Rend. Lincei Mat. Appl. 32 (2021), 819–838
DOI 10.4171/RLM/959

Partial Differential Equations — *Non-homogeneous Dirichlet problems with concave-convex reaction*, by Gabriele Bonanno, Roberto Livrea and Vicenţiu D. Rădulescu, communicated on 2 July 2021.

This paper is dedicated to the memory of Professor Antonio Ambrosetti, a Master of Nonlinear Analysis

Abstract. — The variational methods are adopted for establishing the existence of at least two nontrivial solutions for a Dirichlet problem driven by a non-homogeneous differential operator of p-Laplacian type. A large class of nonlinear terms is considered, covering the concave-convex case. In particular, two positive solutions to the problem are obtained under a $(p-1)$-superlinear growth at infinity, provided that a behaviour less than $(p-1)$-linear of the nonlinear term in a suitable set is requested.

Key words: Variational methods, critical point, nonlinear elliptic problem, p-Laplace operator, multiple solutions

Mathematics Subject Classification (primary; secondary): 35J15; 35J25, 35J62, 35J92, 58E05

1. Introduction

In the present note we deal with the following problem

$$(P_\lambda) \qquad \begin{cases} -\operatorname{div} \boldsymbol{A}(x, \nabla u) = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, $\boldsymbol{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a function admitting a general enough structure in order to cover the simple case $\boldsymbol{A}(x, \xi) = |\xi|^{p-2}\xi$, $p > 1$, namely (P_λ) involves the usual p-Laplacian operator, moreover, λ is a positive parameter, while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable Carathéodory function.

During the last decades a lot of papers have been devoted to the study of several differential problems that are included or strictly related with problem (P_λ), see, e.g., [15] and [14]. In all these manuscripts the case $p \geq 2$ has been investigated, while different asymptotic conditions at zero and/or at infinity of the nonlinear term $f(x, \cdot)$ have been considered. In [13] the more general case $p > 1$ has been treated when the reaction term is a suitable perturbation of the nonlinearity $|u|^{p-2}u$ (see also [6]). More recently, in [8] exploiting the structure of $\boldsymbol{A}$ as introduced in [13], the case when $f(x,\cdot)$ is $(p-1)$-superlinear at zero and $(p-1)$-sublinear at infinity has been studied.

Here, we still consider the same general elliptic operator in divergence form as in [8, 13] and assume that $\boldsymbol{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ admits a potential $\mathscr{A} : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, with

($\mathcal{A}$) *$\mathscr{A} = \mathscr{A}(x,\xi)$ is a continuous function on $\overline{\Omega} \times \mathbb{R}^N$, with a continuous derivative with respect to ξ and $\boldsymbol{A} = \partial_\xi \mathscr{A}$. Moreover*

(i) *$\mathscr{A}(x,0) = 0$ and $\mathscr{A}(x,\xi) = \mathscr{A}(x,-\xi)$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.*

(ii) *$\mathscr{A}(x,\cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$.*

(iii) *There exist two constants a_1, a_2, with $0 < a_1 \le a_2$ such that*

$$\boldsymbol{A}(x,\xi)\cdot\xi \ge a_1|\xi|^p \quad \textit{and} \quad |\boldsymbol{A}(x,\xi)| \le a_2|\xi|^{p-1}$$

for every $(x,\xi) \in \Omega \times \mathbb{R}^N$.

We allow the function f to have a $(p-1)$-superlinear behaviour at infinity that, as a special case, gives back the concave-convex structure. We refer to the seminal papers [1, 3, 4] for existence, multiplicity and non existence results for differential problems involving the p-Laplacian under the combined effects of concave and convex nonlinear terms (see also [7, 16, 17, 18, 22, 23, 29, 33] as well as [25, 26, 27, 28, 31, 32], where a nonhomogeneous operator is considered).

We adopt the variational methods and in the set of assumptions the classical Ambrosetti–Rabinowitz condition is employed in order to assure the existence of an interval of parameters λ for which problem (P_λ) admits at least two non-trivial solutions. In particular, both the solutions are positive when f satisfies a sign condition. The main tool is a general critical point theorem proved in [7] (see Theorem 2.1). We wish to emphasize that the proposed approach permits to consider also more general situations when the multiplicity result can be obtained without requiring any particular asymptotic condition near at zero of $f(x,\cdot)$ (see Theorem 3.1, assumption (jj)) so that we can go further the although meaningful concave-convex case. The autonomous case is also treated (see Corollary 3.1 and Theorem 3.3).

In Section 2 a quite detailed description of some auxiliary results is given. The main result is proved in Section 3.

Some of the main abstract tools used in this paper are developed in the recent monograph [30]. We also refer to [24] for an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

2. Basic notations and auxiliary results

In what follows Ω is a bounded domain of $\mathbb{R}^N$ and $W_0^{1,p}(\Omega)$, with $1 < p < N$, denotes the usual Sobolev space equipped with the norm

$$\|u\| = \|\nabla u\|_p,$$

while $W^{-1,p'}(\Omega)$ is its dual space. It is well known, see [35], that the best constant of the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, with $p^* = \frac{Np}{N-p}$, is explicitly computable by the formula

$$T = \pi^{-1/2} N^{-1/p} \left(\frac{p-1}{N-p}\right)^{1-1/p} \left\{\frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)}\right\}^{1/N}, \tag{2.1}$$

where Γ is the gamma function. In particular, the inequality

$$\|u\|_{p^*} \leq T\|u\|, \tag{2.2}$$

for every $u \in W_0^{1,p}(\Omega)$, can be exploited, together with the Hölder inequality, in such a way that, for every $\tau \in [1, p^*]$, one has

$$\|u\|_\tau \leq c_\tau \|u\| \tag{2.3}$$

for all $u \in X$, where $c_\tau = T|\Omega|^{(p^*-\tau)/(p^*\tau)}$ and $|\Omega|$ is the Lebesgue measure of Ω. Moreover, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\tau(\Omega)$ is compact provided $\tau \in [1, p^*[$.

The condition

$$a_1|\xi|^p \leq p\mathscr{A}(x,\xi) \leq a_2|\xi|^p \tag{2.4}$$

for every $(x,\xi) \in \Omega \times \mathbb{R}^N$, can be derived from assumptions $(\mathcal{A})$(i) and $(\mathcal{A})$(iii); as well as the following lemma has been proved.

Lemma 2.1 ([13, Lemma 2.5]). *Let $\mathscr{A}$ satisfy $(\mathcal{A})$(i)–$(\mathcal{A})$(iii). Then the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Phi(u) = \int_\Omega \mathscr{A}(x, \nabla u(x))\, dx \tag{2.5}$$

is convex, weakly lower semicontinuous and of class C^1 in $W_0^{1,p}(\Omega)$, being

$$\Phi'(u)(v) = \int_\Omega \boldsymbol{A}(x,\nabla u)\cdot\nabla v\, dx$$

for every $u, v \in W_0^{1,p}(\Omega)$.

Moreover, $\Phi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies the $(\mathscr{S}_+)$ condition, i.e., for every sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty} \int_\Omega \boldsymbol{A}(x,\nabla u_n)\cdot(\nabla u_n - \nabla u)\, dx \leq 0,$$

then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

Our nonlinear reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ will be required to be a Carathéodory function such that for some function $a \in L^\alpha(\Omega)$, that is a.e. positive, with $\alpha > \frac{Np}{Np-Nq+pq}$, and $p < q < p^*$, the following growth condition holds

$(G_{f,a,s,q})$ *There exist positive constants s, b_s and b_q, with $1 \le s \le p$, such that*

$$|f(x,t)| \le a(x)(b_s|t|^{s-1} + b_q|t|^{q-1}) \quad \textit{for a.a. } x \in \Omega \textit{ and all } t \in \mathbb{R}.$$

The next lemma will be useful in order to approach problem (P_λ) in the variational setting.

LEMMA 2.2. *Assume that f satisfies condition $(G_{f,a,s,q})$ and put $F(x,t) = \int_0^t f(x,\xi)\,d\xi$. Then, the functional $\Psi_f : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Psi_f(u) = \int_\Omega F(x,u(x))\,dx \tag{2.6}$$

is of class C^1 being

$$\Psi_f'(u)(v) = \int_\Omega f(x,u(x))v(x)\,dx.$$

Moreover the operator $\Psi_f' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is compact and Ψ_f is sequentially weakly continuous in $W_0^{1,p}(\Omega)$.

PROOF. First of all we can observe that from $q < p^*$ one has that $Np - Nq + pq > 0$ and, in particular, $p < N$ implies that $\frac{Np}{Np-Nq+pq} > 1$. Hence, $\alpha > 1$. If α' is the conjugate exponent of α, a direct computation shows that $1 \le \alpha' \le \alpha' s \le \alpha' p < \alpha' q < p^*$. Indeed, it is immediate if $\alpha = +\infty$, being $\alpha' = 1$. Otherwise, if $\alpha < +\infty$, one has

$$\begin{aligned} \alpha' q = \frac{\alpha}{\alpha-1} q < \frac{Np}{N-p} \quad &\Leftrightarrow \quad \alpha q(N-p) < \alpha Np - Np \\ &\Leftrightarrow \quad \alpha(Nq - Np - qp) < -Np \\ &\Leftrightarrow \quad \alpha > \frac{Np}{Np - Nq + pq}. \end{aligned}$$

Thus, since by $(G_{f,a,s,q})$ one has

$$|F(x,u(x))| \le a(x)\left(\frac{b_s}{s}|u(x)|^s + \frac{b_q}{q}|u(x)|^q\right), \tag{2.7}$$

for all $u \in W_0^{1,p}(\Omega)$ and a.e. in Ω, taking in mind that $\alpha' q < p^*$ implies that $|u|^q \in L^{\alpha'}$, the Hölder inequality assures that Ψ_f is well defined.

Classical arguments assure that Ψ_f is differentiable and

$$\Psi_f'(u)(v) = \int_\Omega f(x,u(x))v(x)\,dx$$

for every $u, v \in W_0^{1,p}(\Omega)$. For the reader convenience, we explicitly compute it. Let $u, v \in W_0^{1,p}(\Omega)$. Fixed $x \in \Omega$ and $t \in \,]-1,1[\setminus\{0\}$ there exists $\sigma_t(x) \in$

$[h_t(x), k_t(x)]$, with $h_t(x) = \min\{u(x), u(x) + tv(x)\}$ and $k_t(x) = \max\{u(x), u(x) + tv(x)\}$, such that

$$\frac{F(u(x) + tv(x)) - F(u(x))}{t} = f(x, \sigma_t(x))v(x).$$

In view of $(G_{f,a,s,q})$, if $w(x) = \max\{|u(x)|, |v(x)|\}$, there exists a constant $B > 0$ such that

$$\begin{aligned} |f(x, \sigma_t(x))v(x)| &\le a(x)(b_s(|u(x)| + |v(x)|)^{s-1} + b_q(|u(x)| + |v(x)|)^{q-1})|v(x)| \\ &\le Ba(x)(w(x)^s + w(x)^q) \end{aligned}$$

a.e. in Ω and for every $t \in]-1, 1[\setminus\{0\}$. Since $w \in W_0^{1,p}(\Omega)$, one has $w \in L^{\alpha' q}$ and the dominated convergence theorem assures the announced formula.

For verifying the regularity of Ψ_f we chiefly argue as in the proof of [13, Lemma 3.2]. We report all the details showing that Ψ_f' is weak-to-strong sequentially continuous, namely if $\{u_n\}_n$ and u are in $W_0^{1,p}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then $\|\Psi_f'(u_n) - \Psi_f'(u)\|_{W^{-1,p'}(\Omega)} \to 0$ as $n \to \infty$. Indeed, fix $\{u_n\}_n$ in $W_0^{1,p}(\Omega)$ with $u_n \rightharpoonup u$. The compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\alpha' q}(\Omega)$ assures that $u_n \to u$ in $L^{\alpha' q}(\Omega)$. Hence, from [12, Thm. IV.9], there exist a subsequence, still denoted by $\{u_n\}_n$, and a function $h \in L^{\alpha' q}(\Omega)$ such that

$$u_n \to u \text{ a.e. in } \Omega \quad \text{and} \quad |u_n(x)| \le h(x) \text{ a.e. in } \Omega, \quad \text{for every } n \in \mathbb{N}. \tag{2.8}$$

Again from $(G_{f,a,s,q})$ one has

$$\begin{aligned} &|f(x, u_n(x)) - f(x, u(x))| \\ &\quad \le a(x)(b_s(|u_n(x)|^{s-1} + |u(x)|^{s-1}) + b_q(|u_n(x)|^{q-1} + |u(x)|^{q-1})) \\ &\quad \le 2a(x)(b_s h(x)^{s-1} + b_q h(x)^{q-1}) \end{aligned}$$

for a.a. $x \in \Omega$, that in particular implies

$$\begin{aligned} &|f(x, u_n(x)) - f(x, u(x))|^{q'} a(x)^{1/(1-q)} \\ &\quad \le 2^{q'} a(x)^{q' + 1/(1-q)} (b_s h(x)^{s-1} + b_q h(x)^{q-1})^{q'} \\ &\quad \le 2^{2q'-1} a(x)(b_s^{q'} h(x)^{q'(s-1)} + b_q^{q'} h(x)^{q'(q-1)}) \\ &\quad = 2^{2q'-1} a(x)(b_s^{q'} h(x)^{q'(s-1)} + b_2^{q'} h(x)^q) \end{aligned} \tag{2.9}$$

a.e. in Ω, namely, observing that $1/(1-q) = -q'/q$ and being $ah^{q'(s-1)} \in L^1(\Omega)$, one has

$$|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))| a(\cdot)^{-1/q} \in L^{q'}(\Omega).$$

Thus, fixed $v \in W_0^{1,p}(\Omega)$ with $\|v\| = 1$, taking in mind that $a^{1/q} v \in L^q(\Omega)$ and exploiting the Hölder inequality, one has

$$
\begin{aligned}
|\Psi_f'(u_n)(v) - \Psi_f'(u)(v)| &= \left| \int_\Omega (f(x,u_n(x)) - f(x,u(x)))v(x)\,dx \right| \\
&\leq \int_\Omega |f(x,u_n(x)) - f(x,u(x))||v(x)|\,dx \\
&= \int_\Omega |f(x,u_n(x)) - f(x,u(x))|a(x)^{-1/q}a(x)^{1/q}|v(x)|\,dx \\
&\leq \left(\int_\Omega |f(x,u_n(x)) - f(x,u(x))|^{q'}a(x)^{1/(1-q)}\,dx \right)^{1/q'} \\
&\quad \times \left(\int_\Omega a(x)|v(x)|^q \right)^{1/q}.
\end{aligned}
$$

At this point, observe that from the Hölder inequality and condition (2.3) one has

$$
\int_\Omega a(x)|v(x)|^q\,dx \leq \|a\|_\alpha \|v\|_{\alpha' q}^q \leq c_{\alpha' q}^q \|a\|_\alpha.
$$

In conclusion,

$$
\begin{aligned}
&\|\Psi_f'(u_n)(v) - \Psi_f'(u)\|_{W^{-1,p}(\Omega)} \\
&\quad \leq c_{\alpha' q}\|a\|_\alpha^{1/q} \left(\int_\Omega |f(x,u_n(x)) - f(x,u(x))|^{q'}a(x)^{1/(1-q)}\,dx \right)^{1/q'}
\end{aligned}
$$

and, in view of (2.9), the dominated convergence theorem, condition (2.8) and the continuity of $f(x,\cdot)$, we can conclude that $\Psi_f'(u_n) \to \Psi_f'(u)$ in $W^{-1,p}(\Omega)$, that completes the proof. □

In analogy with [13] and [8], if f is a function of type $(G_{f,a,s,q})$, a weak solution of problem (P_λ) is any $u \in W_0^{1,p}(\Omega)$ such that

$$
\int_\Omega A(x,\nabla u(x)) \cdot \nabla v(x)\,dx - \lambda \int_\Omega f(x,u(x))v(x)\,dx = 0
$$

for every $v \in W_0^{1,p}(\Omega)$. Thus, thanks to the previous lemmas, if for $\lambda > 0$ we define the functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by $I_\lambda(u) = \Phi(u) - \lambda\Psi_f(u)$, the following claim holds

(2.10) The critical points of I_λ are weak solutions of problem (P_λ).

Next theorem, see [7, Theorem 2.1], will be the main tool in order to apply the variational methods and establish our multiplicity results. We recall that the theorem below is based on a local minimum theorem obtained in [5] and the classical mountain pass theorem (see [2] and [10]).

THEOREM 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$ such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[$ the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,1}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. MAIN RESULTS

Here we present some theorems that assure the existence of at least two nontrivial solutions for problem (P_λ).

A technical constant will be used. In particular, if $\rho : \Omega \to]0, +\infty[$ is the function defined by $\rho(x) = d(x, \partial\Omega)$ (observe that for each $x \in \Omega$ obviously $B(x, \rho(x)) = \{y \in \Omega : |y - x_0| < \rho(x)\} \subseteq \Omega$), then, for a fixed $x_0 \in \Omega$ and, for $a \in L^\alpha(\Omega)$ (a.e. positive, with, as usual, $\alpha > Np/(Np - Nq + pq)$), we put

$$K = K(\rho(x_0)) = \frac{a_1}{a_2} \frac{1}{T^p \|a\|_\alpha (2^N - 1)|\Omega|^{(p^* - \alpha' p)/(\alpha' p^*)} |B_1|} \left(\frac{2}{\rho(x_0)}\right)^{N-p}, \tag{3.1}$$

where T is the Talenti constant introduced in (2.1), a_1 and a_2 are the constants considered in $(\mathcal{A})$(iii), α' is the conjugate exponent of α and $|B_1|$ denotes the Lebesgue measure of the N-dimensional unit ball.

We are in the position to state our first main result.

THEOREM 3.1. *Assume that f satisfies condition $(G_{f,a,s,q})$ and that there exist $\mu > \frac{a_2}{a_1} p$, $R > 0$ such that*

$$0 < \mu F(x, t) \le t f(x, t) \tag{AR}$$

for a.a. $x \in \Omega$ and for all $|t| \ge R$. Moreover, suppose that there exist $c, d > 0$ with $c < d$ and $x_0 \in \Omega$ such that

(j) *$F(x, t) \ge 0$ a.e. in $B(x_0, \rho(x_0))$ and for all $t \in [0, c]$;*

(jj) $\dfrac{b_s}{s} d^{s-p} + \dfrac{b_q}{q} d^{q-p} < K \dfrac{\int_{B(x_0, \rho(x_0)/2)} F(x, c)\, dx}{c^p}$.

Put

$$(3.2)\qquad \lambda_* = \lambda_*(c) = \frac{a_2(2^N-1)|B_1|}{p}\left(\frac{\rho(x_0)}{2}\right)^{N-p}\frac{c^p}{\int_{B(x_0,\rho(x_0)/2)}F(x,c)\,dx},$$
$$= \frac{a_1}{\|a\|_\alpha pT^p\|\Omega|^{(p^*-\alpha'p)/(p^*\alpha')}}\frac{1}{K}\frac{c^p}{\int_{B(x_0,\rho(x_0)/2)}F(x,c)\,dx}$$

and

$$(3.3)\qquad \lambda^* = \lambda^*(d) = \frac{a_1}{\|a\|_\alpha pT^p|\Omega|^{(p^*-\alpha'p)/(p^*\alpha')}}\frac{1}{\frac{b_s}{s}d^{s-p}+\frac{b_q}{q}d^{q-p}}.$$

Then, for every $\lambda \in]\lambda_*, \lambda^*[$ *problem* (P_λ) *admits at least two nontrivial weak solutions.*

Proof. We wish to apply Theorem 2.1, with $X = W_0^{1,p}(\Omega)$ and the functionals Φ and Ψ_f as defined in (2.5) and (2.6) respectively, so that, as seen in Lemma 2.1 and Lemma 2.2, they are of class C^1, moreover $\inf_X \Phi = \Phi(0) = \Psi_f(0) = 0$.

Step 1. For every $\lambda > 0$ the functional $I_\lambda = \Phi - \lambda\Psi_f$ is unbounded from below.

Indeed, integrating condition (AR) one has that

$$F(x,t) \geq \frac{|t|^\mu}{R^\mu}\min\{F(x,R),F(x,-R)\}$$

for a.a. $x \in \Omega$ and for every $|t| \geq R$. On the other hand, from $(G_{f,a,s,q})$ one has that

$$|F(x,t)| \leq a(x)\left(\frac{b_s}{s}R^s + \frac{b_q}{q}R^q\right)$$

for a.a. $x \in \Omega$ and for every $|t| \leq R$. Hence, if we put $\beta(x) = \frac{1}{R^\mu}\min\{F(x,R), F(x,-R)\}$ and $\delta(x) = a(x)\left(\frac{b_s}{s}R^s + \frac{b_q}{q}R^q\right)$,

$$\begin{aligned} F(x,t) &\geq \beta(x)|t|^\mu - \beta(x)|t|^\mu - \delta(x) \\ &\geq \beta(x)|t|^\mu - R^\mu\beta(x) - \delta(x) \end{aligned}$$

for a.a. $x \in \Omega$ and for every $|t| \leq R$. Thus,

$$(3.4)\qquad F(x,t) \geq \beta(x)|t|^\mu - R^\mu\beta(x) - \delta(x)$$

for a.a. $x \in \Omega$ and for every $t \in \mathbb{R}$. Moreover, observe that, in view of (AR), $\beta(x) > 0$ for a.a. $x \in \Omega$. In addition, it is clear that $\beta \in L^\alpha(\Omega)$. Then, for every $\lambda > 0$, fixed $u \in W_0^{1,p}(\Omega)$ with $u \neq 0$, exploiting (2.4) and (3.4) one has that for every $\eta > 0$

$$\begin{aligned} I_\lambda(\eta u) &= \Phi(\eta u) - \lambda \Psi_f(\eta u) \\ &\le t^p \frac{a_2}{p} \|u\|^p - \lambda\Big(\eta^\mu \int_\Omega \beta(x)|u(x)|^\mu \, dx - R^\mu \int_\Omega \beta(x)\, dx - \int_\Omega \delta(x)\, dx\Big). \end{aligned}$$

Passing to the limit as $\eta \to +\infty$ in the previous inequality and taking in mind that $\mu > p$ one can conclude the I_λ is unbounded from below.

Step 2. For every $\lambda > 0$ the functional $I_\lambda = \Phi - \lambda \Psi_f$ satisfies the (PS)-condition.

Indeed, let $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ be such that $\{I_\lambda(u_n)\}_n$ is bounded and $I'_\lambda(u_n) \to 0$ in $W^{-1,p}(\Omega)$. A classical argument assures the existence of some $M > 0$ such that for $n \in \mathbb{N}$ large enough

$$\begin{aligned} M + \frac{1}{\mu}\|u_n\| &\ge I_\lambda(u_n) + \frac{1}{\mu}\|I'_\lambda(u_n)\|_{W^{-1,p}(\Omega)}\|u_n\| \\ &\ge I_\lambda(u_n) - \frac{1}{\mu} I'_\lambda(u_n)(u_n) \\ &= \int_\Omega \mathscr{A}(x, \nabla u_n(x)) - \frac{1}{\mu}\int_\Omega \boldsymbol{A}(x, \nabla u_n)\cdot \nabla u_n \, dx \\ &\quad - \lambda \int_\Omega \Big(F(x, u_n) - \frac{1}{\mu} f(x, u_n(x)) u_n(x)\Big)\, dx. \end{aligned}$$

Hence, if we put $h_n(x) = F(x, u_n) - \frac{1}{\mu} f(x, u_n(x)) u_n(x)$, thanks to $(\mathcal{A})$(iii) and (2.4) one achieves

$$\begin{aligned} M + \frac{1}{\mu}\|u\| &\ge \Big(\frac{a_1}{p} - \frac{a_2}{\mu}\Big)\|u\|^p - \lambda \int_{\{x \in \Omega : |u(x)| < R\}} h_n(x)\, dx \\ &\quad - \lambda \int_{\{x \in \Omega : |u(x)| \ge R\}} h_n(x)\, dx. \end{aligned}$$

Observe that condition (AR) implies that the third term in the right hand side is positive, while, in view of $(G_{f,a,s,q})$, the second term is bounded by a constant independent from n. Thus, from $\frac{a_1}{p} - \frac{a_2}{\mu} > 0$ it follows that $\{u_n\}_n$ is bounded. At this point, the reflexivity of $W_0^{1,p}(\Omega)$, the compactness of Ψ'_f Lemma 2.1 and some standard techniques, see for example [8], lead to the existence of a subsequence of $\{u_n\}_n$ that strongly converges, namely the (PS)-condition holds.

Step 3. There exist $r > 0$ and $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\frac{\sup_{\Phi(u) \le r} \Psi_f(u)}{r} < \frac{\Psi_f(\tilde{u})}{\Phi(\tilde{u})}. \tag{3.5}$$

Put

$$r = \frac{a_1 |\Omega|^{p/p^*}}{p T^p} d^p, \tag{3.6}$$

and define the function $\tilde{u} \in W_0^{1,p}(\Omega)$ as follows

$$(3.7) \qquad \tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus \bar{B}(x_0, \rho(x_0)) \\ \frac{2c}{\rho(x_0)}(\rho(x_0) - |x - x_0|) & \text{if } x \in B(x_0, \rho(x_0)) \setminus \bar{B}(x_0, \rho(x_0)/2) \\ c & \text{if } x \in B(x_0, \rho(x_0)/2). \end{cases}$$

Condition (2.4) implies

$$(3.8) \qquad \frac{a_1}{p}\|u\|^p \le \Phi(u) \le \frac{a_2}{p}\|u\|^p$$

for every $u \in W_0^{1,p}(\Omega)$. Hence,

$$(3.9) \qquad \{u \in W_0^{1,p}(\Omega) : \Phi(u) \le r\} \subseteq \left\{u \in W_0^{1,p}(\Omega) : \|u\| \le \left(\frac{pr}{a_1}\right)^{1/p}\right\}.$$

Condition (2.7) and the Hölder inequality lead to

$$\begin{aligned} (3.10) \qquad \Psi_f(u) &\le \frac{b_s}{s}\int_\Omega a(x)|u(x)|^s\,dx + \frac{b_q}{q}\int_\Omega a(x)|u(x)|^q\,dx \\ &\le \frac{b_s}{s}\|a\|_\alpha\|u\|_{\alpha'}^s + \frac{b_q}{q}\|a\|_\alpha\|u\|_{\alpha' q}^q \\ &\le \frac{b_s}{s}\|a\|_\alpha T^s|\Omega|^{(p^*-\alpha' s)/(p^*\alpha')}\|u\|^s + \frac{b_q}{q}\|a\|_\alpha T^q|\Omega|^{(p^*-\alpha' q)/(p^*\alpha')}\|u\|^q \end{aligned}$$

for every $u \in W_0^{1,p}$. Thus, in view of (3.9), condition (3.10) implies

$$\begin{aligned} \sup_{\Phi(u)\le r}\Psi_f(u) \le{}& \frac{b_s}{s}\|a\|_\alpha T^s|\Omega|^{(p^*-\alpha' s)/(p^*\alpha')}\left(\frac{pr}{a_1}\right)^{s/p} \\ &+ \frac{b_q}{q}\|a\|_\alpha T^q|\Omega|^{(p^*-\alpha' q)/(p^*\alpha')}\left(\frac{pr}{a_1}\right)^{q/p}, \end{aligned}$$

from which we deduce that

$$\begin{aligned} (3.11) \qquad \sup_{\Phi(u)\le r}\frac{\Psi_f(u)}{r} &\le \|a\|_\alpha\frac{p}{a_1}T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')} \\ &\quad\times\left[\frac{b_s}{s}\left(\frac{T^p pr}{a_1|\Omega|^{p/p^*}}\right)^{(s-p)/p} + \frac{b_q}{q}\left(\frac{T^p pr}{a_1|\Omega|^{p/p^*}}\right)^{(q-p)/p}\right] \\ &= \|a\|_\alpha\frac{p}{a_1}T^p|\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}\left(\frac{b_s}{s}d^{s-p} + \frac{b_q}{q}d^{q-p}\right) \\ &= \frac{1}{\lambda^*}. \end{aligned}$$

A direct computation shows that

$$\|\tilde{u}\|^p = \left(\frac{2}{\rho(x_0)}\right)^p c^p |B(x_0, \rho(x_0)) \backslash B(x_0, \rho(x_0)/2)| = \left(\frac{\rho(x_0)}{2}\right)^{N-p} (2^N - 1)|B_1| c^p.$$

Hence, in view of (3.8)

$$\Phi(\tilde{u}) \le \frac{a_2}{p}\left(\frac{\rho(x_0)}{2}\right)^{N-p} (2^N - 1)|B_1| c^p. \tag{3.12}$$

Moreover, assumption (j) assures that, one has

$$\Psi_f(\tilde{u}) = \int_\Omega F(x, \tilde{u}(x))\,dx \ge \int_{B(x_0, \rho(x_0)/2)} F(x, c)\,dx. \tag{3.13}$$

The previous (3.12) and (3.13) permit to emphasize that

$$\frac{\Psi_f(\tilde{u})}{\Phi(\tilde{u})} \ge \frac{p}{a_2(2^N - 1)|B_1|}\left(\frac{2}{\rho(x_0)}\right)^{N-p} \frac{\int_{B(x_0, \rho(x_0)/2)} F(x, c)\,dx}{c^p} = \frac{1}{\lambda_*}. \tag{3.14}$$

The proof of Step 3 is concluded in view of (3.11) and (3.14) once observed that assumption (jj) implies that $\frac{1}{\lambda^*} < \frac{1}{\lambda_*}$.

Step 4. For r and $\tilde{u}$ as in (3.6) and (3.7) one has $0 < \Phi(\tilde{u}) < r$.

Indeed, put

$$\tilde{K} = \left[\frac{a_1}{a_2}\frac{2^{N-p}}{[\rho(x_0)]^{N-p}(2^N - 1)|B_1|}\right]^{1/p} \frac{|\Omega|^{1/p^*}}{T},$$

and observe that

$$\tilde{K}^p \ge K \int_{B(x_0, \rho(x_0)/2)} a(x)\,dx.$$

In fact, a direct computation shows that

$$\frac{\tilde{K}^p}{K} = \|a\|_\alpha |\Omega|^{1/\alpha'}$$

and clearly

$$\int_{B(x_0, \rho(x_0)/2)} a(x)\,dx \le \|a\|_1 \le |\Omega|^{1/\alpha'} \|a\|_\alpha.$$

Now we claim that

$$c < \tilde{K} d. \tag{3.15}$$

Suppose (3.15) false. Then, since from $(G_{f,a,s,q})$ we have already observed that $|F(x,c)| \le a(x)\left(\frac{b_s}{s}c^s + \frac{b_q}{q}c^q\right)$ for a.a. $x \in \Omega$, recalling that $c < d$, end exploiting assumption (jj), one has

$$
\begin{aligned}
\frac{\frac{b_s}{s}d^s + \frac{b_q}{q}d^q}{d^p} &\ge \tilde{K}^p \frac{\frac{b_s}{s}d^s + \frac{b_q}{q}d^q}{c^p} > \tilde{K}^p \frac{\frac{b_s}{s}c^s + \frac{b_q}{q}c^q}{c^p} \\
&= \frac{\tilde{K}^p}{\int_{B(x_0,\rho(x_0)/2)} a(x)\,dx} \int_{B(x_0,\rho(x_0)/2)} a(x) \frac{\frac{b_s}{s}c^s + \frac{b_q}{q}c^q}{c^p}\,dx \\
&\ge K \frac{\int_{B(x_0,\rho(x_0)/2)} F(x,c)\,dx}{c^p} \\
&> \frac{\frac{b_s}{s}d^s + \frac{b_q}{q}d^q}{d^p},
\end{aligned}
$$

and we obtain a contradiction. Finally, from (3.12) and (3.15) one has

$$
\Phi(\tilde{u}) < \frac{a_2}{p}\left(\frac{\rho(x_0)}{2}\right)^{N-p}(2^N - 1)|B_1|\tilde{K}^p d^p = r
$$

and the proof of Step 4 is complete.

Putting together Step 1–Step 4 we can apply Theorem 2.1. In particular, from (3.11) and (3.14) it is obvious that $]\lambda_*, \lambda^*[\subseteq \left]\frac{\Phi(\tilde{u})}{\Psi_f(\tilde{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi_f(u)}\right[$. Hence, for every $\lambda \in]\lambda_*, \lambda^*[$ the functional $I_\lambda = \Phi - \lambda\Psi_f$ admits at least two non-zero critical points, namely, in view of (2.10), our proof is complete. □

REMARK 3.1. Condition (AR) is crucial in the proof of Step 1 and Step 2. Namely, as usual, it is the main tool in order to assure that the energy functional associated to the problem is unbounded and satisfies the (PS)-condition.

REMARK 3.2. When f is a non zero function such that $f(x,t) \ge 0$ for a.a. $x \in \Omega$ and every $t \ge 0$ the solutions established in Theorem 3.1 are positive. Indeed, we make use of classical truncation arguments and consider the functions

$$
f_+(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in \Omega \times [0,+\infty[\\ f(x,0) & \text{if } (x,t) \in \Omega \times]-\infty,0[\end{cases}
$$

and $F_+(x,t) = \int_0^t f_+(x,\xi)\,d\xi$ for every $(x,t) \in \Omega \times \mathbb{R}$. At this point, we can apply Theorem 3.1 to the function $f_+(x,t)$. Hence, problem

$$
\begin{cases} -\operatorname{div} \boldsymbol{A}(x,\nabla u) = \lambda f_+(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.16}
$$

admits at least two nontrivial weak solutions. But, since $\tilde{f}$ is non negative, it is simple to verify that every weak solution of (3.16) is non negative. Thus the solutions of (3.16) solve also (P_λ). Finally, the classical regularity theory assures that the weak solutions of (P_λ) are continuous (see [20]) and by the strong maximum principle, [34, Theorem 11.1], we achieve the announced positivity.

REMARK 3.3. Taking into account Remark 3.1, condition (AR) can be required only for positive t provided f, as considered in Remark 3.2, is such that $f(x,t) \geq 0$ for a.a. $x \in \Omega$ and every $t \geq 0$. Indeed, also in this case, because of the definitions of f_+ and F_+, coming back to the proof of Theorem 3.1, the energy functional $\Phi - \lambda\Psi_{f_+}$ is unbounded from below and satisfies the (PS)-condition.

REMARK 3.4. A direct computation shows that if $1 < q < p^*$ and $N > p$, then

$$\frac{Np}{Np - Nq + pq} > \frac{N}{p} \quad \text{if and only if } p < q.$$

Indeed,

$$\frac{Np}{Np - Nq + pq} > \frac{N}{p} \Leftrightarrow p^2 > Np - Nq + pq \Leftrightarrow q(N-p) > p(N-p).$$

Hence, Theorem 3.1 represents a kind of counterpart of [8, Theorem 3.1] where the nonlinear term $f(x,\cdot)$ is assumed to satisfy a global growth condition that, compared with our $(G_{f,a,s,q})$, looks at the complementary case $q < p$ under the condition $\alpha > N/p$.

A simple autonomous version of Theorem 3.1 can be stated as follows.

COROLLARY 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nonnegative function such that*

$$|f(t)| \leq b_s|t|^{s-1} + b_q|t|^{q-1} \tag{3.17}$$

for every $t \in \mathbb{R}$, with $b_s, b_q > 0$, $1 \leq s \leq p$ and $p < q < p^*$. Put $F(t) = \int_0^t f(\xi)\, d\xi$ for every $t \in \mathbb{R}$, and assume that there exist $c, d > 0$ with $c < d$ such that

$$\left(\frac{b_s}{s} d^{s-p} + \frac{b_q}{q} d^{q-p}\right) < H \frac{F(c)}{c^p}, \tag{3.18}$$

where $H = \frac{a_1}{a_2} \frac{1}{T^p(2^N-1)|\Omega|^{(p^*-p)/p^*}} \left(\frac{\rho(x_0)}{2}\right)^p$ and $\rho(x_0) = \max_{x\in\Omega} \rho(x)$. In addition suppose that there exist $\mu > (a_2/a_1)p$ and $R > 0$ such that

$$0 < \mu F(t) \leq tf(t) \tag{AR$'$}$$

for all $t \geq R$.

Then, for every $\lambda \in \left]\frac{a_1}{pT^p|\Omega|^{(p^*-p)/p^*}}\frac{1}{H}\frac{c^p}{F(c)}, \frac{a_1}{pT^p|\Omega|^{(p^*-p)/p^*}}\frac{1}{\frac{b_s}{s}d^{s-p}+\frac{b_q}{q}d^{q-p}}\right[$ *the problem*

$$\begin{cases} -\operatorname{div} \boldsymbol{A}(x, \nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two positive weak solutions.

Proof. Simply apply Theorem 3.1 with $a \equiv 1$, $\alpha = \infty$, $\alpha' = 1$ (see also Remark 3.2 and Remark 3.3). □

As a particular case of Theorem 3.1 a multiplicity result can be derived whenever $F(x, \cdot)$ is p-sublinear at zero.

Theorem 3.2. *Assume that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies both conditions* $(G_{f,a,s,q})$ *and* (AR). *Moreover suppose that there exists* $x_0 \in \Omega$ *such that*

(j′) *there exists* $\delta > 0$ *such that* $F(x, t) \geq 0$ *a.e. in* $B(x_0, \rho(x_0))$ *and for all* $t \in]0, \delta[$;

(jjj) $\limsup_{t\to 0^+} \frac{F(x,t)}{t^p} = +\infty$ *uniformly a.e. in* $B(x_0, \rho(x_0)/2)$.

Put

$$\sigma^* = \frac{a_1}{\|a\|_\alpha pT^p|\Omega|^{(p^*-\alpha'p)/(p^*\alpha')}}\left(\frac{s}{b_s}\right)^{\frac{q-p}{q-s}}\frac{q-p}{q-s}\left(\frac{q}{b_q}\frac{p-s}{q-p}\right)^{(p-s)/(q-s)}.$$

Then, for every $\lambda \in]0, \sigma^*[$ *problem* (P_λ) *admits at least two non trivial weak solutions.*

Proof. First observe that $\sigma^* = \max_{d>0} \lambda^*(d)$, where $\lambda^*(d)$ is defined in (3.3). Indeed, if we put $h(d) = \frac{b_s}{s}d^{s-p} + \frac{b_2}{q}d^{q-p}$ for every $d > 0$, a direct computation shows that

$$h(\bar{d}) = \left(\frac{b_s}{s}\right)^{(q-p)/(q-s)}\frac{q-s}{q-p}\left(\frac{b_q}{q}\frac{q-p}{p-s}\right)^{(p-s)/(q-s)} = \min_{d>0} h(d)$$

with $\bar{d} = \left(\frac{q}{s}\frac{b_s(p-s)}{b_q(q-p)}\right)^{1/(q-s)}$. Hence, in conclusion,

$$\sigma^* = \frac{a_1}{\|a\|_\alpha pT^p|\Omega|^{(p^*-\alpha'p)/(p^*\alpha')}}\frac{1}{h(\bar{d})} = \lambda^*(\bar{d}) = \max_{d>0} \lambda^*(d).$$

Fix $\lambda \in]0, \sigma^*[$, and choose $d > 0$ such that $\lambda < \lambda^*(d)$. From assumption (jjj) one has

$$K\frac{a_1}{\|a\|_\alpha pT^p|\Omega|^{(p^*-\alpha'p)/(p^*\alpha')}}\limsup_{t\to 0^+}\frac{\int_{B(x_0),\rho(x_0)/2} F(x, t)\,dx}{t^p} = +\infty,$$

with K as introduced in (3.1). Hence, there is $c \in]0, \min\{\delta, d\}[$ such that

$$K \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}} \frac{\int_{B(x_0),\rho(x_0)/2} F(x,c)\,dx}{c^p} > \frac{1}{\lambda} > \frac{1}{\lambda^*(d)},$$

that is

$$\lambda_*(c) < \lambda < \lambda^*(d).$$

Thus, taking also in mind (j′) and the choice of c, all the assumptions of Theorem 3.1 are satisfied and the conclusion follows at once. □

REMARK 3.5. I we assume

(jjj′) $\lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = +\infty$

then (j′) and (jjj) hold. So, condition (jjj′) ensures the conclusion of Theorem 3.2.

In the same spirit of Corollary 3.1 we now present an autonomous version of Theorem 3.2.

THEOREM 3.3. *Let $f : [0, +\infty[\to \mathbb{R}$ be a continuous function satisfying the growth condition* (3.17), *as well as* (AR′). *Put $F(t) = \int_0^t f(s)\,ds$ for every $t \in [0, +\infty[$ and assume that*

$$\lim_{t\to 0^+} \frac{F(t)}{t^p} = +\infty. \tag{3.19}$$

Then, if

$$\tau^* = \frac{a_1}{pT^p|\Omega|^{(p^*-p)/(p^*)}} \left(\frac{s}{b_s}\right)^{\frac{q-p}{q-s}} \frac{q-p}{q-s} \left(\frac{q}{b_q}\frac{p-s}{q-p}\right)^{(p-s)/(q-s)},$$

for every $\lambda \in]0, \tau^*[$ the problem

$$\begin{cases} -\operatorname{div} \boldsymbol{A}(x, \nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

admits at least two positive solutions.

PROOF. Apply Theorem 3.2 with $a(x) \equiv 1$, $\alpha = \infty$ and $\alpha' = 1$ (see also Remarks 3.2 and 3.5). □

In [13], the problem

$$\begin{cases} -\operatorname{div} \boldsymbol{A}(x, \nabla u) = \lambda(a(x)|u|^{p-2}u + f(x,u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied when f satisfies condition $(G_{f,a,1,q})$, with $1 < q < p$ in addition to a $(p-1)$-superlinear behaviour at zero, so that the whole nonlinear reaction term is asymptotically $(p-1)$-linear both at zero and at infinity. More recently, in [8] the same problem has been considered under different conditions on f which are compatible with a $(p-1)$-superlinearity at zero and still require its $(p-1)$-sublinearity at infinity. Here, we wish to point out that the complementary case when the nonlinear term is of concave-convex type can be addressed.

THEOREM 3.4. *Let $\gamma \in L^\infty(\Omega)$ and $\zeta \in L^\alpha(\Omega)$ be two functions with $\alpha > \frac{Np}{Np-Nq+pq}$ and $1 \le s < p \le \frac{a_2}{a_1} p < q < q^*$ and $\min\{\operatorname{essinf}_{x\in\Omega}\gamma(x), \operatorname{essinf}_{x\in\Omega}\zeta(x)\} > 0$. Put $a = \max\{1,\zeta\}$ and let*

$$\chi^* = \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}} \left(\frac{s}{\|\gamma\|_\infty}\right)^{\frac{q-p}{q-s}} \frac{q-p}{q-s} \left(q\frac{p-s}{q-p}\right)^{(p-s)/(q-s)}.$$

Then, for every $\lambda \in]0, \chi^*[$ problem

$$\begin{cases} -\operatorname{div} \boldsymbol{A}(x,\nabla u) = \lambda(\gamma(x)|u|^{s-2}u + \zeta(x)|u|^{q-2}u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two positive weak solutions.

PROOF. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x,t) = \begin{cases} \gamma(x)|t|^{s-2}t + \zeta(x)|t|^{q-2}t & \text{if } t \ge 0, \\ 0 & \text{if } t < 0, \end{cases}$$

for every $x \in \Omega$. Clearly, f satisfies $(G_{f,a,s,q})$ with $b_s = \|\gamma\|_\infty$, $b_q = 1$. Indeed

$$|f(x,t)| \le \|\gamma\|_\infty |t|^{s-1} + \zeta(x)|t|^{q-1} \le \max\{1,\zeta(x)\}(\|\gamma\|_\infty |t|^{s-1} + |t|^{q-1})$$

for every $t \in \mathbb{R}$ and a.a. $x \in \Omega$. Moreover, since

$$F(x,t) = \begin{cases} \frac{\gamma(x)}{s}|t|^s + \frac{\zeta(x)}{q}|t|^q & \text{if } t \ge 0, \\ 0 & \text{if } t < 0 \end{cases}$$

a direct computation shows that if $\mu \in \left]\frac{a_1}{a_2}p, q\right[$,

$$\begin{aligned} \frac{tf(x,t) - \mu F(x,t)}{|t|^s} &= \zeta(x)\left(1 - \frac{\mu}{q}\right)|t|^{q-s} - \frac{\mu}{s}\gamma(x) + \gamma(x) \\ &\ge \operatorname{essinf}_{x\in\Omega}\zeta\left(1 - \frac{\mu}{q}\right)|t|^{q-s} - \frac{\mu}{s}\|\gamma\|_\infty \end{aligned}$$

a.e. in Ω and for every $t > 0$. Thus, condition (AR) holds for $t > 0$ large enough.

Observe that

$$\frac{F(x,t)}{t^p} \ge \frac{\gamma(x)t^s}{st^p} \ge (\operatorname{essinf}_\Omega \gamma(x))\frac{1}{s}t^{s-p}$$

a.e. in Ω and for every $t > 0$, namely

$$\lim_{t\to 0^+} \frac{F(x,t)}{t^p} = +\infty$$

uniformly a.e. in Ω. Finally, since $\chi^* = \sigma^*$ (recall the choice of b_s and b_q), for all $\lambda \in]0, \chi^*[$, arguing as in the proof of Theorem 3.2, taking also into account Remarks 3.2 and 3.3, the problem under consideration has at least two nontrivial solutions that, in view of the structure of f and the strong maximum principle are positive. □

We conclude by showing a consequence of the previous result when a concave and convex nonlinearity is still considered.

Corollary 3.2. *Assume that the assumptions of Theorem* 3.4 *are satisfied. Put*

$$\eta^* = \left(\frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}}\right)^{\frac{q-s}{q-p}} \frac{s}{\|\gamma\|_\infty}(q-p) q^{\frac{p-s}{q-p}} \left(\frac{(p-s)^{p-s}}{(q-s)^{q-s}}\right)^{1/(q-p)}.$$

Then, for every $\theta \in]0, \eta^*[$ the problem

$$\begin{cases} -\operatorname{div} \boldsymbol{A}(x, \nabla u) = \theta\gamma(x)|u|^{s-2}u + \zeta(x)|u|^{q-2}u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.21}$$

admits at least two positive weak solutions.

Proof. Fix $\theta \in]0, \eta^*[$. We apply Theorem 3.4 considering the function $\theta\gamma \in L^\infty(\Omega)$. Therefore, the problem

$$\begin{cases} -\operatorname{div} \boldsymbol{A}(x, \nabla u) = \lambda(\theta\gamma(x)|u|^{s-2}u + \zeta(x)|u|^{q-2}u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two positive weak solutions for each $\lambda \in]0, \sigma^*[$, where

$$\sigma^* = \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{(p^*-\alpha' p)/(p^*\alpha')}} \left(\frac{s}{\theta\|\gamma\|_\infty}\right)^{\frac{q-p}{q-s}} \frac{q-p}{q-s} \left(q\frac{p-s}{q-p}\right)^{(p-s)/(q-s)}.$$

Taking into account that from $\theta < \eta^*$ one has that $1 \in]0, \sigma^*[$, the conclusion is achieved. □

Remark 3.6. Starting from [1, 3, 4] a great interest has been devoted to the study of the existence of at least two solutions for differential problems involving the p-Laplacian both for semilinear equations, namely $p = 2$, and nonlinear equations, that is $p \neq 2$ (see also [7, 16, 18, 22, 23] as well as [25, 32], where a non-homogeneous operator is considered). In comparison with the present literature, the previous Theorem 3.2 covers the case when a more general operator than the p-Laplacian is considered. However, we explicitly point out that in presenting Theorem 3.2 our investigation does not consider the best interval for which the problem under examination admits at least to positive solutions. We simply emphasized that the existence of multiple positive solutions can be easily derived from the more general theorems previously proved.

ACKNOWLEDGMENTS. The first two authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017 – Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW). The research of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI–UEFISCDI, project number PCE 137/2021, within PNCDI III.

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Received 30 December 2020,
and in revised form 1 June 2021

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