Combined effects of singular nonlinearities and potentials in elliptic equations

Vicenţiu RĂDULESCU*

Institute of Mathematics "Simion Stoilow" of the Romanian Academy,

P.O. Box 1-764, 014700 Bucharest, Romania

Department of Mathematics, University of Craiova, 200585 Craiova, Romania

E-mail: `vicentiu.radulescu@imar.ro` `http://inf.ucv.ro/~radulescu`

Abstract. In this paper we report on some recent results related to various singular phenomena arising in the study of some classes of nonlinear elliptic equations.We discuss qualitative results on the existence, nonexistence or the uniqueness of solutions and we focus on the following types of problems: (i) blow-up boundary solutions of logistic equations; (ii) Lane-Emden-Fowler equations with singular nonlinearities and subquadratic convection term.

2000 Mathematics Subject Classification: 35A20, 35B32, 35B40, 35B50, 35J60, 47J10, 58J55.

Key words: nonlinear elliptic equation, dingularity, boundary blow-up, bifurcation, asymptotic analysis, maximum principle, Karamata regular variation theory.

1 Motivation and previous results

In this paper I will report on some results contained in the recent papers [1, 2, 3, 4, 5, 6] that are closely related either to the study of blow-up boundary solutions or to elliptic partial differential equations involving singular nonlinearities. I refer to the book [7] for a comprehensive study of such problems and further related results.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary. We are interested in the study of the following classes of nonlinear elliptic problems.

I. Logistic equation

$$\begin{cases} \Delta u = \Phi(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

*Plenary talk delivered at the Second Workshop Series on Applied Mathematics, Piteşti, November 2–4, 2007

II. Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u = \Psi(x,u,\nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Φ is *smooth*, but Ψ may have *singularities.*

Example: Let $\Phi(u,\nabla u) = u^p$, with $p > 1$. Then $v = u^{-1}$ satisfies (1.2) for $\Psi(v,\nabla v) = v^{2-p} - 2v^{-1}|\nabla v|^2$.

We recall the following classical results:

$\Diamond$ 1916, Bieberbach [8]: $\Phi(x,u,\nabla u) = \exp(u)$, $N = 2$: $u(x) = \log(d(x)^{-2}) + O(1)$ as $x \to \partial\Omega$

$\Diamond$ 1943, Rademacher [9]: $\Phi(x,u,\nabla u) = \exp(u)$, $N = 3$ (Lazer and McKenna [10] studied the case of an arbitrary dimension N)

$\Diamond$ 1957, Keller & Osserman [11, 12]: $\Phi(x,u,\nabla u) = f(u)$, where $f \in C^1[0,\infty)$, $f'(s) \geq 0$ for $s \geq 0$, $f(0) = 0$ and $f(s) > 0$ if $s > 0$. Then problem (1.1) has a solution if and only if

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

$\Diamond$ 1974, Loewner and Nirenberg [13]: $f(u) = u^{(N+2)/(N-2)}$, $N > 2$

1) Existence of solutions

M. Crandall, P. H. Rabinowitz, and L. Tartar proved in [14] that the BVP

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$.

2) Non-existence results

M. Coclite and G. Palmieri showed in [15] that the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided $0 < \alpha < 1$ and $\lambda_1 \geq 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

3) Multiplicity and uniqueness

Problems of this type become difficult even in simple cases. In his PhD thesis (2001), J. Shi studied the existence of radial symmetric solutions of the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $0 < \alpha,\, p < 1$, $\lambda > 0$ and B_1 is the unit ball in $\mathbb{R}^N$. He showed that there exists $\lambda_1 > \lambda_0 > 0$ such that the above problem has no solutions for $\lambda < \lambda_0$, one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, two solutions for $\lambda_1 \geq \lambda > \lambda_0$.

We refer to the papers [16]–[22] for related results on stationary equations with singular nonlinearity.

2 A problem of H. Brezis

Consider the problem

$$\Delta u + au = b(x)f(u) \qquad \text{in } \Omega, \tag{2.1}$$

$a \in \mathbb{R}$, $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, $b \geq 0$, $b \not\equiv 0$ dans Ω. Assume that $f \in C^1[0,\infty)$ satisfies

(A_1) $f \geq 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

(A_2) $\displaystyle\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad F(t) = \int_0^t f(s)\,ds.$

Examples:

(i) $f(u) = e^u - 1$; (ii) $f(u) = u^p,\ p > 1$; (iii) $f(u) = u[\ln(u+1)]^p,\ p > 2$.

Problem, H. Brezis, 2001. Find a necessary and sufficient condition such that problem (2.1) has a blow-up boundary solution

Set

$$\Omega_0 = \operatorname{int}\{x \in \Omega :\ b(x) = 0\}$$

and assume that $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$.

Denote $\lambda_1(\Omega_0)$ the first eigenvalue of $(-\Delta)$ in Ω_0 and set $\lambda_1(\Omega_0) = \infty$ if $\Omega_0 = \emptyset$.

Alama & Tarantello (1996) studied in [23] the same equation under the Dirichlet condition $u = 0$ on $\partial\Omega$: there is a positive solution u_a if and only if $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$.

Theorem 2.1. *(Cîrstea & Rădulescu, CRAS, 2003, [24]) Assume f satisfies (A_1) and (A_2). Then problem (2.1) has a blow-up boundary solution if and only if $a \in (-\infty, \lambda_1(\Omega_0))$.*

2.1 Karamata's regular variation theory

Definition 2.1. *$R : [D, \infty) \to [0, \infty)$ measurable has regular variation at $+\infty$ of index $q \in \mathbb{R}$ (notation: $R \in RV_q$) provided that for all $\xi > 0$,*

$$\lim_{u \to \infty} R(\xi u)/R(u) = \xi^q.$$

$q = 0$: weak variation.

$R \in RV_q \implies R(u) = u^q L(u), \quad L \in RV_0.$

Examples: (i) $R(u) = u^q$, $R \in RV_q$.
(ii) The mappings $\ln(1+u)$, $\ln\ln(e+u)$, $\exp\{(\ln u)^\alpha\}$, $\alpha \in (0,1)$ are in RV_0.

Lemma 2.1. *Assume (A_1). The following properties are equivalent:*
a) $f' \in RV_\rho$
b) $\lim_{u\to\infty} uf'(u)/f(u) := \vartheta < \infty$
c) $\lim_{u\to\infty} (F/f)'(u) := \gamma > 0$.

Remark 2.1. *We have:*
(i) $\rho \geq 0$;
(ii) $\gamma = 1/(\rho+2) = 1/(\vartheta+1)$;
(iii) If $\rho \neq 0$, then $(K-O)$. Converse **not** *true: $f(u) = u\ln^4(u+1)$. It may happen that $\rho = 0$ and $(K-O)$ is fulfilled, so Eq. (2.1) does not have blow-up boundary solutions. Examples: $f(u) = u$, $f(u) = u\ln(u+1)$.*

Karamata's class. Let $\mathcal{K}$ denote the class of functions $k : (0,\nu) \to (0,\infty)$ of class C^1, increasing and such that $\lim_{t\to 0^+} \left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i, \; i = \overline{0,1}$.

Then $\ell_0 = 0$ and $\ell_1 \in [0,1]$, for all $k \in \mathcal{K}$.

Lemma 2.2. *Assume $S \in C^1[D,\infty)$ such that $S' \in RV_q$, $q > -1$. Then*
a) If $k(t) = \exp\{-S(1/t)\}$ $\forall t \leq 1/D$, then $k \in \mathcal{K}$ and $\ell_1 = 0$.
b) If $k(t) = 1/S(1/t)$ $\forall t \leq 1/D$, then $k \in \mathcal{K}$ and $\ell_1 = 1/(q+2) \in (0,1)$.
c) If $k(t) = 1/\ln S(1/t)$ $\forall t \leq 1/D$, then $k \in \mathcal{K}$ and $\ell_1 = 1$.

Remark 2.2. *If $S \in C^1[D,\infty)$, then $S' \in RV_q$ with $q > -1$ if and only if $\exists\, m > 0$, $C > 0$ and $B > D$ such that $S(u) = Cu^m \exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, $\forall u \geq B$, where $y \in C[B,\infty)$ satisfies $\lim_{u\to\infty} y(u) = 0$. In such a case, $S' \in RV_q$ with $q = m-1$.*

Theorem 2.2. *(Cîrstea & Rădulescu, 2004, [25]). Assume (A_1) and $f' \in RV_\rho$, with $\rho > 0$. Suppose $b \equiv 0$ on $\partial\Omega$ such that*

(B) $b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *as $d(x) \to 0$, where $c > 0$ and $k \in \mathcal{K}$.*

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, Eq. (2.1) has a unique blow-up boundary solution u_a. Moreover,

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \tag{2.2}$$

where $\xi_0 = \left(\dfrac{2+\ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$ *and h is defined by*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,\nu). \tag{2.3}$$

Examples of admissible functions k:
$k(t) = -1/\ln t$, $k(t) = t^\alpha$, $k(t) = \exp\{-1/t^\alpha\}$,
$k(t) = \exp\left\{-\ln(1+\frac{1}{t})/t^\alpha\right\}$,
$k(t) = \exp\left\{-\left[\arctan\left(\frac{1}{t}\right)\right]/t^\alpha\right\}$,
$k(t) = t^\alpha/\ln(1+\frac{1}{t})$, where $\alpha > 0$.

2.2 An example

$$\begin{cases} -\Delta u = p(d(x))g(u) + \lambda|\nabla u|^a + \mu f(x,u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Notation: $d(x) = \mathrm{dist}(x, \partial\Omega)$, $\lambda \in \mathbb{R}$, $\mu > 0$, and $0 < a \le 2$.

We assume that $g \in C^1(0,\infty)$ verifies

$(g1)$ g is a positive decreasing function such that $\lim_{t\searrow 0} g(t) = +\infty$.

The standard example: $g(t) = t^{-\alpha}$, with $\alpha > 0$.

We also assume that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and such that f is positive in $\Omega \times (0,\infty)$.

The case $a = 2$ is a special one, since by the change of variable (often called *Gelfand transform*) $v = e^{\lambda u} - 1$ we obtain a new singular problem without gradient term. If $f(x,u)$ depends on u, this change of variable does not preserve neither the sublinearity conditions $(f1)-(f2)$ on f, nor the monotonicity of g.

Let

$$m := \lim_{t\to\infty} g(t) \in [0,\infty).$$

Theorem 2.3. *Assume that $a = 2$, $\lambda \ge 0$, $\mu > 0$ and $p \equiv 1$, $f \equiv 1$.*

(i) *The problem (2.4) has a solution if and only if $\lambda(m+\mu) < \lambda_1$;*

(ii) *Assume* $\mu > 0$ *is fixed and let* $\lambda^* = \lambda_1/(m+\mu)$. *Then (2.4) has a unique solution* u_λ *for every* $0 \le \lambda < \lambda^*$ *and the sequence* $(u_\lambda)_{0\le\lambda<\lambda^*}$ *is increasing with respect to* λ. *Moreover, if* $\limsup_{s\searrow 0} s^\alpha g(s) < \infty$, *for some* $\alpha \in (0,1)$, *then the sequence of solutions* $(u_\lambda)_{0\le\lambda<\lambda^*}$ *has the following properties:*

- (ii1) *there exist two positive constants* c_1, c_2 *depending on* λ *such that* $c_1 d(x) \le u_\lambda \le c_2 d(x)$ *in* Ω;
- (ii2) $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;
- (ii3) $\lim_{\lambda\nearrow\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω.

Fig. 1 corresponds to (i) and $a = 0$ (resp., $a > 0$), while Fig. 2 is related to (ii), $\lambda > 0$ and $\mu =$ fixed.

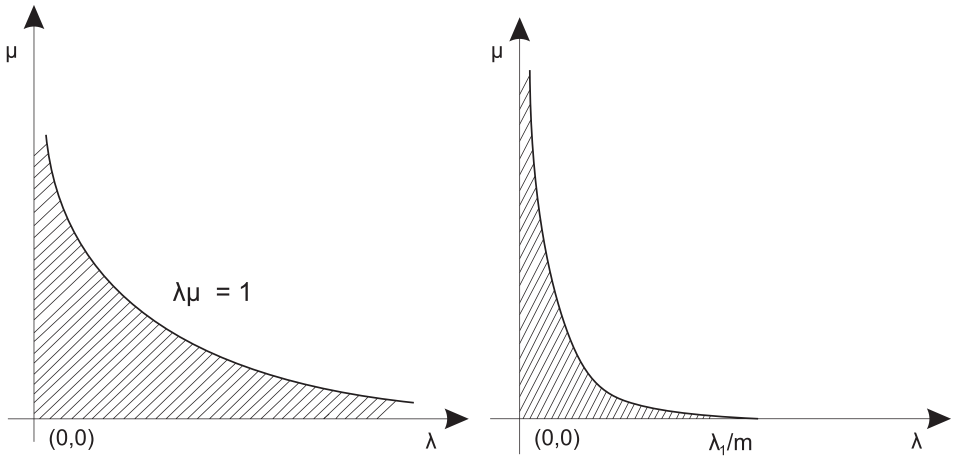

Figure 1: The bifurcation diagrams in Theorem 2.3 (i).

3 Singular Lane-Emden-Fowler equations

Consider the problem

$$\begin{cases} -\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)^\pm$$

where $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) is a bounded domain with smooth boundary, $d(x) = \mathrm{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, and $0 < a \le 2$.

We assume that

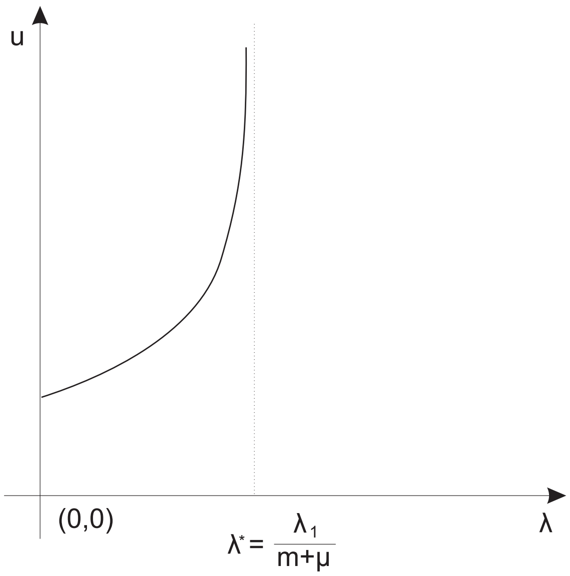

Figure 2: The bifurcation diagram in Theorem 2.3 (ii).

- $g \in C^1(0,\infty)$ is a positive decreasing function and

  $(g1)\qquad \lim_{t\to 0^+} g(t) = +\infty.$

- $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and such that f is positive on $\overline{\Omega} \times (0,\infty)$. Furthermore, f is either linear or f is sublinear with respect to the second variable. This last case means that f fulfills the hypotheses

  $(f1)\qquad (0,\infty) \ni t \longmapsto \dfrac{f(x,t)}{t}$ is nonincreasing,
  for all $x \in \overline{\Omega}$;

  $(f2)\qquad \lim_{t\to 0^+} \dfrac{f(x,t)}{t} = +\infty$ and $\lim_{t\to+\infty} \dfrac{f(x,t)}{t} = 0$, uniformly for $x \in \overline{\Omega}$.

- $p : (0,+\infty) \to (0,+\infty)$ is nonincreasing and Hölder continuous.

Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids. Due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

We show that a necessary condition in order to have classical solution is

$$\int_0^1 p(t)g(t)dt < +\infty. \tag{3.1}$$

In the case where f is sublinear, that is, f fulfills the hypotheses $(f1)$ and $(f2)$, condition (3.1) is also sufficient for existence of a classical solutions of $(P)^+$ provided λ and μ belong to a certain range Obviously, (3.1) implies the following Keller-Osserman type condition around the origin

$$(KO) \qquad \int_0^1 \left(\int_0^t \Phi(s)ds\right)^{-1/2} dt < +\infty\,, \text{ where } \Phi(s) = p(s)g(s), \text{ for all } s > 0.$$

As proved by Bénilan, Brezis and Crandall, condition (KO) is equivalent to the *property of compact support*: for every $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + \Phi(u) = h \qquad \text{a.e. in } \mathbb{R}^N.$$

We prove that assumption

$$\int_0^1 tp(t)dt < +\infty \tag{3.2}$$

is necessary in order that problem $(P)^-$ has classical solutions. Furthermore, the existence of solutions of $(P)^-$ (when f is sublinear) depends on the asymptotic behavior of the gradient term $|\nabla u|^a$. In this sense, if $0 < a < 1$, then $(P)^-$ has at least one classical solution for all $\mu \in \mathbb{R}$. In turn, if $1 < a \le 2$, then $(P)^-$ has no solutions for large values of μ.

Limiting case: $a = 1$. We prove that if Ω is a ball centered at the origin, then $(P)^-$ has at least one solution for all $\mu \in \mathbb{R}$.

3.1 Problem $(P)^+$

Theorem 3.1. *Assume that* $\int_0^1 p(t)g(t)dt = +\infty$. *Let* $\Phi : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ *be a Hölder continuous function. Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + p(d(x))g(u) \le \Phi(x,u) + C\,|\nabla u|^2 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

has no classical solutions.

Corollary 3.1. *Assume that* $\int_0^1 p(t)g(t)dt = +\infty$. *Then problem* $(P)^+$ *has no classical solutions.*

Auxiliary tool in the proof:

Lemma 3.1. *Let* $\Psi : \overline{\Omega} \times (0, +\infty) \to \mathbb{R}$ *be a Hölder continuous function such that the mapping* $(0, +\infty) \ni s \longmapsto \dfrac{\Psi(x,s)}{s}$ *is strictly decreasing for each* $x \in \Omega$. *Assume that there exist* $v,\ w \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*

(a) $\qquad \Delta w + \Psi(x, w) \le 0 \le \Delta v + \Psi(x, v)$ *in* Ω;

(b) $\quad v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$;
(c) $\quad \Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.
Then $v \leq w$ *in* Ω.

The next result shows that (3.1) is sufficient for the existence of a classical solution to $(P)^+$ provided $\mu \leq 0$ and $\lambda > 0$ is sufficiently large.

Theorem 3.2. *Assume that* $\int_0^1 p(t)g(t)dt < +\infty$.

(i) *If* $\mu = -1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^+$ *has at least one classical solution if* $\lambda > \lambda^*$ *and no solution exists if* $0 < \lambda < \lambda^*$.

(ii) *If* $\mu = +1$ *and* $0 < a < 1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^+$ *has at least one classical solution for all* $\lambda > \lambda^*$ *and no solution exists if* $0 < \lambda < \lambda^*$.

Proof. (i) *Step 1: Existence of a solution for* λ *large.*

Lemma 3.2. *There exist two positive constants* $c > 0$ *and* $M > 0$ *such that* $\underline{u}_\lambda := Mh(c\varphi_1)$ *is a sub-solution of* $(P)^+$ *provided* $\lambda > 0$ *is large enough.*

Step 2: Nonexistence for $\lambda > 0$ *small.*

Step 3: Dependence on $\lambda > 0$. Set

$$A = \left\{\lambda > 0;\ (P)^+ \text{ has a classical solution}\right\}.$$

Then A is nonempty and $\lambda^* := \inf A$ is positive. We show that if $\lambda \in A$, then $(\lambda, +\infty) \subseteq A$.

(ii) *Step 1: Existence of a solution for* λ *large.*

Step 2: Nonexistence for $\lambda > 0$ *small.*

We extend Lemma 3.1 in the following way:

Lemma 3.3. *Let* $0 < a < 1$ *and* $\Psi : \overline{\Omega} \times (0, +\infty) \to \mathbb{R}$ *be a Hölder continuous function such that the mapping* $(0, +\infty) \ni s \longmapsto \dfrac{\Psi(x, s)}{s}$ *is strictly decreasing for each* $x \in \Omega$. *Assume that there exist* v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*
(a) $\quad \Delta w + \Psi(x, w) + |\nabla w|^a \leq 0 \leq \Delta v + \Psi(x, v) + |\nabla v|^a$ *in* Ω;
(b) $\quad v, w > 0$ *in* Ω *and* $v < w$ *on* $\partial\Omega$.
Then $v \leq w$ *in* Ω.

3.2 Problem $(P)^-$

Theorem 3.3. *Assume that* $\displaystyle\int_0^1 tp(t)dt = +\infty$. *Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + C|\nabla u|^2 \geq p(d(x))g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

has no classical solutions.

Corollary 3.2. *Assume that* $\displaystyle\int_0^1 tp(t)dt = +\infty$. *Then the problem* $(P)^-$ *has no classical solutions.*

3.3 Problem $(P)^-$ in the sublinear case

In this case the existence of a solution is strongly dependent on the exponent a. To better understand this dependence, we assume $\lambda = 1$ but the same results hold for any $\lambda > 0$ (note only that the bifurcation point μ^* in the following theorem is dependent on λ).

Theorem 3.4. *Assume* $\lambda = 1$, $\int_0^1 tp(t)dt < +\infty$ *and conditions* $(f1)$, $(f2)$, $(g1)$ *and* $0 < a \leq 2$ *are fulfilled.*

(i) *If* $0 < a < 1$, *then problem* $(P)^-$ *has at least one solution, for all* $\mu \in \mathbb{R}$;

(ii) *If* $1 < a \leq 2$, *then there exists* $\mu^* > 0$ *such that* $(P)^-$ *has at least one classical solution for all* $\mu < \mu^*$ *and no solution exists if* $\mu > \mu^*$.

Corollary 3.3. *Assume* $\mu = \pm 1$, $\int_0^1 tp(t)dt < +\infty$ *and conditions* $(f1)$, $(f2)$, $(g1)$ *and* $0 < a \leq 2$ *are fulfilled.*

(i) *If* $0 < a < 1$, *then problem* $(P)^-$ *has at least one solution, for all* $\lambda > 0$;

(ii) *If* $< 1 < a \leq 2$ *and* $\mu = -1$, *then problem* $(P)^-$ *has at least one solution, for all* $\lambda > 0$;

(iii) *If* $1 < a \leq 2$ *and* $\mu = +1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^-$ *has at least one classical solution for all* $\lambda > \lambda^*$ *and no solution exists if* $\lambda < \lambda^*$.

3.4 Critical case: $a = 1$

Assume $\Omega = B_R(0)$ for some $R > 0$. Problem $(P)^-$ becomes

$$\begin{cases} -\Delta u = p(R - |x|)g(u) + f(x,u) + \mu|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{3.5}$$

Theorem 3.5. *Assume that* $\int_0^1 tp(t)dt < +\infty$. *Then the problem (3.5) has at least one solution for all* $\mu \in \mathbb{R}$.

3.5 Existence results for $(P)^-$ in the linear case

Consider the problem

$$\begin{cases} -\Delta u = p(d(x))g(u) + \lambda u + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $\lambda > 0$ and p, g are as in the previous sections. We assume in what follows that $0 < a < 1$.

Theorem 3.6. *Assume that* $\int_0^1 tp(t)dt < +\infty$ *and conditions* $(g1)$, $0 < a < 1$ *are fulfilled. Then for* $\mu \geq 0$ *the problem (3.6) has solutions if and only if* $\lambda < \lambda_1$.

3.6 An application

Consider the problem

$$\begin{cases} -\Delta u = d(x)^{-\alpha}u^{-\beta} + f(x,u) + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Recall that if $\int_0^1 tp(t)dt < +\infty$ and μ belongs to a certain range, then Theorem 3.4 asserts that (3.7) has at least one classical solution u_μ.

Theorem 3.7. *The following properties hold true.*

(i) *If* $\alpha \geq 2$, *then the problem (3.7) has no classical solutions.*

(ii) *If* $\alpha < 2$, *then there exists* $\mu^* \in (0, +\infty]$ *(with* $\mu^* = +\infty$ *if* $0 < a < 1$*) such that problem (3.7) has at least one classical solution* u_μ, *for all* $-\infty < \mu < \mu^*$. *Moreover, for all* $0 < \mu < \mu^*$, *there exist* $0 < \delta < 1$ *and* $C_1, C_2 > 0$ *such that* u_μ *satisfies*

(ii1) *If* $\alpha + \beta > 1$, *then*

$$C_1 d(x)^{\frac{2-\alpha}{1+\beta}} \leq u_\mu(x) \leq C_2 d(x)^{\frac{2-\alpha}{1+\beta}}, \quad \text{for all } x \in \Omega; \tag{3.8}$$

(ii2) *If* $\alpha + \beta = 1$, *then*

$$C_1 d(x)(-\ln d(x))^{\frac{1}{2-\alpha}} \leq u_\mu(x) \leq C_2 d(x)(-\ln d(x))^{\frac{1}{2-\alpha}}, \tag{3.9}$$

for all $x \in \Omega$ *with* $d(x) < \delta$;

(ii3) *If* $\alpha + \beta < 1$, *then*

$$C_1 d(x) \leq u_\mu(x) \leq C_2 d(x), \quad \text{for all } x \in \Omega. \tag{3.10}$$

4 Singular elliptic problems with nonmonotone nonlinearity

Classical framework. Let $f : [0, \infty) \to [0, \infty)$ be a smooth, **increasing**, such that $f(0) = 0$ and $f > 0$ on $(0, \infty)$. Then (Keller, Osserman, 1957) the problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

has a solution if and only if $\int_1^\infty [F(t)]^{-1/2}\, dt < \infty$, where $F(t) = \int_0^t f(s)\, ds$ ($\Longrightarrow f$ "super-linear").

Examples:

(i) $f(u) = e^u - 1$;

(ii) $f(u) = u^p,\ p > 1$;

(ii) $f(u) = u^p \ln(u + 1)$, $p > 1$;

(iv) $f(u) = u^p \arctan u$, $p > 1$;

(v) $f(u) = u[\ln(u + 1)]^p,\ p > 2$.

Let $f : [0, +\infty) \to [0, +\infty)$ be such that $f(0) = 0$.

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

$$\Phi(\alpha) = \frac{1}{\sqrt{2}} \int_\alpha^\infty \frac{ds}{\sqrt{F(s) - F(\alpha)}}\ , F(s) = \int_0^s f(t)\, dt$$

We say that f satisfies the **Keller-Osserman condition** if

$$\exists \alpha > 0 \quad \text{such that} \quad \Phi(\alpha) < \infty. \tag{4.2}$$

We say that f satisfies the **strong Keller-Osserman condition** if

$$\liminf_{\alpha \to \infty} \Phi(\alpha) = 0. \tag{4.3}$$

Example: the function $f(u) = u^2(1 + \cos u)$ satisfies the strong Keller-Osserman condition and $\limsup_{\alpha \to \infty} \Phi(\alpha) = +\infty$.

Theorem 4.1. *The function f satisfies the Keller-Osserman condition if and only if the BVP* (4.1) *admits at least one positive large solution on* **some** *ball.*

Theorem 4.2. *The function f satisfies the strong Keller-Osserman condition if and only if the BVP* (4.1) *has at least one positive large solution on* **each** *smooth bounded domain Ω.*

Theorem 4.3. *Assume that the strong Keller-Osserman condition is fulfilled and let u be a positive large solution of* (4.1). *Then*

$$\lim_{x \to x_0} \frac{\displaystyle\int_{u(x)}^{\infty} \frac{dt}{\sqrt{2F(t)}}}{\delta(x)} = 1,$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

Open problem. Maximum principle with less of monotonicity.

Classical framework: if $u : \overline{\Omega} \to \mathbb{R}$ is a smooth function such that

$$\begin{cases} -\Delta u \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

then $u \geq 0$ in Ω.

Stampacchia's generalized maximum principle: the above result is still true if

$$-\Delta \longmapsto -\Delta + a(x)I \quad \text{coercive} \ , \quad a \in L^{\infty}(\Omega).$$

Let $f : [0, +\infty) \to \mathbb{R}$ be continuous **non-decreasing** such that $f(0) = 0$ and $\int_0^1 (F(t))^{-1/2}\, dt = +\infty$, where $F(t) = \int_0^t f(s)ds$.

Example: $f(u) = u^p$, with $p \geq 1$.

Vázquez' maximum principle: if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + f(u) \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \end{cases}$$

then the following alternative holds:

either $u \equiv 0$ in Ω

or

$u > 0$ in Ω.

Open problem. Establish the Vázquez maximum principle with **no** monotonicity assumption on f.

Acknowledgments. The author has been supported by Grant GAR 315/2007 with the Romanian Academy.

References

[1] V. Rădulescu, Bifurcation and asymptotics for elliptic problems with singular nonlinearity, in *Studies in Nonlinear Partial Differential Equations: In Honor of Haim Brezis, Fifth European Conference on Elliptic and Parabolic Problems: A special tribute to the work of H. Brezis*, Gaeta, Italy, May 30 - June 3, 2004 (C. Bandle *et all.*, Eds), Birkhäuser, 2005, pp. 349-362.

[2] M. Ghergu and V. Rădulescu, Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term, *Proceedings of the Royal Society of Edinburgh: Section A (Mathematics)* **135** (2005), 61-84.

[3] V. Rădulescu, Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities, in *Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 4* (Michel Chipot, Editor), Elsevier, 2007, pp. 483-591.

[4] L. Dupaigne, M. Ghergu, V. Rădulescu, Lane-Emden-Fowler equations with convection and singular potential, *J. Math. Pures Appl. (Journal de Liouville)* **87** (2007), 563-581.

[5] S. Dumont, L. Dupaigne, O. Goubet and V. Rădulescu, Back to the Keller-Osserman condition for boundary blow-up solutions, *Advanced Nonlinear Studies* **7** (2007), 271-298.

[6] F. Cîrstea and V. Rădulescu, Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type, *Transactions Amer. Math. Soc.* **359** (2007), 3275-3286.

[7] M. Ghergu and V. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and its Applications, vol. 37, Oxford University Press, 2008.

[8] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, *Math. Ann.* **77** (1916), 173-212.

[9] H. Rademacher, Einige besondere Probleme der partiellen Differentialgleichungen, in *Die Differential und Integralgleichungen der Mechanik und Physik I*, 2nd. edition, (P. Frank und R. von Mises, eds.), Rosenberg, New York, 1943, p. 838-845.

[10] A. C. Lazer and P. J. McKenna, On a problem of Bieberbach and Rademacher, *Nonlinear Anal., T.M.A.* **21** (1993), 327-335.

[11] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* **10** (1957), 503-510.

[12] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7** (1957), 1641-1647.

[13] C. Loewner and L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in *Contribution to Analysis*, Academic Press, New York, 1974, p. 245-272.

[14] M. G. Crandall, P. H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. P. D. E.* **2** (1977), 193-222.

[15] M. Coclite and G. Palmieri, On a singular nonlinear Dirichlet problem, *Comm. Partial Differential Equations* **14** (1989), 1315-1327.

[16] J. I. Díaz, J. M. Morel, and L. Oswald, An elliptic equation with singular nonlinearity, *Comm. Partial Differential Equations* **12** (1987), 1333-1344.

[17] C. Gui and F. H. Lin, Regularity of an elliptic problem with a singular nonlinearity, *Proc. Royal Soc. Edinburgh Sect. A* **123** (1993), 1021-1029.

[18] Y. Haitao, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, *J. Differential Equations* **189** (2003), 487-512.

[19] J. Hernández, F. J. Mancebo, and J. M. Vega, On the linearization of some singular nonlinear elliptic problems and applications, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **19** (2002), 777-813.

[20] A. Meadows, Stable and singular solutions of the equation $\Delta u = 1/u$, *Indiana Univ. Math. J.* **53** (2004), 1681-1703.

[21] J. Shi and M. Yao, On a singular nonlinear semilinear elliptic problem, *Proc. Royal Soc. Edinburgh Sect. A* **128** (1998), 1389-1401.

[22] Z. Zhang, The existence and asymptotic behaviour of the unique solution near the boundary to a singular Dirichlet problem with a convection term, *Proc. Royal Soc. Edinburgh Sect. A* **136** (2006), 209-222.

[23] S. Alama and G. Tarantello, On the solvability of a semilinear elliptic equation via an associated eigenvalue problem, *Math. Z.* **221** (1996), 467-493.

[24] F. Cîrstea and V. Rădulescu, Solutions with boundary blow-up for a class of nonlinear elliptic problems, *Houston J. Math.* **29** (2003), 821-829.

[25] F. Cîrstea and V. Rădulescu, Extremal singular solutions for degenerate logistic-type equations in anisotropic media, *C. R. Acad. Sci. Paris, Ser. I* **339** (2004), 119-124.