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Let a be a positive real number. Assume that f is a twice differentiable function from $\mathbb{R}$ into $\mathbb{R}$ such that $f(0) = a$ and satisfying the differential equation $f'' = -f(1-f^2)$. Moreover, we suppose that f is periodic with principal period $T = T(a)$ and that f^2 achieves its maximum at the origin.

(a) *Prove that $a \le 1$ and deduce that f is well defined on the real axis.*

(b) *Prove that $T > 2\pi$.*

(c) *Denote $t_0(a) := \sup\{t > 0;\ f > 0 \text{ in } (0,t)\}$. Show that f is decreasing on $(0, t_0(a))$ and express T in terms of $t_0(a)$.*

(d) *Prove that $a^2 + \left(\frac{2\pi}{T}\right)^2 > 1$.*

(e) *Show that the mapping $t_0 : (0,1) \to \mathbb{R}$ is increasing and compute $\lim_{a \searrow 0} t_0(a)$ and $\lim_{a \nearrow 1} t_0(a)$.*

(f) *Deduce that for any $T > 2\pi$, there exists a unique periodic function $f : \mathbb{R} \to \mathbb{R}$ with principal period T such that $f'' = -f(1-f^2)$ and f^2 achieves its maximum at the origin.*

Solution. **(a)** Arguing by contradiction, let us assume that $a > 1$. Using the differential equation satisfied by f, it follows that $f''(0) > 0$ which contradicts our hypothesis that f^2 achieves its maximum at the origin. If $a = 1$, we get only the trivial solution $f \equiv 1$. That is why we shall assume in what follows that $a \in (0,1)$. Multiplying by f' in the differential equation satisfied by f and integrating, we find

$$f'^2 = -f^2 + \frac{1}{2}f^4 + a^2 - \frac{1}{2}a^4\,. \tag{1}$$

It follows that, as far as a function f with the required properties exists, we have $|f(x)| \le a$ and $|f'(x)| \le \left(a^2 - \frac{a^4}{2}\right)^{1/2}$, for all $x \in \mathbb{R}$. Hence f is globally defined.

(b) We first observe that f cannot be positive (resp, negative) on an infinite interval, provided that f is periodic. Indeed, in this case, f would be a periodic concave (resp., convex) function, that is, a constant function. But this is impossible, due to our choice of a.

Let x_1, x_2 be two consecutive zeros of f. We may suppose that $f(x) > 0$ if $x_1 < x < x_2$, so that $f'(x_1) > 0$ and $f'(x_2) < 0$. If x_3 denotes the smallest $x > x_2$ such that $f(x_3) = 0$, it follows that $f(x) < 0$, for any $x \in (x_2, x_3)$. If we prove that $x_2 - x_1 > \pi$, it will also follow that $x_3 - x_1 > 2\pi$ and that there is no $x \in (x_1, x_3)$ such that $f(x) = 0$ and $f'(x) > 0$. This means that the principal period of f **must** be greater than 2π. This will be done in the following auxiliary result.

Lemma 1. *Let $\Psi : \mathbb{R} \to [0,1]$ be such that the set $\{x;\ \Psi(x) = 0 \ \text{ or } \ \Psi(x) = 1\}$ contains only isolated points. Let f be a real function such that $f(x_1) = f(x_2) = 0$, and $f > 0$ in (x_1, x_2). Assume that $-f'' = f\Psi$ in $[x_1, x_2]$. Then $x_2 - x_1 > \pi$.*

Proof of Lemma. We may assume that $x_1 = 0$. Multiplying by $\varphi(x) := \sin\frac{\pi x}{x_2}$ in the differential equation $-f'' = f\Psi$ and integrating by parts, we obtain

$$\int_0^{x_2} f\varphi dx > \int_0^{x_2} f\Psi\varphi dx = \frac{\pi^2}{x_2^2}\int_0^{x_2} f\varphi dx\,,$$

that is, $x_2 > \pi$.

(c) Since $f'(0) = 0$ and $f''(0) < 0$, it follows that f decreases for small $x > 0$. Moreover, $f'(x) < 0$ for $0 < x < t_0(a)$. Indeed, suppose the contrary. Then, taking into account relation (1), we obtain the existence of some $\tau > 0$ with $\tau < t_0(a)$ and such that $f(\tau) = a$. If we consider the smallest $\tau > 0$ such that the above equality holds true, then $f(x) < a$ for any $0 < x < \tau$. Since $f(0) = f(\tau) = a$, it follows that there exists some $0 < t_1 < \tau$ such that $f'(t_1) = 0$, which is the desired contradiction. Hence

$$f' = -\sqrt{a^2 - \frac{a^4}{2} - f^2 + \frac{f^4}{2}} < 0 \qquad \text{in } \ (0, t_0(a))\,.$$

It follows that, for any $0 < x < t_0(a)$,

$$\int_{f(x)}^{a} \frac{dt}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} = x\,, \tag{2}$$

which yields

$$t_0(a) = \int_0^a \frac{dt}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)[1 - \frac{a^2}{2}(1+\xi^2)]}}\,. \tag{3}$$

Taking into account the differential equation satisfied by f we first deduce that

$$f(t_0(a) + x) = -f(t_0(a) - x)\,. \tag{4}$$

Indeed, both functions $g(x) = f(t_0(a) + x)$ and $h(x) = -f(t_0(a) - x)$ are solutions of the Sturm-Liouville problem

$$\begin{cases} z'' = -z(1 - z^2), & \text{in } (0, t_0(a)) \\ z(0) = 0,\ z'(0) = f'(t_0(a))\,. \end{cases}$$

Using now the uniqueness of the solution to the above boundary value problem we deduce relation (4). Next, similar arguments imply $f(2t_0(a) - x) = -f(x)$ and $f(4t_0(a) + x) = f(x)$. It follows that f is periodic and its principal period is $T(a) = 4t_0(a)$.

We observe that **(b)** easily follows from the above results. Indeed, relation (3) yields

$$t_0(a) > \int_0^1 (1-\xi^2)^{-1/2} d\xi = \frac{\pi}{2}.$$

So, by $T(a) = 4t_0(a)$, we obtain **(b)**.

We may give the following alternative proof in order to justify that f decreases on the interval $(0, t_0(a))$. Using the differential equation $f'' = -f(1 - f^2)$ in conjunction with $f > 0$ on $(0, t_0(a))$ and $f^2 \le a^2 < 1$, it follows that $f'' < 0$ on $(0, t_0(a))$. Hence f' is decreasing on $(0, t_0(a))$, that is, $f'(x) < f'(0) = 0$ for any $x \in (0, t_0(a))$.

(d) Since $T(a) = 4t_0(a)$, it is enough to show that $\sqrt{1-a^2}\, t_0(a) < \frac{\pi}{2}$. Relation (3) yields

$$\sqrt{1-a^2}\, t_0(a) = \int_0^1 \frac{\sqrt{1-a^2}}{\sqrt{(1-\xi^2)[1-\frac{a^2}{2}(1+\xi^2)]}} d\xi < \int_0^1 \frac{1}{\sqrt{1-\xi^2}} d\xi = \frac{\pi}{2}\,.$$

(e) Relation (3) implies that the mapping $a \longmapsto t_0(a)$ is increasing and $\lim_{a \searrow 0} t_0(a) = \frac{\pi}{2}$, $\lim_{a \nearrow 1} t_0(a) = +\infty$.

(f) Since $t_0'(a) > 0$, it follows that the mapping $T(a) \longmapsto a := a(T)$ is analytic. Taking into account relation (2), we conclude the proof. Moreover, we have $\lim_{T \searrow 2\pi} a(T) = 0$ and $\lim_{T \nearrow +\infty} a(T) = 1$.