BOUNDARY BLOW-UP IN NONLINEAR ELLIPTIC EQUATIONS OF BIEBERBACH–RADEMACHER TYPE

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ABSTRACT. We establish the uniqueness of the positive solution for equations of the form $-\Delta u = au - b(x)f(u)$ in Ω, $u|_{\partial\Omega} = \infty$. The special feature is to consider nonlinearities f whose variation at infinity is *not regular* (e.g., $\exp(u) - 1$, $\sinh(u)$, $\cosh(u) - 1$, $\exp(u)\log(u+1)$, $u^\beta \exp(u^\gamma)$, $\beta \in \mathbb{R}$, $\gamma > 0$ or $\exp(\exp(u)) - e$) and functions $b \geq 0$ in Ω vanishing on $\partial\Omega$. The main innovation consists of using Karamata's theory not only in the statement/proof of the main result but also to link the nonregular variation of f at infinity with the blow-up rate of the solution near $\partial\Omega$.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a smooth bounded domain. We consider semilinear elliptic problems under the following form:

$$(1.1) \qquad \Delta u = g(x, u) \quad \text{in } \Omega,$$

subject to the singular boundary condition

$$(1.2) \qquad u(x) \to \infty \ \text{ as } d(x) := \text{dist}\,(x, \partial\Omega) \to 0 \ \text{ (for short, } u = \infty \text{ on } \partial\Omega).$$

The nonnegative solutions of (1.1)+(1.2) are called *large* (or *blow-up*) *solutions*.

The study of large solutions was initiated in 1916 by Bieberbach [5] for the particular case $g(x, u) = \exp(u)$ and $N = 2$. He showed that there exists a unique solution of (1.1) such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Problems of this type arise in Riemannian geometry; if a Riemannian metric of the form $|ds|^2 = \exp(2u(x))|dx|^2$ has constant Gaussian curvature $-c^2$, then $\Delta u = c^2 \exp(2u)$. Motivated by a problem in mathematical physics, Rademacher [28] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$. Lazer–McKenna [23] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniformal external sphere condition and for nonlinearities $g(x, u) = b(x)\exp(u)$, where b is continuous and strictly positive on $\overline{\Omega}$.

The interest in large solutions extended to N-dimensional domains and to other classes of nonlinearities (see e.g., [2], [3], [8], [9], [11], [14], [19], [22], [24]–[27]).

Let $g(x, u) = f(u)$ where f satisfies

$$(A) \qquad f \in C^1[0, \infty), \ f'(s) \geq 0 \text{ for } s \geq 0, \ f(0) = 0 \text{ and } f(s) > 0 \text{ for } s > 0.$$

In this case, Keller [22] and Osserman [27] proved that large solutions of (1.1) exist if and only if

$$(A_0) \qquad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

In a celebrated paper, Loewner and Nirenberg [26] linked the uniqueness of the blow-up solution to the growth rate at the boundary. Motivated by certain geometric problems, they established the uniqueness for the case $f(u) = u^{\frac{N+2}{N-2}}$ ($N > 2$). Bandle and Marcus [3] give results on asymptotic behaviour and uniqueness of the large solution for more general nonlinearities including $f(u) = u^p$ for any $p > 1$. Theorem 2.3 in [3] proves that when (A) holds and

$$(B) \qquad \exists \mu > 0 \text{ and } s_0 \geq 1 \text{ such that } f(\tau s) \leq \tau^{\mu+1} f(s)\ \forall \tau \in (0,1)\ \forall s \geq s_0/\tau$$

then for any large solution of $\Delta u = f(u)$ we have

$$(1.3) \qquad \lim_{d(x)\to 0} \frac{u(x)}{Z(d(x))} = 1$$

where Z is a chosen solution of

$$(1.4) \qquad \begin{cases} Z''(r) = f(Z(r)), & r \in (0,\delta) \text{ for some } \delta > 0, \\ Z(r) \to \infty \text{ as } r \to 0^+. \end{cases}$$

If, in addition, $f(\tau s) \leq \tau f(s)$, for all $\tau \in (0,1)$ and $s > 0$, then the uniqueness of large solutions takes place. Lazer and McKenna [24] consider the case when the C^1-function f is either defined and positive on $\mathbb{R}$ or is defined on $[a_0, \infty)$ with $f(a_0) = 0$ and $f(s) > 0$ for $s > a_0$. They prove the uniqueness of large solutions to $\Delta u = f(u)$ in $\Omega \subset \mathbb{R}^N$, $N > 1$, under the assumptions (see [24, Theorem 3.1]):

$$(1.5) \qquad \begin{array}{l} \Omega \text{ satisfies both a uniform internal sphere condition and a uniform} \\ \text{external sphere condition with the same constant } R_1 > 0; \end{array}$$

$$(1.6) \qquad f'(s) \geq 0 \text{ for } s \text{ in the domain of } f;$$

$$(1.7) \qquad \text{there exists } a_1 \text{ such that } f'(s) \text{ is nondecreasing for } s \geq a_1;$$

$$(1.8) \qquad \lim_{s\to\infty} f'(s)/\sqrt{F(s)} = \infty.$$

Moreover, the asymptotics of the large solution is found in terms of a difference

$$\lim_{d(x)\to 0} [u(x) - Z(d(x))] = 0, \quad \text{for any } Z \text{ satisfying } (1.4).$$

We are interested in large solutions of (1.1) when $g(x,u) = b(x)f(u) - au$, i.e.,

$$(P) \qquad -\Delta u = au - b(x)f(u) \quad \text{in } \Omega,$$

where $f \in C^1[0,\infty)$, $a \in \mathbb{R}$ and $b \in C^{0,\mu}(\overline{\Omega})$ ($0 < \mu < 1$) satisfies $b \geq 0$, $b \not\equiv 0$ in Ω.

Many papers (see e.g., [1], [12]–[19]) have been written about Eq. (P), on a bounded domain or $\mathbb{R}^N$, when $f(u) = u^p$ ($p > 1$). For this case of nonlinearity and $b > 0$ on $\overline{\Omega}$, Eq. (P) subject to $u = 0$ on $\partial\Omega$ is referred to as the logistic equation. It is known that it has a unique positive solution if and only if $a > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $(-\Delta)$ in Ω. We mention that the logistic equation has been proposed as a model for population density of a steady-state single species $u(x)$ when Ω is fully surrounded by inhospitable areas. However, not until recently was the case of a *degenerate* logistic type equation considered, which allows b to vanish on $\overline{\Omega}$ (see [1], [18] and [13]). The understanding of the

asymptotics for positive solutions of the degenerate logistic equation leads to the study of large solutions (we refer to [18] and [19]).

Let Ω_0 denote the interior of the zero set of b in Ω, i.e.,

$$\Omega_0 = \text{int}\,\{x \in \Omega : \ b(x) = 0\}.$$

We assume throughout that Ω_0 is connected, $\partial\Omega_0$ satisfies the exterior cone condition (possibly, $\Omega_0 = \emptyset$), $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$. Note that $b \geq 0$ on $\partial\Omega$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in Ω_0. Set $\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$.

Alama and Tarantello [1] find the maximal interval I for the parameter a such that (P), subject to $u = 0$ on $\partial\Omega$, has a positive solution u_a, provided that

(A_1) $\qquad\qquad\qquad f \geq 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

Moreover, for each $a \in I$, the solution u_a is unique (see [1, Theorem A (bis)]).

Theorem 1.1 in [9] proves that if (A_0) and (A_1) are fulfilled, then Eq. (P) has large solutions if and only if $a \in (-\infty, \lambda_{\infty,1})$. The uniqueness and asymptotic behaviour near $\partial\Omega$ prove to be very challenging in the above generality.

In [8] we advance for the first time the idea of using the regular variation theory arising in applied probability to study the uniqueness of large solutions. There we consider the case when f' varies regularly at infinity (see Definition 2.1).

Note that there are many nonlinearities $f(u)$, such as $\exp(u) - 1$, $\sinh(u)$, $\exp(\exp(u)) - e$, $\exp(u)\log(u+1)$, which do not fall in the category treated by Theorem 1 in [8]. Although some examples might fit into the framework of [3, Theorem 2.3] or [24, Theorem 3.1], the uniqueness and growth rate at the boundary for large solutions of (P) have not yet been studied when $a \neq 0$ and b vanishes in Ω with $b \equiv 0$ on $\partial\Omega$.

Our purpose is to fill in this gap by analysing a wide range of functions f and b. Here we develop the research line opened up in [8] to treat the case when f does not vary regularly at infinity. Thus our approach for the uniqueness is different from that of Bandle–Marcus and Lazer–McKenna, being based on Karamata's theory.

2. Framework and main result

We first recall some results from the Karamata regular variation theory (see [6]).

Definition 2.1. A measurable function $R : [A, \infty) \to (0, \infty)$, for some $A > 0$, is called *regularly varying at infinity of index* $\rho \in \mathbb{R}$, for short $R \in RV_\rho$, provided that

$$\lim_{u\to\infty} \frac{R(\xi u)}{R(u)} = \xi^\rho, \qquad \forall \xi > 0.$$

When the index ρ is zero, we say that the function is *slowly varying*.

From now on, we do not write at infinity when the regular variation occurs there. Notice that the transformation $R(u) = u^\rho L(u)$ reduces regular variation to slow variation. Examples of slowly varying functions are given by:

(i) Every measurable function on $[A, \infty)$ which has a positive limit at ∞.
(ii) The logarithm $\log u$, its iterates $\log_m u$, and powers of $\log_m u$.
(iii) $\exp\{(\log u)^{\alpha_1}(\log_2 u)^{\alpha_2}\ldots(\log_m u)^{\alpha_m}\}$ where $\alpha_i \in (0,1)$ and $\exp\{\frac{\log u}{\log\log u}\}$.

Proposition 2.2 (Representation Theorem). *The function $L(u)$ is slowly varying if and only if it can be written in the form*

$$L(u) = M(u)\exp\left\{\int_B^u \frac{\phi(t)}{t}\, dt\right\}, \quad \forall u \geq B$$

for some $B > 0$, where $\phi \in C[B,\infty)$ satisfies $\lim_{u\to\infty}\phi(u) = 0$ and $M(u)$ is measurable on $[B,\infty)$ such that $\lim_{u\to\infty} M(u) = \widehat{M} \in (0,\infty)$.

If $M(u)$ is replaced by $\widehat{M}$, then the new function, say $\widehat{L}(u)$, is referred to as a *normalised* slowly varying function. We see that $\phi(u) = \frac{u\widehat{L}'(u)}{\widehat{L}(u)}$, $\forall u \geq B$. Conversely, any function $\widehat{L} \in C^1[B,\infty)$ which is positive and satisfies $\lim_{u\to\infty}\frac{u\widehat{L}'(u)}{\widehat{L}(u)} = 0$ is a normalised slowly varying function.

Note that *any* slowly varying function $L(u)$ is asymptotic equivalent to some normalised slowly varying function $\widehat{L}(u)$ (i.e., $\lim_{u\to\infty} L(u)/\widehat{L}(u) = 1$).

The notion of regular variation can be extended to any real number. For instance, we say that $R(u)$ is regularly varying (on the right) at the origin with index $\rho \in \mathbb{R}$ (and write $R \in RV_\rho(0+)$) if $R(1/u) \in RV_{-\rho}$. Let $NRV_\rho(0+)$ (resp., NRV_ρ) denote the set of all normalised regularly varying functions at 0 (resp., ∞) of index ρ.

By $f_1(x) \sim f_2(x)$ as $d(x) \to 0$ we mean that $\lim_{d(x)\to 0}\frac{f_1(x)}{f_2(x)} = 1$.

Our main result is

Theorem 2.3. *Let (A_1) hold and $f \circ \mathfrak{L} \in RV_\rho$ $(\rho > 0)$ for some $\mathfrak{L} \in C^2[A,\infty)$ satisfying $\lim_{u\to\infty} \mathfrak{L}(u) = \infty$ and $\mathfrak{L}' \in NRV_{-1}$. Suppose that*

(H) $b(x) \sim \mathfrak{K}^2(d(x))$ *as $d(x) \to 0$, where $\mathfrak{K} \in NRV_\theta(0+)$, for some $\theta \geq 0$, and $\mathfrak{K}$ is nondecreasing near the origin if $\theta = 0$.*

Then, for any $a < \lambda_{\infty,1}$, Eq. (P) has a unique large solution u_a. In addition, the blow-up rate of u_a at $\partial\Omega$ can be expressed by

$$(2.1)\qquad u_a(x) \sim (\mathfrak{L} \circ \Phi)(d(x)) \quad as\ d(x) \to 0, \quad \forall a < \lambda_{\infty,1}.$$

The function Φ is defined as follows:

$$(2.2)\qquad \int_{\Phi(t)}^\infty \frac{[\mathfrak{L}'(y)]^{\frac{1}{2}}}{y^{\frac{\rho+1}{2}}[\boldsymbol{L}_f(y)]^{\frac{1}{2}}}\, dy = \int_0^t \mathfrak{K}(s)\, ds, \quad \forall t \in (0,\beta)\ with\ \beta > 0\ small,$$

where $\boldsymbol{L}_f$ is a normalised slowly varying function such that $\lim_{u\to\infty}\frac{f(\mathfrak{L}(u))}{u^\rho \boldsymbol{L}_f(u)} = 1$.

Note that Theorem 2.3 brings a new insight into the asymptotics of the large solution of (P) even in the case $a = 0$ and $b = 1$. For instance, the function which is used in (2.1) to estimate the blow-up rate of the solution near $\partial\Omega$ is *not chosen as a solution of* (1.4). This fact will allow us, through Corollary 2.7, to illustrate the explosion pattern followed by the large solution when the nonlinearity f is of the form (2.8) at infinity and satisfies (A_1). In particular, if $\lim_{u\to\infty}\frac{f(u)}{[\exp_m(u^{\frac{1}{\alpha}})]^\rho} = 1$, ($\alpha, \rho > 0$ and $m \geq 1$ an integer), then the unique large solution of $\Delta u = f(u)$ satisfies $\frac{u(x)}{\Psi(d(x))} \to 1$ as $d(x) \to 0$, where

$$\Psi(d(x)) = \begin{cases} [\log(d(x)^{-\frac{2}{\rho}})]^\alpha, & \text{if } m = 1, \\ [\log_m(d(x)^{-1})]^\alpha, & \text{if } m \geq 2. \end{cases}$$

We set $\log_m(\cdot) = (\underbrace{\log \circ \ldots \circ \log}_{m \text{ times}})(\cdot)$ and $\exp_m(\cdot) = (\underbrace{\exp \circ \ldots \circ \exp}_{m \text{ times}})(\cdot)$, $\mathbb{Z} \ni m \geq 1$.
If $f(u) = \exp_2(u) + \cos(\exp_2(u))$ for u large and (A_1) holds, the uniqueness of large solutions for $\Delta u = f(u)$ cannot be inferred from the Lazer–McKenna result, since condition (1.7) fails. Nevertheless, the uniqueness is valid as we can derive from either [3, Theorem 2.3] or Theorem 2.3. But it is not transparent through (1.3) that the large solution fulfills $\lim_{d(x)\to 0} \frac{u(x)}{\log_2\left(\frac{1}{d(x)}\right)} = 1$, as Corollary 2.7 proves.

Remark 2.4. We point out that $\mathfrak{L}' \in NRV_{-1}$ with $\lim_{u\to\infty} \mathfrak{L}(u) = \infty$ if and only if

$$(2.3) \qquad \mathfrak{L}(u) = C\exp\left\{\int_B^u \frac{\ell(t)}{t}\,dt\right\}, \quad \forall u \geq B > 0$$

where $C > 0$ is a constant and ℓ is a normalised slowly varying function satisfying $\lim_{u\to\infty} \ell(u) = 0$ and $\lim_{u\to\infty} \int_B^u \frac{\ell(t)}{t}\,dt = \infty$. Nontrivial examples of functions $\mathfrak{L}$ are: $\exp\{(\log u)^\gamma\}$, where $\gamma \in (0,1)$, $\exp\left\{\frac{\log u}{\log\log u}\right\}$, and $(\log_m u)^\alpha$ with $\alpha > 0$.

The hypothesis $f \circ \mathfrak{L} \in RV_\rho$ $(\rho > 0)$ is equivalent to the existence of $g \in RV_\rho$ so that $f(u) = g(\mathfrak{L}^{\leftarrow}(u))$, for u large (where $\mathfrak{L}^{\leftarrow}$ denotes the inverse of $\mathfrak{L}$). By Proposition 0.8 (v) in [29], $\mathfrak{L}^{\leftarrow}$ is *rapidly varying with index* ∞ ($\mathfrak{L}^{\leftarrow} \in RV_\infty$), i.e.,

$$\lim_{u\to\infty} \frac{\mathfrak{L}^{\leftarrow}(\lambda u)}{\mathfrak{L}^{\leftarrow}(u)} = \begin{cases} 0 & \text{if } \lambda \in (0,1), \\ 1 & \text{if } \lambda = 1, \\ \infty & \text{if } \lambda > 1. \end{cases}$$

Therefore, for $g(u) = u^\rho$, $f(u) = [\mathfrak{L}^{\leftarrow}(u)]^\rho$ is rapidly varying with index ∞.

If $g \in NRV_\rho$, then $\boldsymbol{L}_f$ (which appears in (2.2)) can be taken as $\frac{f(\mathfrak{L}(u))}{u^\rho}$. Moreover, $\frac{f(u)}{u}$ is increasing in a neighbourhood of infinity. For this, it is enough to see that $\lim_{u\to\infty} \frac{uf'(u)}{f(u)} > 1$. Indeed, using (2.3), we derive that

$$\lim_{y\to\infty} \frac{f'(\mathfrak{L}(y))\mathfrak{L}(y)}{f(\mathfrak{L}(y))} = \lim_{y\to\infty} \frac{yg'(y)}{g(y)} \frac{\mathfrak{L}(y)}{y\mathfrak{L}'(y)} = \rho \lim_{y\to\infty} \frac{\mathfrak{L}(y)}{y\mathfrak{L}'(y)} = \infty.$$

Proposition 2.2 will provide countless functions $g \in NRV_\rho$ and $\mathfrak{L}$ as in (2.3). Hence, by taking $f(u) = g(\mathfrak{L}^{\leftarrow}(u))$ $(u \geq B > 0)$, the assumptions of Theorem 2.3 are fulfilled. It remains only to extend the definition of f to the remaining part of $(0,\infty)$ such that the smoothness of f and (A_1) hold.

Regarding the assumption (H), $\mathfrak{K} \in NRV_\theta(0+)$ if and only if there exists a normalised slowly varying function $L_{\mathfrak{K}}$ such that

$$(2.4) \qquad \mathfrak{K}(t) = t^\theta L_{\mathfrak{K}}(1/t), \quad t \in (0,\nu) \text{ with } \nu > 0.$$

Therefore (2.4) is equivalent to saying that for some constants $c, d > 0$ and $\varphi \in C(0,\nu)$ with $\lim_{t\to 0^+} \varphi(t) = 0$ we have

$$\mathfrak{K}(t) = ct^\theta \exp\left(\int_t^d \frac{\varphi(y)}{y}\,dy\right), \quad \text{for } t \in (0,d).$$

Some examples of $\mathfrak{K}$ as in (H) are: t^θ, $(\sin t)^\theta$, $t^\theta / \exp\left[\frac{\log(1/t)}{\log\log(1/t)}\right]$, $t^\theta / \exp[(-\log t)^\gamma]$ with $\gamma \in (0,1)$, $t^\theta[\log(t+1)]^\alpha$ or $t^\theta[\log_m(1/t)]^{-\alpha}$ with $\alpha > 0$ and $m \geq 1$ an integer.

Remark 2.5. If in Theorem 2.3 we replace $f \circ \mathfrak{L} \in RV_\rho$ by the hypothesis $f' \in RV_\rho$ ($\rho > 0$), then (P) still has a unique large solution u_a, $\forall a < \lambda_{\infty,1}$. However, the blow-up rate of u_a near $\partial\Omega$ is as follows (see [8, Theorem 1]):

$$(2.5) \qquad \lim_{d(x)\to 0} \frac{u_a(x)}{\left(\frac{2\theta+\rho+2}{(2+\rho)(\theta+1)}\right)^{1/\rho} h(d(x))} = 1, \quad \forall a < \lambda_{\infty,1}$$

where h is defined by

$$(2.6) \qquad \int_{h(t)}^\infty \frac{ds}{\sqrt{2F(s)}} = \int_0^t \mathfrak{K}(s)\,ds, \quad \forall t \in (0,\nu).$$

Remark 2.6. The variation of f at ∞ is *not regular* in Theorem 2.3 (i.e., $f \notin RV_\gamma$, for any $\gamma \in \mathbb{R}$), in contrast to Remark 2.5 where $f \in NRV_{\rho+1}$. This fact will bring a significant change in the explosion speed of the large solution of (P).

By Lemma 3.4 we know that $\Phi \in NRV_{\frac{-2(\theta+1)}{\rho}}(0+)$. Since $\mathfrak{L}$ varies slowly at infinity, we can invoke [29, Proposition 0.8 (iv)] to conclude that $\mathfrak{L} \circ \Phi \in RV_0(0+)$.

We show that, in the setting of Remark 2.5, $h \in RV_{\frac{-2(\theta+1)}{\rho}}(0+)$. It is easy to check that $T(u) = (\int_0^{1/u} \mathfrak{K}(s)\,ds)^{-1} \in RV_{\theta+1}$. Set $Y(u) = (\int_u^\infty \frac{ds}{\sqrt{2F(s)}})^{-1}$, for $u > 0$. Clearly, Y is increasing on $(0,\infty)$, $Y(\infty) = \infty$ and $Y \in RV_{\rho/2}$. By (2.6), we find $h(1/u) = Y^\leftarrow(T(u))$ for u sufficiently large, where $Y^\leftarrow(u)$ is the inverse of $Y(u)$. By Proposition 0.8(v) in [29], $Y^\leftarrow \in RV_{2/\rho}$ so that $h(1/u) \in RV_{\frac{2(1+\theta)}{\rho}}$.

As a consequence of Theorem 2.3 and (3.1), we obtain

Corollary 2.7. *Let (A_1) and (H) hold. Assume that there exists $\alpha, \rho > 0$ and an integer $m \geq 1$ such that $f((\log_m u)^\alpha) \in RV_\rho$.*

Then Eq. (P) has a unique large solution u_a, for any $a < \lambda_{\infty,1}$. Moreover,

$$(2.7) \qquad \lim_{d(x)\to 0} \frac{u_a(x)}{\left[\log_m\left(\frac{1}{d(x)}\right)\right]^\alpha} = \begin{cases} \left(\frac{2(1+\theta)}{\rho}\right)^\alpha & \text{if } m = 1, \\ 1 & \text{if } m \geq 2. \end{cases}$$

Remark 2.8. For $m = 1$ the influence of f (resp., $\mathfrak{K}$) into the blow-up rate (2.7) of the large solution can be seen through α and ρ (resp., θ). Nevertheless, if $m \geq 2$, then the order of iteration for logarithm changes accordingly in the asymptotic behaviour (2.7) that proves to be *independent of the index of regular variation ρ* (for $f((\log_m u)^\alpha)$) and θ (for $\mathfrak{K}$).

The assumption $f((\log_m u)^\alpha) \in RV_\rho$ holds if and only if there exists a slowly varying function L such that

$$(2.8) \qquad f(u) = \left[\exp_m(u^{\frac{1}{\alpha}})\right]^\rho L(\exp_m(u^{\frac{1}{\alpha}})), \quad u \geq B > 0.$$

Such examples are given below:

 (i) $f(u) = u^\beta \exp\{\rho u^{\frac{1}{\alpha}}\}$, $f(u) = (\log u)^\beta \exp\{\rho u^{\frac{1}{\alpha}}\}$, where $\beta \in \mathbb{R}$ is arbitrary;
 (ii) $f(u) = \exp\left\{u^{\frac{1}{\alpha}}(\rho + \frac{\alpha}{\log u})\right\}$, $f(u) = \exp\left\{u^{\frac{1}{\alpha}}[\rho + u^{\frac{-2}{3\alpha}}\cos(u^{\frac{1}{3\alpha}})]\right\}$;
 (iii) $f(u) = \exp\left\{u^{\frac{1}{\alpha}}(\rho + u^{\frac{\alpha_1-1}{\alpha}})\right\}$ with $\alpha_1 \in (0,1)$;
 (iv) $f(u) = \exp\{u^{\frac{1}{\alpha}} + \rho\exp\{u^{\frac{1}{\alpha}}\}\}$, $f(u) = \exp\{(u^{-\frac{1}{\alpha}} + \rho)\exp\{u^{\frac{1}{\alpha}}\}\}$

($m = 1$ in (i)–(iii) and $m = 2$ in (iv)).

Example 2.9. Among functions f which fulfill the hypotheses of Corollary 2.7, we illustrate: $f(u) = \exp\{u\} - 1$, $f(u) = \sinh(u)$, $f(u) = \cosh(u) - 1$, $f(u) = \exp\{u\} \log(u+1)$, $f(u) = u^\beta \exp\{\rho u^{\frac{1}{\alpha}}\}$ with $\beta \geq 1$, $\alpha, \rho > 0$, $f(u) = u^\beta \exp(\exp\{u\})$ with $\beta \geq 1$ and $f(u) = \exp(\exp\{u\}) - e$.

Boundary blow-up phenomena for (P) with $a = 0$, $b = 1$ and $f(u) = u^p$, $1 < p \leq 2$, appear in the analytical theory of a Markov process called superdiffusion. In this case, the uniqueness of the large solution was studied in Dynkin [16, 17] by probabilistic techniques. It is remarkable that Dynkin's papers realize, on one hand, a connection between superprocesses and singularity phenomena and, on the other hand, they contain a probabilistic representation of the minimal large solution. By means of a probabilistic representation, a uniqueness result in domains with nonsmooth boundary was established by Le Gall [25] in the case $p = 2$. The existence of large solutions is usually deduced by comparison methods combined with Keller-Osserman *a priori* bounds, Calderon-Zygmund estimates, Agmon-Douglis-Nirenberg's theory, or Alexandrov and Krylov-Safonov techniques.

Our interest here falls on the uniqueness of large solutions to (P) when f does not vary regularly at infinity (thus excluding the power case). Note that if $f(u) = \exp(u) - 1$ or $f(u) = \exp(u) - u - 1$, then by Corollary 2.7 the equation $\Delta u = f(u)$ in Ω has a unique large solution which satisfies $\lim_{d(x)\to 0} \frac{u(x)}{\log[d(x)]^{-2}} = 1$. This asymptotic behaviour is exactly the same as for the unique large solution of $\Delta u = \exp(u)$ in Ω, going back to the pioneering works of Bieberbach [5] and Rademacher [28]. For the two-term asymptotic expansion of the large solution of $\Delta u = \exp(u)$ we refer to [4]. We point out that our approach is completely different from the above papers, for it relies exclusively on the regular variation theory (see [6] for details) not only in the statement, but also in the proof of the main result.

3. Auxiliary results

For details about Propositions 3.1 and 3.3 we refer the reader to [6] ([29] or [30]).

Proposition 3.1 (Elementary properties of slowly varying functions). *Assume that L is a slowly varying function. Then the following hold:*

  (i) $\log L(u)/\log u \to 0$ *as* $u \to \infty$.
  (ii) *For any* $m > 0$, $u^m L(u) \to \infty$, $u^{-m} L(u) \to 0$ *as* $u \to \infty$.
  (iii) $(L(u))^m$ *varies slowly for every* $m \in \mathbb{R}$.
  (iv) *If* L_1 *varies slowly, so do* $L(u)L_1(u)$ *and* $L(u) + L_1(u)$.

Remark 3.2. If $g \in RV_\rho$ with $\rho > 0$ ($\rho < 0$), then $\lim_{u\to\infty} g(u) = \infty$ (0). However, the behaviour at infinity for a slowly varying function cannot be predicted. We see that $L(u) = \exp\{(\log u)^{1/3} \cos((\log u)^{1/3})\}$ is a (normalised) slowly varying function (use $\lim_{u\to\infty} \frac{uL'(u)}{L(u)} = 0$) for which $\liminf_{u\to\infty} L(u) = 0$ and $\limsup_{u\to\infty} L(u) = \infty$.

Proposition 3.3 (Karamata's Theorem). *Let* $R \in RV_\rho$ *be locally bounded in* $[A, \infty)$. *Then, for any* $j < -(\rho+1)$ *(resp.,* $j = -(\rho+1)$ *if* $\int^\infty x^{-(\rho+1)} R(x)\, dx < \infty$)

$$\lim_{u\to\infty} \frac{u^{j+1} R(u)}{\int_u^\infty x^j R(x)\, dx} = -(j + \rho + 1).$$

Under the assumptions of Theorem 2.3, we prove

Lemma 3.4. *The function Φ given by (2.2) is well defined on some interval $(0, \beta)$. Furthermore, $\Phi \in NRV_{\frac{-2(\theta+1)}{\rho}}(0+)$ satisfies*

$$(3.1) \qquad \lim_{t \to 0^+} \frac{\log_m \Phi(t)}{\log_m(\frac{1}{t})} = \begin{cases} \frac{2(1+\theta)}{\rho} & \text{if } m = 1, \\ 1 & \text{if } m \geq 2, \end{cases}$$

$$(3.2) \qquad \lim_{t \to 0^+} \frac{\Phi(t)\Phi''(t)}{[\Phi'(t)]^2} = 1 + \frac{\rho}{2(\theta+1)} \quad and \quad \lim_{t \to 0^+} \frac{\mathcal{L}(\Phi(t))}{\mathcal{L}'(\Phi(t))} \frac{\Phi(t)}{[\Phi'(t)]^2} = 0.$$

Proof. Let $b > 0$ be such that $\boldsymbol{L}_f$, resp. $\mathcal{L}'$, is positive on $[b, \infty)$. Since $\mathcal{L}' \in RV_{-1}$ and $\boldsymbol{L}_f$ is slowly varying, Proposition 3.1 yields

$$\lim_{u \to \infty} \frac{[\mathcal{L}'(u)]^{\frac{1}{2}}}{u^{\frac{\rho+1}{2}}[\boldsymbol{L}_f(u)]^{\frac{1}{2}}} u^{1+\tau} = 0, \quad \text{for any } \tau \in (0, \rho/2).$$

Therefore, there exists $B > b$ large so that

$$\zeta(x) = \int_x^\infty \frac{[\mathcal{L}'(y)]^{\frac{1}{2}}}{y^{\frac{\rho+1}{2}}[\boldsymbol{L}_f(y)]^{\frac{1}{2}}} \, dy < \infty, \quad \forall x > B.$$

It follows that Φ is well defined on $(0, \beta)$, for some $\beta > 0$. Moreover, $\Phi \in C^2(0, \beta)$ and $\lim_{t \to 0^+} \Phi(t) = \infty$. Using (2.2), we find

$$(3.3) \qquad \frac{-\Phi'(t)[\mathcal{L}'(\Phi(t))]^{\frac{1}{2}}}{[\Phi(t)]^{\frac{\rho+1}{2}}[\boldsymbol{L}_f(\Phi(t))]^{\frac{1}{2}}} = \mathfrak{K}(t), \quad \forall t \in (0, \beta).$$

In view of Proposition 3.3, we have

$$\lim_{u \to \infty} \frac{[\mathcal{L}'(u)]^{\frac{1}{2}}}{u^{\frac{\rho-1}{2}}[\boldsymbol{L}_f(u)]^{\frac{1}{2}}\zeta(u)} = \frac{\rho}{2}$$

which, together with (2.2), produces

$$(3.4) \qquad \lim_{t \to 0^+} \frac{[\mathcal{L}'(\Phi(t))]^{\frac{1}{2}}[\Phi(t)]^{\frac{-\rho+1}{2}}}{[\boldsymbol{L}_f(\Phi(t))]^{\frac{1}{2}} \int_0^t \mathfrak{K}(s) \, ds} = \frac{\rho}{2}.$$

By (3.3), (3.4) and L'Hospital's rule, we find

$$(3.5) \qquad \lim_{t \to 0^+} \frac{\log \Phi(t)}{\log(\int_0^t \mathfrak{K}(s) \, ds)} = \lim_{t \to 0^+} \frac{\Phi'(t)}{\Phi(t)} \frac{\int_0^t \mathfrak{K}(s) \, ds}{\mathfrak{K}(t)} = -\frac{2}{\rho}.$$

We differentiate (3.3) to obtain

$$(3.6) \qquad \Phi''(t) = \frac{-\mathfrak{K}(t)\Phi'(t)[\boldsymbol{L}_f(\Phi(t))]^{\frac{1}{2}}}{[\mathcal{L}'(\Phi(t))]^{\frac{1}{2}}[\Phi(t)]^{\frac{1-\rho}{2}}} \left\{ \frac{\rho+1}{2} + \frac{\mathfrak{K}'(t)\Phi(t)}{\mathfrak{K}(t)\Phi'(t)} + \frac{\Phi(t)\boldsymbol{L}_f'(\Phi(t))}{2\boldsymbol{L}_f(\Phi(t))} \right.$$

$$\left. - \frac{\Phi(t)\mathcal{L}''(\Phi(t))}{2\mathcal{L}'(\Phi(t))} \right\}$$

for each $t \in (0, \beta)$. By $\mathfrak{K} \in NRV_\theta(0+)$ we mean $\widetilde{\mathfrak{K}}(u) = \mathfrak{K}(1/u) \in NRV_{-\theta}$. Hence, $\lim_{t \to 0^+} \frac{t\mathfrak{K}'(t)}{\mathfrak{K}(t)} = \theta$ and $\lim_{t \to 0^+} \frac{\int_0^t \mathfrak{K}(s) \, ds}{t\mathfrak{K}(t)} = \frac{1}{\theta+1}$. This, combined with (3.5), yields

$$(3.7) \qquad \lim_{t \to 0^+} \frac{\mathfrak{K}'(t)}{\mathfrak{K}(t)} \frac{\Phi(t)}{\Phi'(t)} = -\frac{\rho\theta}{2(\theta+1)} \quad and \quad \lim_{t \to 0^+} \frac{t\Phi'(t)}{\Phi(t)} = -\frac{2(\theta+1)}{\rho}.$$

Thus, $\Phi \in NRV_{-\frac{2(\theta+1)}{\rho}}(0+)$. By (3.7) and L'Hospital's rule, we obtain

$$(3.8) \qquad \lim_{t \to 0^+} \frac{\log \Phi(t)}{\log t} = \lim_{t \to 0^+} \frac{t\Phi'(t)}{\Phi(t)} = -\frac{2}{\rho}(1+\theta).$$

Proceeding by induction, we conclude (3.1). Since $\boldsymbol{L}_f$ is a normalised slowly varying function and $\mathcal{L}' \in NRV_{-1}$, we have

$$(3.9) \qquad \lim_{t \to 0^+} \frac{\Phi(t)\boldsymbol{L}'_f(\Phi(t))}{\boldsymbol{L}_f(\Phi(t))} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{\Phi(t)\mathcal{L}''(\Phi(t))}{\mathcal{L}'(\Phi(t))} = -1.$$

By (3.6), (3.7) and (3.9), we infer that

$$\lim_{t \to 0^+} \frac{\Phi''(t)[\mathcal{L}'(\Phi(t))]^{\frac{1}{2}}}{\mathfrak{K}(t)\Phi'(t)[\Phi(t)]^{\frac{\rho-1}{2}}[\boldsymbol{L}_f(\Phi)]^{\frac{1}{2}}} = -\left(1 + \frac{\rho}{2(\theta+1)}\right).$$

Replacing $\mathfrak{K}(t)$ by its value in (3.3), we obtain the first assertion of (3.2). Moreover,

$$(3.10) \qquad \lim_{t \to 0^+} \frac{\log(-\Phi'(t))}{\log \Phi(t)} = \lim_{t \to 0^+} \frac{\Phi''(t)\Phi(t)}{[\Phi'(t)]^2} = 1 + \frac{\rho}{2(1+\theta)}.$$

Since $\mathcal{L}$ varies slowly at ∞ and $\mathcal{L}' \in RV_{-1}$, we use Proposition 3.1 (i) to obtain

$$(3.11) \qquad \lim_{t \to 0^+} \frac{\log \mathcal{L}(\Phi(t))}{\log \Phi(t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{\log \mathcal{L}'(\Phi(t))}{\log \Phi(t)} = -1.$$

We notice that

$$\log \left(\frac{\mathcal{L}(\Phi(t))\Phi(t)}{\mathcal{L}'(\Phi(t))[\Phi'(t)]^2} \right) = \log \Phi(t) \left[1 + \frac{\log \mathcal{L}(\Phi(t))}{\log \Phi(t)} - \frac{2\log|\Phi'(t)|}{\log \Phi(t)} - \frac{\log \mathcal{L}'(\Phi(t))}{\log \Phi(t)} \right]$$

which, together with (3.10) and (3.11), leads to

$$\lim_{t \to 0^+} \log \left(\frac{\mathcal{L}(\Phi(t))\Phi(t)}{\mathcal{L}'(\Phi(t))[\Phi'(t)]^2} \right) = -\infty.$$

Thus the second claim of (3.2) is proved. $\qquad \square$

4. Proof of Theorem 2.3

Let us first remark that the Keller–Osserman condition (A_2) holds. Indeed, by using Proposition 3.1, we arrive at

$$\lim_{z \to \infty} \frac{f(z)}{z^p} = \lim_{u \to \infty} \frac{f(\mathcal{L}(u))}{u^\rho \boldsymbol{L}_f(u)} \frac{u^\rho \boldsymbol{L}_f(u)}{[\mathcal{L}(u)]^p} = \lim_{u \to \infty} \frac{u^\rho \boldsymbol{L}_f(u)}{[\mathcal{L}(u)]^p} = \infty, \quad \forall p > 1.$$

Thus, Eq. (P) has at least a large solution when $a < \lambda_{\infty,1}$ and no large solution provided that $a \geq \lambda_{\infty,1}$ (see [9, Theorem 1.1]).

We now prove that, for $a < \lambda_{\infty,1}$ fixed, every large solution of (P) exhibits the same asymptotic behaviour near $\partial\Omega$, namely (2.1). Set $\vartheta^\pm = \left(\frac{\rho}{2(1+\theta)(1 \mp 2\epsilon_0)} \right)^{\frac{1}{\rho}}$, where $\epsilon_0 \in (0, 1/2)$ is arbitrary. Let $\delta \in (0, \beta/2)$ be small such that

   (i) $d(x)$ is a C^2-function on the set $\{x \in \mathbb{R}^N : d(x) < 2\delta\}$.
   (ii) $\mathfrak{K}$ is nondecreasing on $(0, 2\delta)$.
   (iii) $(1 - \epsilon_0)\mathfrak{K}^2(d(x)) < b(x) < (1 + \epsilon_0)\mathfrak{K}^2(d(x))$, for all $x \in \Omega$ with $d(x) < 2\delta$.
   (iv) $\mathcal{L}(\vartheta^\pm \Phi(2\delta)) > 0$.

Let $\sigma \in (0,\delta)$ be arbitrary. We define $u_\sigma^\pm(x) = \mathfrak{L}(\vartheta^\pm \Phi(d(x) \mp \sigma))$, where $d(x) \in (\sigma, 2\delta)$ (resp., $d(x) + \sigma < 2\delta$) for $u_\sigma^+(x)$ (resp., $u_\sigma^-(x)$). It follows that

$$\Delta u_\sigma^\pm = \operatorname{div}\left(\vartheta^\pm \mathfrak{L}'(\vartheta^\pm \Phi(d(x) \mp \sigma))\Phi'(d(x) \mp \sigma)\nabla d(x)\right)$$
$$= (\vartheta^\pm)^2 \mathfrak{L}''(\vartheta^\pm \Phi(d(x) \mp \sigma))[\Phi'(d(x) \mp \sigma)]^2 |\nabla d(x)|^2$$
$$+ \vartheta^\pm \mathfrak{L}'(\vartheta^\pm \Phi(d(x) \mp \sigma))\Phi''(d(x) \mp \sigma)|\nabla d(x)|^2$$
$$+ \vartheta^\pm \mathfrak{L}'(\vartheta^\pm \Phi(d(x) \mp \sigma))\Phi'(d(x) \mp \sigma)\Delta d(x).$$

In view of (i)–(iii), when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| = 1$)

(4.1)
$$\Delta u_\sigma^+ + a u_\sigma^+ - b(x)f(u_\sigma^+) \leq \frac{\vartheta^+ \mathfrak{L}'(\vartheta^+ \Phi(d(x) - \sigma))[\Phi'(d(x) - \sigma)]^2}{\Phi(d(x) - \sigma)}$$
$$\times \left\{ \frac{\Phi(d(x) - \sigma)}{\Phi'(d(x) - \sigma)} \Delta d + \mathcal{E}^+(d(x) - \sigma) \right\},$$

respectively, when $d(x) + \sigma < 2\delta$,

(4.2)
$$\Delta u_\sigma^- + a u_\sigma^- - b(x)f(u_\sigma^-) \geq \frac{\vartheta^- \mathfrak{L}'(\vartheta^- \Phi(d(x) + \sigma))[\Phi'(d(x) + \sigma)]^2}{\Phi(d(x) + \sigma)}$$
$$\times \left\{ \frac{\Phi(d(x) + \sigma)}{\Phi'(d(x) + \sigma)} \Delta d + \mathcal{E}^-(d(x) + \sigma) \right\}.$$

Here $\mathcal{E}^\pm$ are real functions defined on $(0, 2\delta)$ as follows:

(4.3) $\mathcal{E}^\pm(t) := \dfrac{\Phi''(t)\Phi(t)}{[\Phi'(t)]^2} + \dfrac{\vartheta^\pm \Phi(t)\, \mathfrak{L}''(\vartheta^\pm \Phi(t))}{\mathfrak{L}'(\vartheta^\pm \Phi(t))} + \dfrac{a\mathfrak{L}(\vartheta^\pm \Phi(t))}{\vartheta^\pm \mathfrak{L}'(\vartheta^\pm \Phi(t))} \dfrac{\Phi(t)}{[\Phi'(t)]^2} - \mathcal{D}^\pm(t)$

where we denote

$$\mathcal{D}^\pm(t) = (1 \mp \epsilon_0)\, \frac{\mathfrak{K}^2(t)f(\mathfrak{L}(\vartheta^\pm \Phi(t)))}{\vartheta^\pm \mathfrak{L}'(\vartheta^\pm \Phi(t))}\, \frac{\Phi(t)}{[\Phi'(t)]^2}.$$

By virtue of (3.3), we may rewrite $\mathcal{D}^\pm(t)$ as

$$\mathcal{D}^\pm(t) = (1 \mp \epsilon_0)(\vartheta^\pm)^\rho\, \frac{f(\mathfrak{L}(\vartheta^\pm \Phi(t)))}{(\vartheta^\pm \Phi(t))^\rho \boldsymbol{L}_f(\vartheta^\pm \Phi(t))}\, \frac{\boldsymbol{L}_f(\vartheta^\pm \Phi(t))}{\boldsymbol{L}_f(\Phi(t))}\, \frac{\mathfrak{L}'(\Phi(t))(\vartheta^\pm)^{-1}}{\mathfrak{L}'(\vartheta^\pm \Phi(t))}.$$

It follows that $\lim_{t \to 0^+} \mathcal{D}^\pm(t) = (1 \mp \epsilon_0)(\vartheta^\pm)^\rho$. Note that $\lim_{u \to \infty} \frac{u\mathfrak{L}''(u)}{\mathfrak{L}'(u)} = -1$ (since $\mathfrak{L}' \in NRV_{-1}$). Moreover, by (4.3) and Lemma 3.4, we find

$$\lim_{t \to 0^+} \mathcal{E}^\pm(t) = \frac{\rho}{2(1 + \theta)} - (1 \mp \epsilon_0)(\vartheta^\pm)^\rho = -\frac{\rho}{2(1 + \theta)}\, \frac{\pm \epsilon_0}{1 \mp 2\epsilon_0}.$$

Hence, using (4.1) and (4.2), we can choose $\delta > 0$ small enough so that

(4.4)
$$\Delta u_\sigma^+ + a u_\sigma^+ - b(x)f(u_\sigma^+) \leq 0, \quad \forall x \text{ with } \sigma < d(x) < 2\delta,$$
$$\Delta u_\sigma^- + a u_\sigma^- - b(x)f(u_\sigma^-) \geq 0, \quad \forall x \text{ with } d(x) + \sigma < 2\delta.$$

For $\eta > 0$, we define $C_\eta = \{x \in \Omega : d(x) > \eta\}$ and $D_\eta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < \eta\}$.

Let $\eta > 0$ be small such that $a < \lambda_1(-\Delta, D_\eta \setminus \overline{\Omega})$, where $\lambda_1(-\Delta, D_\eta \setminus \overline{\Omega})$ denotes the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $D_\eta \setminus \overline{\Omega}$. Let $p \in C^{0,\mu}(\overline{D}_{2\eta})$ satisfy $0 < p(x) \leq b(x)$ for $x \in \Omega \setminus C_{2\delta}$, $p = 0$ on $\overline{D}_\eta \setminus \Omega$ and $p > 0$ on $\overline{D}_{2\eta} \setminus \overline{D}_\eta$.

We denote by w a positive large solution of $\Delta w + aw = p(x)f(w)$ in $D_{2\eta} \setminus \overline{C}_\delta$ (see [9, Theorem 1.1]). Set $U := u_a + w$ and $V_\sigma := u_\sigma^+ + w$, where u_a is an arbitrary large solution of (P). It follows that

$$\Delta U + aU - b(x)f(U) \leq 0 \text{ in } \Omega \setminus \overline{C}_\delta \text{ and } \Delta V_\sigma + aV_\sigma - b(x)f(V_\sigma) \leq 0 \text{ in } C_\sigma \setminus \overline{C}_\delta.$$

Notice that $U|_{\partial\Omega} = \infty > u_\sigma^-|_{\partial\Omega}$, $U|_{\partial C_\delta} = \infty > u_\sigma^-|_{\partial C_\delta}$, resp., $V_\sigma|_{\partial C_\sigma} = \infty > u_a|_{\partial C_\sigma}$ and $V_\sigma|_{\partial C_\delta} = \infty > u_a|_{\partial C_\delta}$. Thus, by [9, Lemma 2.1], we deduce $u_a + w \geq u_\sigma^-$ on $\Omega \setminus \overline{C}_\delta$ and $u_\sigma^+ + w \geq u_a$ on $C_\sigma \setminus \overline{C}_\delta$. Letting $\sigma \to 0^+$, we find

$$\mathfrak{L}(\vartheta^- \Phi(d)) - w \leq u_a \leq \mathfrak{L}(\vartheta^+ \Phi(d)) + w \quad \text{on } \Omega \setminus \overline{C}_\delta.$$

Since w is uniformly bounded on $\partial\Omega$ and $\mathfrak{L}$ varies slowly at ∞, we conclude (2.1).

Thus, $\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1$ for any two large solutions u_1, u_2 of (P). From now on, we can use the same line of reasoning as in the proof of [8, Theorem 1] to obtain $u_1 \equiv u_2$ on Ω. □

Acknowledgment

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