# BOUNDARY VALUE PROBLEMS ON KLEIN SURFACES

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ABSTRACT. This survey paper deals with the unitary treatment of some classes of linear partial differential equations on Klein surfaces. We are mainly concerned with the study of harmonic functions with Dirichlet or Neumann boundary condition. In such a way, the present paper extends several classical results to the abstract setting of dianalytic manifolds of complex dimension 1. The analysis developed in this paper offers perspectives to the qualitative analysis of other classes of linear or nonlinear elliptic equations on Klein surfaces.

## 1. INTRODUCTION

In the preface to the first edition of Courant-Hilbert's "Methoden der mathematischen Physik" (see [COH]), R. Courant noted the danger that mathematical research would lose the initial link between the problems and methods of analysis and the physical and geometric intuition, the tendencies being to refine the methods and to extreme generalize the existing concepts.

Over the years, these trends led to an increasing distinction between pure and applied mathematicians, who severely criticized each other. This constructive criticism gave rise to the theory of real numbers and to many topological concepts including non-orientable surfaces. It is obvious that some areas that use mathematical methods but their object is derived from physical and geometric intuition are disadvantaged in such a discussion.

The present paper is a piece of the bridge between the theoretical approach of the pure mathematician and the practical interest of the engineer, physicist and applied mathematician. The main purpose is to bring together various geometrical and physical concepts relating to surfaces that have motivated the development of the theory of Klein surfaces.

Riemann surfaces, in the form of domains spread out over the complex plane were introduced in Riemann's dissertation whose methods were developed much further in the first edition of Riemann's paper on Abelian functions "Theorie der Abel'schen Functionen" (see [RIE]), in 1857. Riemann's works provided the basic tools to classify all compact orientable surfaces and, more generally, to study the topology of manifolds. They are equally important for the development of algebraic geometry and the geometric treatment of complex analysis. As for the importance that was attached to this topic, it suffices to say that Albert Einstein's "general theory of relativity" is wholly based on Riemann's ideas.

In his "Extremale quasikonforme Abbildungen und quadratische Differentiale" (see [TEI]) Teichmüller considered the cases of oriented bordered Riemann surfaces and non-orientable Riemann surfaces. He defined the double of a oriented bordered Riemann surfaces or of a non-orientable Riemann surfaces. These are closed Riemann

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surfaces with genus depending on the original surfaces. He introduced the notions of meromorphic functions, n-differentials and divisors on bordered non-orientable Riemann surfaces.

Teichmüller defined, through two examples, the notion of conformal invariant for non-orientable regions. It is important to note that according to Teichmüller's definition, a conformal mapping (even in the orientable case) preserves only the angles, but not necessarily the orientation. Thus, such a mapping is also defined in the non-orientable case. Teichmüller exhibited complete systems of conformal invariants for some special surfaces. For example, a simply connected domain with two distinguished points in the interior has one conformal invariant, namely the Green function.

Teichmüller considered special cases of the fact that one deals with non-orientable surfaces by passing to the orientation double cover. To treat the most general surfaces, he considered a symmetrization process on the corresponding doubles. This idea was used to solve the main problem of extremality, known as the Teichmüller theorem. By using the two-sheeted covering (an annulus) of the Möbius strip, he gets the Teichmüller distance. He showed that problems on the projective plane with two distinguished points can be reduced to similar problems on the sphere with four distinguished points. In two examples of non-orientable surfaces, Möbius strip and the projective plane, the problem of finding the conformal invariants is lifted to the oriented double cover. Teichmuller considered problems on the Klein bottle and lifted them to problems concerning the case of the torus. The torus is the two-sheeted orientation covering of the Klein bottle.

The genesis and development of the idea of symmetry are related to Lie's and Klein's research which were inspired by their deep interest in the theory of groups and in various aspects of the notion of symmetry. According to Klein's Erlangen program, a geometry is determined by a “domain of action” (the plane, space, etc.) and a “group of automorphisms” (or a symmetry group) acting on the domain. When we change the symmetry group we change the geometric scheme under consideration, namely we obtain a new “geometry”.

Thus, the main difference between, say, Euclidean and hyperbolic geometry is not the possibility of constructing one or more lines passing through a point and not intersecting a given line, but the difference in the structure of the respective groups of symmetries of Euclidean and hyperbolic geometry. Therefore the object of the geometry is the study of those properties of a domain which are preserved by the transformations in a symmetry group. The description of all possible geometries is an open problem.

In the same way, classifying non-classical topological compact surfaces is the same thing as classifying all orientation reversing involutions of a classical compact surface.

This brings us to the interesting question of the possible global forms of various (say, two-dimensional) geometric systems (Euclidean, hyperbolic, elliptic) first stated (in connection with Euclidean geometry) by the outstanding geometer W. K. Clifford. Today this question is known as the Clifford-Klein problem and the possible global forms of geometries are called Clifford-Klein forms.

It is known that there are only two spatial forms of two-dimensional elliptic geometry (the sphere and the elliptic plane), but there are as many as five forms of two-dimensional Euclidean geometry (the ordinary Euclidean plane, the infinite Möbius strip, the infinite cylinder, the torus and the so-called Klein bottle). Finally, there are infinitely many forms of two-dimensional hyperbolic geometry.

In this context, non-orientable surfaces are a possible geometric system capable of "modelling" the real shape of the universe surrounding us. For more details, see Munzner's video [MUN], about the different topological structures that a three-dimensional universe could have and Weeks's book [WEE], which fills the gap between the simplest examples, such as the Mobius strip and the Klein bottle, and the sophisticated mathematics found in upper-level college courses.

Spencer and Schiffer in their advanced monograph "Functionals of Finite Riemann Surfaces", extended the investigation of finite Riemann surfaces from the point of view of functional analysis, that is, the study of the various Abelian differentials of the surface in their dependence on the surface itself.

The methodology that Schiffer and Spencer employed is characterized by Ahlfors' next comment: "such a surface has a double, obtained by reflection across the boundary, and one of the main features of the book is the systematic use of this symmetrization process".

The notion of *Klein surface* goes back to Felix Klein due to his closing remarks in [KLE1882], even though one does not find a definition of a Klein surface there. Klein surfaces generalize Riemann surfaces and they are dianalytic manifolds of complex dimension 1. Roughly speaking, a Klein surface is a surface on which the notion of angle between two tangent vectors at a given point is well-defined, and so is the angle between two intersecting curves on the surface.

Basic function theory on Klein surfaces and the relation between compact Klein surfaces and real algebraic function fields were developed in the monograph "Foundations of the Theory of Klein surfaces" (see [ALG]) by N. Alling and N. Greenleaf. They showed that every Klein surface can be represented as the quotient of a Riemann surface by a conjugate analytic involution. Thus, it is natural to extend on Klein surfaces the most fundamental problems in engineering, physics and other sciences. Alling and Greenleaf were the ones who introduced the name "Klein surface".

Our approach is an alternative theory to the standard theory given by Alling and Greenleaf and aims at the natural imbedded of calculus on Klein surfaces in the well known Cartan's model of calculus on manifolds. We have developed this theory because of the unusual behavior from the analytical point of view of the Alling and Greenleaf's results. For instance, functions are not usual functions but equivalence classes of families of meromorphic functions relative to dianalytic atlases. Such a family defines an usual function if and only if all its members are the same real constant. The meromorphic differentials are also equivalence classes of families of functions satisfying some compatibility conditions that lead to the impossibility of defining a consistent integral on Klein surfaces (see [ALG, Theorem 1.10.4]).

We follow Schiffer and Spencer's method to study the objects on Klein surfaces by means of the complex double, whose existence and uniqueness are demonstrated in [ALG].

We are enabled to bring together systematically and concisely the concepts of the Green and Neumann functions, the harmonic kernel function and the harmonic measure and to build from them an elegant generalization for the basic ideas of boundary value problems on Klein surfaces.

The main objectives of study in this paper are the Dirichlet problem and the Neumann problem for harmonic functions on Klein surfaces. The technique is based on the fact that according to a classical result due to Klein, the boundary value problems on a Klein surface can be reduced to similar problems on its complex double. This process has many advantages, starting from the fact that complex double is a symmetric Riemann surface, that is, a Riemann surface endowed with a fixed point free antianalytic involution. Consequently, we obtain harmonic functions on a Klein surface by adding together a pair of harmonic functions on the symmetric Riemann surface, whose singularities lye at symmetric points. In our study, we use methods that have wide applicability in function theory and partial differential equations.

The symmetric conditions on the boundary determine symmetric solutions on the complex double, which lead to solutions for the similar problems on the Klein surface. Specifically, in the case of Klein surfaces, the formula for the solution of the Dirichlet problem is expressed in terms of an analogue of the Green function, which has the symmetry in argument and parameter. In these terms, we extend the use of the Green function to the study of the harmonic measure on a Klein surface. That is why we distinguish the method to solve the Dirichlet problem for harmonic functions on a Klein surface, once the harmonic measure on a symmetric Riemann surface is known. This procedure generates an explicit formula for the solution of the Dirichlet problem on a Klein surface, which is similar to the Poisson integral. At the same time, we rewrite the Radon–Nikodym derivative of harmonic measure against the symmetric arc length.

The corresponding solution of the Neumann problem for harmonic functions on a Klein surface is expressed in terms of an analogue of a Neumann function.

The harmonic kernel function is related to the classical domain functions, such as the Green function and the Neumann function on a Klein surface. In such a way it is possible to solve both boundary value problems of potential theory on a Klein surface, once the harmonic kernel function on a symmetric Riemann surface is known.

We refer to Krantz [KRA1, KRA3, KRA4, KRA5] for an excellent exposition of various topics at the interplay between complex analysis and partial differential equations.

The study of objects on Klein surfaces is an important part of surface topology due to the applications of these surfaces in several fields of science such as quantum physics, chemistry and biology. Indeed, for a physicist, Möbius's band and Klein's bottle are essential elements in the so-called annulment of divergences (see [KAT]). In chemistry, the recent synthesis and the "half-cutting" of a molecular Möbius strip (see [WHRH]) was considered as a spectacular event, described as "the most topologically stimulating molecular structure synthesized to date" it catalyzed the birth of extrinsic graph theory, dealing with topological chirality, a field now burgeoning in mathematics.

The study of liquid crystals is another field where Klein surfaces have surprisingly materialized themselves. In the so-called nematic liquids, the molecules form ribbons which may or may not be orientable (see [BOU]). A systematic topological analysis highlighted the double topological character of distortions in liquid crystals differentiated for "energetic reasons" (see [BDPPT]). If we consider the potential function of some form of internal energy of the ribbons, then the normal derivative on the border characterizes the flow of energy across the border. It may be necessary to determine this potential knowing the respective flow or the values of the potential on the border. These are boundary value problems which will be solved in this paper.

The natural tendency of some macromolecules to store energy through distortions, a fact well known to chemists, might be the cause itself for the formation of non-orientable strings, thus making obvious the practical need of dealing with boundary value problems related to them.

A unified principle for science that works with dualism is presented in terms of torsion fields and the non-orientable surfaces, namely the Klein Bottle, the Möbius strip and the projective plane, in (see [RAP]). This principle is applied to the complex numbers and cosmology, to non-linear systems integrating the issue of hyperbolic divergences with the change of orientability, to the biomechanics of vision and the mammal heart, to the morphogenesis of crustal shapes on Earth in connection to the wavefronts of gravitation, elasticity and electromagnetism, to pattern recognition of artificial images and visual recognition, to neurology and the topographic maps of the sensorium, to perception, in particular of music.

As it is noticed in (see [JSS]), these are the types of problems which contribute to a unifying treatment of orientable and non-orientable surfaces, not only from the topological point of view (see [BEGG] and [SES]) but also from an analytical point of view (see [SCS] and [ALG]).

## 2. Klein surfaces and symmetric Riemann surfaces

Klein surfaces are the most general two-manifolds that support harmonic functions. In order to be able to extend results about boundary value problems for harmonic functions on Riemann surfaces to Klein surfaces, we have to review some results on the topology of surfaces, Klein surfaces, and the uniformization of Riemann surfaces. The history of Klein surfaces is going back to Klein (see [KLE1882]) who considered the group of conformal maps of the Klein bottle and other non-orientable surfaces. In their monograph, Schiffer and Spencer (see [SCS]) did the first modern study of the surfaces endowed with dianalytic structures. Much of the material of this Section will be presented without proofs and will be completed with references to proofs. The main reference to topology of surfaces is the monograph of Ahlfors and Sario (see [AHS]).

A connected topological Hausdorff space $\mathcal{X}$ is a *surface with boundary* if every point $\widetilde{P} \in \mathcal{X}$ has an open neighborhood $\widetilde{U}$, which is homeomorphic to a relatively open subset of the closed upper half-plane. A homeomorphism $h : \widetilde{U} \to h(\widetilde{U})$ is called a *local parameter* at the point $\widetilde{P} \in \widetilde{U}$. The boundary $\partial\mathcal{X}$ of $\mathcal{X}$ consists of those points $\widetilde{P} \in \mathcal{X}$, such that $h(\widetilde{P}) \in \mathbb{R}$, for all the local parameters $h$ at the point $\widetilde{P}$. The pair $(\widetilde{U}, h)$ is called a *chart*. Let $h_i : \widetilde{U}_i \to h_i(\widetilde{U}_i)$ and $h_j : \widetilde{U}_j \to h_j(\widetilde{U}_j)$ be two local parameters, such that $\widetilde{U}_i \cap \widetilde{U}_j \neq \emptyset$, $i, j \in I$. The mapping $h_i \circ h_j^{-1} : h_j(\widetilde{U}_i \cap \widetilde{U}_j) \to h_i(\widetilde{U}_i \cap \widetilde{U}_j)$ is called a *transition function*.

Let $A$ and $B$ be nonempty open sets in the closed upper half-plane. A continuous map of $A$ into $B$ is analytic on $A$ (resp., antianalytic on $A$) if it extends to an analytic (resp., antianalytic) function on some neighborhood of $A$ in $\mathbb{C}$ into $\mathbb{C}$. If $f$ or the complex conjugate of $f$ is analytic on each connected component of the set $A$, then $f$ is called *dianalytic* on $A$.

An *atlas* of the surface $\mathcal{X}$ is a family $\mathcal{A} = \{(\widetilde{U}_i, h_i) \,|\, i \in I\}$ of charts, where $(\widetilde{U}_i)_{i\in I}$ is an open cover of $\mathcal{X}$. The atlas $\mathcal{A}$ is *dianalytic* if all of its transition functions are dianalytic. Two dianalytic atlases $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if $\mathcal{A} \cup \mathcal{B}$ is a dianalytic atlas as well. An equivalence class $A$ of dianalytic atlases of $\mathcal{X}$ is called a *dianalytic structure* on $X$.

A *Klein surface* is a surface $\mathcal{X}$ with boundary endowed with a dianalytic structure $A$ and will be denoted by $X$. Observe that a classical Riemann surface is an orientable Klein surface with empty boundary.

The main tool in our study is the complex double of a Klein surface. Details about the history of this concept may be found in [SCS] and for some of its applications see [ALG], [BEGG] and [SES].

Let $X$ be a Klein surface endowed with the maximal atlas $A = \{(\widetilde{U}_i, h_i)\,|i \in I\}$. We recall the construction of the complex double of $X$, which we shall use in order to apply results about Riemann surfaces to Klein surfaces. We consider the disjoint union $S = \underset{i\in I}{\cup} \widetilde{U}_i$. Let $(\widetilde{U}_i, h_i)$ and $(\widetilde{U}_j, h_j)$ be two charts, such that $\widetilde{U}_i \cap \widetilde{U}_j \neq \emptyset$. For a point $\widetilde{P} \in \widetilde{U}_i \cap \widetilde{U}_j$, the set $S$ has two points which both correspond to the point $\widetilde{P}$, namely the point $\widetilde{P} \in \widetilde{U}_i$ and the same point $\widetilde{P} \in \widetilde{U}_j$.We denote the latter one with $\widetilde{P}^*$. Next, we identify the points $\widetilde{P}$ and $\widetilde{P}^*$, if the corresponding transition function is analytic. If $\partial X \neq \emptyset$, then there are two points lying over each boundary point of $X$. Identifying these two points, we obtain a surface $O_2$ which is called the *complex double* of the surface $X$. The two points of $O_2$ which lie over the same point of $X$ are called *symmetric points* of $O_2$. For more details, see Alling and Greenleaf [ALG].

Similar to the orientable case (see [LEH]), it is obtained that the cover group of the double cover $\pi : O_2 \to X$ is generated by an orientation reversing involution. For details, see Seppala and Sorvali [SES].

The next theorem relates a Klein surface to its complex double. We refer to [ALG] for the proof and more details.

**Theorem 1.** *Given a Klein surface $X$, there exist a double cover $\pi : O_2 \to X$ of the Klein surface $X$ by a Riemann surface $O_2$ and an antianalytic involution $k : O_2 \to O_2$, with $\pi \circ k = \pi$, such that $X$ is dianalytically equivalent with $O_2/\langle k\rangle$, where $\langle k\rangle$ is the group generated by $k$. Conversely, given a pair $(O_2, k)$ consisting of a Riemann surface $X$ and an antianalytic involution $k$, the orbit space $O_2/\langle k\rangle$ admits a unique structure of Klein surface, such that $f : O_2 \to O_2/\langle k\rangle$ is a morphism of Klein surfaces, provided that one regards $O_2$ as a Klein surface.*

The mapping $\pi$ is a local homeomorphism at all points $P \in O_2$, for which $\pi(P) \notin \partial X$. At points lying over the boundary of $X$, the mapping $\pi$ is a folding map similar to the mapping $x + iy \to x + i\,|y|$ at the real axes. For more details about the folding map and the morphisms of Klein surfaces, see [ALG] and [ANC].

By Poincaré's uniformization theorem, each compact Riemann surface of algebraic genus $g \geqslant 2$ can be represented as an orbit space $H/\Gamma^+$ of the upper half complex plane $H$. Next, $H$ is endowed with the conformal structure induced by the group $M$ of the Möbius transformations and the acting group $\Gamma^+$ is a Fuchsian group, that is, a discrete subgroup of $M$. The group $\Gamma^+$ can be chosen with no elements of finite order. For details, we refer to Poincaré [POI].

In his unpublished thesis, Preston proved the real counterpart of Poincaré's uniformization theorem: for a Klein surface $X$ of algebraic genus $g \geqslant 2$, there exists a non-euclidean crystallographic ($NEC$ in short) group $\Gamma$, that is, a discrete subgroup of the extended modular group, such that $X$ and $H/\Gamma$ are isomorphic as Klein surfaces. This $NEC$ group can be assumed having no orientation preserving mapping of finite order. For details, see [PRE] and [ROS1].

By Klein's definition, a *symmetric Riemann surface,* $(O_2, k)$, is a Riemann surface $O_2$, together with an orientation reversing involution $k : O_2 \to O_2$. The involution $k$ is called a *symmetry* of $O_2$. For more details about symmetries of a topologic surface, see [SES].

A set $D$ of $O_2$ is called *symmetric* if $k(D) = D$. Thus, given $\Omega$ a subset of $X$, then $\pi^{-1}(\Omega) = D$ is a symmetric subset of $O_2$.

A function $f$ defined on a symmetric set is called a *symmetric function* if it is $k$-invariant, that is, $f = f \circ k$.

Next, we identify $X$ with the orbit space $O_2/\langle k\rangle$ obtained by identifying $P$ with $k(P)$, for all $P \in O_2$. If $\widetilde{U}$ is a parametric disk on $X$, then $\pi^{-1}(\widetilde{U}) = U \cup k(U)$ is a pair of symmetric disks of $O_2$, hence it is natural to consider restrictions on $U \cup k(U)$ for the local study of the objects on $O_2$. Since $k$ is an involution without fixed points, one can suppose that $U \cap k(U) = \emptyset$.

We identify the points of $O_2$, respectively $X$, with their images on $\mathbb{C}$ from the corresponding local parameters, with respect to the relation between the dianalytic atlas on $X$ and the analytic atlases on $O_2$. Let $z$ be the local parameter on $U$. Then $k(z)$ is the local parameter on $k(U)$ and $\widetilde{z} = \widetilde{k(z)} = \pi(z) = \pi(k(z)) = \{z, k(z)\}$ is the local parameter on $\widetilde{U}$.

Let $\mathcal{F}(X)$ be the vector space of the complex functions on the Klein surface $X$ and $\mathcal{F}_s(O_2)$ the vector space of the symmetric functions on $O_2$. By Theorem 1, we conclude that there exists an isomorphism $\pi^* : F(X) \to F_s(O_2)$, between the vector spaces $F(X)$ and $F_s(O_2)$. Indeed, let $F : X \to \overline{\mathbb{C}}$ be a complex function on $X$, that can take the value $\infty$ only on finite sets. Its lifting $f$ to $O_2$ is given by

$$f(z) = f(k(z)) = F(\widetilde{z}),\ z \in O_2,\ \widetilde{z} = \pi(z). \tag{1}$$

Then, it is easy to see that the function $\pi^*$, defined by $\pi^*(F) = f$ is an isomorphism.

Also, to any function $g : O_2 \to \overline{\mathbb{C}}$, we can associate a function $f = g + g \circ k$ which is a symmetric function on $O_2$. Thus, (1) defines a function $F$ on $X$.

Let $\widetilde{\gamma}$ be a piecewise smooth Jordan curve on a parametric disk $\widetilde{U}$. The curve $\widetilde{\gamma}$ has exactly two lifts from $\pi^{-1}(\widetilde{U})$. If $\widetilde{\gamma}(0) = \widetilde{z}_0 = \{z_0, k(z_0)\}$ and if $\gamma$ is the lift of $\widetilde{\gamma}$ on $O_2$ from $z_0$, then $k \circ \gamma$ is the lift of $\widetilde{\gamma}$ on $O_2$ from $k(z_0)$. We refer to [AHL] for details about covering surfaces. By definition of $\gamma$, we obtain $\pi \circ \gamma = \pi \circ k \circ \gamma$, hence for any continuous real-valued function $F$ defined on $\widetilde{\gamma}$, the function $f = F \circ \pi$ is a continuous real-valued symmetric function on $\gamma \cup k(\gamma)$.

The Euclidean lengths of the two curves $\gamma$ and its symmetric $k \circ \gamma$, that is their lengths with respect to the metric $ds = |dz|$, may be different. We modify this metric and get a new metric $d\sigma$ on $O_2$, such that the lengths of $\gamma$ and $k \circ \gamma$, with respect to the metric $d\sigma$, will be the same. We define a symmetric metric on $O_2$ by

$$d\sigma = \frac{1}{2}\left(ds + ds \circ k\right).$$

Then the $d\sigma$-lengths of $\gamma$ and $k \circ \gamma$ are equal. By definition, the length of $\widetilde{\gamma}$ is the common $d\sigma$-length of $\gamma$ and $k \circ \gamma$. Then

$$d\Sigma(\widetilde{z}) = d\sigma(z) = d\sigma(k(z)), \widetilde{z} = \pi(z) \in X$$

is a metric on $X$. The metric $d\Sigma$ is invariant with respect to the group of conformal or anticonformal transition functions of $X$.

By definition,

$$\int_{\widetilde{\gamma}} F d\Sigma = \int_{\gamma} f d\sigma = \int_{k\circ\gamma} f d\sigma.$$

For more details about measure and integration on Klein surfaces, see [BAR].

Any Riemann surface $O_2$ of class $C^1$ is endowed with a *Riemannian metric* determined by the line element

$$ds = \nu \left| dz + \mu d\overline{z} \right| ,$$

where $\nu$ is a positive function. If $\mu$ is identically zero, then the metric

$$ds(z) = \nu(z) \left| dz \right|$$

and the local parameter $z$ are called *isothermal*.

It is known that the isothermal metric $ds$ defines a natural analytic structure on $O_2$. Similar to the orientable case, the isothermal metric $d\sigma$ defines a dianalytic structure on the Klein surface $X$. See [AHS] and [LEH], for details.

Next, we give an example of a symmetric isothermal metric (see Schiffer and Spencer [SCS]).

**Example 1.** *The simplest example of a Klein surface is provided by the Möbius strip. Consider $R > 1$ and the annulus*

$$A_R = \left\{ z \in \mathbb{C} \left| \frac{1}{R} < |z| < R \right. \right\}$$

*of the $z$-plane. The Möbius strip, denoted by $M$, is obtained from $A_R$ by identifying the points $z$ and $-1/\overline{z}$. Let $k : A_R \to A_R$ defined by $k(z) = -1/\overline{z}$. Then $(A_R, k)$ is a symmetric Riemann surface and the quotient space $A_R / \langle k \rangle$ is a Möbius strip. The Möbius strip is obtained by cutting the ring along the real axis in the $z$-plane and joining the two halves together along corresponding boundaries. Thus, the annulus $A_R$ with points $z$ and $-1/\overline{z}$ identified is a canonical form for the Möbius strip. The Euclidean metric*

$$ds = |dz|$$

*is not symmetric. We define a symmetric isothermal metric on $A_R$ by*

$$\begin{aligned} d\sigma &= \frac{1}{2} \left( |dz| + \left| d\left( -\frac{1}{\overline{z}} \right) \right| \right) \\ &= \frac{1}{2} \left( 1 + \frac{1}{|z|^2} \right) |dz| . \end{aligned}$$

*By definition, the metric on the Möbius strip is*

$$d\Sigma(\widetilde{z}) = d\sigma(z) = d\sigma(k(z)$$

*thus,*

$$d\Sigma(\widetilde{z}) = \frac{1}{2} \left( 1 + \frac{1}{|z|^2} \right) |dz| .$$

*The area element $da$ on $A_R$ is*

$$da(z) = \frac{1}{4} \left( 1 + \frac{1}{|z|^2} \right)^2 dm(x, y),$$

*where $m$ is the Lebesgue measure in the complex plane. Then the area element $d\mathcal{A}$ on the Möbius strip is*

$$d\mathcal{A}(\widetilde{z}) = da(z) = da(k(z)),$$

*hence*

$$d\mathcal{A}(\widetilde{z}) = \frac{1}{4} \left( 1 + \frac{1}{|z|^2} \right)^2 dm(x, y).$$

*Let $\gamma : [a,b] \to A_R$ be a piecewise continuously differentiable curve and let $f : \gamma([a,b]) \to \mathbb{C}$ be a continuous function. The integral of $f$ on the curve $\gamma$, denoted by $\int_\gamma f d\sigma$, is defined by*

$$\int_\gamma f d\sigma = \frac{1}{2}\int_a^b f(\gamma(t))\left(1+\frac{1}{|\gamma(t)|^2}\right)|\gamma'(t)|\,dt$$

*and*

$$\int\int_M F d\mathcal{A} = \frac{1}{8}\int\int_{A_R} f(z)\left(1+\frac{1}{|z|^2}\right)^2 dm(x,y).$$

Let $\gamma$ be a $\sigma$-rectifiable Jordan arc $\gamma$, parametrized in terms of the arc $\sigma$-length. Therefore, $\gamma : z = z(s) = x(s)+iy(s)$, $s \in [0,l]$, where $l$ is the $\sigma$-length of $\gamma$. Then the unit inward normal vector to $\gamma$ at $z(s)$ is $n_\sigma = \left(-\frac{dy}{d\sigma}, \frac{dx}{d\sigma}\right)$ and we denote by $\frac{\partial}{\partial n_\sigma}$ the inward normal derivative, with respect to the symmetric metric $d\sigma$. In this way, our approach is consistent with Nevanlinna [NEV2], Bergman [BER] and Schiffer and Spencer [SCS]. For more details about the normal derivative and Green's identities in terms of $d\sigma$, see [BAG1].

## 3. The Dirichlet problem for harmonic functions

This section is devoted to the study of harmonic functions with Dirichlet boundary condition on a Klein surface. The similar analysis in the complex plane has been developed in Krantz [KRA1, Section 1.2].

The notion of harmonic function, as being a solution of the Laplace equation, makes sense on a Klein surface. Moreover, a Klein surface is the most general two-manifold in which this notion of harmonic function makes sense. For details, see [ALG]. We notice that the notion of analytic function is meaningless on a Klein surface.

The Dirichlet problem on an arbitrary Riemann surface can be solved because the property that a function that is harmonic remains invariant under bi-holomorphic mappings. For the existence of a harmonic function which vanishes on the boundary and has a finite number of isolated singularities with given singular parts in a relatively compact region, which is contained in a chart of a Riemann surface, we refer to Ahlfors and Sario [AHS].

Any Klein surface $X$ can be regularly imbedded in a border free surface using a duplication process (see [AHS]). Therefore, for the boundary problems involving a part of $\partial X$ we can consider it as a part of the boundary of a region on a border free surface.

Let $O_2$ be a region in the complex plane, bounded by a finite number of analytic Jordan curves. Then $\overline{O_2} = O_2 \cup \partial O_2$ can be conceived as a bordered Riemann surface (see [AHL], [SCS]). Because the Klein surfaces $X$ and $O_2/\langle k\rangle$ are dianalytically equivalent, a boundary value problem on a region $\Omega$ of the Klein surface $X$, can be replaced by a similar problem on a symmetric region $D$ of its double $O_2$, as follows.

Consider the Dirichlet problem on $X$ for harmonic functions

$$\begin{cases} \Delta U = 0 \text{ on } \Omega \\ U = F \text{ on } \partial\Omega. \end{cases} \tag{2}$$

where $\Omega$ is a region of $X$ bounded by a finite number of $\sigma$- rectifiable Jordan curves and $F$ is a continuous real-valued function on $\partial\Omega$.

We define $D = \pi^{-1}(\Omega)$ and $f = F \circ \pi$ on $\partial D$. Then $D$ is a symmetric region of $O_2$, bounded by a finite number of $\sigma$-rectifiable Jordan curves on $O_2$, some of which may contain part of $\partial O_2$. Since $\pi \circ k = \pi$, we obtain $f = f \circ k$ on $\partial D$, hence $f$ is a symmetric, continuous real-valued function on $\partial D$. The Dirichlet problem (2) on $X$ is equivalent with the following Dirichlet problem for harmonic functions on $O_2$

$$\begin{cases} \Delta u = 0 \text{ on } D \\ u = f \text{ on } \partial D. \end{cases} \tag{3}$$

For details about the Dirichlet problem on bordered Riemann surfaces, see Ahlfors and Sario [AHS].

The Dirichlet problem turned out to be fundamental in many areas of mathematics and physics. For example, if $D$ is a thin, heat-conducting metal plate and $f$ is a continuous temperature distribution on $\partial D$, then the solution $u$ of problem (3) represents the resulting steady-state heat distribution on $D$ (see [COH], [KRA2]).

Using the maximum principle for harmonic functions, it follows that the Dirichlet problem (3) with continuous boundary values has a unique solution for any region $D$ with only regular points. For some basic monotonicity, analytic and variational methods of the theory of partial differential equations of elliptic type, we refer to [RAD].

The symmetric conditions on the boundary imply symmetric solutions for the problem (3). For more details, see Schiffer and Spencer [SCS].

**Proposition 2.** *A solution $u$ of the problem* (3) *is a symmetric function on $D$.*

*Proof.* Let $u$ be a solution of the problem (3). We define $\widetilde{u} : \overline{D} \to \mathbb{R}$ by $\widetilde{u} = \frac{1}{2}(u + u \circ k)$. Then $\Delta\widetilde{u} = 0$ on $D$. By hypothesis, $f = f \circ k$ on $\partial D$, hence

$$\widetilde{u} = \frac{1}{2}(f + f \circ k) = f \quad \text{on } \partial D.$$

Thus, $\widetilde{u}$ is also a solution of the problem (3). The uniqueness of the solution yields $\widetilde{u} = u$ on $D$, therefore $u = u \circ k$ on $D$. □

3.1. **The symmetric Green function.** Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves on the symmetric Riemann surface $O_2$.

Fix a point $\zeta \in D$. The function $\upsilon(z, \zeta) = -\ln|z - \zeta|$ is harmonic at all points $z \neq \zeta$. Let $w$ be the solution of the Dirichlet problem on $D$, with the boundary condition $w(z) = \upsilon(z, \zeta)$ on $\partial D$. The unique function $G_D(z, \zeta) = \upsilon(z, \zeta) - w(z)$ defined on $\overline{D} \setminus \{\zeta\}$ is called the *Green function* of the region $D$, with singularity at $\zeta$ (see [AHS]).

We assume that $u$ and $v$ are continuously twice differentiable in $D$ and once on the boundary $\partial D$. We will use the following Green formula:

$$\int_{\partial D} \left( u\frac{\partial v}{\partial n_\sigma} - v\frac{\partial u}{\partial n_\sigma} \right) d\sigma = -\int\int_{D} (u\Delta v - v\ \Delta u)\, dxdy,$$

where $d\sigma$ is the arc $\sigma$-length element on $\partial D$ and the derivatives on the left are taken with respect to the inward normal on $\partial D$. For more details, see [NEV2].

The next theorem is similar to the Cauchy integral formula for harmonic functions in terms of the metric $d\sigma$.

**Proposition 3.** *(Green representation formula) Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves and let $u$ be a harmonic function in $D$ and continuously differentiable on its boundary $\partial D$. Then, for all $\zeta$ in $D$,*

$$u(\zeta) = \frac{1}{2\pi} \int\limits_{\partial D} \left( u(z) \frac{\partial \upsilon(z,\zeta)}{\partial n_\sigma} - \upsilon(z,\zeta) \frac{\partial u(z)}{\partial n_\sigma} \right) d\sigma, \tag{4}$$

*where the derivatives are taken with respect to the inward normal on $\partial D$.*

*Proof.* Fix a point $\zeta \in D$ and a positive number $\varepsilon$ that is less than the Euclidean distance of $\zeta$ to $\partial D$. Define $D_\varepsilon = D \setminus \overline{D}(\zeta, \varepsilon)$. Let $C_\varepsilon$ be the negatively oriented circle of radius $\varepsilon$, centered at $\zeta$. We apply the Green formula for $D_\varepsilon$, with the harmonic functions $u$ and $\upsilon$. It follows that

$$\int\limits_{\partial D} \left( u \frac{\partial \upsilon}{\partial n_\sigma} - \upsilon \frac{\partial u}{\partial n_\sigma} \right) d\sigma = - \int\limits_{C_\varepsilon} \left( u \frac{\partial \upsilon}{\partial n_\sigma} - \upsilon \frac{\partial u}{\partial n_\sigma} \right) d\sigma. \tag{5}$$

The curve $-C_\varepsilon$ is parameterized by $z = z(\theta) = \zeta + \varepsilon e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. We deduce that

$$- \int\limits_{C_\varepsilon} \upsilon \frac{\partial u}{\partial n_\sigma} d\sigma = \int\limits_{-C_\varepsilon} \upsilon \frac{\partial u}{\partial n_\sigma} d\sigma = -\varepsilon \int\limits_0^{2\pi} \upsilon(z(\theta), \zeta) \frac{\partial u(z(\theta))}{\partial \rho} d\theta.$$

As the function $u$ has continuous partial derivatives in $D$, there is a constant $C$ such that $\left| \frac{\partial u}{\partial \rho} \right| \leqslant C$ on $C_\varepsilon$. Then, on $C_\varepsilon$, we obtain

$$\left| \int\limits_{-C_\varepsilon} \upsilon \frac{\partial u}{\partial n_\sigma} d\sigma \right| \leqslant 2\pi C \varepsilon \left| \ln \varepsilon \right|.$$

We observe that the right-hand side of the last inequality tends to zero as $\varepsilon$ tends to zero. Therefore

$$\lim_{\varepsilon \to 0} \int\limits_{-C_\varepsilon} \upsilon \frac{\partial u}{\partial n_\sigma} d\sigma = 0.$$

Using the mean value property, we have

$$\int\limits_{C_\varepsilon} u \frac{\partial \upsilon}{\partial n_\sigma} d\sigma = - \int\limits_0^{2\pi} u(z(\theta)) d\theta = -2\pi u(\zeta).$$

Then relation (5) becomes

$$\int\limits_{\partial D} \left( u \frac{\partial \upsilon}{\partial n_\sigma} - \upsilon \frac{\partial u}{\partial n_\sigma} \right) d\sigma = 2\pi u(\zeta).$$

The proof is now complete. □

Following Nevannlina (see [NEV2]), we obtain that the values of $u$ inside $D$ are determined from its values and the values of the normal derivative of the Green function on the boundary $\partial D$.

**Theorem 4.** *Let $D$ be a symmetric region, whose boundary $\partial D$ consists of a finite number of $\sigma$-rectifiable Jordan curves. If $u$ is harmonic on $D$ and continuously differentiable on $\partial D$, then for all $\zeta$ in $D$,*

$$u(\zeta) = \frac{1}{2\pi} \int\limits_{\partial D} u(z) \frac{\partial G_D(z,\zeta)}{\partial n_\sigma} d\sigma. \tag{6}$$

*Proof.* Applying Green's formula for $D$ with the harmonic functions $u$ and $w$, we obtain

$$\int\limits_{\partial D} \left( \upsilon \frac{\partial u}{\partial n_\sigma} - u \frac{\partial w}{\partial n_\sigma} \right) d\sigma = 0. \tag{7}$$

Dividing (7) by $2\pi$, and adding this identity to the Green representation formula, we obtain (6). ◻

The function

$$P_\zeta(z) = \frac{1}{2\pi} \frac{\partial G_D(z,\zeta)}{\partial n_\sigma}$$

is called the *Poisson kernel* of the Laplace operator and the Dirichlet problem on the region $D$.

We define $G_D^{(k)}(z,\widetilde{\zeta})$ as

$$G_D^{(k)}(z,\widetilde{\zeta}) = \frac{1}{2} \left[ G_D(z,\zeta) + G_D(z,k(\zeta)) \right]$$

on $\overline{D} \setminus \{\zeta, k(\zeta)\}$.

Let $w_s$ be the solution of the Dirichlet problem on $D$, with the boundary condition $w_s(z) = \frac{1}{2} \left[ \upsilon(z,\zeta) + \upsilon(z,k(\zeta)) \right]$ on $\partial D$. Then

$$G_D^{(k)}(z,\widetilde{\zeta}) = \frac{1}{2} \left[ \upsilon(z,\zeta) + \upsilon(z,k(\zeta)) \right] - w_s(z).$$

Therefore, $G_D^{(k)}(z,\widetilde{\zeta})$ is a harmonic function of $z$ in $D \setminus \{\zeta, k(\zeta)\}$, with singularities $-\frac{1}{2} \ln |z-\zeta|$ and $-\frac{1}{2} \ln |z-k(\zeta)|$ at $\zeta$ and $k(\zeta)$, respectively. Also, $G_D^{(k)}(z,\widetilde{\zeta}) = 0$ for all $z$ on $\partial D$.

We can derive the following result (see [BAG1]):

**Proposition 5.** *For every symmetric region $D$, the function $G_D^{(k)}(\cdot,\widetilde{\zeta})$ is symmetric on $\overline{D}$, that is, for all $z \in \overline{D}$,*

$$G_D^{(k)}(z,\widetilde{\zeta}) = G_D^{(k)}(k(z),\widetilde{\zeta}).$$

Consequently, the function $G_D^{(k)}(z,\widetilde{\zeta})$ is called the *symmetric Green function* of the region $D$, with singularities at $\zeta$ and $k(\zeta)$.

An explicit form for the symmetric Green function of the annulus is obtained in [BAG1]. For additional information on this topic we refer to [SCS].

3.2. **The symmetric harmonic measure.** Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves on $O_2$ and $\mathcal{B}(\partial D)$ the $\sigma$-algebra of Borel sets of $\partial D$. The $\sigma$-algebra of symmetric Borel sets of $\partial D$ is denoted by $\mathcal{B}_s(\partial D)$ and $\mathcal{B}_s(\partial D) = \{U \cup k(U) \,|U \in \mathcal{B}(\partial D)\}$.

The *harmonic measure* for $D$ is a function $\omega_D : D \times \mathcal{B}_s(\partial D) \to [0,1]$ such that:

(1) for each $\zeta \in D$, the map $B \mapsto \omega_D(\zeta, B)$ is a Borel probability measure on $\partial D$;

(2) if $f : \partial D \to \mathbb{R}$ is a continuous function, then the solution of the Dirichlet problem, for $D$ and the boundary function $f$, is the generalized Poisson integral of $f$ on $D$ given by

$$P_D f(\zeta) = \int_{\partial D} f(z) d\omega_D(\zeta, z), \zeta \in D. \tag{8}$$

For details, see [RAN].

**Remark 1.** *The uniqueness of $\omega_D$ is a consequence of the Riesz representation theorem.*

An extensive study of the harmonic measure is developed in [GAM].

A method of determining the harmonic measure is given by the following characterization (see [RAN]):

**Proposition 6.** *The function $\omega_D(\cdot, B)$, is the solution of the generalized Dirichlet problem with boundary function $f = 1_B$.*

The harmonic measure for $D$ is related to another conformal invariant, the Green function for the symmetric region $D$.

Using Theorem 4 and the fact that Borel measures are determined by their actions on continuous functions, we obtain a representation of the harmonic measure in terms of the inward normal derivative of the Green function with respect to $d\sigma$.

**Proposition 7.** *Let $D$ be a symmetric region, whose boundary $\partial D$ consists of a finite number of $\sigma$-rectifiable Jordan curves. If $\zeta \in D$, then for any $z \in \partial D$,*

$$d\omega_D(\zeta, z) = \frac{\partial G_D(z, \zeta)}{\partial n_\sigma} \cdot \frac{d\sigma(z)}{2\pi}.$$

Thus, the harmonic measure for $\zeta \in D$ is absolutely continuous to arc $\sigma$-length on $\partial D$ and on $\partial D$, the density being

$$\frac{d\omega_D}{d\sigma} = \frac{1}{2\pi} \frac{\partial G_D(z, \zeta)}{\partial n_\sigma} = P_\zeta(z).$$

Let $\omega_D^{(k)} : D \times \mathcal{B}_s(\partial D) \to [0, 1]$ be the function defined by

$$\omega_D^{(k)}(\widetilde{\zeta}, B) = \frac{1}{2} \left[ \omega_D(\zeta, B) + \omega_D(k(\zeta), B) \right],$$

where $\widetilde{\zeta} = \{\zeta, k(\zeta)\}$, $\zeta \in D$, $B \in \mathcal{B}_s(\partial D)$; see [ROS2].

**Remark 2.** *The symmetry of the region $D$ implies that the function $\omega_D^{(k)}(\widetilde{\zeta}, B)$ is symmetric with respect to $B$ on $\mathcal{B}_s(\partial D)$, that is, for any $B \in \mathcal{B}_s(\partial D)$,*

$$\omega_D^{(k)}(\widetilde{\zeta}, B) = \omega_D^{(k)}(\widetilde{\zeta}, k(B)).$$

The function $\omega_D^{(k)}(\widetilde{\zeta}, B)$ is called the *symmetric harmonic measure* for $D$.

The function

$$P_{\widetilde{\zeta}}^{(k)}(z) = \frac{1}{2\pi} \frac{\partial G_D^{(k)}(z, \widetilde{\zeta})}{\partial n_\sigma}, \ z \in D$$

is called the *symmetric Poisson kernel* for the region $D$.

3.3. **The Dirichlet problem on the complex double.** The following Poisson integral formula both reproduces and creates harmonic functions on the complex double. Roughly speaking, the next theorem yields the formula for the solution of the Dirichlet problem (3) on a symmetric region $D$, in terms of the symmetric Green function.

**Theorem 8.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves and let $f$ be a symmetric, continuous function on $\partial D$. There is a unique symmetric function $u$ on $\overline{D}$, which is harmonic in $D$, continuous on $\overline{D}$, such that $u = f$ on $\partial D$. Moreover, for all $\zeta$ in $D$,*

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial D} f(z)\frac{\partial G_D^{(k)}(z,\zeta)}{\partial n_\sigma}d\sigma,\ \zeta \in D. \tag{9}$$

*Proof.* By Theorem 4, for all $\zeta \in D$,

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial G_D(z,\zeta)}{\partial n_\sigma}d\sigma.$$

Replacing $\zeta$ with $k(\zeta)$ we obtain

$$u(k(\zeta)) = \frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial G_D(z,k(\zeta))}{\partial n_\sigma}d\sigma.$$

Adding the last two equations and dividing by 2, we obtain

$$\frac{u(\zeta)+u(k(\zeta))}{2} = \frac{1}{4\pi}\int_{\partial D} u(z)\left[\frac{\partial G_D(z,\zeta)}{\partial n_\sigma} + \frac{\partial G_D(z,k(\zeta))}{\partial n_\sigma}\right]d\sigma,$$

for all $\zeta \in D$. By Proposition 2, $u$ is a symmetric function on $D$, then the left-hand side of the last equality is $u(\zeta)$. We conclude that for all $\zeta$ in $D$,

$$u(\zeta) = \frac{1}{4\pi}\int_{\partial D} u(z)\left[\frac{\partial G_D(z,\zeta)}{\partial n_\sigma} + \frac{\partial G_D(z,k(\zeta))}{\partial n_\sigma}\right]d\sigma.$$

The uniqueness of the solution of the Dirichlet problem for harmonic functions implies relation (9). □

Theorem 8 is the equivalent of the Poisson formula for the solution of the Dirichlet problem on the disc in the complex plane, see Krantz [KRA5, Section 7.3]. In such a way, Theorem 8 creates a function that agrees with $f$ on the boundary of the domain $D$ and is harmonic inside.

The formula for the solution to the Dirichlet problem on the annulus is obtained in [BAG2].

In a similar way, we obtain the following representation of the solution of the problem (3) on a symmetric region $D$, in terms of the symmetric harmonic measure.

**Theorem 9.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves and let $f$ be a symmetric, continuous function on $\partial D$. There exists a unique symmetric function $u$ on $\overline{D}$, which is harmonic on $D$, continuous on $\overline{D}$, such that $u = f$ on $\partial D$. For all $\zeta$ in $D$, we have*

$$u(\zeta) = \int_{\partial D} f(z)d\omega_D^{(k)}(\widetilde{\zeta},z). \tag{10}$$

*Proof.* Let $\zeta$ be a point in $D$. By (8), for all $\zeta \in D$,

$$u(\zeta) = \int_{\partial D} f(z) d\omega(\zeta, z) d\sigma.$$

Replacing $\zeta$ with $k(\zeta)$ we get

$$u(k(\zeta)) = \int_{\partial D} u(z) d\omega(k(\zeta), z) d\sigma.$$

Adding the last two equations and dividing by 2, we obtain

$$\frac{u(\zeta) + u(k(\zeta))}{2} = \frac{1}{2} \int_{\partial D} f(z) \left[ d\omega(\zeta, z) + d\omega(k(\zeta), z) \right],$$

for all $\zeta$ in $D$. By Proposition 2, $u$ is a symmetric function on $D$, then the left-hand side of the last equality is $u(\zeta)$ and we conclude that for all $\zeta$ in $D$,

$$u(\zeta) = \int_{\partial D} f(z) d\omega_D^{(k)}(\widetilde{\zeta}, z).$$

The proof is now complete. □

By Proposition 7, we obtain the Radon-Nikodym derivative of symmetric harmonic measure for $D$ against $\sigma$-arc length.

**Proposition 10.** *Let $D$ be a symmetric region whose boundary $\partial D$ consists of a finite number of $\sigma$-rectifiable Jordan curves. If $\zeta \in D$, then for any $z \in \partial D$,*

$$d\omega_D^{(k)}(\widetilde{\zeta}, z) = \frac{\partial G_D^{(k)}(z, \widetilde{\zeta})}{\partial n_\sigma} \cdot \frac{d\sigma(z)}{2\pi}.$$

This result shows that the symmetric harmonic measure for $D$ is absolutely continuous to arc $\sigma$-length on $\partial D$ and on $\partial D$, the density being

$$\frac{d\omega_D^{(k)}}{d\sigma} = \frac{1}{2\pi} \frac{\partial G_D^{(k)}(z, \widetilde{\zeta})}{\partial n_\sigma} = P_{\widetilde{\zeta}}^{(k)}(z).$$

3.4. **The Dirichlet problem on the Klein surface.** Let $X$ be a Klein surface and let $\Omega$ be a region bounded by a finite number of $\sigma$-rectifiable Jordan curves. The Klein surface $X$ is the factor manifold of the symmetric Riemann surface $O_2$ with respect to the group $\langle k \rangle$. Then, $\Omega$ is obtained from the symmetric region $D$ by identifying the corresponding symmetric points.

The Green function of $\Omega$ with singularity at $\widetilde{\zeta}$ is defined by

$$G_\Omega(\widetilde{z}, \widetilde{\zeta}) = G_D^{(k)}(z, \widetilde{\zeta}) = G_D^{(k)}(k(z), \widetilde{\zeta}),$$

where $\widetilde{z} = \pi(z)$.

By definition, the function $G_\Omega(\widetilde{z}, \widetilde{\zeta})$ is continuous on $\overline{\Omega}$, harmonic on $\Omega \setminus \left\{ \widetilde{\zeta} \right\}$ and has the singularity at $\widetilde{\zeta} = \pi(\zeta)$.

**Remark 3.** *By Proposition 5, it follows that $G_\Omega(\widetilde{z}, \widetilde{\zeta})$ is well-defined on $\Omega$.*

An explicit form for the Green function of the Möbius strip is obtained in [BAG1].

The harmonic measure for $\Omega$, $\omega_\Omega : \Omega \times \mathcal{B}(\partial\Omega) \to [0,1]$, is defined by

$$\omega_\Omega(\widetilde{\zeta}, \widetilde{B}) = \omega_D^{(k)}(\widetilde{\zeta}, B) = \omega_D^{(k)}(\widetilde{\zeta}, k(B)).$$

for all $\widetilde{\zeta} \in \Omega$ and $\widetilde{B} = \pi(B) \in \mathcal{B}(\partial\Omega)$.

The function

$$P_{\widetilde{\zeta}}(\widetilde{z}) = P_{\widetilde{\zeta}}^{(k)}(z) = P_{\widetilde{\zeta}}^{(k)}(k(z)),\ z \in D$$

is called the *Poisson kernel* for the region $\Omega$.

**Remark 4.** *By Remark 2, it follows that the function $\omega_\Omega$ is well-defined. By Proposition 5, it follows that the function $P_{\widetilde{\zeta}}$ is well-defined, too.*

The symmetric solutions on $O_2$ determine the solutions of the similar problems on the Klein surface $X$.

Consequently, we obtain the solution of the Dirichlet problem on the region $\Omega$, with respect to the Green function of $\Omega$.

**Theorem 11.** *Let $F$ be a continuous real-valued function on the border $\partial\Omega$. The solution of the Dirichlet problem (2) with the boundary function $F$ is the function $U$ defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where $\pi$ is the canonical projection of $O_2$ on $X$ and $u$ is the solution (9) of the Dirichlet problem (3) on the symmetric region $D$, with the boundary function $f$, given by $f = F \circ \pi$.*

*Proof.* By definition, $\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0$, for all $\widetilde{\zeta} \in \Omega$, where $\widetilde{\zeta} = \pi(\zeta)$. Thus, $U$ is a harmonic function on $\Omega$. The symmetry of the function $f$ on $\partial D$ implies

$$U(\widetilde{\zeta}) = u(\zeta) = f(\zeta) = f(k(\zeta)) = F(\widetilde{\zeta}),$$

for all $\widetilde{\zeta} \in \partial\Omega$. Due to the uniqueness of the solution, the function $U$ defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

for all $\widetilde{\zeta}$ in $\overline{\Omega}$, where $\widetilde{\zeta} = \pi(\zeta)$, is the solution of the Dirichlet problem (2) on $\Omega$. □

In a similar way, we obtain the solution of the problem (2) on the region $\Omega$, with respect to the harmonic measure for the region $\Omega$.

**Theorem 12.** *Let $F$ be a continuous real-valued function on the border $\partial\Omega$. The solution of the problem (2) with the boundary function $F$ is the function $U$ defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where $\pi$ is the canonical projection of $O_2$ on $X$ and $u$ is the solution (10) of the problem (3) on the symmetric region $D$, with the boundary function $f$, given by $f = F \circ \pi$.*

By Proposition 10, we obtain the Radon-Nikodym derivative of harmonic measure for $\Omega$ against $\Sigma$-arc length.

**Proposition 13.** *Let $\Omega$ be a region bounded by a finite number of $\sigma$-rectifiable Jordan curves. If $\widetilde{\zeta} \in \Omega$, then for all $\widetilde{z} \in \partial\Omega$,*

$$d\omega_\Omega(\widetilde{\zeta}, \widetilde{z}) = d\omega_D^{(k)}(\widetilde{\zeta}, z) = d\omega_D^{(k)}(\widetilde{\zeta}, k(z)).$$

This result implies that the harmonic measure for $\Omega$ is absolutely continuous to arc $\Sigma$-length on $\partial\Omega$ and on $\partial\Omega$, the density being

$$\frac{d\omega_\Omega}{d\Sigma} = P_{\widetilde{\zeta}}(\widetilde{z}).$$

## 4. The Neumann problem for harmonic functions

This section is devoted to the study of harmonic functions with Neumann boundary condition on a Klein surface. The similar analysis in the complex place has been developed by Schiffer and Spencer [SCS].

Consider the Neumann problem for harmonic functions

$$\begin{cases} \Delta U = 0 \text{ on } \Omega \\ \dfrac{\partial U}{\partial n_\Sigma} = G \text{ on } \partial\Omega, \end{cases} \tag{11}$$

where $\Omega$ is a region of $X$ bounded by a finite number of $\sigma$-rectifiable Jordan curves and $G$ is a continuous real-valued function on $\partial\Omega$.

We define $D = \pi^{-1}(\Omega)$ and $g = G \circ \pi$ on $\partial D$. Since $\pi \circ k = \pi$, we obtain that $D$ is a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves on $O_2$, some of which may contain part of $\partial O_2$ and $g$ is a symmetric, continuous real-valued function on the boundary $\partial D$.

The Neumann problem on $X$ is equivalent with the following Neumann problem on $O_2$

$$\begin{cases} \Delta u = 0 \text{ on } D \\ \dfrac{\partial u}{\partial n_\sigma} = g \text{ on } \partial D. \end{cases} \tag{12}$$

Since $k$ is an antianalytic involution, the symmetry of $D$ and the symmetry of $g$ on $\partial D$, imply that the prescribed values of the normal derivative satisfy the compatibility condition

$$\int\limits_{\partial D} g d\sigma = 0.$$

Therefore the Neumann problem on $O_2$ for the region $D$ and the boundary function $g$ has solutions. For details, see [NEV1].

**Proposition 14.** *If the problem* (12) *admits a solution, then it is unique up to an additive constant.*

*Proof.* Let $u_1$ and $u_2$ be solutions of the problem (12). If $u = u_1 - u_2$, then $u$ is harmonic on $D$ and $\dfrac{\partial u}{\partial n_\sigma} = 0$ on $\partial D$. Applying Green's first identity, we get

$$\int\int\limits_{D} \left(u_x^2 + u_y^2\right) dxdy = 0.$$

Therefore $u$ is constant on $D$. □

**Proposition 15.** *The solution of the problem* (12) *is a symmetric function on $D$.*

*Proof.* Let $u$ be a solution of the problem (12). We define $\widetilde{u} : \overline{D} \to \mathbb{R}$ by $\widetilde{u} = \dfrac{1}{2}(u + u \circ k)$. By hypothesis $g = g \circ k$ on $\partial D$, then $\dfrac{\partial \widetilde{u}}{\partial n_\sigma} = \dfrac{\partial u}{\partial n_\sigma} = g$ on $\partial D$ and $\Delta\widetilde{u} = 0$ on $D$. Thus, $\widetilde{u}$ is also a solution of the problem (12). By Proposition 14, there is a constant $c$ such that $\widetilde{u} = u + c$ on $D$. Thus $u \circ k = u + 2c$ on $D$ and using the symmetry of the region $D$, we obtain $u = u \circ k + 2c$ on $D$. Hence $c = 0$, that is, $u \circ k = u$ on $D$. □

4.1. **The symmetric Neumann function.** Let $\zeta$ be a point inside $D$. A *Neumann function* $N_D(z,\zeta)$ for the region $D$, with singularity at $\zeta$, in terms of the metric $d\sigma$, is the function

$$N_D(z,\zeta) = \upsilon(z,\zeta) - h(z,\zeta),\ z \in D,\ z \neq \zeta,$$

where $h(z,\zeta)$ is a solution of the following Neumann problem in terms of the metric $d\sigma$:

$$\begin{cases} \Delta h(z,\zeta) = 0\ , & z \in D \\ \dfrac{\partial h}{\partial n_\sigma}(z,\zeta) = \dfrac{\partial \upsilon}{\partial n_\sigma}(z,\zeta) - \dfrac{2\pi}{l}, & z \in \partial D, \end{cases}$$

where $l = \int\limits_{\partial D} d\sigma$ is the $\sigma$-length of $\partial D$.

**Remark 5.** *The boundary value of the inward normal derivative of the Neumann function is a constant equal to* $\frac{2\pi}{l}$.

**Theorem 16.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves. If $u$ is harmonic in $D$ and continuously differentiable on $\partial D$ then, up to an additive constant,*

$$u(\zeta) = -\frac{1}{2\pi}\int\limits_{\partial D} \frac{\partial u(z)}{\partial n_\sigma} N_D(z,\zeta) d\sigma,\ \zeta \in D.$$

*Proof.* Fix a point $\zeta \in D$ and a positive number $\varepsilon$ that is less than the Euclidean distance of $\zeta$ to $\partial D$. Define $D_\varepsilon = D \setminus \overline{D}(\zeta,\varepsilon)$. Let $C_\varepsilon$ be the negatively oriented circle of radius $\varepsilon$, centered at $\zeta$. Applying Green formula for $D_\varepsilon$ with the harmonic functions $h$ and $u$, we obtain

$$\int\limits_{\partial D} \left( h\frac{\partial u}{\partial n_\sigma} - u\frac{\partial h}{\partial n_\sigma} \right) d\sigma = 0. \tag{13}$$

Dividing (13) by $2\pi$ and adding this identity to the Green representation formula, it follows that

$$u(\zeta) = -\frac{1}{2\pi}\int\limits_{\partial D} N(z,\zeta)\frac{\partial u}{\partial n_\sigma} d\sigma + \frac{1}{l}\int\limits_{\partial D} u d\sigma.$$

Thus, $u$ is determined up to the additive constant $\frac{1}{l}\int\limits_{\partial D} u(z)d\sigma$. □

Let $N_D^{(k)}(z,\widetilde{\zeta})$ be the function defined by

$$N_D^{(k)}(z,\widetilde{\zeta}) = \frac{1}{2}\left[N_D(z,\zeta) + N_D(z,k(\zeta))\right],\ z \in D \setminus \{\zeta, k(\zeta)\},$$

where $N_D(z,k(\zeta))$ is a Neumann function for the region $D$, with singularity at $k(\zeta)$ and $\widetilde{\zeta} = \{\zeta, k(\zeta)\}$. Therefore

$$N_D^{(k)}(z,\widetilde{\zeta}) = \frac{1}{2}\left[\upsilon(z,\zeta) + \upsilon(z,k(\zeta))\right] - h_s(z,\widetilde{\zeta}),\ z \neq \zeta,\ z \neq k(\zeta),$$

where $h_s$ is a harmonic function on $D$ that satisfies

$$\frac{\partial h_s}{\partial n_\sigma}(z,\widetilde{\zeta}) = \frac{1}{2}\left[\frac{\partial \upsilon}{\partial n_\sigma}(z,\zeta) + \frac{\partial \upsilon}{\partial n_\sigma}(z,k(\zeta))\right] - \frac{2\pi}{l}.$$

Therefore, $N_D^{(k)}(z,\widetilde{\zeta})$ is a harmonic function of $z$ in $D\setminus\{\zeta,k(\zeta)\}$, with singularities at $\zeta$ and $k(\zeta)$ and $\dfrac{\partial N_D^{(k)}}{\partial n_\sigma}(z,\widetilde{\zeta})=\dfrac{2\pi}{l}$, for all $z\in\partial D$.

An explicit form for the function $N_D^{(k)}(z,\widetilde{\zeta})$ of the annulus and of the Möbius strip are obtained in [ROS3].

**Proposition 17.** *If $D$ is a symmetric region, then the function $N_D^{(k)}(z,\widetilde{\zeta})$ is symmetric with respect to $z$ on $D$, that is, for any $z\in D$,*

$$N_D^{(k)}(z,\widetilde{\zeta})=N_D^{(k)}(k(z),\widetilde{\zeta}).$$

*Proof.* Let $h^*(\cdot,\zeta)$ be a harmonic function in $D$, such that

$$\frac{\partial h^*}{\partial n_\sigma}(z,\zeta)=\frac{1}{2}\left(\frac{\partial \upsilon}{\partial n_\sigma}(z,\zeta)+\frac{\partial \upsilon}{\partial n_\sigma}(k(z),\zeta)\right)-\frac{2\pi}{l},\ z\in\partial D.$$

Therefore

$$\frac{\partial h^*}{\partial n_\sigma}(z,\zeta)=\frac{\partial h^*}{\partial n_\sigma}(k(z),\zeta),\quad \text{for all } z\in\partial D.$$

By Proposition 15, $h^*(\cdot,\zeta)$ is a symmetric function. Hence the function

$$M_D^{(k)}(z,\widetilde{\zeta})=\frac{1}{2}\left[\upsilon(z,\zeta)+\upsilon(k(z),\zeta)\right]-h^*(z;\zeta)$$

is a symmetric function, harmonic in $D\setminus\left\{\widetilde{\zeta}\right\}$ and $\dfrac{\partial M_D^{(k)}}{\partial n}(z,\widetilde{\zeta})=\dfrac{2\pi}{l}$. So, $N_D^{(k)}(z,\widetilde{\zeta})$ and $M_D^{(k)}(z,\widetilde{\zeta})$ are solutions of the same Neumann problem. Thus, by Proposition 14, there is a constant $c$ such that $N_D^{(k)}(z,\widetilde{\zeta})=M_D^{(k)}(z,\widetilde{\zeta})+c$. Since $M_D^{(k)}(z,\widetilde{\zeta})$ is a symmetric function, we obtain that $N_D^{(k)}(z,\widetilde{\zeta})$ is also a symmetric function. □

Let $\zeta_0$ be a point of $D$. A Neumann function $N_D(z,\zeta)$ is not a conformal invariant, but the difference $N_D(z,\zeta)-N_D(z,\zeta_0)$ is a Neumann function and has a vanishing normal derivative on $\partial D$, hence it is a conformal invariant. We redefine the difference

$$N_D(z,\zeta,\zeta_0)=N_D(z,\zeta)-N_D(z,\zeta_0)$$

to be a *Neumann function* for the region $D$ on the Riemann surface $O_2$, see [SCS].

The function $N_D^{(k)}(z,\widetilde{\zeta},\widetilde{\zeta_0})$ defined by

$$N_D^{(k)}(z,\widetilde{\zeta},\widetilde{\zeta_0})=\frac{1}{2}\left[N_D(z,\zeta,\zeta_0)+N_D(z,k(\zeta),k\left(\zeta_0\right))\right],$$

for all $z\in\overline{D}\setminus\{\widetilde{\zeta},\widetilde{\zeta_0}\}$, is called *a symmetric Neumann function* for the region $D$.

4.2. **The symmetric harmonic kernel function.** Let $D$ be a symmetric region in the complex plane, bounded by a finite number of $\sigma$-rectifiable Jordan curves. In this section, we introduce closed systems $(\varphi_i)_i$ of harmonic functions in $D$, which are orthonormal with respect to the Dirichlet integral

$$D\{\varphi_i,\varphi_j\}=\int\!\!\int_D\left(\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial y}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial x}\right)dxdy.$$

We recall some notions and results about orthogonal harmonic functions. For more details, see [BER].

Let $\Lambda^2(D)$ be the set of harmonic functions $\varphi$ in $D$ with a finite Dirichlet integral

$$D\{\varphi\}=D\{\varphi,\varphi\}<\infty$$

such that

$$D\{N_D(z,\zeta),\varphi(\zeta)\} = 2\pi\varphi(\zeta),$$

where $N_D(z,\zeta)$ is the Neumann function of $D$ with its singularity at the fixed point $\zeta$, $\zeta \in D$.

**Remark 6.** *The second condition is imposed to normalize $D\{\varphi,\varphi\}$ to be zero if and only if $\varphi$ vanishes identically.*

**Proposition 18.** *There exists a closed system $(\varphi_i)_i$ for the class $\Lambda^2(D)$, which is orthonormal with respect to the Dirichlet integral, that is,*

$$D\{\varphi_i,\varphi_j\} = \delta_{ij},\ \ \delta_{ii} = 1,\ \ \delta_{ij} = 0,\ \ i \neq j.$$

Let $\zeta$ be a point inside $D$. The *harmonic kernel function* $K_D(z,\zeta)$ of the closed orthonormal system $(\varphi_i)_i$ for the region $D$, with respect to the point $\zeta$, is the function defined by

$$K_D(z,\zeta) = \sum_{i=1}^{\infty}\varphi_i(z)\varphi_i(\zeta),\ \ z \in \overline{D}.$$

The harmonic kernel function is uniquely characterized by the following properties:

$$K_D(z,\zeta) = K_D(\zeta,z)$$

and

$$D\{K_D(z,\zeta),\varphi(\zeta)\} = \varphi(\zeta),\ \ \varphi \in \Lambda^2(D).$$

An extensive study of the harmonic kernel function is due to [BER].

The representation of the harmonic kernel function in terms of a closed orthonormal system gives the opportunity to solve numerically the Dirichlet problem for arbitrarily multiply connected regions. This is an important tool in physics, in particular in fluid mechanics, elasticity and electricity.

It is known that the harmonic kernel function $K_D(z,\zeta)$, the Green function $G_D(z,\zeta)$ and the Neumann function $N_D(z,\zeta)$ satisfy the relation

$$K_D(z,\zeta) = \frac{1}{2\pi}\left[N_D(z,\zeta) - G_D(z,\zeta)\right],\ \ z \in \overline{D}. \tag{14}$$

We first derive a formula that solves the Dirichlet problem (3). We prove that if $u$ is harmonic inside a region $D$ and continuous on $\partial D$, then we can determine the values of $u$ inside of $D$ by integrating on $\partial D$ the product of $u$ times the inward normal derivative of the harmonic kernel function for the region $D$, which is a fixed function that depends only on $D$.

**Theorem 19.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves. If $u$ is harmonic in $D$ and continuous on $\overline{D}$, then, up to an additive constant,*

$$u(\zeta) = -\int_{\partial D} u(z)\frac{\partial K_D(z,\zeta)}{\partial n_\sigma}d\sigma,\ \ \zeta \in D. \tag{15}$$

*Proof.* From (6), the solution of the Dirichlet problem (3) is

$$u(\zeta) = \frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial G_D(z,\zeta)}{\partial n_\sigma}d\sigma,\ \ \zeta \in D. \tag{16}$$

Using (14), we obtain

$$\frac{\partial K_D(z,\zeta)}{\partial n_\sigma} = \frac{1}{2\pi}\frac{\partial N_D(z,\zeta)}{\partial n_\sigma} - \frac{1}{2\pi}\frac{\partial g_D(z,\zeta)}{\partial n_\sigma}$$
$$= \frac{1}{l} - \frac{1}{2\pi}\frac{\partial G_D(z,\zeta)}{\partial n_\sigma},$$

for any $z \in \partial D$, where $l$ is the length of $\partial D$ (see [NEV1]). Combining this relation with (16), we find

$$u(\zeta) = -\int\limits_{\partial D} u(z)\frac{\partial K_D(z,\zeta)}{\partial n_\sigma}d\sigma + \frac{1}{l}\int\limits_{\partial D} u(z)d\sigma.$$

Thus, $u$ is determined up to the additive constant $\frac{1}{l}\int\limits_{\partial D} u(z)d\sigma$. □

Next, we derive a formula that solves the Neumann problem (12).

**Theorem 20.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves. If $u$ is harmonic in $D$ and continuously differentiable on $\partial D$ then, up to an additive constant,*

$$u(\zeta) = -\int\limits_{\partial D} \frac{\partial u(z)}{\partial n_\sigma} K_D(z,\zeta)d\sigma,\ \zeta \in D. \tag{17}$$

*Proof.* By Theorem 16, using Green formula, it follows that, up to an additive constant, a solution of the Neumann problem is given by

$$u(\zeta) = -\frac{1}{2\pi}\int\limits_{\partial D} \frac{\partial u(z)}{\partial n_\sigma} N_D(z,\zeta)d\sigma,\ \zeta \in D. \tag{18}$$

The constant is chosen such that $u(z)$ is in $\Lambda^2(D)$.
By (14), for $\zeta \in \partial D$, we have

$$K_D(z,\zeta) = \frac{1}{2\pi}N_D(z,\zeta).$$

Substituting this in (18), we obtain (17). □

Let $K_D^{(k)}(z,\widetilde{\zeta})$ be the function defined by

$$K_D^{(k)}(z,\widetilde{\zeta}) = \frac{1}{2}\left[K_D(z,\zeta) + K_D(z,k(\zeta))\right],\ z \in \overline{D},$$

where $K_D(z,k(\zeta))$ is the harmonic kernel function of the closed orthonormal system $(\varphi_i)_i$, for the region $D$, with respect to the point $k(\zeta)$. The function $K_D^{(k)}(z,\widetilde{\zeta})$ is in $\Lambda^2(D)$ (see [BER], [ROS4]).

**Proposition 21.** *If $D$ is a symmetric region, then the function $K_D^{(k)}(z,\widetilde{\zeta})$ is symmetric with respect to $z$ on $\overline{D}$, that is, for every $z \in \overline{D}$,*

$$K_D^{(k)}(z,\widetilde{\zeta}) = K_D^{(k)}(k(z),\widetilde{\zeta}).$$

*Proof.* We use (14) and the symmetric properties of the symmetric Green function and symmetric Neumann function. □

Let $\zeta_0$ be a point of $D$. Let $\Lambda_0^2(D)$ be the class of harmonic functions $\varphi$ that satisfy the conditions:

$$D\{\varphi, \varphi\} < \infty$$

and

$$\varphi(\zeta_0) = 0.$$

The harmonic kernel function $K_D(z, \zeta, \zeta_0)$ of the class $\Lambda_0^2(D)$ is related to the harmonic kernel function $K_D(z, \zeta)$ of the class $\Lambda_0^2(D)$ by the following identity:

$$K_D(z, \zeta, \zeta_0) = K_D(z, \zeta) - K_D(\zeta, \zeta_0).$$

The harmonic kernel function $K_D(z, \zeta)$ for the region $D$, with respect to the point $\zeta$ is not a conformal invariant but the harmonic kernel function $K_D(z, \zeta, \zeta_0)$ is invariant under conformal mapping (see [BER]), therefore $K_D(z, \zeta, \zeta_0)$ is well-defined on the Riemann surface $O_2$.

The function $K_D^{(k)}(z, \widetilde{\zeta}, \widetilde{\zeta_0})$ defined by

$$K_D^{(k)}(z, \widetilde{\zeta}, \widetilde{\zeta_0}) = \frac{1}{2}\left[K_D(z, \zeta, \zeta_0) + K_D(z, k(\zeta), k\left(\zeta_0\right))\right],$$

for all $z \in \overline{D} \backslash \{\widetilde{\zeta}, \widetilde{\zeta_0}\}$, is called the *symmetric harmonic kernel function* for the region $D$.

4.3. **Integral representations on the double cover.** We first express the solution of the Neumann problem (12) for harmonic functions in terms of $d\sigma$ as a line integral involving the boundary function and a symmetric Neumann function.

**Theorem 22.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves and let $g$ be a symmetric, continuous function on $\partial D$. If $u$ is harmonic in $D$ and $g$ is its inward normal derivative on $\partial D$, then up to an additive constant*

$$u(\zeta) = -\frac{1}{2\pi} \int_{\partial D} g(z) N_D^{(k)}(z, \widetilde{\zeta}) d\sigma, \ \zeta \in D. \tag{19}$$

*Proof.* By Theorem 16, up to the additive constant $\frac{1}{l} \int_{\partial D} u(z) d\sigma$, we have for all $\zeta \in D$,

$$u(\zeta) = -\frac{1}{2\pi} \int_{\partial D} g(z) N_D(z, \zeta) d\sigma\,.$$

Replacing $\zeta$ with $k(\zeta)$ we get

$$u(k(\zeta)) = -\frac{1}{2\pi} \int_{\partial D} g(z) N_D(z, k(\zeta)) d\sigma.$$

Adding the last two equations and dividing by 2, we obtain, up to the additive constant $\frac{1}{l} \int_{\partial D} u(z) d\sigma$,

$$\frac{u(\zeta) + u(k(\zeta))}{2} = -\frac{1}{2\pi} \int_{\partial D} g(z) \frac{N_D(z, \zeta) + N_D(z, k(\zeta))}{2} d\sigma.$$

By Proposition 15, $u$ is a symmetric function on $D$, then the left-hand side of the last equality is $u(\zeta)$. Therefore

$$u(\zeta) = -\frac{1}{4\pi}\int\limits_{\partial D} g(z)\left[N_D(z,\zeta) + N_D(z,k(\zeta))\right] d\sigma,$$

up to the additive constant $\frac{1}{l}\int\limits_{\partial D} u(z)d\sigma$. □

Similarly, we obtain a formula for the symmetric solution of the Neumann problem (12) on a symmetric region $D$, in terms of the symmetric harmonic kernel function.

**Theorem 23.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves. Let $g$ be a symmetric, continuous function on $\partial D$. If $u$ is harmonic in $D$ and $g$ is its inward normal derivative on $\partial D$, then up to an additive constant,*

$$u(\zeta) = -\int\limits_{\partial D} g(z)K_D^{(k)}(z,\widetilde{\zeta})d\sigma,\ \zeta \in D. \tag{20}$$

*Proof.* It is similar with the proof of the Theorem 22. Here we use Theorem 20. □

The next theorem yields a formula for the symmetric solution of the Dirichlet problem (3) on a symmetric region $D$, in terms of the symmetric harmonic kernel function.

**Theorem 24.** *Let $D$ be a symmetric region bounded by a finite number of $\sigma$-rectifiable Jordan curves. Let $f$ be a symmetric, continuous function on $\partial D$. There is a unique symmetric function $u$ on $\overline{D}$, which is harmonic on $D$, continuous on $\overline{D}$, such that $u = f$ on $\partial D$. For all $\zeta$ in $D$,*

$$u(\zeta) = -\int\limits_{\partial D} f(z)\frac{\partial K_D^{(k)}(z,\widetilde{\zeta})}{\partial n_\sigma}d\sigma. \tag{21}$$

*Proof.* By Theorem 19, for all $\zeta \in D$,

$$u(\zeta) = -\int\limits_{\partial D} u(z)\frac{\partial K_D(z,\zeta)}{\partial n_\sigma}d\sigma.$$

Replacing $\zeta$ with $k(\zeta)$ we get, for all $\zeta \in D$,

$$u(k(\zeta)) = -\int\limits_{\partial D} u(z)\frac{\partial K_D(z,k(\zeta))}{\partial n_\sigma}d\sigma.$$

Adding the last two equations and dividing by 2, it follows that

$$\frac{u(\zeta)+u(k(\zeta))}{2} = -\frac{1}{2}\int\limits_{\partial D} u(z)\left[\frac{\partial K_D(z,\zeta)}{\partial n_\sigma} + \frac{\partial K_D(z,k(\zeta))}{\partial n_\sigma}\right] d\sigma,$$

for all $\zeta \in D$.

By Proposition 2, $u$ is a symmetric function on $D$, then the left-hand side of the last equality is $u(\zeta)$ and we conclude that for all $\zeta$ in $D$,

$$u(\zeta) = -\frac{1}{2}\int\limits_{\partial D} u(z)\left[\frac{\partial K_D(z,\zeta)}{\partial n_\sigma} + \frac{\partial K_D(z,k(\zeta))}{\partial n_\sigma}\right] d\sigma.$$

The uniqueness of the solution of the Dirichlet problem for harmonic functions implies (21). □

4.4. **Integral representations on the Klein surface.** Let $\widetilde{\zeta}$ be a point inside $\Omega$. A *Neumann function* $N_\Omega(\widetilde{z},\widetilde{\zeta})$ for the region $\Omega$, with singularity at $\widetilde{\zeta}$ is defined by

$$N_\Omega(\widetilde{z},\widetilde{\zeta}) = N_D^{(k)}(z,\widetilde{\zeta}) = N_D^{(k)}(k(z),\widetilde{\zeta}), \tag{22}$$

where $\widetilde{z} = \pi(z)$.

**Remark 7.** *By Proposition 17, it follows that* $N_\Omega(\widetilde{z},\widetilde{\zeta})$ *is well-defined on* $\Omega$*.*

Therefore $N_\Omega(\widetilde{z},\widetilde{\zeta})$ is a harmonic function on $\Omega\setminus\left\{\widetilde{\zeta}\right\}$, which has a constant normal derivative $\dfrac{\partial N_\Omega}{\partial n_\Sigma}$ on the boundary $\partial\Omega$ and has a logarithmic pole at the point $\widetilde{\zeta} = \pi(\zeta)$.

Next, we derive the solution of the Neumann problem (11) on the region $\Omega$.

**Theorem 25.** *Let $G$ be a continuous real-valued function on $\partial\Omega$. Then, up to an additive constant, the solution of problem* (11) *is the function $U$ defined by the relation $u = U\circ\pi$, where $\pi$ is the canonical projection of $O_2$ on $X$ and $u$ is the solution* (19) *of the problem* (12) *on the symmetric region $D$, with the inward normal derivative $g$ given by $g = G\circ\pi$ on $\partial D$.*

*Proof.* The symmetry of the function $u$ on $D$, yields

$$\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = \Delta u(k(\zeta)) = 0 \quad \text{for all } \widetilde{\zeta}\in\Omega,$$

where $\widetilde{\zeta} = \pi(\zeta)$.

Using the symmetry of the function $g$ on $\partial D$, we obtain

$$\frac{\partial U}{\partial n_\Sigma}(\widetilde{\zeta}) = \frac{\partial U}{\partial n_\sigma}(\zeta) = g(\zeta) = g(k(\zeta)) = G(\widetilde{\zeta}),$$

for all $\widetilde{\zeta}\in\partial\Omega$. Then, up to an additive constant, the function $U$ defined on $\Omega$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

for all $\widetilde{\zeta}$ in $\Omega$, is the solution of problem (11). □

Let $\widetilde{\zeta}$ be a point inside $\Omega$. The harmonic kernel function $K_\Omega(\widetilde{z},\widetilde{\zeta})$ of the closed orthonormal system $(\varphi_i)_i$, for the region $\Omega$, with respect to the point $\widetilde{\zeta} = \{\zeta, k(\zeta)\}$ is defined by

$$K_\Omega(\widetilde{z},\widetilde{\zeta}) = K_D^{(k)}(z,\widetilde{\zeta}) = K_D^{(k)}(k(z),\ \widetilde{\zeta}), \widetilde{z} = \pi(z)\in\Omega.$$

**Remark 8.** *By Proposition 21, it follows that* $K_\Omega(\widetilde{z},\widetilde{\zeta})$*, is well-defined on* $\Omega$*.*

The symmetric solutions on $O_2$ determine the solutions of the similar problems on the Klein surface $X$. Thus, we obtain the solution of the Dirichlet problem (2) on the region $\Omega$, with respect to the harmonic kernel function, for the region $\Omega$.

**Theorem 26.** *Let $F$ be a continuous real-valued function on the border $\partial\Omega$. The solution of the problem* (2) *with the boundary function $F$ is the function $U$ defined on $\overline{\Omega}$, by the relation $u = U\circ\pi$, where $\pi$ is the canonical projection of $O_2$ on $X$ and $u$ is the solution* (21) *of the problem* (3) *on the symmetric region $D$, with the boundary function $f$ given by $f = F\circ\pi$.*

*Proof.* By definition,

$$\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0 \quad \text{for all } \widetilde{\zeta} \in \Omega,$$

where $\widetilde{\zeta} = \pi(\zeta)$, thus $U$ is a harmonic function. The symmetry of the function $f$ on $\partial D$, implies

$$U(\widetilde{\zeta}) = u(\zeta) = f(\zeta) = f(k(\zeta)) = F(\widetilde{\zeta}) \quad \text{for all } \widetilde{\zeta} \in \partial\Omega.$$

Due to the uniqueness of the solution, the function $U$ defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

for all $\widetilde{\zeta}$ in $\overline{\Omega}$, where $\widetilde{\zeta} = \pi(\zeta)$, is the solution of problem (2) on $\Omega$. □

The next theorem gives the solution of the Neumann problem (11) on the region $\Omega$, with respect to the harmonic kernel function, for the region $\Omega$.

**Theorem 27.** *Let $G$ be a continuous real-valued function on the border $\partial\Omega$. Then, up to an additive constant, the solution of the problem* (11) *with the normal derivative $G$ on $\partial\Omega$ is the function $U$ defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where $\pi$ is the canonical projection of $O_2$ on $X$ and $u$ is the solution* (20) *of the problem* (12) *on the symmetric region $D$, with the normal derivative function $g$ given by $g = G \circ \pi$ on $\partial D$.*

*Proof.* By definition,

$$\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0 \quad \text{for all } \widetilde{\zeta} \in \Omega,$$

where $\widetilde{\zeta} = \pi(\zeta)$, thus $U$ is a harmonic function. The symmetry of the function $g$ on $\partial D$, implies

$$\frac{\partial U(\widetilde{\zeta})}{\partial n_\Sigma} = \frac{\partial u(\zeta)}{\partial n_\sigma} = g(\zeta) = g(k(\zeta)) = G(\widetilde{\zeta}),$$

for all $\widetilde{\zeta} \in \partial\Omega$. Thus, up to an additive constant, the function $U$ defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

is the solution of the problem (11) on $\Omega$. □

**Concluding remarks.** The methods developed in this paper remain valid in the case of all differential operators associated to *conformal invariant metrics*. Such an example corresponds to the *invariant Laplacian* (or sometimes the *Laplace-Beltrami operator* for the Poincaré-Bergman metric), see Krantz [KRA1, Section 6.5]. We also refer to the *pseudo-hyperbolic metric*, which is conformally invariant, but it does not arise from integrating an infinitesimal metric (that is, lengths of tangent vectors at a point). A comprehensive analysis of the pseudo-hyperbolic metric on the disc may be found in Krantz [KRA3].

To the best of our knowledge, there are not further results involving either *linear* or *nonlinear* elliptic equations on Klein surfaces. This study can include qualitative and quantitative properties of solutions but also related singular or degenerate phenomena. We consider that the mathematical analysis of these classes of PDEs on *Klein surfaces* is a very rich and attractive research field at the interplay between complex analysis and nonlinear analysis.

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