, a_1, a_2)$ is achieved for $a_1 = -a_2 = 5^{-\frac{1}{4}}$. A straightforward calculation gives

$$\inf \widehat{W}(0, a_1, a_2) < \inf \widehat{W}(0, a_1)$$

which means that $d_0 = \frac{p}{4}$.

We next turn to the case of general G, g .

Theorem 9. *Let G be a smooth bounded domain in \mathbb{R}^2 , $g : \partial G \rightarrow S^1$ of topological degree d and $w : \overline{G} \rightarrow \mathbb{R}$, $w > 0$ in $\overline{G} \setminus \{x_0\}$, $w(x) = C |x - x_0|^p + f(x) |x - x_0|^{p+1}$ in a small neighbourhood of x_0 , where f is a C^1 function. If $d > \frac{p}{4} + 1$ then the limit configuration $a = (0, a_1, \dots, a_k)$ with degrees $\bar{d} = (d_0, +1, \dots, +1)$, $d_0 > 0$, minimizes the functional $\widehat{W}(a, g)$.*

The proof is similar as of Theorem 8. ■

We shall now give an example which shows that if p is an integer multiple of 4 and for a general weight w that is like $|x|^p$ in a neighbourhood of 0, then one can not obtain a general result, in the sense that the zero of the weight might have different degrees at the limit. This example shows that not only the behavior of the weight around its zero is important in the determination of degrees, but also the form of the weight w away from 0.

Example 3. Let $h : [0, 1] \rightarrow (0, 1]$ be a C^1 function which equals 1 on $[0, \delta_0]$ and $h(a_1) = \min_{[0, 1]} h = \delta > 0$, which will be suitable chosen. We take

$w(x) = h(|x|) |x|^p$, p an integer multiple of 4 and $g(x) = x^d$ on ∂B_1 , where $d = \frac{p}{4} + 1$. We shall choose δ such that

$$W\left((0), (d)\right) > W\left((0, a_1), (d-1, +1)\right) + \frac{\pi}{2} \log(\delta a_1^p) .$$

Taking into account Theorems 8 and 9, it follows that this choice of δ gives $d_0 = \frac{p}{4}$.

3 Remarks for the case of a weight with several zeroes

For the sake of simplicity assume w has two zeroes a_1 and a_2 in G and, in small neighbourhoods of a_j ,

$$w(x) = |x - a_j|^{p_j} \quad \text{with } p_j > 0, j = 1, 2 .$$

We also suppose that each p_j is not an integer multiple of 4. If $d > \left\lceil \frac{p_1}{4} \right\rceil + \left\lceil \frac{p_2}{4} \right\rceil + 2$ it can be proved using the same techniques that u_{ε_n} converges to u_\star which has singularities a_1, a_2, \dots, a_k of corresponding degrees $d_1 = \left\lceil \frac{p_1}{4} \right\rceil + 1, d_2 = \left\lceil \frac{p_2}{4} \right\rceil + 1, d_3 = \dots = d_k = +1$. Moreover, the configuration $a = (a_1, a_2, a_3, \dots, a_k)$ with $\bar{d} = (d_1, d_2, +1, \dots, +1)$ minimizes the renormalized energy

$$\widehat{W}(a, \bar{d}, g) = W(a, \bar{d}, g) + \frac{\pi}{2} \sum_{j=3}^k \log w(a_j) .$$

The case $d \leq \left\lceil \frac{p_1}{4} \right\rceil + \left\lceil \frac{p_2}{4} \right\rceil + 2$ yields a delicate discussion. For example, if $d = 1$, then there is only one singularity at the limit. This is a_1 if

$$\frac{2}{p_1 + 2} < \frac{2}{p_2 + 2}, \quad \text{that is } p_1 > p_2 .$$

The case $p_1 = p_2$ is more difficult. If

$$(29) \quad W(a_1, 1, g) < W(a_2, 1, g)$$

then the singularity at the limit is a_1 . We cannot conclude when equality holds in (29).

Suppose now $d = 2$ and $p_1 > p_2$. If

$$(30) \quad \frac{8}{p_1 + 2} < \frac{2}{p_1 + 2} + \frac{2}{p_2 + 2}$$

then, at the limit, there is one singularity, namely a_1 , of degree $+2$. If

$$\frac{8}{p_1 + 2} > \frac{2}{p_1 + 2} + \frac{2}{p_2 + 2}$$

then there are two singularities at the limit, namely a_1 and a_2 of corresponding degrees $+1$. If the equality holds in (30) we argue in terms of renormalized energy as above.

The discussion may be similarly continued for greater values of d .

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