Proceedings of the Royal Society of Edinburgh, **151**, 601–651, *2021*
DOI:10.1017/prm.2020.32

Concentration of positive solutions for a class of fractional p-Kirchhoff type equations

Vincenzo Ambrosio and Teresa Isernia
Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 12, 60131 Ancona, Italy (v.ambrosio@univpm.it; t.isernia@univpm.it)

Vicenţiu D. Radulescu
Faculty of Applied Mathematics, AGH University of Science and Technology al. Mickiewicza 30, 30-059 Kraków (Poland) & Department of Mathematics University of Craiova, 200585 Craiova, Romania (radulescu@inf.ucv.ro)

(MS received 6 October 2019; accepted 5 April 2020)

We study the existence and concentration of positive solutions for the following class of fractional p-Kirchhoff type problems:

$$\begin{cases} \left(\varepsilon^{sp}a + \varepsilon^{2sp-3}b\,[u]^p_{s,p}\right)(-\Delta)^s_p u + V(x)u^{p-1} = f(u) & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where ε is a small positive parameter, a and b are positive constants, $s \in (0,1)$ and $p \in (1,\infty)$ are such that $sp \in (\frac{3}{2}, 3)$, $(-\Delta)^s_p$ is the fractional p-Laplacian operator, $f : \mathbb{R} \to \mathbb{R}$ is a superlinear continuous function with subcritical growth and $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous potential having a local minimum. We also prove a multiplicity result and relate the number of positive solutions with the topology of the set where the potential V attains its minimum values. Finally, we obtain an existence result when $f(u) = u^{q-1} + \gamma u^{r-1}$, where $\gamma > 0$ is sufficiently small, and the powers q and r satisfy $2p < q < p^*_s \leqslant r$. The main results are obtained by using some appropriate variational arguments.

Keywords: Fractional p-Kirchhoff equation; variational methods; critical and supercritical growth

2010 Mathematics Subject Classification: 47G20; 35R11; 35A15; 35B33; 55M30

1. Introduction

In this paper, we focus on the following class of fractional p-Kirchhoff problems:

$$\begin{cases} \left(\varepsilon^{sp}\,a + \varepsilon^{2sp-3}\,b\,[u]^p_{s,p}\right)(-\Delta)^s_p u + V(x)u^{p-1} = f(u) & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants, $s \in (0,1)$ and $p \in (1,\infty)$ are such that $sp \in (\frac{3}{2}, 3)$, $(-\Delta)^s_p$ is the fractional p-Laplacian operator which, up to normalization factors, may be defined for every function $u \in \mathcal{C}^\infty_c(\mathbb{R}^3)$ as

$$(-\Delta)^s_p u(x) = 2 \lim_{r\to 0} \int_{\mathbb{R}^3\setminus\mathcal{B}_r(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{3+sp}}\, \mathrm{d}y \quad (x \in \mathbb{R}^3),$$

and $W^{s,p}(\mathbb{R}^3)$ is the fractional Sobolev space of functions $u \in L^p(\mathbb{R}^3)$ such that

$$[u]^p_{s,p} := \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^p}{|x-y|^{3+sp}}\, \mathrm{d}x\, \mathrm{d}y < \infty$$

endowed with the natural norm

$$\|u\|^p_{s,p} = [u]^p_{s,p} + |u|^p_p.$$

We recall that in these years a tremendous popularity has achieved the investigation of nonlinear problems involving fractional and nonlocal operators due to their fundamental role in describing several phenomena such as phase transition, game theory, finance, image processing, Lévy processes and optimization; see for instance [**24**] for more details.

When $a = \varepsilon = 1$, $b = 0$ and $p = 2$, equation (P_ε) becomes a fractional Schrödinger equation of the type

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3 \tag{1.1}$$

introduced by Laskin [**39**] in the study of fractional quantum mechanics; see [**18**] for more details. Equation (1.1) has been widely considered in these last years, and several existence and multiplicity results to (1.1) have been established by applying suitable techniques and assuming different conditions on the potential V and on the nonlinearity f; see [**3**, **5**, **7**, **19**, **25**, **27**, **31**] and the references therein.

On the other hand, when $s = \varepsilon = 1$ and $p = 2$, problem (P_ε) boils down to a classical Kirchhoff equation of the type

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\, \mathrm{d}x\right)\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

which is related to the stationary analogue of the well-known Kirchhoff equation

$$\rho u_{tt} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2\, \mathrm{d}x\right) u_{xx} = 0 \tag{1.3}$$

proposed by Kirchhoff [**38**] as an extension of the classical D'Alembert's wave equation for describing the transversal oscillations of a stretched string. The parameters appearing in (1.3) have the following meaning: L is the length of the string, h is the area of the cross-section, E is the young modulus (elastic modulus) of the material, ρ is the mass density and p_0 is the initial tension. We refer to [**13**, **50**] for the early classical studies dedicated to (1.3). We also note that nonlocal boundary value problems like (1.2) model several physical and biological systems where u describes a process which depends on the average of itself, as for example, the population density; see [**2**, **17**].

Anyway, only after the Lions' work [41], where a functional analysis approach was proposed to attack a general Kirchhoff equation in arbitrary dimension with an external force term, problem (1.2) began to catch the attention of several mathematicians; see [1, 30, 35, 36, 49, 56] and the references therein. Concerning perturbed Kirchhoff problems, He and Zou [36] proved a multiplicity result for the following Kirchhoff equation

$$-\left(a\varepsilon^2 + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right) \Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

provided that $\varepsilon > 0$ is sufficiently small, under the following condition on V introduced by Rabinowitz [53]:

$$V_\infty = \liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^3} V(x), \quad \text{where } V_\infty \leqslant \infty,$$

and g is a subcritical nonlinearity. Wang et al. [56] extended the results in [36] considering critical nonlinearities. After that, Figueiredo and Santos Junior [30] applied the generalized Nehari manifold method and Ljusternik–Schnirelmann theory to deduce a multiplicity result for a class of Kirchhoff equations requiring that the potential V fulfils (V_1)–(V_2). Later, He et al. [35] obtained the existence and multiplicity of solutions to (1.4), when (V_1)–(V_2) are in force, and $g(u) = f(u) + u^5$, where $f \in \mathcal{C}^1$ is a subcritical nonlinearity which does not satisfy the Ambrosetti–Rabinowitz condition [4].

In the nonlocal framework, Fiscella and Valdinoci [33] proposed for the first time a stationary fractional Kirchhoff variational model in a bounded domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions and involving a critical nonlinearity:

$$\begin{cases} M\left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous positive Kirchhoff function whose typical example is given by $M(t) = a + bt$, with $a > 0$ and $b \geqslant 0$, f is a superlinear function with subcritical growth at infinity, and $\lambda > 0$ is a parameter. Their model takes care of the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string; see the Appendix in [33] for more details. After the pioneering work [33], several authors dealt with the existence and multiplicity of solutions for (1.5); see [8, 9, 11, 29, 44, 46] and the references therein. We stress that, in [9], the authors obtained a multiplicity result for a perturbed fractional Kirchhoff problem under assumptions (V_1)–(V_2).

On the other hand, a great attention has been devoted to the study of fractional Kirchhoff problems involving $(-\Delta)_p^s$. For instance, Pucci et al. [51] obtained a multiplicity result for a nonhomogeneous fractional Kirchhoff–Schrödinger equation assuming that the potential V satisfies a Bartsch–Wang type condition. Fiscella and Pucci [32] dealt with stationary fractional Kirchhoff p-Laplacian equations involving a Hardy potential and different critical nonlinearities. Mingqi et al. [43] proved some existence result for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian. We also refer to [40, 42, 47] for other interesting results.

We note that fractional p-Laplacian problems have received a great attention in these years since two phenomena are present in $(-\Delta)^s_p$: the nonlinearity of the operator and its nonlocal character; see [20, 23, 34, 48] and the references therein. Moreover, some useful techniques developed to study fractional Laplacian problems are not available to attack problems like (P_ε). Indeed, we can make use neither of the powerful framework provided by the Caffarelli–Silvestre s-harmonic extension [15] nor of various tools as, e.g., commutators estimates, strong barriers and density estimates.

Particularly motivated by [3, 5, 9, 10, 30] and by the interest shared by the mathematical community on fractional p-Laplacian problems, the goal of this paper is to study the existence, multiplicity and concentration of solutions to (P_ε). In order to state precisely our results, we first introduce the main assumptions on the potential V and the nonlinearity f. Along the paper, we suppose that $V \in \mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$ satisfies the following assumptions introduced by del Pino and Felmer [22]:

(V_1) there exists $V_1 > 0$ such that $V_1 := \inf_{x\in\mathbb{R}^3} V(x)$,

(V_2) there exists an open bounded set $\Lambda \subset \mathbb{R}^3$ such that

$$0 < V_0 := \inf_{\Lambda} V < \min_{\partial\Lambda} V,$$

while we assume that $f \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$, vanishes in $(-\infty, 0)$, and fulfils the following conditions:

(f_1) $f(t) = o(t^{2p-1})$ as $t \to 0^+$,

(f_2) there exists $\nu \in (2p, p^*_s)$, with $p^*_s = 3p/3 - sp$, such that

$$\lim_{t\to\infty} \frac{f(t)}{t^{\nu-1}} = 0,$$

(f_3) there exists $\vartheta \in (2p, p^*_s)$ such that $0 < \vartheta F(t) := \vartheta \int_0^t f(\tau)\, \mathrm{d}\tau \leqslant t f(t)$ for all $t > 0$,

(f_4) the map $t \mapsto f(t)/t^{2p-1}$ is increasing in $(0, \infty)$.

We emphasize that under the control on fractional parameter $s \in (0, 1)$, the condition $sp \in (\frac{3}{2}, 3)$ forces $p \in (\frac{3}{2}, \infty)$. In particular, the restriction $sp \in (\frac{3}{2}, 3)$ implies that $2p < p^*_s$, therefore (f_3) makes sense. A typical example of function which satisfies (f_1)–(f_4) is given by $f(t) = \sum_{i=1}^k \alpha_i t^{r_i-1}$ with $\alpha_i \geqslant 0$ not all null and $2p < r_i < p^*_s$ for all $i \in \{1, \ldots, k\}$.

Now, we are in the position to state our first main result of this work:

Theorem 1.1. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *are in force. Then, there exists* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, ($P_\varepsilon$) *has a positive solution* u_ε. *Moreover, if* η_ε *denotes a global maximum point of* u_ε, *then* $\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0$, *and there exists* $C > 0$ *such that*

$$u_\varepsilon(x) \leqslant \frac{C\,\varepsilon^{3+sp}}{\varepsilon^{3+sp} + |x - \eta_\varepsilon|^{3+sp}} \quad \forall x \in \mathbb{R}^3.$$

The proof of theorem 1.1 relies on suitable variational arguments. We first adapt in a suitable way the penalization argument in [**22**] which consists in modifying the nonlinearity f outside Λ, considering an auxiliary problem and then we check that, for $\varepsilon > 0$ small enough, the solutions of the modified problem are solutions of the original one. In order to achieve our purpose, we look for critical points of the corresponding energy functional $\mathcal{I}_\varepsilon$. The main difficulties that arise in the study of $\mathcal{I}_\varepsilon$ are related to the presence of the Kirchhoff term $[u]_{s,p}^p(-\Delta)_p^s$ and the lack of compactness caused by the unboundedness of the domain $\mathbb{R}^3$. Indeed, in general, it is not trivial to prove that the weak limits of bounded Palais–Smale sequences are critical points of $\mathcal{I}_\varepsilon$ when we consider Kirchhoff problems. Moreover, the non-Hilbertian structure of the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^3)$ when $p \neq 2$ makes our study rather tough. To circumvent these difficulties, we will develop some clever arguments which take care of the nonlocal character of the leading operator $(-\Delta)_p^s$ and that allow us to recover the compactness of the functional $\mathcal{I}_\varepsilon$; see lemma 2.4. After that, we show that the solution of the modified problem is also a solution of the original one by combining a Moser iteration argument [**45**] with the Hölder continuity result established for $(-\Delta)_p^s$; see [**23**, **37**]. We also prove a decay estimate for the solutions of (P_ε) exploiting some recent results obtained in [**10**, **21**].

In the second part of this work, we deal with the multiplicity of positive solutions to (P_ε). In this case, we replace (V_2) by the following assumption:

(V_2') $V_0 = V_1$ and $M = \{x \in \Lambda : V(x) = V_0\} \neq \emptyset$.

We recall that if Y is a given closed set of a topological space X, we denote by $cat_Y(X)$ the Ljusternik–Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y; see [**57**] for more details.

Now, we state the second result of this paper:

Theorem 1.2. *Assume that* (V_1)–(V_2') *and* (f_1)–(f_4) *hold. Then, for any* $\delta > 0$ *such that*

$$M_\delta = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, M) \leqslant \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *problem* (P_ε) *has at least* $cat_{M_\delta}(M)$ *positive solutions. Furthermore, if* u_ε *denotes one of these positive solutions and* $\eta_\varepsilon \in \mathbb{R}^3$ *is a global maximum point of* u_ε, *then*

$$\lim_{\varepsilon \to 0} V(\eta_\varepsilon) = V_0.$$

In order to prove theorem 1.2, we need to use some suitable variational and topological arguments. More precisely, to obtain multiple solutions, we study the modified functional $\mathcal{I}_\varepsilon$ on the associated Nehari manifold $\mathcal{N}_\varepsilon$. Anyway, due to the fact that f is only continuous, the Nehari manifold $\mathcal{N}_\varepsilon$ is not differentiable and some well-known arguments for $\mathcal{C}^1$-Nehari manifolds do not work in our situation. To overcome this obstacle, we take inspiration by some results due to Szulkin and Weth [**55**]. After that, we use a technique introduced by Benci and Cerami in [**12**], in which the main ingredient is to make fine comparisons between the category of some sublevel sets of the modified functional $\mathcal{I}_\varepsilon$ and the category of the set M.

In the last part of this work, we consider the case when the nonlinearity f has a critical or supercritical growth. More precisely, we study the following nonlocal problem

$$\begin{cases} \left(\varepsilon^{sp}\, a + \varepsilon^{2sp-3}\, b\, [u]_{s,p}^{p}\right)(-\Delta)_p^s u + V(x)u^{p-1} = u^{q-1} + \gamma u^{r-1} & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $\varepsilon, \gamma > 0$ and the powers q and r are such that $2p < q < p_s^* \leqslant r$. Also in this case, we are able to obtain the following result:

Theorem 1.3. *Assume that* (V_1)–(V_2) *hold. Then, there exists* $\gamma_0 > 0$ *such that, for any* $\gamma \in (0, \gamma_0)$, *there exists* $\varepsilon_\gamma > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\gamma)$, *problem* (1.6) *has a positive solution* u_ε. *Moreover, if* η_ε *denotes a global maximum point of* u_ε, *then* $\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0$.

Remark 1.1. If we assume (V_1)–(V_2'), then the conclusion of theorem 1.2 holds true for ε and γ sufficiently small.

Differently from the study of (P_ε), an additional difficulty arises in the study of (1.6). Indeed, when $r > p_s^*$, problem (1.6) becomes supercritical, and we cannot directly apply variational techniques because the corresponding energy functional is not well-defined on the fractional Sobolev space $W^{s,p}(\mathbb{R}^3)$. To overcome this hitch, inspired by [**16**, **28**, **52**], we truncate the nonlinearity involving the critical or supercritical power, and we consider a new subcritical problem for which it is possible to apply the existence result given by theorem 1.1. Then, after proving a priori bound (independent of γ) for the solution of the truncated problem, we use a suitable Moser's iteration argument [**45**] to prove that, if γ is small enough, this solution also satisfies the original problem (1.6).

To our knowledge, the results presented here are new also in the case $s = 1$, and they complement and improve the once obtained in [**9**, **30**] because we are considering the case $p \in (\frac{3}{2}, \infty)$ and nonlinearities with subcritical, critical or supercritical growth.

The paper is organized as follows: in § 2 we introduce the modified problem and we give the proof of theorem 1.1. Section 3 is devoted to the multiplicity of solutions to (P_ε). In § 4 we are interested in critical or supercritical problem.

Notations: If $A \subset \mathbb{R}^3$, we denote by $A^c = \mathbb{R}^3 \setminus A$. We use the notation $|u|_{L^q(A)}$ to indicate the $L^q(A)$-norm of a function $u : \mathbb{R}^3 \to \mathbb{R}$, and by $|u|_q$ its $L^q(\mathbb{R}^3)$-norm. We write $\mathcal{B}_r(x)$ to denote the ball centred at $x \in \mathbb{R}^3$ with radius $r > 0$, and, when $x = 0$, we put $\mathcal{B}_r = \mathcal{B}_r(0)$ and $\mathcal{B}_r^c := \mathcal{B}_r^c(0)$.

2. The modified problem

2.1. Work space stuff

We define $D^{s,p}(\mathbb{R}^3)$ as the closure of $\mathcal{C}_c^\infty(\mathbb{R}^3)$ with respect to

$$[u]_{s,p}^p := \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^p}{|x - y|^{3+sp}} \, dx \, dy.$$

Let us indicate by $W^{s,p}(\mathbb{R}^3)$ the set of functions $u \in L^p(\mathbb{R}^3)$ such that $[u]_{s,p} < \infty$, endowed with the natural norm

$$\|u\|_{s,p}^p := [u]_{s,p}^p + |u|_p^p.$$

We have the following well-known embeddings (see [24]).

THEOREM 2.1 [24]. *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $sp < 3$. Then there exists a constant $C_* := C_*(s,p) > 0$ such that, for any $u \in D^{s,p}(\mathbb{R}^3)$, we have*

$$|u|_{p_s^*}^p \leqslant C_*[u]_{s,p}^p.$$

Moreover, $W^{s,p}(\mathbb{R}^3)$ is continuously embedded in $L^q(\mathbb{R}^3)$ for any $q \in [p, p_s^*]$, and compactly in $L^q_{loc}(\mathbb{R}^3)$ for any $q \in [1, p_s^*)$.

We will often use the following vanishing-Lions type result (see lemma 2.2 in [10]).

LEMMA 2.1 [10]. *Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < 3$, and $r \in [p, p_s^*)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^3)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{\mathcal{B}_R(y)} |u_n|^r \,\mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^\sigma(\mathbb{R}^3)$ for all $\sigma \in (p, p_s^*)$.

We also recall the following useful technical result (see lemma 2.3 in [10]).

LEMMA 2.2 [10]. *Let $u \in W^{s,p}(\mathbb{R}^3)$ and let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ be such that $0 \leqslant \phi \leqslant 1$, $\phi = 1$ in $\mathcal{B}_1$ and $\phi = 0$ in $\mathcal{B}_2^c$. Set $\phi_r(x) = \phi(x/r)$. Then,*

$$[u\phi_r - u]_{s,p} \to 0 \quad \text{and} \quad |u\phi_r - u|_p \to 0.$$

In order to study (P_ε), we use the change of variable $x \mapsto \varepsilon\, x$ and we look for solutions to

$$\begin{cases} (a + b\,[u]_{s,p}^p)(-\Delta)_p^s u + V(\varepsilon\, x)u^{p-1} = f(u) & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{$\hat{P}_\varepsilon$}$$

Now, we introduce a penalized function in the spirit of [22] which will be fundamental to obtain our main result. First of all, without loss of generality, we may assume that

$$0 \in \Lambda \quad \text{and} \quad V(0) = V_0.$$

Let $K > 2p$ and $a_0 > 0$ be such that

$$f(a_0) = \frac{V_1}{K} a_0^{p-1} \tag{2.1}$$

and we define

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \leqslant a_0, \\ \dfrac{V_1}{K} t^{p-1} & \text{if } t > a_0, \end{cases}$$

and

$$g(x,t) := \begin{cases} \chi_\Lambda(x) f(t) + (1-\chi_\Lambda(x))\tilde{f}(t) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

It is easy to check that g satisfies the following properties:

(g_1) $\lim_{t\to 0^+} g(x,t)/t^{2p-1} = 0$ uniformly with respect to $x \in \mathbb{R}^3$,

(g_2) $g(x,t) \leqslant f(t)$ for all $x \in \mathbb{R}^3$, $t > 0$,

(g_3) (i) $0 < \vartheta G(x,t) \leqslant g(x,t)t$ for all $x \in \Lambda$ and $t > 0$,
(ii) $0 \leqslant pG(x,t) \leqslant g(x,t)t \leqslant V_1/K t^p$ for all $x \in \Lambda^c$ and $t > 0$,

(g_4) for each $x \in \Lambda$ the function $g(x,t)/t^{2p-1}$ is increasing in $(0,\infty)$, and for each $x \in \Lambda^c$ the function $g(x,t)/t^{2p-1}$ is increasing in $(0, a_0)$.

We point out that if u_ε is a solution to

$$\begin{cases} (a + b\,[u]_{s,p}^p)(-\Delta)_p^s u + V(\varepsilon\, x) u^{p-1} = g(\varepsilon\, x, u) & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

with $u_\varepsilon(x) \leqslant a_0$ for all $x \in \Lambda_\varepsilon^c$, where $\Lambda_\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon\, x \in \Lambda\}$, then $g(\varepsilon\, x, u_\varepsilon) = f(u_\varepsilon)$. Therefore $v_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a solution to (P_ε)

For any $\varepsilon > 0$ we consider the following fractional space

$$\mathcal{H}_\varepsilon := \left\{ u \in W^{s,p}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon\, x)|u|^p \,\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon^p := a[u]_{s,p}^p + \int_{\mathbb{R}^3} V(\varepsilon\, x)|u|^p \,\mathrm{d}x.$$

In order to study (2.2), we look for critical points of the $\mathcal{C}^1$-functional $\mathcal{I}_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$ defined as

$$\mathcal{I}_\varepsilon(u) = \frac{1}{p}\|u\|_\varepsilon^p + \frac{b}{2p}[u]_{s,p}^{2p} - \int_{\mathbb{R}^3} G(\varepsilon\, x, u)\,\mathrm{d}x.$$

For any $u, \varphi \in \mathcal{H}_\varepsilon$ the differential of $\mathcal{I}_\varepsilon$ is given by

$$\begin{aligned}\langle \mathcal{I}'_\varepsilon(u), \varphi\rangle = a \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ + b[u]_{s,p}^p \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ + \int_{\mathbb{R}^3} V(\varepsilon x)|u|^{p-2}u\varphi\,\mathrm{d}x - \int_{\mathbb{R}^3} g(\varepsilon x, u)\varphi\,\mathrm{d}x.\end{aligned}$$

Next we show that $\mathcal{I}_\varepsilon$ has a mountain pass geometry [4].

LEMMA 2.3. *The functional $\mathcal{I}_\varepsilon$ satisfies the following conditions*:

(i) *there exist $\alpha, \rho > 0$ such that $\mathcal{I}_\varepsilon(u) \geqslant \alpha$ with $\|u\|_\varepsilon = \rho$;*

(ii) *there exists $e \in \mathcal{H}_\varepsilon$ with $\|e\|_\varepsilon > \rho$ and $\mathcal{I}_\varepsilon(e) < 0$.*

Proof. (i) Using assumptions (g_1), (g_2), (f_1) and (f_2), for any given $\zeta > 0$ there exists a positive constant C_ζ such that

$$|g(x,t)| \leqslant \zeta |t|^{p-1} + C_\zeta |t|^{\nu-1}.$$

Thus, using Sobolev embeddings we have

$$\mathcal{I}_\varepsilon(u) = \frac{1}{p}\|u\|_\varepsilon^p + \frac{b}{2p}[u]_{s,p}^{2p} - \int_{\mathbb{R}^3} G(\varepsilon x, u)\,\mathrm{d}x \geqslant \frac{1}{p}\|u\|_\varepsilon^p - C\zeta\|u\|_\varepsilon^p - \bar{C}_\zeta\|u\|_\varepsilon^\nu.$$

Consequently, we can choose α, ρ such that $I_\varepsilon(u) \geqslant \alpha$ and $\|u\|_\varepsilon = \rho$.

(ii) Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ be such that $u \geqslant 0$, $u \not\equiv 0$ and $\mathrm{supp}(u) \subset \Lambda_\varepsilon$. In view of (g_3)-(i) and $\vartheta \in (2p, p_s^*)$, we can see that, for some constants $C_1, C_2 > 0$ and for any $t > 0$

$$\mathcal{I}_\varepsilon(tu) \leqslant \frac{t^p}{p}\|u\|_\varepsilon^p + \frac{bt^{2p}}{2p}[u]_{s,p}^{2p} - C_1 t^\vartheta \int_{\Lambda_\varepsilon} u^\vartheta\,\mathrm{d}x + C_2 \to -\infty \quad \text{as } t \to \infty.$$

□

Invoking a variant of the mountain-pass theorem without (PS)-condition (see [57]), we deduce the existence of a Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\varepsilon$ such that

$$\mathcal{I}_\varepsilon(u_n) = c_\varepsilon + o_n(1) \quad \text{and} \quad \mathcal{I}'_\varepsilon(u_n) = o_n(1), \tag{2.3}$$

where

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{I}_\varepsilon(\gamma(t)) \quad \text{and} \quad \Gamma_\varepsilon := \left\{\gamma \in \mathcal{C}^0([0,1], \mathcal{H}_\varepsilon) : \gamma(0) = 0,\, \mathcal{I}_\varepsilon(\gamma(1)) \leqslant 0\right\}. \tag{2.4}$$

As in [57], we can use the following equivalent characterization of c_ε more appropriate for our aim:

$$c_\varepsilon = \inf_{u\in\mathcal{H}_\varepsilon\setminus\{0\}} \max_{t\geqslant 0} \mathcal{I}_\varepsilon(tu).$$

Moreover, from the monotonicity of g, it is easy to check that for all $u \in \mathcal{H}_\varepsilon \setminus \{0\}$ there exists a unique $t_0 = t_0(u) > 0$ such that

$$\mathcal{I}_\varepsilon(t_0 u) = \max_{t \geqslant 0} \mathcal{I}_\varepsilon(tu).$$

LEMMA 2.4. *$\mathcal{I}_\varepsilon$ fulfils the Palais–Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence satisfying (2.3). Let us prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$. Using (g_3) we obtain that

$$\begin{aligned}
c_\varepsilon + o_n(1) &= \mathcal{I}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{I}_\varepsilon'(u_n), u_n\rangle \\
&= \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\|u_n\|_\varepsilon^p + b\left(\frac{1}{2p} - \frac{1}{\vartheta}\right)[u_n]_{s,p}^{2p} \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon^c} [g(\varepsilon\, x, u_n)u_n - \vartheta G(\varepsilon\, x, u_n)]\,\mathrm{d}x \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon} [g(\varepsilon\, x, u_n)u_n - \vartheta G(\varepsilon\, x, u_n)]\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\|u_n\|_\varepsilon^p \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon^c} [g(\varepsilon\, x, u_n)u_n - \vartheta G(\varepsilon\, x, u_n)]\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\|u_n\|_\varepsilon^p - \frac{(\vartheta - p)}{\vartheta p}\frac{1}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon\, x)|u_n|^p\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\left(1 - \frac{1}{K}\right)\|u_n\|_\varepsilon^p.
\end{aligned}$$

Thanks to $\vartheta > 2p$ and $K > 2p > 3$, we deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$. From theorem 2.1 we may assume that $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$.

Now, we prove that for any $\eta > 0$ there exists $R = R(\eta) > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathcal{B}_R^c} \left(a\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{3+sp}}\,\mathrm{d}y + V(\varepsilon\, x)|u_n|^p\right)\mathrm{d}x < \eta. \tag{2.5}$$

For any $R > 0$, let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^3)$ be such that $\psi_R = 0$ in $\mathcal{B}_R$, $\psi_R = 1$ in $\mathcal{B}_{2R}^c$, $0 \leqslant \psi_R \leqslant 1$, and $|\nabla\psi_R| \leqslant C/R$, where C is a constant independent of R. Since $\{\psi_R u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, it follows that $\langle \mathcal{I}_\varepsilon'(u_n), \psi_R u_n\rangle = o_n(1)$, that is

$$\begin{aligned}
&(a + b[u_n]_{s,p}^p)\iint_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{3+sp}}\psi_R(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} V(\varepsilon\, x)|u_n|^p\psi_R\,\mathrm{d}x \\
&\quad = o_n(1) + \int_{\mathbb{R}^3} g(\varepsilon\, x, u_n)\psi_R u_n\,\mathrm{d}x
\end{aligned}$$

$$- (a + b\,[u_n]_{s,p}^p) \iint_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{3+sp}} \times u_n(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Take $R > 0$ such that $\Lambda_\varepsilon \subset \mathcal{B}_R$. By the definition of ψ_R and (g_3)-(ii) we get

$$\begin{aligned} & a \iint_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{3+sp}} \psi_R\,\mathrm{d}x\,\mathrm{d}y + \left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^3} V(\varepsilon\, x)|u_n|^p \psi_R\,\mathrm{d}x \\ & \quad \leqslant o_n(1) - (a + b[u_n]_{s,p}^p) \\ & \qquad \times \iint_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{3+sp}} u_n(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{2.6}$$

Let us note that, from the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_\varepsilon$, we can suppose that $a + b\,[u_n]_{s,p}^p \to \ell \in (0, \infty)$. Now, using the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_\varepsilon$ we have

$$\begin{aligned} & \left| \iint_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{3+sp}} u_n(y)\,\mathrm{d}x\,\mathrm{d}y \right| \\ & \quad \leqslant C \left(\iint_{\mathbb{R}^6} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{3+sp}} |u_n(y)|^p\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}. \end{aligned} \tag{2.7}$$

On the other hand, by the definition of ψ_R, and using polar coordinates and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_\varepsilon$, we obtain

$$\begin{aligned} & \iint_{\mathbb{R}^6} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{3+sp}} |u_n(x)|^p\,\mathrm{d}x\,\mathrm{d}y \\ & \quad = \int_{\mathbb{R}^3} \int_{|y-x|>R} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{3+sp}} |u_n(x)|^p\,\mathrm{d}x\,\mathrm{d}y \\ & \qquad + \int_{\mathbb{R}^3} \int_{|y-x|\leqslant R} \frac{|\psi_R(x) - \psi_R(y)|^p}{|x-y|^{3+sp}} |u_n(x)|^p\,\mathrm{d}x\,\mathrm{d}y \\ & \quad \leqslant C \int_{\mathbb{R}^3} |u_n(x)|^p \left(\int_{|y-x|>R} \frac{\mathrm{d}y}{|x-y|^{3+sp}} \right) \mathrm{d}x \\ & \qquad + \frac{C}{R^p} \int_{\mathbb{R}^3} |u_n(x)|^p \left(\int_{|y-x|\leqslant R} \frac{\mathrm{d}y}{|x-y|^{3+sp-p}} \right) \mathrm{d}x \\ & \quad \leqslant C \int_{\mathbb{R}^3} |u_n(x)|^p \left(\int_{|z|>R} \frac{\mathrm{d}z}{|z|^{3+sp}} \right) \mathrm{d}x \\ & \qquad + \frac{C}{R^p} \int_{\mathbb{R}^3} |u_n(x)|^p \left(\int_{|z|\leqslant R} \frac{\mathrm{d}z}{|z|^{3+sp-p}} \right) \mathrm{d}x \\ & \quad \leqslant C \int_{\mathbb{R}^3} |u_n(x)|^p\,\mathrm{d}x \left(\int_R^\infty \frac{\mathrm{d}\rho}{\rho^{sp+1}} \right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{C}{R^p}\int_{\mathbb{R}^3}|u_n(x)|^p\,\mathrm{d}x\left(\int_0^R\frac{\mathrm{d}\rho}{\rho^{sp-p+1}}\right)\\
&\leqslant \frac{C}{R^{sp}}\int_{\mathbb{R}^3}|u_n(x)|^p\,\mathrm{d}x+\frac{C}{R^p}R^{-sp+p}\int_{\mathbb{R}^3}|u_n(x)|^p\,\mathrm{d}x\\
&\leqslant \frac{C}{R^{sp}}\int_{\mathbb{R}^3}|u_n(x)|^p\,\mathrm{d}x\leqslant\frac{C}{R^{sp}}\to 0 \quad \text{as } R\to\infty. \qquad (2.8)
\end{aligned}
$$

Gathering (2.6), (2.7) and (2.8) we infer that (2.5) is satisfied. In particular, by Fatou's lemma we have

$$
\int_{\mathcal{B}_R^c}\left(a\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^p}{|x-y|^{3+sp}}\,\mathrm{d}y+V(\varepsilon\, x)|u|^p\right)\mathrm{d}x<\eta. \qquad (2.9)
$$

Moreover, from (2.5) and (2.9), we can deduce that

$$
u_n\to u \text{ in } L^\sigma(\mathbb{R}^3) \quad \text{for any } \sigma\in[p,p_s^*). \qquad (2.10)
$$

Indeed, for any $\eta>0$ there exits $R=R(\eta)>0$ such that, for any n large enough, we have

$$
\begin{aligned}
|u_n-u|_p^p&=|u_n-u|_{L^p(\mathcal{B}_R)}^p+|u_n-u|_{L^p(\mathcal{B}_R^c)}^p\\
&\leqslant \eta+|u_n-u|_{L^p(\mathcal{B}_R^c)}^p\\
&\leqslant \eta+\frac{1}{V_1}\int_{\mathcal{B}_R^c}V(\varepsilon\, x)|u_n-u|^p\,\mathrm{d}x\\
&\leqslant \eta+C\int_{\mathcal{B}_R^c}\left(a\int_{\mathbb{R}^3}\frac{|(u_n(x)-u(x))-(u_n(y)-u(y))|^p}{|x-y|^{3+sp}}\,\mathrm{d}y\right.\\
&\qquad\left.+V(\varepsilon\, x)|u_n-u|^p\right)\mathrm{d}x\\
&\leqslant (1+C)\eta,
\end{aligned}
$$

that is $u_n\to u$ in $L^p(\mathbb{R}^3)$. Then, by interpolation, it follows that (2.10) is verified.

Now, from the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ and the growth assumptions on g we get

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}(g(\varepsilon\, x,u_n)u_n-g(\varepsilon\, x,u)u)(u_n-u)\,\mathrm{d}x\right|\\
&\quad\leqslant\left(|u_n|_p^{p-1}+|u|_p^{p-1}\right)|u_n-u|_p+C\left(|u_n|_\nu^{\nu-1}+|u|_\nu^{\nu-1}\right)|u_n-u|_\nu\\
&\quad\leqslant C|u_n-u|_p+C|u_n-u|_\nu
\end{aligned}
$$

which together with (2.10) implies that

$$
\lim_{n\to 0}\int_{\mathbb{R}^3}(g(\varepsilon\, x,u_n)u_n-g(\varepsilon\, x,u)u)(u_n-u)\,\mathrm{d}x=0. \qquad (2.11)
$$

In what follows, we prove that $\|u_n - u\|_\varepsilon \to 0$ as $n \to \infty$. Let $\varphi \in \mathcal{H}_\varepsilon$ be fixed and denote by $B_\varphi : \mathcal{H}_\varepsilon \to \mathbb{R}$ the linear functional on $\mathcal{H}_\varepsilon$ defined as

$$B_\varphi(v) := \iint_{\mathbb{R}^6} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(v(x)-v(y))}{|x-y|^{3+sp}}\, \mathrm{d}x\, \mathrm{d}y$$

for all $v \in \mathcal{H}_\varepsilon$. Moreover, by Hölder's inequality, it is clear that B_φ is continuous on $\mathcal{H}_\varepsilon$.

It follows from $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$ that

$$\lim_{n\to\infty} \left((a + b[u_n]^p_{s,p}) - (a + b[u]^p_{s,p})\right) B_u(u_n - u) = 0, \tag{2.12}$$

where we used the fact that $(a + b[u_n]^p_{s,p}) - (a + b[u]^p_{s,p})$ is bounded in $\mathbb{R}$.

On the other hand, by $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$, $\mathcal{I}'_\varepsilon(u_n) \to 0$, and (2.10), we know that $\langle \mathcal{I}'_\varepsilon(u_n) - \mathcal{I}'_\varepsilon(u), u_n - u\rangle \to 0$ as $n \to \infty$. Then, by (2.11) and (2.12) we obtain

$$\begin{aligned}
o_n(1) &= \langle \mathcal{I}'_\varepsilon(u_n) - \mathcal{I}'_\varepsilon(u), u_n - u\rangle \\
&= (a + b[u_n]^p_{s,p})B_{u_n}(u_n - u) - (a + b[u_n]^p_{s,p})B_u(u_n - u) \\
&\quad + \left((a + b[u_n]^p_{s,p}) - (a + b[u]^p_{s,p})\right) B_u(u_n - u) \\
&\quad + \int_{\mathbb{R}^3} V(\varepsilon\, x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} (g(\varepsilon\, x, u_n)u_n - g(\varepsilon\, x, u)u)(u_n - u)\, \mathrm{d}x \\
&= (a + b[u_n]^p_{s,p})(B_{u_n}(u_n - u) - B_u(u_n - u)) \\
&\quad + \int_{\mathbb{R}^3} V(\varepsilon\, x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\, \mathrm{d}x + o_n(1),
\end{aligned}$$

that is

$$\begin{aligned}
\lim_{n\to\infty} \Bigg(&(a + b[u_n]^p_{s,p})(B_{u_n}(u_n - u) - B_u(u_n - u)) \\
&+ \int_{\mathbb{R}^3} V(\varepsilon\, x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\, \mathrm{d}x \Bigg) = 0.
\end{aligned}$$

Since $(|z|^{p-2}z - |w|^{p-2}w)(z - w) \geqslant 0$ for all $z, w \in \mathbb{R}$, we can see that

$$(a + b[u_n]^p_{s,p})(B_{u_n}(u_n - u) - B_u(u_n - u)) \geqslant 0,$$

and by (V_1) we also have

$$V(\varepsilon\, x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \geqslant 0.$$

Therefore, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} (a + b[u_n]^p_{s,p})(B_{u_n}(u_n - u) - B_u(u_n - u)) = 0, \\
&\lim_{n\to\infty} \int_{\mathbb{R}^3} V(\varepsilon\, x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\, \mathrm{d}x = 0.
\end{aligned} \tag{2.13}$$

Now, we recall the following useful Simon inequalities [54]:

$$
\begin{aligned}
&|\xi-\eta|^{p} \leqslant c_{p}(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta) \quad \text{for } p \geqslant 2,\\
&|\xi-\eta|^{p} \leqslant C_{p}[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)]^{p/2}(|\xi|^{p}+|\eta|^{p})^{(2-p)/2} \quad \text{for } 1<p<2
\end{aligned}
\tag{2.14}
$$

for all $\xi,\eta\in\mathbb{R}^{N}$, where c_p and C_p are positive constants depending only on p.

We first suppose that $p\geqslant 2$. Then, by (2.13) and (2.14), it holds

$$
\begin{aligned}
[u_n-u]^{p}_{s,p} &= \iint_{\mathbb{R}^6} |u_n(x)-u_n(y)-u(x)+u(y)|^{p}|x-y|^{-(3+sp)}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant c_p \iint_{\mathbb{R}^6} [|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\\
&\quad -|u(x)-u(y)|^{p-2}(u(x)-u(y))]\\
&\quad \times (u_n(x)-u_n(y)-u(x)+u(y))|x-y|^{-(3+sp)}\,\mathrm{d}x\,\mathrm{d}y\\
&= c_p[B_{u_n}(u_n-u)-B_u(u_n-u)] = o_n(1).
\end{aligned}
$$

In a similar fashion, by (V_1) and (2.13), we get

$$
\int_{\mathbb{R}^3} V(\varepsilon x)|u_n-u|^{p}\,\mathrm{d}x \leqslant c_p \int_{\mathbb{R}^3} V(\varepsilon x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,\mathrm{d}x = o_n(1).
$$

In conclusion, $\|u_n-u\|_{\varepsilon}\to 0$ as $n\to\infty$.

Now, we consider the case when $1<p<2$. Since $u_n \rightharpoonup u$ in $\mathcal{H}_{\varepsilon}$, there exists $\kappa>0$ such that $\|u_n\|_{\varepsilon}\leqslant\kappa$ for all $n\in\mathbb{N}$. Hence, by (2.13), (2.14) and Hölder's inequality we can see that

$$
\begin{aligned}
[u_n-u]^{p}_{s,p} &\leqslant C_p\left(B_{u_n}(u_n-u)-B_u(u_n-u)\right)^{p/2}\left([u_n]^{p}_{s,p}+[u]^{p}_{s,p}\right)^{2-p/2}\\
&\leqslant C_p\left(B_{u_n}(u_n-u)-B_u(u_n-u)\right)^{p/2}\left([u_n]^{p(2-p)/2}_{s,p}+[u]^{p(2-p)/2}_{s,p}\right)\\
&\leqslant C'_p\left(B_{u_n}(u_n-u)-B_u(u_n-u)\right)^{p/2} = o_n(1),
\end{aligned}
$$

where we used the following inequality

$$
(a+b)^{2-p/2}\leqslant a^{2-p/2}+b^{2-p/2} \quad \forall a,b\geqslant 0,\ 1<p<2.
$$

In a similar way, we obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} V(\varepsilon x)|u_n-u|^{p}\,\mathrm{d}x\\
&\quad\leqslant C''_p\left(\int_{\mathbb{R}^3} V(\varepsilon x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,\mathrm{d}x\right)^{p/2} = o_n(1).
\end{aligned}
$$

Then we can infer that $\|u_n-u\|_{\varepsilon}\to 0$ as $n\to\infty$. This ends the proof of lemma. □

As a byproduct of lemma 2.3, lemma 2.4 and mountain pass theorem [4], we can deduce that for all $\varepsilon>0$ there exists $u_{\varepsilon}\in\mathcal{H}_{\varepsilon}$ such that $\mathcal{I}_{\varepsilon}(u_{\varepsilon})=c_{\varepsilon}$ and $\mathcal{I}'_{\varepsilon}(u_{\varepsilon})=0$, that is u_{ε} is a weak solution to (2.2).

Let $u^- := \min\{u, 0\}$. Using $\langle \mathcal{I}'_\varepsilon(u), u^-\rangle = 0$, $g(x,t) = 0$ for $t \leqslant 0$, and

$$|x-y|^{p-2}(x-y)(x^- - y^-) \geqslant |x^- - y^-|^p \quad \forall x, y \in \mathbb{R},$$

we can infer

$$\begin{aligned}\|u^-\|_\varepsilon^p \leqslant a \iint_{\mathbb{R}^6} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\mathbb{R}^3} V(\varepsilon x)|u|^{p-2}uu^-\,\mathrm{d}x + b[u]_{s,p}^p \\ &\times \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= 0,\end{aligned}$$

which implies that $u^- = 0$, that is $u \geqslant 0$ in $\mathbb{R}^3$. By a Moser iteration argument [**45**], we can prove that $u \in L^\infty(\mathbb{R}^3) \cap \mathcal{C}^0(\mathbb{R}^N)$ (see lemma 2.8 below). From maximum principle [**20**] we can infer that $u > 0$ in $\mathbb{R}^3$.

2.2. The limiting problem

Let us consider the following family of autonomous problems, with $\mu > 0$,

$$\begin{cases}(a + b\,[u]_{s,p}^p)(-\Delta)_p^s u + \mu u^{p-1} = f(u) & \text{in } \mathbb{R}^3,\\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3.\end{cases} \tag{2.15}$$

The energy functional associated with (2.15) is given by

$$\mathcal{E}_\mu(u) = \frac{1}{p}a[u]_{s,p}^p + \frac{b}{2p}[u]_{s,p}^{2p} + \mu|u|_p^p - \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x$$

which is well defined on the space $\mathcal{H}_\mu := W^{s,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\mu^p := a[u]_{s,p}^p + \mu|u|_p^p.$$

It is easy to check that $\mathcal{E}_\mu \in \mathcal{C}^1(\mathcal{H}_\mu, \mathbb{R})$ and for any $u, \varphi \in \mathcal{H}_\mu$ its differential is given by

$$\begin{aligned}\langle \mathcal{E}'_\mu(u), \varphi\rangle = a \iint_{\mathbb{R}^6} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ b[u]_{s,p}^p \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \mu \int_{\mathbb{R}^3} |u|^{p-2}u\varphi\,\mathrm{d}x - \int_{\mathbb{R}^3} f(u)\varphi\,\mathrm{d}x.\end{aligned}$$

We denote by $\mathcal{N}_\mu$ the Nehari manifold associated with $\mathcal{E}_\mu$, that is

$$\mathcal{N}_\mu := \left\{u \in \mathcal{H}_\mu \setminus \{0\} : \langle \mathcal{E}'_\mu(u), u\rangle = 0\right\},$$

and

$$d_\mu := \inf_{u\in\mathcal{N}_\mu} \mathcal{E}_\mu(u).$$

Arguing as in lemma 2.3, it is easy to show that $\mathcal{E}_\mu$ has a mountain-pass geometry, and as in [**57**], it is standard to verify that

$$d_\mu = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} \mathcal{E}_\mu(\gamma(t)) = \inf_{u\in\mathcal{H}_\mu\setminus\{0\}} \max_{t\geqslant 0} \mathcal{E}_\mu(tu),$$

where

$$\Gamma_\mu := \left\{\gamma \in \mathcal{C}^0([0,1],\mathcal{H}_\mu) : \gamma(0) = 0,\, \mathcal{E}_\mu(\gamma(1)) \leqslant 0\right\}.$$

Lemma 2.5. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\mu$ be a $(PS)_c$ sequence for $\mathcal{E}_\mu$ such that $u_n \rightharpoonup 0$. Then we have either*

(a) *$u_n \to 0$ in $\mathcal{H}_\mu$, or*

(b) *there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and constant $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^p \,\mathrm{d}x \geqslant \beta.$$

Proof. Assume that (b) is not true. Then, by lemma 2.1 we can deduce that

$$u_n \to 0 \quad \text{in } L^\sigma(\mathbb{R}^3) \quad \text{for all } \sigma \in (p, p_s^*). \tag{2.16}$$

From (2.16) and (f_1)–(f_2) we can infer that

$$\int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x = \int_{\mathbb{R}^3} f(u_n)u_n\,\mathrm{d}x = o_n(1) \quad \text{as } n \to \infty.$$

On the other hand, arguing as in lemma 2.4, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\mu$, and we may assume that $u_n \rightharpoonup u$ in $\mathcal{H}_\mu$. Taking into account that $\langle \mathcal{E}'_\mu(u_n), u_n\rangle = 0$, we get

$$\|u_n\|_\mu^p + b\,[u_n]_{s,p}^{2p} = \int_{\mathbb{R}^3} f(u_n)u_n\,\mathrm{d}x = o_n(1),$$

which implies that $\|u_n\|_\mu \to 0$ as $n \to \infty$, that is (a) holds true. □

Now, we prove the following existence result for (2.15):

Theorem 2.2. *For all $\mu > 0$, problem (2.15) admits a positive ground state solution.*

Proof. Using a variant of the mountain-pass theorem without (PS)-condition [**57**], there exists a Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\mu$ for $\mathcal{E}_\mu$ at the level d_μ. Arguing as in lemma 2.4, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\mu$, so we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } \mathcal{H}_\mu,\\ &u_n \to u \quad \text{in } L^\sigma_{loc}(\mathbb{R}^3) \text{ for all } \sigma\in[1,p_s^*). \end{aligned}$$

Moreover, by lemma 2.5, it follows that u is nontrivial. Now, for any $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ we have

$$\begin{aligned} &a\iint_{\mathbb{R}^6}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y+\mu\int_{\mathbb{R}^3}|u|^{p-2}u\varphi\,\mathrm{d}x\\ &\quad+bB^p\iint_{\mathbb{R}^6}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad-\int_{\mathbb{R}^3}f(u)\varphi\,\mathrm{d}x=0, \end{aligned}\tag{2.17}$$

where $B^p := \lim_{n\to\infty}[u_n]^p_{s,p}$. By Fatou's Lemma we have

$$[u]^p_{s,p}\leqslant B^p. \tag{2.18}$$

Our aim is to prove that the equality holds in (2.18). Assume by contradiction that $[u]^p_{s,p}<B^p$. Taking $\varphi=u$ in (2.17) we have that $\langle\mathcal{E}'_\mu(u),u\rangle<0$. From assumptions (f_1) and (f_2) we can see that $\langle\mathcal{E}'_\mu(t_1u),t_1u\rangle<0$ for some $0<t_1\ll 1$. Thus, there exists $\tau\in(t_1,1)$ such that $\langle\mathcal{E}'_\mu(\tau u),\tau u\rangle=0$. Now, using $\tau\in(0,1)$, the fact that $t\mapsto(1/2p)f(t)t-F(t)$ is increasing for any $t\geqslant 0$, and Fatou's lemma we can infer that

$$\begin{aligned} d_\mu &\leqslant \mathcal{E}_\mu(\tau u)-\frac{1}{2p}\langle\mathcal{E}'_\mu(\tau u),\tau u\rangle\\ &<\mathcal{E}_\mu(u)-\frac{1}{2p}\langle\mathcal{E}'_\mu(u),u\rangle\\ &\leqslant\liminf_{n\to\infty}\left(\mathcal{E}_\mu(u_n)-\frac{1}{2p}\langle\mathcal{E}'_\mu(u_n),u_n\rangle\right)=d_\mu, \end{aligned}\tag{2.19}$$

which gives a contradiction. Hence $[u]^p_{s,p}=B^p$. Therefore, by (2.17), we deduce that $\mathcal{E}'_\mu(u)=0$, that is $\mathcal{E}_\mu$ admits a nontrivial critical point $u\in\mathcal{H}_\mu$. Arguing as at the end of § 2.1, we can deduce that $u>0$ in $\mathbb{R}^3$. Finally, proceeding as in (2.19) with $\tau=1$, we can show that u is a ground state solution to (2.15). □

REMARK 2.1. We suspect that under the assumptions that $s\in(0,1)$ and $p\in(1,\infty)$ are such that $sp<3$, $f(u)=u^{q-1}$ with $q\in(p,p_s^*)$, it is possible to obtain an existence result to (2.15) for small $b>0$. The idea is to apply the Struwe–Jeanjean monotonicity trick as in [**6**] by considering the family of truncated functional $\mathcal{E}^k_{b,\lambda}:W^{s,p}_{rad}(\mathbb{R}^3)\to\mathbb{R}$, with $k\in\mathbb{N}$, $\lambda\in[\lambda_0,1]$, defined by

$$\mathcal{E}^k_{b,\lambda}(u)=\frac{\|u\|^p_\mu}{p}+\frac{b}{2p}\chi\left(\frac{\|u\|^p_\mu}{k^p}\right)[u]^{2p}_{s,p}-\lambda\int_{\mathbb{R}^N}F(u)\,\mathrm{d}x,$$

where χ is a cut-off function with support in the ball $\mathcal{B}_2$. Then, once proved that there exists a sequence $\{\lambda_j\}_{j\in\mathbb{N}}\subset[\lambda_0,1]$, $\lambda_j\to 1$, and $\{u_j\}_{j\in\mathbb{N}}\subset W^{s,p}_{rad}(\mathbb{R}^3)$ such that each u_j is a critical point of $\mathcal{E}^k_{b,\lambda_j}$, one has to show that for all $k > 0$ large, there exists $b_0 > 0$ such that, for all $b \in (0, b_0)$, $\|u_j\|_\mu \leqslant k$ for all $j \in \mathbb{N}$. After that, arguing as in [6], the existence result for (2.15) follows.

We conclude this section by proving a very interesting relation between c_ε and d_{V_0}.

Lemma 2.6. *It holds* $\limsup_{\varepsilon\to 0} c_\varepsilon \leqslant d_{V_0}$.

Proof. Let ω be a positive ground state of (2.15) given by theorem 2.2 with $\mu = V_0$. For any $\varepsilon > 0$ let $\psi_\varepsilon(x) := \psi(\varepsilon x)$ where $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^3)$ is such that

$$\begin{cases} \psi(x) = 1 & \text{if } x \in \mathcal{B}_1, \\ \psi(x) = 0 & \text{if } x \in \mathcal{B}_2^c, \\ 0 \leqslant \psi \leqslant 1, \end{cases}$$

and consider the function $\omega_\varepsilon(x) = \psi_\varepsilon(x)\omega(x)$. For simplicity, let us assume that $\mathrm{supp}(\psi) \subset \mathcal{B}_2 \subset \Lambda$.

By lemma 2.2 and the Dominated Convergence Theorem we can infer that, as $\varepsilon \to 0$,

$$\omega_\varepsilon \to \omega \in W^{s,p}(\mathbb{R}^3) \quad \text{and} \quad \mathcal{E}_{V_0}(\omega_\varepsilon) \to \mathcal{E}_{V_0}(\omega) = d_{V_0}. \tag{2.20}$$

Now, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\mathcal{I}_\varepsilon(t_\varepsilon\omega_\varepsilon) = \max_{t\geqslant 0} \mathcal{I}_\varepsilon(t\omega_\varepsilon).$$

Hence, $\langle \mathcal{I}'_\varepsilon(t_\varepsilon\omega_\varepsilon), \omega_\varepsilon\rangle = 0$ and we have

$$t_\varepsilon^p \|\omega_\varepsilon\|_\varepsilon^p + b\, t_\varepsilon^{2p} [\omega_\varepsilon]_{s,p}^{2p} = \int_{\mathbb{R}^3} f(t_\varepsilon\omega_\varepsilon) t_\varepsilon\omega_\varepsilon \,\mathrm{d}x,$$

which implies that

$$\frac{a}{t_\varepsilon^p}[\omega_\varepsilon]_{s,p}^p + \frac{1}{t_\varepsilon^p}\int_{\mathbb{R}^3} V(\varepsilon x)|\omega_\varepsilon|^p \,\mathrm{d}x + b[\omega_\varepsilon]_{s,p}^{2p} = \int_{\mathbb{R}^3} \frac{f(t_\varepsilon\omega_\varepsilon)}{(t_\varepsilon\omega_\varepsilon)^{2p-1}}\omega_\varepsilon^{2p}\,\mathrm{d}x. \tag{2.21}$$

By the growth assumptions on f it follows that $t_\varepsilon \to t_0 > 0$. Our aim is to prove that $t_0 = 1$. Taking the limit as $\varepsilon \to 0$ in (2.21) and using (2.20) we get

$$\frac{a}{t_0^p}[\omega]_{s,p}^p + \frac{1}{t_0^p}\int_{\mathbb{R}^3} V_0|\omega|^p \,\mathrm{d}x + b[\omega]_{s,p}^{2p} = \int_{\mathbb{R}^3} \frac{f(t_0\omega)}{(t_0\omega)^{2p-1}}\omega^{2p}\,\mathrm{d}x.$$

From the above relation, $\omega \in \mathcal{N}_{V_0}$ and (f_4) we deduce that $t_0 = 1$. On the other hand, we can note that

$$c_\varepsilon \leqslant \max_{t\geqslant 0} \mathcal{I}_\varepsilon(t\omega_\varepsilon) = \mathcal{I}_\varepsilon(t_\varepsilon\omega_\varepsilon) = \mathcal{E}_{V_0}(t_\varepsilon\omega_\varepsilon) + \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^3} (V(\varepsilon x) - V_0)\omega_\varepsilon^p \,\mathrm{d}x.$$

Hence, using (2.20), $t_\varepsilon \to 1$, and that $V(\varepsilon x)$ is bounded on the support of ω_ε, we deduce the thesis. □

2.3. Proof of theorem 1.1

For $\varepsilon > 0$, let u_ε be the mountain pass solution to (2.2). For any $\varepsilon_n \to 0^+$ we denote by

$$u_n := u_{\varepsilon_n}, \quad \mathcal{I}_n := \mathcal{I}_{\varepsilon_n}, \quad \mathcal{H}_n := \mathcal{H}_{\varepsilon_n} \quad \text{and} \quad c_n := c_{\varepsilon_n}.$$

Then, u_n satisfies

$$(a + b[u_n]_{s,p}^p)(-\Delta)_p^s u_n + V(\varepsilon\, x)u_n^{p-1} = g(\varepsilon\, x, u_n) \quad \text{in } \mathbb{R}^3.$$

LEMMA 2.7. *Let $\varepsilon_n \to 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_n$ be such that $\mathcal{I}_n(u_n) = c_{\varepsilon_n}$ and $\mathcal{I}_n'(u_n) = 0$. Then there exists $\{\tilde{y}_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that the translated sequence*

$$\tilde{u}_n(x) := u_n(x + \tilde{y}_{\varepsilon_n})$$

has a subsequence which converges in $W^{s,p}(\mathbb{R}^3)$. Moreover, up to a subsequence, $\{y_{\varepsilon_n}\}_{n\in\mathbb{N}} := \{\varepsilon_n\, \tilde{y}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is such that $y_{\varepsilon_n} \to y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$.

Proof. Using $\langle \mathcal{I}_n'(u_n), u_n\rangle = 0$ and assumptions (g_1) and (g_2), it is easy to see that there is $\kappa > 0$ such that

$$\|u_n\|_{\varepsilon_n} \geqslant \kappa > 0 \quad \text{for any } n \in \mathbb{N}.$$

Taking into account $\mathcal{I}_n(u_n) = c_n$, $\langle \mathcal{I}_n'(u_n), u_n\rangle = 0$ and lemma 2.6, we can argue as in the proof of lemma 2.4 to deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_n$.

Now, proceeding as in lemma 2.5, it is easy to prove that there are a sequence $\{\tilde{y}_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(\tilde{y}_{\varepsilon_n})} |u_n|^p \, \mathrm{d}x \geqslant \beta.$$

Hereafter we denote by $\{\tilde{y}_n\}_{n\in\mathbb{N}}$ the sequence $\{\tilde{y}_{\varepsilon_n}\}_{n\in\mathbb{N}}$. Set $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$. Then $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}(\mathbb{R}^3)$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{weakly in } W^{s,p}(\mathbb{R}^3). \tag{2.22}$$

Moreover, $\tilde{u} \neq 0$ in view of

$$\int_{\mathcal{B}_R} |\tilde{u}|^p \, \mathrm{d}x \geqslant \beta. \tag{2.23}$$

Now, we set $y_n := \varepsilon_n\, \tilde{y}_n$. Let us begin by proving that $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. To this end, it is enough to show the following claim:

CLAIM 1. $\lim_{n\to\infty} \operatorname{dist}(y_n, \overline{\Lambda}) = 0$.

Indeed, if the claim does not hold, there is $\delta > 0$ and a subsequence of $\{y_n\}_{n\in\mathbb{N}}$, still denoted by itself, such that

$$\operatorname{dist}(y_n, \overline{\Lambda}) \geqslant \delta \quad \forall n \in \mathbb{N}.$$

Then we can find $r > 0$ such that $\mathcal{B}_r(y_n) \subset \Lambda^c$ for all $n \in \mathbb{N}$. Since $\tilde{u} \geqslant 0$ and $\mathcal{C}_c^\infty(\mathbb{R}^3)$ is dense in $W^{s,p}(\mathbb{R}^3)$, we can find a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that

$\psi_j \geqslant 0$ and $\psi_j \to \tilde{u}$ in $W^{s,p}(\mathbb{R}^3)$. Fixed $j \in \mathbb{N}$ and using $\psi = \psi_j$ as test function in $\langle \mathcal{I}_n'(u_n), \psi \rangle = 0$ we get

$$\begin{aligned} & a \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \\ & \quad + b[\tilde{u}_n]_{s,p}^p \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \\ & \quad + \int_{\mathbb{R}^3} V(\widetilde{\xi}_n)|\tilde{u}_n|^{p-2}\tilde{u}_n \psi_j \, \mathrm{d}x = \int_{\mathbb{R}^3} g(\widetilde{\xi}_n, \tilde{u}_n)\psi_j \, \mathrm{d}x, \end{aligned} \tag{2.24}$$

where $\widetilde{\xi}_n := \varepsilon_n \, x + \varepsilon_n \, \tilde{y}_n$. Taking into account that $u_n \geqslant 0$, $\psi_j \geqslant 0$ and the definition of g, we can see that

$$\begin{aligned} \int_{\mathbb{R}^3} g(\widetilde{\xi}_n, \tilde{u}_n)\psi_j \, \mathrm{d}x &= \int_{\mathcal{B}_{r/\varepsilon_n}} g(\widetilde{\xi}_n, \tilde{u}_n)\psi_j \, \mathrm{d}x + \int_{\mathcal{B}^c_{r/\varepsilon_n}} g(\widetilde{\xi}_n, \tilde{u}_n)\psi_j \, \mathrm{d}x \\ &\leqslant \int_{\mathcal{B}_{r/\varepsilon_n}} \frac{V_1}{K}|\tilde{u}_n|^{p-2}\tilde{u}_n\psi_j \, \mathrm{d}x + \int_{\mathcal{B}^c_{r/\varepsilon_n}} f(\tilde{u}_n)\psi_j \, \mathrm{d}x. \end{aligned} \tag{2.25}$$

Gathering (2.24) and (2.25) we have

$$\begin{aligned} & a \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \\ & \quad + b[\tilde{u}_n]_{s,p}^p \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \\ & \quad + V_1\left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^3} |\tilde{u}_n|^{p-2}\tilde{u}_n\psi_j \, \mathrm{d}x \leqslant \int_{\mathcal{B}^c_{r/\varepsilon_n}} f(\tilde{u}_n)\psi_j \, \mathrm{d}x. \end{aligned} \tag{2.26}$$

From (2.22) and the facts that ψ_j as compact support in $\mathbb{R}^3$ and $\varepsilon_n \to 0$ we can see that

$$\begin{aligned} & \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \\ & \quad \to \iint_{\mathbb{R}^6} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2}(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \quad \text{as } n \to \infty \end{aligned}$$

and

$$\int_{\mathcal{B}^c_{r/\varepsilon_n}} f(\tilde{u}_n)\psi_j \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

The above limits together with (2.26) and $[\tilde{u}_n]_{s,p}^p \to B^p$ imply that

$$
\begin{aligned}
a &\iint_{\mathbb{R}^6} \frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}(\tilde{u}(x)-\tilde{u}(y))(\psi_j(x)-\psi_j(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ bB^p \iint_{\mathbb{R}^6} \frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}(\tilde{u}(x)-\tilde{u}(y))(\psi_j(x)-\psi_j(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ V_1\left(1-\frac{1}{K}\right)\int_{\mathbb{R}^3} |\tilde{u}|^{p-2}\tilde{u}\psi_j\,\mathrm{d}x \leqslant 0.
\end{aligned}
$$

Taking the limit as $j \to \infty$ we have

$$
a[\tilde{u}]_{s,p}^p + bB^p[\tilde{u}]_{s,p}^p + V_1\left(1-\frac{1}{K}\right)|\tilde{u}|_p^p \leqslant 0.
$$

This gives a contradiction in view of (2.23). Hence there exists a subsequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \to y_0 \in \overline{\Lambda}$.

Claim 2. $y_0 \in \Lambda$.

Using the definition of g and (2.24) we can see that

$$
\begin{aligned}
a &\iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\psi_j(x)-\psi_j(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ b[\tilde{u}_n]_{s,p}^p \iint_{\mathbb{R}^6} \frac{|\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\psi_j(x)-\psi_j(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^3} V(\widetilde{\xi_n})|\tilde{u}_n|^{p-2}\tilde{u}_n\psi_j\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} f(\tilde{u}_n)\psi_j\,\mathrm{d}x.
\end{aligned}
$$

Taking the limit as $n \to \infty$ we get

$$
\begin{aligned}
a &\iint_{\mathbb{R}^6} \frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}(\tilde{u}(x)-\tilde{u}(y))(\psi_j(x)-\psi_j(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ b[\tilde{u}]_{s,p}^p \iint_{\mathbb{R}^6} \frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}(\tilde{u}(x)-\tilde{u}(y))(\psi_j(x)-\psi_j(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^3} V(y_0)|\tilde{u}|^{p-2}\tilde{u}\psi_j\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} f(\tilde{u})\psi_j\,\mathrm{d}x.
\end{aligned}
$$

Passing to the limit as $j \to \infty$ we obtain

$$
a[\tilde{u}]_{s,p}^p + bB^p[\tilde{u}]_{s,p}^p + V(y_0)|\tilde{u}|_p^p \leqslant \int_{\mathbb{R}^3} f(\tilde{u})\tilde{u}\,\mathrm{d}x.
$$

Using Fatou's Lemma we have $[\tilde{u}]_{s,p}^p = [u]_{s,p}^p \leqslant B^p$, which combined with the above inequality yields

$$
a[\tilde{u}]_{s,p}^p + b[\tilde{u}]_{s,p}^{2p} + V(y_0)|\tilde{u}|_p^p \leqslant \int_{\mathbb{R}^3} f(\tilde{u})\tilde{u}\,\mathrm{d}x.
$$

Hence, there exists $\tau \in (0,1)$ such that $\tau\tilde{u} \in \mathcal{N}_{V(y_0)}$. Let $d_{V(y_0)}$ be the mountain pass level associated with $\mathcal{E}_{V(y_0)}$. By lemma 2.6 we can see that

$$d_{V(y_0)} \leqslant \mathcal{E}_{V(y_0)}(\tau\tilde{u}) \leqslant \liminf_{n\to\infty} \mathcal{I}_n(u_n) = \liminf_{n\to\infty} c_n \leqslant d_{V_0}.$$

Thereby, $d_{V(y_0)} \leqslant d_{V_0}$, and this implies that $V(y_0) \leqslant V_0 = V(0)$. This together with the definition of V_0 yields that $V(y_0) = V_0$. From assumption (V_2) we have that $y_0 \notin \partial\Lambda$, thus $y_0 \in \Lambda$.

Claim 3. *$\tilde{u}_n \to \tilde{u}$ in $W^{s,p}(\mathbb{R}^3)$ as $n \to \infty$.*

Consider the set $\widetilde{\Lambda}_n = \Lambda/\,\varepsilon_n - \tilde{y}_n$ and define the functions

$$\widetilde{\chi}_n^1(x) := \begin{cases} 1 & \text{if } x \in \widetilde{\Lambda}_n \\ 0 & \text{if } x \in \widetilde{\Lambda}_n^c \end{cases} \qquad \text{and} \quad \widetilde{\chi}_n^2(x) := 1 - \widetilde{\chi}_n^1(x).$$

Introduce the following functions:

$$\begin{aligned}
h_n^1(x) &:= \left(\frac{1}{p} - \frac{1}{\vartheta}\right) V(\widetilde{\xi}_n)|\tilde{u}_n|^p \widetilde{\chi}_n^1(x), \\
h^1(x) &:= \left(\frac{1}{p} - \frac{1}{\vartheta}\right) V(y_0)|\tilde{u}|^p, \\
h_n^2(x) &:= \left[\left(\frac{1}{p} - \frac{1}{\vartheta}\right) V(\widetilde{\xi}_n)|\tilde{u}_n(x)|^p + \frac{1}{\vartheta} g(\widetilde{\xi}_n, \tilde{u}_n(x))\tilde{u}_n(x) - G(\widetilde{\xi}_n, \tilde{u}_n(x))\right] \widetilde{\chi}_n^2(x) \\
&\geqslant \left[\left(\frac{1}{p} - \frac{1}{\vartheta}\right) - \frac{1}{K}\right] V(\widetilde{\xi}_n)|\tilde{u}_n|^p \widetilde{\chi}_n^2(x), \\
h_n^3(x) &:= \left(\frac{1}{\vartheta} g(\widetilde{\xi}_n, \tilde{u}_n(x))\tilde{u}_n(x) - G(\widetilde{\xi}_n, \tilde{u}_n(x))\right) \widetilde{\chi}_n^1(x) \\
&= \left[\frac{1}{\vartheta} f(\tilde{u}_n(x))\tilde{u}_n(x) - F(\tilde{u}_n(x))\right] \widetilde{\chi}_n^1(x), \\
h^3(x) &:= \left[\frac{1}{\vartheta} f(\tilde{u}(x))\tilde{u}(x) - F(\tilde{u}(x))\right].
\end{aligned}$$

In the light of (f_3), $K > 2p > (\vartheta p)/\vartheta - p$, and (g_3), we can see that the above functions are nonnegative.

By (2.22) and claim 2, we can deduce that $\tilde{u}_n(x) \to \tilde{u}(x)$ a.e. $x \in \mathbb{R}^3$ and $\varepsilon_n\,\tilde{y}_n \to y_0 \in \Lambda$, from which we can infer that $\widetilde{\chi}_n^1(x) \to 1$, $h_n^1(x) \to h^1(x)$, $h_n^2(x) \to 0$ and $h_n^3(x) \to h^3(x)$ a.e. $x \in \mathbb{R}^3$.

Now, using lemma 2.6, the invariance of $\mathbb{R}^3$ by translation and Fatou's lemma we can deduce that

$$\begin{aligned}
d_{V_0} &\geqslant \limsup_{n\to\infty} c_n = \limsup_{n\to\infty} \left(\mathcal{I}_n(u_n) - \frac{1}{\vartheta}\langle \mathcal{I}_n'(u_n), u_n\rangle\right) \\
&= \limsup_{n\to\infty} \left[\left(\frac{1}{p} - \frac{1}{\vartheta}\right) a[\tilde{u}_n]_{s,p}^p + \left(\frac{1}{2p} - \frac{1}{\vartheta}\right) b\,[\tilde{u}_n]_{s,p}^{2p}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}(h_n^1(x)+h_n^2(x)+h_n^3(x))\,\mathrm{d}x\Bigg]\\
&\geqslant \liminf_{n\to\infty}\Bigg[\left(\frac{1}{p}-\frac{1}{\vartheta}\right)a[\tilde u_n]_{s,p}^p+\left(\frac{1}{2p}-\frac{1}{\vartheta}\right)b[\tilde u_n]_{s,p}^{2p}\\
&+\int_{\mathbb{R}^3}(h_n^1(x)+h_n^2(x)+h_n^3(x))\,\mathrm{d}x\Bigg]\\
&\geqslant \left(\frac{1}{p}-\frac{1}{\vartheta}\right)a[\tilde u]_{s,p}^p+\left(\frac{1}{2p}-\frac{1}{\vartheta}\right)b[\tilde u]_{s,p}^{2p}+\int_{\mathbb{R}^3}(h^1(x)+h^3(x))\,\mathrm{d}x\\
&\geqslant d_{V_0}.
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty}[\tilde u_n]_{s,p}^p=[\tilde u]_{s,p}^p \tag{2.27}$$

and $h_n^1\to h^1$, $h_n^2\to 0$, and $h_n^3\to h^3$ in $L^1(\mathbb{R}^3)$. Hence we can infer that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}V(\widetilde{\xi}_n)|\tilde u_n|^p\,\mathrm{d}x=\int_{\mathbb{R}^3}V(y_0)|\tilde u|^p\,\mathrm{d}x$$

and thus

$$\lim_{n\to\infty}|\tilde u_n|_p^p=|\tilde u|_p^p. \tag{2.28}$$

Combining (2.27) with (2.28), and using the Brezis–Lieb lemma [**14**] we get

$$\|\tilde u_n-\tilde u\|_{s,p}^p\to 0 \quad \text{as } n\to\infty.$$

□

LEMMA 2.8. *Let $\{\tilde u_n\}_{n\in\mathbb{N}}$ be the sequence given in lemma 3.4. Then, $\tilde u_n\in L^\infty(\mathbb{R}^3)$ and there exists $C>0$ such that*

$$|\tilde u_n|_\infty\leqslant C \quad \forall n\in\mathbb{N}.$$

Moreover,

$$\tilde u_n(x)\to 0 \text{ as } |x|\to\infty \text{ uniformly in } n\in\mathbb{N}. \tag{2.29}$$

Proof. For any $L>0$, let $\tilde u_{L,n}:=\min\{\tilde u_n,L\}$. Let $\sigma>1$ and define the function

$$\ell(\tilde u_n):=\ell_{L,\sigma}(\tilde u_n)=\tilde u_n\,\tilde u_{L,n}^{p(\sigma-1)}\in\mathcal{H}_\varepsilon.$$

We note that ℓ is increasing, so for any $a,b\in\mathbb{R}$ it holds $(a-b)(\ell(a)-\ell(b))\geqslant 0$. We introduce the functions

$$\mathcal{Q}(t):=\frac{|t|^p}{p} \quad \text{and} \quad \mathcal{L}(t):=\int_0^t(\ell'(\tau))^{1/p}\,\mathrm{d}\tau.$$

Let us point out that

$$\mathcal{L}(\tilde u_n)\geqslant\frac{1}{\sigma}\tilde u_n\tilde u_{L,n}^{\sigma-1}.$$

Hence, from theorem 2.1 and the above inequality we get

$$[\mathcal{L}(\tilde{u}_n)]_{s,p}^p \geqslant C_*^{-1}|\mathcal{L}(\tilde{u}_n)|_{p_s^*}^p \geqslant C_*^{-1}\frac{1}{\sigma^p}|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{p_s^*}^p. \tag{2.30}$$

On the other hand, for any $a, b \in \mathbb{R}$, it holds

$$\mathcal{Q}'(a-b)(\ell(a)-\ell(b)) \geqslant |\mathcal{L}(a)-\mathcal{L}(b)|^p.$$

Indeed, when $a > b$ we have

$$\begin{aligned}\mathcal{Q}'(a-b)(\ell(a)-\ell(b)) &= (a-b)^{p-1}(\ell(a)-\ell(b)) = (a-b)^{p-1}\int_b^a \ell'(\tau)\,\mathrm{d}\tau \\ &= (a-b)^{p-1}\int_b^a (\mathcal{L}'(\tau))^p\,\mathrm{d}\tau \geqslant \left(\int_b^a \mathcal{L}'(\tau)\,\mathrm{d}\tau\right)^p \\ &= (\mathcal{L}(a)-\mathcal{L}(b))^p,\end{aligned}$$

where in the last inequality we used the Jensen inequality. A similar argument works when $a \leqslant b$. Therefore we deduce that

$$\begin{aligned}&|\mathcal{L}(\tilde{u}_n)(x)-\mathcal{L}(\tilde{u}_n)(y)|^p \\ &\quad\leqslant |\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\tilde{u}_n(x)\tilde{u}_{L,n}^{p(\sigma-1)}(x)-\tilde{u}_n(y)\tilde{u}_{L,n}^{p(\sigma-1)}(y)).\end{aligned} \tag{2.31}$$

Using $\ell(\tilde{u}_n)$ as test function in (2.2) we get

$$\begin{aligned}&a\iint_{\mathbb{R}^6}\frac{|\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\tilde{u}_n(x)\tilde{u}_{L,n}^{p(\sigma-1)}(x)-\tilde{u}_n(y)\tilde{u}_{L,n}^{p(\sigma-1)}(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad+ b[\tilde{u}_n]_{s,p}^p\iint_{\mathbb{R}^6}\frac{\begin{array}{c}|\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\tilde{u}_n(x)\tilde{u}_{L,n}^{p(\sigma-1)}(x)\\ -\tilde{u}_n(y)\tilde{u}_{L,n}^{p(\sigma-1)}(y))\end{array}}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad+\int_{\mathbb{R}^3}V(\widetilde{\xi}_n)|\tilde{u}_n|^p\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x = \int_{\mathbb{R}^3}g(\widetilde{\xi}_n,\tilde{u}_n)\tilde{u}_n\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x,\end{aligned} \tag{2.32}$$

where $\widetilde{\xi}_n := \varepsilon_n\, x + \varepsilon_n\, \tilde{y}_n$. Putting together (2.31) and (2.32) we get

$$\begin{aligned}&a[\mathcal{L}(\tilde{u}_n)]_{s,p}^p + \int_{\mathbb{R}^3}V(\widetilde{\xi}_n)|\tilde{u}_n|^p\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x \\ &\quad\leqslant a\iint_{\mathbb{R}^6}\frac{\begin{array}{c}|\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\tilde{u}_n(x)\tilde{u}_{L,n}^{p(\sigma-1)}(x)\\ -\tilde{u}_n(y)\tilde{u}_{L,n}^{p(\sigma-1)}(y))\end{array}}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$
+ b[\tilde{u}_n]_{s,p}^p \iint_{\mathbb{R}^6} \frac{\begin{array}{c}|\tilde{u}_n(x)-\tilde{u}_n(y)|^{p-2}(\tilde{u}_n(x)-\tilde{u}_n(y))(\tilde{u}_n(x)\tilde{u}_{L,n}^{p(\sigma-1)}(x) \\ -\tilde{u}_n(y)\tilde{u}_{L,n}^{p(\sigma-1)}(y))\end{array}}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y
$$

$$
+ \int_{\mathbb{R}^3} V(\widetilde{\xi}_n)|\tilde{u}_n|^p \tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x = \int_{\mathbb{R}^3} g(\widetilde{\xi}_n,\tilde{u}_n)\tilde{u}_n\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x. \tag{2.33}
$$

From (2.30) and (2.33) we can infer that

$$
\begin{aligned}
|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{p_s^*}^p \leqslant \sigma^p C_*[\mathcal{L}(\tilde{u}_n)]_{s,p}^p &\leqslant \frac{\sigma^p C_*}{a}\int_{\mathbb{R}^3} g(\widetilde{\xi}_n,\tilde{u}_n)\tilde{u}_n\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x \\
&\leqslant \sigma^p C\int_{\mathbb{R}^3} g(\widetilde{\xi}_n,\tilde{u}_n)\tilde{u}_n\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x.
\end{aligned} \tag{2.34}
$$

Using the growth assumptions on g we have that for all $\zeta > 0$ there exists $C_\zeta > 0$ such that

$$
|g(x,t)| \leqslant \zeta|t|^{p-1} + C_\zeta|t|^{p_s^*-1} \quad \text{for all } x\in\mathbb{R}^3 \text{ and } t\in\mathbb{R},
$$

which together with (2.34) implies

$$
|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{p_s^*}^p \leqslant C\sigma^p\left(\int_{\mathbb{R}^3}\zeta|\tilde{u}_n|^p\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x + \int_{\mathbb{R}^3} C_\zeta|\tilde{u}_n|^{p_s^*}\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x\right).
$$

Choosing ζ sufficiently small we deduce that

$$
|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{p_s^*}^p \leqslant C\sigma^p\int_{\mathbb{R}^3}|\tilde{u}_n|^{p_s^*}\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x. \tag{2.35}
$$

Now, let $\sigma = p_s^*/p$ and fix $R > 0$. Using the fact that $0 \leqslant \tilde{u}_{L,n} \leqslant \tilde{u}_n$ and the Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3}\tilde{u}_n^{p_s^*}\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x &= \int_{\mathbb{R}^3}\tilde{u}_n^{p_s^*-p}(\tilde{u}_n\tilde{u}_{L,n}^{(p_s^*-p)/p})^p\,\mathrm{d}x \\
&= \int_{\{\tilde{u}_n>R\}}\tilde{u}_n^{p_s^*-p}(\tilde{u}_n\tilde{u}_{L,n}^{(p_s^*-p)/p})^p\,\mathrm{d}x \\
&\quad + \int_{\{\tilde{u}_n<R\}}\tilde{u}_n^{p_s^*-p}(\tilde{u}_n\tilde{u}_{L,n}^{(p_s^*-p)/p})^p\,\mathrm{d}x \\
&\leqslant \left(\int_{\{\tilde{u}_n>R\}}\tilde{u}_n^p\,\mathrm{d}x\right)^{(p_s^*-p)/p}\left(\int_{\mathbb{R}^3}(\tilde{u}_n\tilde{u}_{L,n}^{(p_s^*-p)/p})^{p_s^*}\,\mathrm{d}x\right)^{p/p_s^*} \\
&\quad + R^{p_s^*-p}\int_{\{\tilde{u}_n<R\}}\tilde{u}_n^{p_s^*}\,\mathrm{d}x.
\end{aligned}
$$

Since $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ strongly converges in $L^{p_s^*}(\mathbb{R}^3)$, we can see that for any R sufficiently large

$$\left(\int_{\{\tilde{u}_n>R\}} \tilde{u}_n^p \,\mathrm{d}x\right)^{(p_s^*-p)/p} \leqslant \frac{1}{2C\sigma^p},$$

and thus we deduce

$$\begin{aligned}\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*}\tilde{u}_{L,n}^{p(\sigma-1)}\,\mathrm{d}x \leqslant \frac{1}{2C\sigma^p}\left(\int_{\mathbb{R}^3} (\tilde{u}_n\tilde{u}_{L,n}^{(p_s^*-p)/p})^{p_s^*}\,\mathrm{d}x\right)^{p/p_s^*}\\ + R^{p_s^*-p}\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*}\,\mathrm{d}x.\end{aligned} \tag{2.36}$$

Putting together (2.35) and (2.36) we have

$$|\tilde{u}_n\tilde{u}_{L,n}^{\sigma-1}|_{p_s^*}^p \leqslant C\sigma^p R^{p_s^*-p}\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*}\,\mathrm{d}x < \infty,$$

and letting $L\to\infty$ we deduce that $\tilde{u}_n \in L^{(p_s^*)^2/p}(\mathbb{R}^3)$.

Now, taking the limit as $L\to\infty$ in (2.35) we have

$$|\tilde{u}_n|_{\sigma p_s^*}^{\sigma p} \leqslant C\sigma^p\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*+p(\sigma-1)}\,\mathrm{d}x$$

which implies

$$\left(\int_{\mathbb{R}^3} \tilde{u}_n^{\sigma p_s^*}\,\mathrm{d}x\right)^{1/(p_s^*(\sigma-1))} \leqslant (C\sigma)^{1/(\sigma-1)}\left(\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*+p(\sigma-1)}\,\mathrm{d}x\right)^{1/(p(\sigma-1))}.$$

For $m\geqslant 1$ we define σ_{m+1} inductively so that $p_s^*+p(\sigma_{m+1}-1)=p_s^*\sigma_m$ and $\sigma_1 = p_s^*/p$. Then we have

$$\left(\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*\sigma_{m+1}}\,\mathrm{d}x\right)^{1/(p_s^*(\sigma_{m+1}-1))} \leqslant (C\sigma_{m+1})^{1/(\sigma_{m+1}-1)}\left(\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*\sigma_m}\,\mathrm{d}x\right)^{1/(p_s^*(\sigma_m-1))}.$$

Set

$$\mathcal{D}_m := \left(\int_{\mathbb{R}^3} \tilde{u}_n^{p_s^*\sigma_m}\,\mathrm{d}x\right)^{1/(p_s^*(\sigma_m-1))}.$$

Using an iteration argument, we can find $C_0>0$ independent of m such that

$$\mathcal{D}_{m+1} \leqslant \prod_{k=1}^{m}(C\sigma_{k+1})^{1/(\sigma_{k+1}-1)}\mathcal{D}_1 \leqslant C_0\mathcal{D}_1.$$

Taking the limit as $m\to\infty$ we get $|\tilde{u}_n|_\infty \leqslant C$ for all $n\in\mathbb{N}$.

Now, we note that $\tilde{u}_n$ is a solution to

$$\begin{aligned}(-\Delta)_p^s\tilde{u}_n = [g(\varepsilon_n\, x+\varepsilon_n\,\tilde{y}_n, \tilde{u}_n)\\ - V(\varepsilon_n\, x+\varepsilon_n\,\tilde{y}_n)\tilde{u}_n^{p-1}](a+b[\tilde{u}_n]_{s,p}^p)^{-1} =: h_n \quad \text{in } \mathbb{R}^3.\end{aligned}$$

Moreover, $h_n \in L^\infty(\mathbb{R}^3)$ and $|h_n|_\infty \leqslant C$ for all $n \in \mathbb{N}$. Indeed, this last inequality is a consequence of the growth assumptions on g, $|\tilde{u}_n|_\infty \leqslant C$ and $a \leqslant a + b[\tilde{u}_n]^p_{s,p} \leqslant C$ for all $n \in \mathbb{N}$. Then, using corollary 5.5 in [**37**] we can deduce that $\tilde{u}_n \in \mathcal{C}^{0,\alpha}(\mathbb{R}^3)$ for some $\alpha > 0$. From this fact and (2.22) we infer that (2.29) holds true. $\square$

Corollary 2.1. *There is $n_0 \in \mathbb{N}$ such that*

$$u_n(x) < a_0 \quad \forall n \geqslant n_0 \text{ and } \forall x \in \Lambda^c_{\varepsilon_n}.$$

Hence, u_n is a solution to $(\hat{P}_{\varepsilon_n})$.

Proof. By lemma 3.4 we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $\tilde{u}_n = u_n(\cdot + \tilde{y}_n) \to \tilde{u}$ in $W^{s,p}(\mathbb{R}^3)$ and $y_n = \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$.

Now, if we choose $r > 0$ such that $\mathcal{B}_r(y_0) \subset \mathcal{B}_{2r}(y_0) \subset \Lambda$, then $\mathcal{B}_{r/\varepsilon_n}(y_0/\varepsilon_n) \subset \Lambda_{\varepsilon_n}$. Hence, there exists $n_1 \in \mathbb{N}$ such that for any $y \in \mathcal{B}_{r/\varepsilon_n}(\tilde{y}_n)$ we have

$$\left| y - \frac{y_0}{\varepsilon_n} \right| \leqslant |y - \tilde{y}_n| + \left| \tilde{y}_n - \frac{y_0}{\varepsilon_n} \right| < \frac{2r}{\varepsilon_n} \quad \forall n \geqslant n_1,$$

and consequently

$$\Lambda^c_{\varepsilon_n} \subset \mathcal{B}^c_{r/\varepsilon_n}(\tilde{y}_n) \quad \text{for any } n \geqslant n_1.$$

In the light of lemma 2.8 there is $R > 0$ such that

$$\tilde{u}_n(x) < a_0 \quad \text{for } |x| \geqslant R \text{ and } \forall n \in \mathbb{N},$$

from which

$$u_n(x) = \tilde{u}_n(x - \tilde{y}_n) < a_0 \quad \text{for } x \in \mathcal{B}^c_R(\tilde{y}_n) \text{ and } \forall n \in \mathbb{N}.$$

On the other hand, there exists $n_2 \in \mathbb{N}$ such that

$$\mathcal{B}^c_{r/\varepsilon_n}(\tilde{y}_n) \subset \mathcal{B}^c_R(\tilde{y}_n) \quad \forall n \geqslant n_2.$$

Hence, choosing $n_0 = \max\{n_1, n_2\}$, we can infer that

$$\Lambda^c_{\varepsilon_n} \subset \mathcal{B}^c_{r/\varepsilon_n}(\tilde{y}_n) \subset \mathcal{B}^c_R(\tilde{y}_n) \quad \forall n \geqslant n_0,$$

and then

$$u_n(x) < a_0 \quad \forall x \in \Lambda^c_{\varepsilon_n} \text{ and } \forall n \geqslant n_0.$$

$\square$

Proof of theorem 1.1. Let u_ε be a nonnegative solution to (2.2). Then, there is $\varepsilon_0 > 0$ such that

$$u_\varepsilon(x) < a_0 \quad \forall x \in \Lambda^c_\varepsilon \text{ and } \forall\, \varepsilon \in (0, \varepsilon_0),$$

that is u_ε is a solution to $(\hat{P}_\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$. Consider $v_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_0)$ and note that v_ε is a solution to (P_ε). Let η_ε be a global maximum point of v_ε. It is easy to see that there exists $\tau_0 > 0$ such that $v_\varepsilon(\eta_\varepsilon) \geqslant \tau_0$ for any $\varepsilon > 0$.

Set $z_\varepsilon := \eta_\varepsilon/\varepsilon - \tilde{y}_\varepsilon$. Then z_ε is a global maximum point of $\tilde{u}_\varepsilon(x) = u_\varepsilon(x+\tilde{y}_\varepsilon)$ and $\tilde{u}_\varepsilon(z_\varepsilon) \geqslant \tau_0$ for any $\varepsilon > 0$. We claim that

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

If the above limit is not true, then there exist $\varepsilon_n \to 0$ and $\delta > 0$ such that

$$V(\eta_{\varepsilon_n}) \geqslant V_0 + \delta \quad \forall n \in \mathbb{N}. \tag{2.37}$$

Using lemma 2.8 we know that $\tilde{u}_{\varepsilon_n}(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. Therefore $\{z_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded. Moreover, for some subsequence, there is $y_0 \in \Lambda$ such that $\varepsilon_n \tilde{y}_{\varepsilon_n} \to y_0$ and $V(y_0) = V_0$. Therefore $\eta_{\varepsilon_n} = \varepsilon_n z_{\varepsilon_n} + \varepsilon_n \tilde{y}_{\varepsilon_n} \to y_0$ which combined with the continuity of V yields $V(\eta_{\varepsilon_n}) \to V_0$. This contradicts (2.37). Accordingly, $V(\eta_\varepsilon) \to V_0$ as $\varepsilon \to 0$.

We conclude this section by proving a decay estimate for v_ε. Using (2.29) and (g_1), there exists $R_1 > 0$ such that

$$g(\varepsilon x, \tilde{u}_\varepsilon(x)) \leqslant \frac{V_1}{2}\tilde{u}_\varepsilon(x)^{p-1} \quad \forall x \in \mathcal{B}^c_{R_1}.$$

Therefore,

$$\begin{aligned}
&(-\Delta)^s_p \tilde{u}_\varepsilon + \frac{V_1}{2(a+bA_1^p)}\tilde{u}_\varepsilon^{p-1} \\
&\quad \leqslant (-\Delta)^s_p \tilde{u}_\varepsilon + \frac{V_1}{2(a+b[\tilde{u}_\varepsilon]^p_{s,p})}\tilde{u}_\varepsilon^{p-1} \\
&\quad = \frac{1}{a+b[\tilde{u}_\varepsilon]^p_{s,p}}\left[g(\varepsilon x+\varepsilon\tilde{y}_\varepsilon, \tilde{u}_\varepsilon) - \left(V(\varepsilon x+\varepsilon\tilde{y}_\varepsilon) - \frac{V_1}{2}\right)\tilde{u}_\varepsilon^{p-1}\right] \\
&\quad \leqslant \frac{1}{a+b[\tilde{u}_\varepsilon]^p_{s,p}}\left[g(\varepsilon x+\varepsilon\tilde{y}_\varepsilon, \tilde{u}_\varepsilon) - \frac{V_1}{2}\tilde{u}_\varepsilon^{p-1}\right] \leqslant 0 \quad \text{in } \mathcal{B}^c_{R_1},
\end{aligned} \tag{2.38}$$

where $A_1 > 0$ is such that $a + b[u_\varepsilon]^p_{s,p} \leqslant a + bA_1^p$, for any $\varepsilon \in (0, \varepsilon_0)$. Applying lemma 7.1 in [**21**], we can find a continuous positive function w and a positive constant C such that

$$0 < w(x) \leqslant \frac{C}{1+|x|^{3+sp}} \tag{2.39}$$

and

$$(-\Delta)^s_p w + \frac{V_1}{2(a+bA_1^p)} w^{p-1} \geqslant 0 \quad \text{in } \mathcal{B}^c_{R_2}, \tag{2.40}$$

for some $R_2 > 0$. Thanks to the continuity of $\tilde{u}_\varepsilon$ and w, there exists $C_1 > 0$ such that

$$\psi_\varepsilon := \tilde{u}_\varepsilon - C_1 w \leqslant 0 \quad \text{for } |x| = R_3,$$

where $R_3 := \max\{R_1, R_2\}$.

Taking $\phi = \max\{\psi_\varepsilon, 0\} \in W_0^{s,p}(\mathcal{B}_R^c)$ as test function in (2.38) and using (2.40) with $\tilde{w} = C_1 w$, we can deduce that

$$\begin{aligned}
0 &\geqslant \iint_{\mathbb{R}^6} \frac{|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)|^{p-2}(\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y))(\phi(x) - \phi(y))}{|x-y|^{3+sp}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \frac{V_1}{2(a + bA_1^p)} \int_{\mathbb{R}^3} \tilde{u}_\varepsilon^{p-1} \phi \, \mathrm{d}x \\
&\geqslant \iint_{\mathbb{R}^6} \frac{\mathcal{G}_\varepsilon(x,y)}{|x-y|^{3+sp}} (\phi(x) - \phi(y)) \, \mathrm{d}x \, \mathrm{d}y + \frac{V_1}{2(a + bA_1^p)} \int_{\mathbb{R}^3} [\tilde{u}_\varepsilon^{p-1} - \tilde{w}^{p-1}] \phi \, \mathrm{d}x,
\end{aligned} \tag{2.41}$$

where

$$\mathcal{G}_\varepsilon(x,y) := |\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)|^{p-2}(\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)) - |\tilde{w}(x) - \tilde{w}(y)|^{p-2}(\tilde{w}(x) - \tilde{w}(y)).$$

Therefore, if we prove that

$$\iint_{\mathbb{R}^6} \frac{\mathcal{G}_\varepsilon(x,y)}{|x-y|^{N+sp}} (\phi(x) - \phi(y)) \, \mathrm{d}x \, \mathrm{d}y \geqslant 0, \tag{2.42}$$

it follows from (2.41) that

$$0 \geqslant \frac{V_1}{2(a + bA_1^p)} \int_{\{\tilde{u}_\varepsilon \geqslant \tilde{w}\}} [\tilde{u}_\varepsilon^{p-1} - \tilde{w}^{p-1}](\tilde{u}_\varepsilon - \tilde{w}) \, \mathrm{d}x \geqslant 0$$

which yields that

$$\{x \in \mathbb{R}^3 : |x| \geqslant R_3 \text{ and } \tilde{u}_\varepsilon(x) \geqslant \tilde{w}\} = \emptyset.$$

To achieve our purpose, we first note that for all $c, d \in \mathbb{R}$ it holds

$$|d|^{p-2}d - |c|^{p-2}c = (p-1)(d-c) \int_0^1 |c + t(d-c)|^{p-2} \, \mathrm{d}t.$$

Taking $d = \tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)$ and $c = \tilde{w}(x) - \tilde{w}(y)$ we can see that

$$|d|^{p-2}d - |c|^{p-2}c = (p-1)(d-c)I(x,y),$$

where $I(x,y) \geqslant 0$ stands for the integral. Now, recalling that

$$(x-y)(x^+ - y^+) \geqslant |x^+ - y^+|^2 \quad \text{for all } x, y \in \mathbb{R},$$

we have

$$\begin{aligned}
(d-c)(\phi(x) - \phi(y)) &= [(\tilde{u}_\varepsilon - \tilde{w})(x) - (\tilde{u}_\varepsilon - \tilde{w})(y)][(\tilde{u}_\varepsilon - \tilde{w})^+(x) - (\tilde{u}_\varepsilon - \tilde{w})^+(y)] \\
&\geqslant |(\tilde{u}_\varepsilon - \tilde{w})^+(x) - (\tilde{u}_\varepsilon - \tilde{w})^+(y)|^2,
\end{aligned}$$

which gives $(|d|^{p-2}d - |c|^{p-2}c)(\phi(x) - \phi(y)) \geqslant 0$, that is (2.42) holds true.

Therefore, $\psi_\varepsilon \leqslant 0$ in $\mathcal{B}^c_{R_3}$, which implies that $\tilde{u}_\varepsilon \leqslant C_1 w$ in $\mathcal{B}^c_{R_3}$, that is $\tilde{u}_\varepsilon(x) \leqslant C(1+|x|^{3+sp})^{-1}$ in $\mathcal{B}^c_{R_3}$. Consequently,

$$\begin{aligned} v_\varepsilon(x) &= u_\varepsilon\left(\frac{x}{\varepsilon}\right) = \tilde{u}_\varepsilon\left(\frac{x}{\varepsilon} - \tilde{y}_\varepsilon\right) \leqslant C_1 w\left(\frac{x}{\varepsilon} - \tilde{y}_\varepsilon\right) \\ &\leqslant \frac{C}{1+|\frac{x}{\varepsilon} - \tilde{y}_\varepsilon|^{3+sp}} = \frac{C\,\varepsilon^{3+sp}}{\varepsilon^{3+sp} + |x - \varepsilon\,\tilde{y}_\varepsilon|^{3+sp}} \\ &\leqslant \frac{C\,\varepsilon^{3+sp}}{\varepsilon^{3+sp} + |x - \eta_\varepsilon|^{3+sp}} \end{aligned}$$

and this ends the proof of theorem 1.1. □

3. Multiple solutions for (P_ε)

3.1. The generalized Nehari method

In this section we deal with the multiplicity of positive solutions to (P_ε). To achieve our result, we need to introduce some fundamental tools.

Le us denote by

$$\mathcal{N}_\varepsilon := \{u \in \mathcal{H}_\varepsilon : \langle \mathcal{I}'_\varepsilon(u), u\rangle = 0\}$$

the Nehari manifold associated with (2.2), and define

$$\mathcal{H}^+_\varepsilon := \{u \in \mathcal{H}_\varepsilon : |\operatorname{supp}(u^+) \cap \Lambda_\varepsilon| > 0\} \subset \mathcal{H}_\varepsilon.$$

Let $\mathbb{S}_\varepsilon$ be the unit sphere of $\mathcal{H}_\varepsilon$ and set $\mathbb{S}^+_\varepsilon := \mathbb{S}_\varepsilon \cap \mathcal{H}^+_\varepsilon$. By the definition of $\mathbb{S}^+_\varepsilon$ and using the fact that $\mathcal{H}^+_\varepsilon$ is open in $\mathcal{H}_\varepsilon$, it follows that $\mathbb{S}^+_\varepsilon$ is a incomplete $\mathcal{C}^1$-manifold of codimension 1, modelled on $\mathcal{H}_\varepsilon$ and contained in the open $\mathcal{H}^+_\varepsilon$. Thus, $\mathcal{H}_\varepsilon = T_u\mathbb{S}^+_\varepsilon \oplus \mathbb{R}u$ for each $u \in \mathbb{S}^+_\varepsilon$, where

$$T_u\mathbb{S}^+_\varepsilon := \left\{ v \in \mathcal{H}_\varepsilon : B_u(v) + \int_{\mathbb{R}^3} V(\varepsilon\, x)|u|^{p-2}uv\,\mathrm{d}x = 0 \right\}$$

and

$$B_u(v) = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

The next results will be fundamental to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}^+_\varepsilon$.

Lemma 3.1. *Assume that* (V_1)–(V'_2) *and* (f_1)–(f_4) *hold. Then, we have the following results*:

(i) *For each* $u \in \mathcal{H}^+_\varepsilon$, *let* $h_u : \mathbb{R}^+ \to \mathbb{R}$ *be defined by* $h_u(t) := \mathcal{I}_\varepsilon(tu)$. *Then there exists a unique* $t_u > 0$ *such that* $h'_u(t) > 0$ *for all* $t \in (0, t_u)$ *and* $h'_u(t) < 0$ *for all* $t \in (t_u, \infty)$.

(ii) *There exists* $\tau > 0$ *independent of* u *such that* $t_u \geqslant \tau$ *for any* $u \in \mathbb{S}^+_\varepsilon$. *Moreover, for each compact set* $\mathbb{K} \subset \mathbb{S}^+_\varepsilon$ *there is a constant* $C_{\mathbb{K}} > 0$ *such that* $t_u \leqslant C_{\mathbb{K}}$ *for any* $u \in \mathbb{K}$.

(iii) *The map* $\hat{m}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ *given by* $\hat{m}_\varepsilon(u) = t_u u$ *is continuous and* $m_\varepsilon := \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ *is a homeomorphism between* $\mathbb{S}_\varepsilon^+$ *and* $\mathcal{N}_\varepsilon$. *Moreover,* $m_\varepsilon^{-1}(u) = u/\|u\|_\varepsilon$.

(iv) *If there is a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ *such that* $\operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$, *then* $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ *and* $\mathcal{I}_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof. (i) We know that $h_u \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$, and by lemma 2.3, we have that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Then there exists a global maximum point $t_u > 0$ for h_u such that $h_u'(t_u) = 0$, that is $t_u u \in \mathcal{N}_\varepsilon$.

Now, we aim to prove the uniqueness of a such t_u. Assume by contradiction that there exist $t_1 > t_2 > 0$ such that $h_u'(t_1) = h_u'(t_2) = 0$, or equivalently

$$t_1^{p-1}\|u\|_\varepsilon^p + b t_1^{2p-1}[u]_{s,p}^{2p} = \int_{\mathbb{R}^3} g(\varepsilon x, t_1 u) u \,\mathrm{d}x \tag{3.1}$$

$$t_2^{p-1}\|u\|_\varepsilon^p + b t_2^{2p-1}[u]_{s,p}^{2p} = \int_{\mathbb{R}^3} g(\varepsilon x, t_2 u) u \,\mathrm{d}x. \tag{3.2}$$

Dividing both members of (3.1) by t_1^{2p-1} we get

$$\frac{\|u\|_\varepsilon^p}{t_1^p} + b[u]_{s,p}^{2p} = \int_{\mathbb{R}^3} \frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2p-1}} u^{2p} \,\mathrm{d}x,$$

and similarly, dividing both members of (3.2) by t_2^{2p-1}, we obtain

$$\frac{\|u\|_\varepsilon^p}{t_2^p} + b[u]_{s,p}^{2p} = \int_{\mathbb{R}^3} \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2p-1}} u^{2p} \,\mathrm{d}x.$$

Subtracting the above identities, and taking into account the definition of g we can see that

$$\begin{aligned}
\left(\frac{1}{t_1^p} - \frac{1}{t_2^p}\right)\|u\|_\varepsilon^p &= \int_{\mathbb{R}^3} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2p-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2p-1}}\right] u^{2p} \,\mathrm{d}x \\
&\geqslant \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a_0\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2p-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2p-1}}\right] u^{2p} \,\mathrm{d}x \\
&\quad + \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leqslant a_0 < t_1 u\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2p-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2p-1}}\right] u^{2p} \,\mathrm{d}x \\
&\quad + \int_{\Lambda_\varepsilon^c \cap \{t_1 u < a_0\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2p-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2p-1}}\right] u^{2p} \,\mathrm{d}x \\
&\geqslant \frac{1}{K}\left(\frac{1}{t_1^p} - \frac{1}{t_2^p}\right) \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a_0\}} V_0 u^p \,\mathrm{d}x \\
&\quad + \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leqslant a_0 < t_1 u\}} \left[\frac{V_0}{K}\frac{1}{(t_1 u)^p} - \frac{f(t_2 u)}{(t_2 u)^{2p-1}}\right] u^{2p} \,\mathrm{d}x.
\end{aligned}$$

Accordingly, in view of $t_1 > t_2$, we have

$$\begin{aligned}
\|u\|_\varepsilon^p &\leqslant \frac{1}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a_0\}} V_0 u^p \,\mathrm{d}x \\
&\quad + \frac{t_1^p t_2^p}{t_2^p - t_1^p}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leqslant a_0<t_1u\}} \left[\frac{V_0}{K}\frac{1}{(t_1u)^p} - \frac{f(t_2u)}{(t_2u)^{2p-1}}\right] u^{2p}\,\mathrm{d}x \\
&= \frac{1}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a_0\}} V_0 u^p \,\mathrm{d}x \\
&\quad - \frac{t_2^p}{t_1^p - t_2^p}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leqslant a_0<t_1u\}} \frac{V_0}{K}u^p\,\mathrm{d}x \\
&\quad + \frac{t_1^p}{t_1^p - t_2^p}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leqslant a_0<t_1u\}} \frac{f(t_2u)}{(t_2u)^{p-1}}u^p\,\mathrm{d}x \\
&\leqslant \frac{1}{K}\int_{\Lambda_\varepsilon^c} V_0u^p\,\mathrm{d}x \leqslant \frac{1}{K}\|u\|_\varepsilon^p.
\end{aligned}$$

Since $u \neq 0$ and $K > 1$, we get a contradiction.

(ii) Let $u \in \mathbb{S}_\varepsilon^+$. By (i) there exists $t_u > 0$ such that $h_u'(t_u) = 0$, or equivalently

$$t_u^{p-1} \leqslant t_u^{p-1}\|u\|_\varepsilon^p + bt_u^{2p-1}[u]_{s,p}^{2p} = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_u u)u\,\mathrm{d}x.$$

From (g_1)–(g_2) and theorem 2.1, for all $\xi > 0$ we obtain

$$t_u^{p-1} \leqslant \int_{\mathbb{R}^3} g(\varepsilon\, x, t_u u)u\,\mathrm{d}x \leqslant \xi t_u^{p-1}C_1 + C_\xi t_u^{\nu-1}C_2,$$

and choosing ξ sufficiently small, we can find $\tau > 0$, independent of u, such that $t_u \geqslant \tau$.

Now, let $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ be a compact set, and assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ such that $t_n := t_{u_n} \to \infty$. Therefore, there exists $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathcal{H}_\varepsilon$. From the proof of (ii) in lemma 2.3 we get

$$\mathcal{I}_\varepsilon(t_n u_n) \to -\infty. \tag{3.3}$$

On the other hand, fixed $v \in \mathcal{N}_\varepsilon$, by $\langle \mathcal{I}_\varepsilon'(v), v\rangle = 0$ and (g_3) we can infer

$$\begin{aligned}
\mathcal{I}_\varepsilon(v) &= \mathcal{I}_\varepsilon(v) - \frac{1}{\vartheta}\langle \mathcal{I}_\varepsilon'(v), v\rangle \\
&\geqslant \left(\frac{\vartheta - p}{p\vartheta}\right)\|v\|_\varepsilon^p \\
&\quad + \left(\frac{\vartheta - 2p}{2p\vartheta}\right) b[v]_{s,p}^{2p} + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon^c} [g(\varepsilon\, x, v)v - \vartheta G(\varepsilon\, x, v)]\,\mathrm{d}x
\end{aligned}$$

$$\geqslant \left(\frac{\vartheta - p}{p\vartheta}\right)\|v\|_{\varepsilon}^{p} - \left(\frac{\vartheta - p}{p\vartheta}\right)\frac{1}{K}\int_{\Lambda_{\varepsilon}^{c}} V(\varepsilon\, x)v^{p}\,\mathrm{d}x$$
$$\geqslant \left(1 - \frac{1}{K}\right)\left(\frac{\vartheta - p}{p\vartheta}\right)\|v\|_{\varepsilon}^{p}.$$

Taking into account that $\{t_{u_n}u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon}$, $K > 2p > 3$, $\vartheta > 2p$, from the above inequality and (3.3) we obtain a contradiction.

(iii) Firstly, we note that $\hat{m}_{\varepsilon}$, m_{ε} and m_{ε}^{-1} are well defined. Indeed, by (i), for each $u \in \mathcal{H}_{\varepsilon}^{+}$ there exists a unique $m_{\varepsilon}(u) \in \mathcal{N}_{\varepsilon}$. On the other hand, if $u \in \mathcal{N}_{\varepsilon}$ then $u \in \mathcal{H}_{\varepsilon}^{+}$. Otherwise, if $u \notin \mathcal{H}_{\varepsilon}^{+}$, we get

$$|\operatorname{supp}(u^{+}) \cap \Lambda_{\varepsilon}| = 0,$$

which together with $(g3)$-(ii) yields

$$\begin{aligned}
\|u\|_{\varepsilon}^{p} &\leqslant \int_{\mathbb{R}^3} g(\varepsilon\, x, u)u\,\mathrm{d}x \\
&= \int_{\Lambda_{\varepsilon}^{c}} g(\varepsilon\, x, u)u\,\mathrm{d}x + \int_{\Lambda_{\varepsilon}} g(\varepsilon\, x, u)u\,\mathrm{d}x \\
&= \int_{\Lambda_{\varepsilon}^{c}} g(\varepsilon\, x, u^{+})\,u^{+}\,\mathrm{d}x \\
&\leqslant \frac{1}{K}\int_{\Lambda_{\varepsilon}^{c}} V(\varepsilon\, x)|u|^{p}\,\mathrm{d}x \leqslant \frac{1}{K}\|u\|_{\varepsilon}^{p}
\end{aligned} \tag{3.4}$$

and this leads to a contradiction because $K > 1$. Consequently, $m_{\varepsilon}^{-1}(u) = u/\|u\|_{\varepsilon} \in \mathbb{S}_{\varepsilon}^{+}$, m_{ε}^{-1} is well defined and continuous. From $u \in \mathbb{S}_{\varepsilon}^{+}$, we can see that

$$m_{\varepsilon}^{-1}(m_{\varepsilon}(u)) = m_{\varepsilon}^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{\varepsilon}} = \frac{u}{\|u\|_{\varepsilon}} = u$$

which implies that m_{ε} is a bijection. Next, we prove that $\hat{m}_{\varepsilon}$ is a continuous function. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\varepsilon}^{+}$ and $u \in \mathcal{H}_{\varepsilon}^{+}$ such that $u_n \to u$ in $\mathcal{H}_{\varepsilon}$. Since $\hat{m}(tu) = \hat{m}(u)$ for all $t > 0$, we may assume that $\|u_n\|_{\varepsilon} = \|u\|_{\varepsilon} = 1$ for all $n \in \mathbb{N}$. By (ii) there exists $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Since $t_n u_n \in \mathcal{N}_{\varepsilon}$, we have

$$t_n^{p}\|u_n\|_{\varepsilon}^{p} + bt_n^{2p}[u_n]_{s,p}^{2p} = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_n u_n)\,t_n u_n\,\mathrm{d}x.$$

Letting $n \to \infty$ we obtain

$$t_0^{p}\|u\|_{\varepsilon}^{p} + bt_0^{2p}[u]_{s,p}^{2p} = \int_{\mathbb{R}^3} g(\varepsilon\, x, t_0 u)\,t_0 u\,\mathrm{d}x,$$

which implies that $t_0 u \in \mathcal{N}_{\varepsilon}$. By (i), we deduce that $t_u = t_0$, and this shows that $\hat{m}_{\varepsilon}(u_n) \to \hat{m}_{\varepsilon}(u)$ in $\mathcal{H}_{\varepsilon}^{+}$. Therefore, $\hat{m}_{\varepsilon}$ and m_{ε} are continuous functions.

(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^+_\varepsilon$ be such that $\operatorname{dist}(u_n, \partial\mathbb{S}^+_\varepsilon) \to 0$. Observing that for each $r \in [p, p^*_s]$ and $n \in \mathbb{N}$ it holds

$$\begin{aligned}|u_n^+|_{L^r(\Lambda_\varepsilon)} &\leqslant \inf_{v\in\partial\mathbb{S}^+_\varepsilon} |u_n - v|_{L^r(\Lambda_\varepsilon)}\\ &\leqslant C_r \inf_{v\in\partial\mathbb{S}^+_\varepsilon} \|u_n - v\|_\varepsilon,\end{aligned}$$

by (g_1), (g_2), (g_3)-(ii), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} G(\varepsilon\, x, tu_n)\,\mathrm{d}x &= \int_{\Lambda^c_\varepsilon} G(\varepsilon\, x, tu_n)\,\mathrm{d}x + \int_{\Lambda_\varepsilon} G(\varepsilon\, x, tu_n)\,\mathrm{d}x\\ &\leqslant \frac{t^p}{K}\int_{\Lambda^c_\varepsilon} V(\varepsilon\, x)|u_n|^p\,\mathrm{d}x + \int_{\Lambda_\varepsilon} F(tu_n)\,\mathrm{d}x\\ &\leqslant \frac{t^p}{K}\|u_n\|^p_\varepsilon + C_1 t^{2p}\int_{\Lambda_\varepsilon}(u_n^+)^{2p}\,\mathrm{d}x + C_2 t^\nu \int_{\Lambda_\varepsilon}(u_n^+)^\nu\,\mathrm{d}x\\ &\leqslant \frac{t^p}{K} + C_1' t^{2p}\mathrm{dist}(\mathrm{u_n}, \partial\mathbb{S}^+_\varepsilon)^{2\mathrm{p}} + \mathrm{C}_2'\mathrm{t}^\nu \mathrm{dist}(\mathrm{u_n}, \partial\mathbb{S}^+_\varepsilon)^\nu\end{aligned}$$

from which

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} G(\varepsilon\, x, tu_n)\,\mathrm{d}x \leqslant \frac{t^p}{K} \quad \forall t > 0. \tag{3.5}$$

Taking in mind the definition of $m_\varepsilon(u_n)$ and using (3.5), we have

$$\begin{aligned}\liminf_{n\to\infty}\mathcal{I}_\varepsilon(m_\varepsilon(u_n)) &\geqslant \liminf_{n\to\infty}\mathcal{I}_\varepsilon(tu_n)\\ &\geqslant \liminf_{n\to\infty}\left[\frac{t^p}{p}\|u_n\|^p_\varepsilon + \frac{bt^{2p}}{2p}[u_n]^{2p}_{s,p}\right] - \frac{t^p}{K}\\ &\geqslant \left(\frac{K-p}{pK}\right)t^p.\end{aligned}$$

This implies that for all $t > 0$

$$\liminf_{n\to\infty}\left\{\frac{1}{p}\|m_\varepsilon(u_n)\|^p_\varepsilon + \frac{b}{2p}[m_\varepsilon(u_n)]^{2p}_{s,p}\right\} \geqslant \liminf_{n\to\infty}\mathcal{I}_\varepsilon(m_\varepsilon(u_n)) \geqslant \left(\frac{K-p}{pK}\right)t^p,$$

and recalling that $K > 2p > p$, from the arbitrariness of t, we get $\mathcal{I}_\varepsilon(m_\varepsilon(u_n)) = \infty$ and $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ as $n \to \infty$. □

Let us define the maps

$$\hat{\psi}_\varepsilon : \mathcal{H}^+_\varepsilon \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}^+_\varepsilon \to \mathbb{R},$$

by $\hat{\psi}_\varepsilon(u) := \mathcal{I}_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon := \hat{\psi}_\varepsilon|_{\mathbb{S}^+_\varepsilon}$. The next result is a consequence of lemma 3.1 and corollary 2.3 in [55].

Proposition 3.1. *Assume that* (V_1)–(V_2') *and* (f_1)–(f_4) *are satisfied. Then,*

(a) $\hat{\psi}_\varepsilon \in \mathcal{C}^1(\mathcal{H}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}_\varepsilon'(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon}\langle \mathcal{I}_\varepsilon'(\hat{m}_\varepsilon(u)), v\rangle \quad \forall u \in \mathcal{H}_\varepsilon^+,\ \forall v \in \mathcal{H}_\varepsilon.$$

(b) $\psi_\varepsilon \in \mathcal{C}^1(\mathbb{S}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \psi_\varepsilon'(u), v\rangle = \|m_\varepsilon(u)\|_\varepsilon \langle \mathcal{I}_\varepsilon'(m_\varepsilon(u)), v\rangle, \quad \forall v \in T_u\mathbb{S}_\varepsilon^+.$$

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_ε, *then* $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* $\mathcal{I}_\varepsilon$. *If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_d$ *sequence for* $\mathcal{I}_\varepsilon$, *then* $\{m_\varepsilon^{-1}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_ε.

(d) u *is a critical point of* ψ_ε *if and only if* $m_\varepsilon(u)$ *is a critical point for* $\mathcal{I}_\varepsilon$. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_\varepsilon^+} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u).$$

Remark 3.1. As in [**55**], we can see that the infimum of $\mathcal{I}_\varepsilon$ over $\mathcal{N}_\varepsilon$ has the following minimax characterization:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u) = \inf_{u\in\mathcal{H}_\varepsilon^+} \max_{t>0} \mathcal{I}_\varepsilon(tu) = \inf_{u\in\mathbb{S}_\varepsilon^+} \max_{t>0} \mathcal{I}_\varepsilon(tu).$$

Now, we prove the following result:

Corollary 3.1. *Let* $d \in \mathbb{R}$. *Then* ψ_ε *satisfies the* $(PS)_d$ *condition on* $\mathbb{S}_\varepsilon^+$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be a (PS) sequence for ψ_ε at the level d. Then

$$\psi_\varepsilon(u_n) \to d \quad \text{and} \quad \psi_\varepsilon'(u_n) \to 0 \text{ in } (T_{u_n}\mathbb{S}_\varepsilon^+)'.$$

By proposition 3.1(c) it follows that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{I}_\varepsilon$ in $\mathcal{H}_\varepsilon$. Then, by lemma 2.4, we can see that $\mathcal{I}_\varepsilon$ fulfils the $(PS)_d$ condition in $\mathcal{H}_\varepsilon$, so there exists $u \in \mathbb{S}_\varepsilon^+$ such that, up to a subsequence,

$$m_\varepsilon(u_n) \to m_\varepsilon(u) \text{ in } \mathcal{H}_\varepsilon.$$

Applying lemma 3.1(iii), we conclude that $u_n \to u$ in $\mathbb{S}_\varepsilon^+$. □

Now, we deal with the autonomous problem (2.15). We denote by $\mathcal{H}_\mu^+$ the open subset of $\mathcal{H}_\mu$ defined as

$$\mathcal{H}_\mu^+ := \{u \in \mathcal{H}_\mu : |\operatorname{supp}(u^+)| > 0\},$$

and $\mathbb{S}_\mu^+ := \mathbb{S}_\mu \cap \mathcal{H}_\mu^+$, where $\mathbb{S}_\mu$ is the unit sphere of $\mathcal{H}_\mu$. We note that $\mathbb{S}_\mu^+$ is a incomplete $\mathcal{C}^1$-manifold of codimension 1 modelled on $\mathcal{H}_\mu$ and contained in $\mathcal{H}_\mu^+$. Thus, $\mathcal{H}_\mu = T_u\mathbb{S}_\mu^+ \oplus \mathbb{R}u$ for each $u \in \mathbb{S}_\mu^+$, where $T_u\mathbb{S}_\mu^+ := \{u \in \mathcal{H}_\mu : B_u(v) + \mu\int_{\mathbb{R}^3} |u|^{p-2}uv\,dx = 0\}$.

Arguing as before, we can see that the following results hold.

LEMMA 3.2. *Assume that* (f_1)–(f_4) *hold. Then,*

(i) *For each* $u \in \mathcal{H}_\mu^+$, *let* $h : \mathbb{R}^+ \to \mathbb{R}$ *be defined by* $h_u(t) := \mathcal{E}_\mu(tu)$. *Then there exists a unique* $t_u > 0$ *such that* $h_u'(t) > 0$ *for all* $t \in (0, t_u)$ *and* $h_u'(t) < 0$ *for all* $t \in (t_u, \infty)$.

(ii) *There exists* $\tau > 0$ *independent of* u *such that* $t_u \geqslant \tau$ *for any* $u \in \mathbb{S}_\mu^+$. *Moreover, for each compact set* $\mathbb{K} \subset \mathbb{S}_\mu^+$ *there is a constant* $C_\mathbb{K} > 0$ *such that* $t_u \leqslant C_\mathbb{K}$ *for any* $u \in \mathbb{K}$.

(iii) *The map* $\hat{m}_\mu : \mathcal{H}_\mu^+ \to \mathcal{N}_\mu$ *given by* $\hat{m}_\mu(u) = t_u u$ *is continuous and* $m_\mu := \hat{m}_\mu|_{\mathbb{S}_\mu^+}$ *is a homeomorphism between* $\mathbb{S}_\mu^+$ *and* $\mathcal{N}_\mu$. *Moreover,* $m_\mu^{-1}(u) = u/\|u\|_\mu$.

(iv) *If there is a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\mu^+$ *such that* $\mathrm{dist}(u_n, \partial\mathbb{S}_\mu^+) \to 0$, *then* $\|m_\mu(u_n)\|_\mu \to \infty$ *and* $\mathcal{E}_\mu(m_\mu(u_n)) \to \infty$.

Let us define the maps

$$\hat{\psi}_\mu : \mathcal{H}_\mu^+ \to \mathbb{R} \quad \text{and} \quad \psi_\mu : \mathbb{S}_\mu^+ \to \mathbb{R},$$

by $\hat{\psi}_\mu(u) := \mathcal{E}_\mu(\hat{m}_0(u))$ and $\psi_\mu := \hat{\psi}_\mu|_{\mathbb{S}_\mu^+}$.

PROPOSITION 3.2. *Assume that* (f_1)–(f_4) *are satisfied. Then,*

(a) $\hat{\psi}_\mu \in \mathcal{C}^1(\mathcal{H}_\mu^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}_\mu'(u), v\rangle = \frac{\|\hat{m}_\mu(u)\|_\mu}{\|u\|_\mu}\langle \mathcal{E}_\mu'(\hat{m}_\mu(u)), v\rangle \quad \forall u \in \mathcal{H}_\mu^+, \forall v \in \mathcal{H}_\mu.$$

(b) $\psi_\mu \in \mathcal{C}^1(\mathbb{S}_\mu^+, \mathbb{R})$ *and*

$$\langle \psi_\mu'(u), v\rangle = \|m_\mu(u)\|_\mu \langle \mathcal{E}_\mu'(m_\mu(u)), v\rangle, \quad \forall v \in T_u\mathbb{S}_\mu^+.$$

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_μ, *then* $\{m_\mu(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* $\mathcal{E}_\mu$. *If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\mu$ *is a bounded* $(PS)_d$ *sequence for* $\mathcal{E}_\mu$, *then* $\{m_\mu^{-1}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_μ.

(d) u *is a critical point of* ψ_μ *if and only if* $m_\mu(u)$ *is a critical point for* $\mathcal{E}_\mu$. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_\mu^+} \psi_\mu(u) = \inf_{u\in\mathcal{N}_\mu} \mathcal{E}_\mu(u).$$

REMARK 3.2. As in [**55**], we can see that the infimum of $\mathcal{E}_\mu$ over $\mathcal{N}_\mu$ has the following minimax characterization:

$$d_\mu = \inf_{u\in\mathcal{N}_\mu} \mathcal{E}_\mu(u) = \inf_{u\in\mathcal{H}_\mu^+} \max_{t>0} \mathcal{E}_\mu(tu) = \inf_{u\in\mathbb{S}_\mu^+} \max_{t>0} \mathcal{E}_\mu(tu).$$

Next we give a compactness result for the autonomous problem which we will use later.

LEMMA 3.3. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\mu$ *be a sequence such that* $\mathcal{E}_\mu(u_n) \to d_\mu$. *Then,* $\{u_n\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* $\mathcal{H}_\mu$.

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\mu$ and $\mathcal{E}_\mu(u_n) \to d_\mu$, we can apply lemma 3.2(iii), proposition 3.2(d) and the definition of d_μ to infer that

$$v_n := m^{-1}(u_n) = \frac{u_n}{\|u_n\|_\mu} \in \mathbb{S}_\mu^+ \quad \forall n \in \mathbb{N}$$

and

$$\psi_\mu(v_n) = \mathcal{E}_\mu(u_n) \to d_\mu = \inf_{v\in\mathbb{S}_\mu^+} \psi_\mu(v).$$

Let us introduce the map $\mathcal{F} : \overline{\mathbb{S}}_\mu^+ \to \mathbb{R} \cup \{\infty\}$ defined by setting

$$\mathcal{F}(u) := \begin{cases} \psi_\mu(u) & \text{if } u \in \mathbb{S}_\mu^+, \\ \infty & \text{if } u \in \partial\mathbb{S}_\mu^+. \end{cases}$$

We note that

- $(\overline{\mathbb{S}}_\mu^+, \delta_\mu)$, where $\delta_\mu(u,v) := \|u - v\|_\mu$, is a complete metric space;
- $\mathcal{F} \in \mathcal{C}(\overline{\mathbb{S}}_\mu^+, \mathbb{R} \cup \{\infty\})$, by lemma 3.2-$(iv)$;
- $\mathcal{F}$ is bounded below, by proposition 3.2(d).

Hence, invoking the Ekeland variational principle [**26**] to $\mathcal{F}$, we can find $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\mu^+$ such that $\{\hat{v}_n\}_{n\in\mathbb{N}}$ is a $(PS)_{d_\mu}$ sequence for ψ_μ on $\mathbb{S}_\mu^+$ and $\|\hat{v}_n - v_n\|_\mu = o_n(1)$. Then, using proposition 3.2, theorem 2.2 and arguing as in the proof of corollary 3.1, we obtain the thesis. □

REMARK 3.3. By lemma 2.6, (V_1) and (V_2'), we obtain that $\lim_{\varepsilon\to 0} c_\varepsilon = d_{V_0}$.

3.2. The barycenter map

In this subsection, our main purpose is to apply the Ljusternik–Schnirelmann category theory to prove a multiplicity result for (2.2). We begin by proving some technical results.

LEMMA 3.4. *Let* $\varepsilon_n \to 0$ *and* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$ *be such that* $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. *Then there exists* $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ *such that the translated sequence*

$$\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$$

has a subsequence which converges in $\mathcal{H}_{V_0}$. *Moreover, up to a subsequence,* $\{y_n\}_{n\in\mathbb{N}} := \{\varepsilon_n \tilde{y}_n\}_{n\in\mathbb{N}}$ *is such that* $y_n \to y_0 \in M$.

Proof. Since $\langle \mathcal{I}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$, it is easy to see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Let us observe that $\|u_n\|_{\varepsilon_n} \nrightarrow 0$ since $d_{V_0} > 0$. Therefore, arguing as in lemma 2.5, we can find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and constants $R, \alpha > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^p \, \mathrm{d}x \geqslant \alpha.$$

Set $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$. Then, it is clear that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{V_0}$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } \mathcal{H}_{V_0},$$

for some $\tilde{u} \neq 0$. Let $\{t_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ be such that $\tilde{v}_n := t_n \tilde{u}_n \in \mathcal{N}_{V_0}$ (see lemma 3.2(i)), and set $y_n := \varepsilon_n \tilde{y}_n$. Then, from $u_n \in \mathcal{N}_{\varepsilon_n}$ and (g_2), we can see that

$$\begin{aligned} d_{V_0} \leqslant \mathcal{E}_{V_0}(\tilde{v}_n) &\leqslant \frac{a}{p}[\tilde{v}_n]^p_{s,p} + \frac{1}{p}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)|\tilde{v}_n|^p \, \mathrm{d}x + \frac{b}{2p}[\tilde{v}_n]^{2p}_{s,p} - \int_{\mathbb{R}^3} F(\tilde{v}_n) \, \mathrm{d}x \\ &\leqslant \frac{a t_n^p}{p}[u_n]^p_{s,p} + \frac{t_n^p}{p}\int_{\mathbb{R}^3} V(\varepsilon_n x)|u_n|^p \, \mathrm{d}x + \frac{b}{2p}[u_n]^{2p}_{s,p} \\ &\quad - \int_{\mathbb{R}^3} G(\varepsilon_n x, t_n u_n) \, \mathrm{d}x \\ &= \mathcal{I}_{\varepsilon_n}(t_n u_n) \leqslant \mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1), \end{aligned}$$

which gives

$$\mathcal{E}_{V_0}(\tilde{v}_n) \to d_{V_0} \quad \text{and} \quad \{\tilde{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{V_0}. \tag{3.6}$$

In particular, (3.6) implies that $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{V_0}$, so we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}$. Obviously, $\{t_n\}_{n\in\mathbb{N}}$ is bounded and we have $t_n \to t_0 \geqslant 0$. If $t_0 = 0$, from the boundedness of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, we get $\|\tilde{v}_n\|_{V_0} = t_n\|\tilde{u}_n\|_{V_0} \to 0$, that is $\mathcal{E}_{V_0}(\tilde{v}_n) \to 0$ in contrast with the fact $d_{V_0} > 0$. Then, $t_0 > 0$. From the uniqueness of the weak limit we have $\tilde{v} = t_0\tilde{u}$ and $\tilde{u} \neq 0$. By lemma 3.3, we deduce that

$$\tilde{v}_n \to \tilde{v} \quad \text{in } \mathcal{H}_{V_0}, \tag{3.7}$$

which implies that $\tilde{u}_n = \tilde{v}_n/t_n \to \tilde{v}/t_0 = \tilde{u}$ in $\mathcal{H}_{V_0}$, and

$$\mathcal{E}_{V_0}(\tilde{v}) = d_{V_0} \quad \text{and} \quad \langle \mathcal{E}'_{V_0}(\tilde{v}), \tilde{v}\rangle = 0.$$

Next, we show that $\{y_n\}_{n\in\mathbb{N}}$ has a subsequence such that $y_n \to y_0 \in M$. Assume by contradiction that $\{y_n\}_{n\in\mathbb{N}}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}_{n\in\mathbb{N}}$, such that $|y_n| \to +\infty$. Take $R > 0$ such that $\Lambda \subset \mathcal{B}_R$. We may suppose that $|y_n| > 2R$ for n large enough, so, for any $x \in \mathcal{B}_{R/\varepsilon_n}$, we get $|\varepsilon_n x + y_n| \geqslant |y_n| - |\varepsilon_n x| > R$. Then, we deduce that

$$\begin{aligned} \|\tilde{u}_n\|^p_{V_0} \leqslant \|\tilde{u}_n\|^p_{V_0} + b[\tilde{u}_n]^{2p}_{s,p} &= \int_{\mathbb{R}^3} g(\varepsilon_n x + y_n, \tilde{u}_n)\tilde{u}_n \, \mathrm{d}x \\ &\leqslant \int_{\mathcal{B}_{R/\varepsilon_n}} \tilde{f}(\tilde{u}_n)\tilde{u}_n \, \mathrm{d}x + \int_{\mathcal{B}^c_{R/\varepsilon_n}} f(\tilde{u}_n)\tilde{u}_n \, \mathrm{d}x. \end{aligned}$$

Since $\tilde{u}_n \to \tilde{u}$ in $\mathcal{H}_{V_0}$, from the Dominated Convergence Theorem we can see that

$$\int_{\mathcal{B}^c_{R/\varepsilon_n}} f(\tilde{u}_n)\tilde{u}_n \,\mathrm{d}x = o_n(1).$$

Recalling that $\tilde{f}(\tilde{u}_n)\tilde{u}_n \leqslant V_0/K|\tilde{u}_n|^p$, we get

$$\|\tilde{u}_n\|^p_{V_0} \leqslant \frac{1}{K}\int_{\mathcal{B}_{R/\varepsilon_n}} V_0|\tilde{u}_n|^p \,\mathrm{d}x + o_n(1),$$

which yields

$$\left(1-\frac{1}{K}\right)\|\tilde{u}_n\|^p_{V_0} \leqslant o_n(1).$$

Then we obtain a contradiction thanks to $\tilde{u}_n \to \tilde{u} \neq 0$. Thus, $\{y_n\}_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, we may assume that $y_n \to y_0$. If $y_0 \notin \overline{\Lambda}$, then there exists $r > 0$ such that $y_n \in \mathcal{B}_{r/2}(y_0) \subset \overline{\Lambda}^c$ for any n large enough. Reasoning as before, we get a contradiction. Hence, $y \in \overline{\Lambda}$. Next, we prove that $V(y_0) = V_0$. Assume by contradiction that $V(y_0) > V_0$. Taking into account (3.7), Fatou's Lemma and the invariance of $\mathbb{R}^3$ by translations, we have

$$\begin{aligned}
d_{V_0} &= \mathcal{E}_{V_0}(\tilde{v}) \\
&< \liminf_{n\to\infty}\left[\frac{1}{p}[\tilde{v}_n]^p_{s,p} + \frac{1}{p}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)|\tilde{v}_n|^p \,\mathrm{d}x + \frac{b}{2p}[\tilde{v}_n]^{2p}_{s,p} - \int_{\mathbb{R}^3} F(\tilde{v}_n)\,\mathrm{d}x\right] \\
&\leqslant \liminf_{n\to\infty}\mathcal{I}_{\varepsilon_n}(t_n u_n) \leqslant \liminf_{n\to\infty}\mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0}
\end{aligned}$$

which gives a contradiction. □

Now, we aim to relate the number of positive solutions of (2.2) to the topology of the set M. For this reason, we take $\delta > 0$ such that

$$M_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \leqslant \delta\} \subset \Lambda,$$

and we consider $\eta \in \mathcal{C}^\infty_c(\mathbb{R}_+, [0,1])$ such that $\eta(t) = 1$ if $0 \leqslant t \leqslant \delta/2$ and $\eta(t) = 0$ if $t \geqslant \delta$.

For any $y \in M$, we define

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

where $w \in \mathcal{H}_{V_0}$ is a positive ground state solution to the autonomous problem (2.15) (whose existence is guaranteed by theorem 2.2). Let $t_\varepsilon > 0$ be the unique number such that

$$\max_{t\geqslant 0}\mathcal{I}_\varepsilon(t\Psi_{\varepsilon,y}) = \mathcal{I}_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}).$$

Finally, we consider $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ defined by setting

$$\Phi_\varepsilon(y) := t_\varepsilon\Psi_{\varepsilon,y}.$$

Lemma 3.5. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y\in M.$$

Proof. Assume by contradiction that there exist $\delta_0>0$, $\{y_n\}_{n\in\mathbb{N}}\subset M$ and $\varepsilon_n\to 0$ such that

$$|\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - d_{V_0}| \geqslant \delta_0. \tag{3.8}$$

Let us observe that, by using the change of variable $z=(\varepsilon_n x - y_n)/\varepsilon_n$, if $z\in\mathcal{B}_{\delta/\varepsilon_n}$, it follows that $\varepsilon_n z\in\mathcal{B}_\delta$ and then $\varepsilon_n z + y_n\in\mathcal{B}_\delta(y_n)\subset M_\delta\subset\Lambda$.

Then, recalling that $G=F$ in Λ and $\eta(t)=0$ for $t\geqslant\delta$, we have

$$\begin{aligned}\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p}\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^p + b\frac{t_{\varepsilon_n}^{2p}}{2p}[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} - \int_{\mathbb{R}^3} G(\varepsilon_n x, t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\,\mathrm{d}x\\ &= \frac{t_{\varepsilon_n}^p}{p}\left([\eta(|\varepsilon_n z|)w]_{s,p}^p + \int_{\mathbb{R}^3} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^p\,\mathrm{d}z\right)\\ &\quad + b\frac{t_{\varepsilon_n}^{2p}}{2p}[\eta(|\varepsilon_n z|)w]_{s,p}^{2p} - \int_{\mathbb{R}^3} F(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))\,\mathrm{d}z. \end{aligned}\tag{3.9}$$

Now, we verify that the sequence $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ satisfies $t_{\varepsilon_n}\to 1$ as $\varepsilon_n\to 0$. By the definition of t_{ε_n}, it follows that $\langle\mathcal{I}'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)),\Phi_{\varepsilon_n}(y_n)\rangle = 0$, which gives

$$\frac{1}{t_{\varepsilon_n}^p}\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^p + b[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} = \int_{\mathbb{R}^3}\left[\frac{f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))}{(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^{2p-1}}\right](\eta(|\varepsilon_n z|)w(z))^{2p}\,\mathrm{d}z, \tag{3.10}$$

where we used the fact that $g=f$ on Λ. Since $\eta=1$ in $\mathcal{B}_{\delta/2}\subset\mathcal{B}_{\delta/\varepsilon_n}$ for all n large enough, from (3.10) it follows that

$$\frac{1}{t_{\varepsilon_n}^p}\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^p + b[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} \geqslant \int_{\mathcal{B}_{\delta/2}}\left[\frac{f(t_{\varepsilon_n}w(z))}{(t_{\varepsilon_n}w(z))^{2p-1}}\right]|w(z)|^{2p}\,\mathrm{d}z.$$

Since w is continuous, we can find a vector $\hat{z}\in\mathbb{R}^3$ such that

$$w(\hat{z}) = \min_{z\in\bar{\mathcal{B}}_{\delta/2}} w(z) > 0.$$

Then, by (f_4), we deduce that

$$\frac{1}{t_{\varepsilon_n}^p}\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^p + b[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} \geqslant \left[\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{2p-1}}\right]|w(\hat{z})|^{2p}|\mathcal{B}_{\delta/2}|. \tag{3.11}$$

Now, assume by contradiction that $t_{\varepsilon_n} \to \infty$. Let us observe that lemma 2.2 yields

$$\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n} \to \|w\|_{V_0} \in (0,\infty). \tag{3.12}$$

From $t_{\varepsilon_n} \to \infty$ and (3.12), it follows that

$$\frac{1}{t_{\varepsilon_n}^p}\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^p + b[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} \to b[w]_{s,p}^{2p}. \tag{3.13}$$

On the other hand, by (f_3), we have

$$\lim_{n\to\infty} \frac{f(t_{\varepsilon_n} w(\hat{z}))}{(t_{\varepsilon_n} w(\hat{z}))^{2p-1}} = \infty. \tag{3.14}$$

Putting together (3.11), (3.13) and (3.14) we get a contradiction. Therefore, $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, we may assume that $t_{\varepsilon_n} \to t_0$ for some $t_0 \geqslant 0$. Indeed, from (3.10), (3.12), (f_1)–(f_2) we can see that $t_0 > 0$. Hence, letting $n \to \infty$ in (3.10), we deduce from (3.12) and the Dominated Convergence Theorem that

$$\frac{1}{t_0^p}\|w\|_{V_0}^p + b[w]_{s,p}^{2p} = \int_{\mathbb{R}^3} \frac{f(t_0 w)}{(t_0 w)^{2p-1}}\, w^{2p}\, \mathrm{d}x. \tag{3.15}$$

Since $w \in \mathcal{N}_{V_0}$, we can see that

$$\|w\|_{V_0}^p + b[w]_{s,p}^{2p} = \int_{\mathbb{R}^3} f(w)w\, \mathrm{d}x. \tag{3.16}$$

In the light of (3.15), (3.16) and (f_4), we deduce that $t_0 = 1$. Accordingly, taking the limit as $n \to \infty$ in (3.9), we obtain

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \mathcal{E}_{V_0}(w) = d_{V_0},$$

which contradicts (3.8). □

At this point, we are in the position to define the barycenter map. For any $\delta > 0$ given by lemma 3.5, we take $\rho = \rho(\delta) > 0$ such that $M_\delta \subset \mathcal{B}_\rho$, and we consider $\Upsilon : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\Upsilon(x) := \begin{cases} x & \text{if } |x| < \rho \\ \frac{\rho x}{|x|} & \text{if } |x| \geqslant \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ as follows

$$\beta_\varepsilon(u) := \frac{\displaystyle\int_{\mathbb{R}^3} \Upsilon(\varepsilon\, x)|u(x)|^p\, \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^3} |u(x)|^p\, \mathrm{d}x}.$$

Arguing as in lemma 3.14 in [**10**] we can prove the following result:

Lemma 3.6. *The function β_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y\in M.$$

Now, we introduce the following subset of $\mathcal{N}_\varepsilon$:

$$\widetilde{\mathcal{N}}_\varepsilon := \{u\in\mathcal{N}_\varepsilon : \mathcal{I}_\varepsilon(u)\leqslant d_{V_0}+h_1(\varepsilon)\},$$

where $h_1(\varepsilon) := \sup_{y\in M}|\mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) - d_{V_0}|\to 0$ as $\varepsilon\to 0$ by lemma 3.5. By the definition of $h_1(\varepsilon)$, it follows that, for all $y\in M$ and $\varepsilon>0$, $\Phi_\varepsilon(y)\in\widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon\neq\emptyset$. Moreover, as in lemma 3.15 in [**10**], we can see that the following lemma holds true.

Lemma 3.7. *For any $\delta>0$ there holds*

$$\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}}_\varepsilon}\operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Before proving our multiplicity result for the modified problem (2.2), we recall the following useful abstract result whose proof can be found in [**12**].

Lemma 3.8. *Let I, I_1 and I_2 be closed sets with $I_1\subset I_2$, and let $\pi : I\to I_2$ and $\psi : I_1\to I$ be two continuous maps such that $\pi\circ\psi$ is homotopically equivalent to the embedding $j : I_1\to I_2$. Then $cat_I(I)\geqslant cat_{I_2}(I_1)$.*

Theorem 3.1. *Assume that (V_1)–(V_2') and (f_1)–(f_4) hold. Then, given $\delta>0$ there exists $\bar{\varepsilon}_\delta>0$ such that, for any $\varepsilon\in(0,\bar{\varepsilon}_\delta)$, problem (2.2) has at least $cat_{M_\delta}(M)$ positive solutions.*

Proof. For any $\varepsilon>0$, we consider the map $\alpha_\varepsilon : M\to\mathbb{S}_\varepsilon^+$ defined as $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$.

By lemma 3.5, we can see that

$$\lim_{\varepsilon\to 0}\psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y\in M. \tag{3.17}$$

Set

$$\widetilde{\mathcal{S}}_\varepsilon^+ := \{w\in\mathbb{S}_\varepsilon^+ : \psi_\varepsilon(w)\leqslant d_{V_0}+h_1(\varepsilon)\},$$

where $h_1(\varepsilon) := \sup_{y\in M}|\psi_\varepsilon(\alpha_\varepsilon(y)) - d_{V_0}|\to 0$ as $\varepsilon\to 0$ by (3.17). Since $\psi_\varepsilon(\alpha_\varepsilon(y))\in\widetilde{\mathcal{S}}_\varepsilon^+$ we deduce that $\widetilde{\mathcal{S}}_\varepsilon^+\neq\emptyset$ for all $\varepsilon>0$. From lemma 3.5, lemma 3.1(iii), lemma 3.7 and lemma 3.6, we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta>0$ such that the following diagram is well defined for any $\varepsilon\in(0,\bar{\varepsilon})$.

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \alpha_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta.$$

In view of lemma 3.6, and decreasing $\bar{\varepsilon}$ if necessary, we can see that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y+\theta(\varepsilon,y)$ for all $y\in M$, for some function $\theta(\varepsilon,y)$ such that $|\theta(\varepsilon,y)|<\delta/2$ uniformly in $y\in M$ and for all $\varepsilon\in(0,\bar{\varepsilon})$. Then, we can see that $H(t,y) := y+(1-t)\theta(\varepsilon,y)$

with $(t, y) \in [0,1] \times M$, is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $id : M \to M_\delta$. This fact together with lemma 3.8 implies that

$$cat_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M) \geqslant cat_{M_\delta}(M). \tag{3.18}$$

Therefore, by corollary 3.1 and corollary 28 in [**55**], with $c = c_\varepsilon \leqslant d_{V_0} + h_1(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, we can see that Ψ_ε has at least $cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Taking into account proposition 3.1-(d) and (3.18), we can infer that $\mathcal{I}_\varepsilon$ admits at least $cat_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

Now, we are able to give the proof of our second main result of this work.

Proof of theorem 1.2. Take $\delta > 0$ such that $M_\delta \subset \Lambda$. We begin by proving that there exists $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (2.2), it holds

$$|u_\varepsilon|_{L^\infty(\Lambda_\varepsilon^c)} < a_0. \tag{3.19}$$

Suppose by contradiction that for some subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, we can find $u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $\mathcal{I}'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_{\varepsilon_n}|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geqslant a_0. \tag{3.20}$$

Since $\mathcal{I}_{\varepsilon_n}(u_{\varepsilon_n}) \leqslant d_{V_0} + h_1(\varepsilon_n)$ and $h_1(\varepsilon_n) \to 0$, we can proceed as in the first part of the proof of lemma 3.4, to deduce that $\mathcal{I}_{\varepsilon_n}(u_{\varepsilon_n}) \to d_{V_0}$. Then, by lemma 3.4, we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $\tilde{u}_n = u_{\varepsilon_n}(\cdot + \tilde{y}_n) \to \tilde{u}$ in $W^{s,p}(\mathbb{R}^3)$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

Now, if we choose $r > 0$ such that $\mathcal{B}_r(y_0) \subset \mathcal{B}_{2r}(y_0) \subset \Lambda$, we can see that $\mathcal{B}_{r/\varepsilon_n}(y_0/\varepsilon_n) \subset \Lambda_{\varepsilon_n}$. In particular, for any $y \in \mathcal{B}_{r/\varepsilon_n}(\tilde{y}_n)$ it holds

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leqslant |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \quad \text{for } n \text{ sufficiently large.}$$

Therefore, for any n big enough we have $\Lambda_{\varepsilon_n}^c \subset \mathcal{B}_{r/\varepsilon_n}^c(\tilde{y}_n)$. On the other hand, using (2.29), we know that

$$\tilde{u}_n(x) \to 0 \quad \text{as } |x| \to \infty$$

uniformly in $n \in \mathbb{N}$. Hence, there exists $R > 0$ such that

$$\tilde{u}_n(x) < a_0 \quad \text{for all } |x| \geqslant R, n \in \mathbb{N}.$$

Consequently, $u_{\varepsilon_n}(x) < a_0$ for any $x \in \mathcal{B}_R^c(\tilde{y}_n)$ and $n \in \mathbb{N}$.

Since there exists $\nu \in \mathbb{N}$ such that for any $n \geqslant \nu$ it holds

$$\Lambda_{\varepsilon_n}^c \subset \mathcal{B}_{r/\varepsilon_n}^c(\tilde{y}_n) \subset \mathcal{B}_R^c(\tilde{y}_n),$$

we deduce that $u_{\varepsilon_n}(x) < a_0$ for any $x \in \Lambda_{\varepsilon_n}^c$ and $n \geqslant \nu$, which is in contrast with (3.20).

Let $\bar{\varepsilon}_\delta > 0$ given by theorem 3.1 and we fix $\varepsilon \in (0, \varepsilon_\delta)$, where $\varepsilon_\delta := \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. In view of theorem 3.1, we know that the problem (2.2) admits at least $cat_{M_\delta}(M)$ nontrivial solutions. Let us denote by u_ε one of these solutions. Since $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ satisfies (3.19), by the definition of g it follows that u_ε is a solution of $(\hat{P}_\varepsilon)$. Then $\hat{u}(x) := u(x/\varepsilon)$ is a solution to (P_ε), and we can conclude that (P_ε) has at least $cat_{M_\delta}(M)$ solutions.

Finally, we study the behaviour of the maximum points of solutions to the problem $(\hat{P}_\varepsilon)$. Take $\varepsilon_n \to 0^+$ and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\varepsilon_n}$ of solutions to $(\hat{P}_\varepsilon)$. Let us observe that (g_1) implies that we can find $\gamma > 0$ such that

$$g(\varepsilon\, x, t)t \leqslant \frac{V_0}{K} t^p \quad \text{for any } x \in \mathbb{R}^3, t \leqslant \gamma. \tag{3.21}$$

Arguing as before, we can find $R > 0$ such that

$$|u_n|_{L^\infty(B^c_R(\tilde{y}_n))} < \gamma. \tag{3.22}$$

Moreover, up to extract a subsequence, we may assume that

$$|u_n|_{L^\infty(B_R(\tilde{y}_n))} \geqslant \gamma. \tag{3.23}$$

Indeed, if (3.23) does not hold, in view of (3.22) we can see that $|u_n|_\infty < \gamma$. Then, thanks to $\langle \mathcal{I}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and (3.21), we get

$$\|u_n\|^p_{\varepsilon_n} \leqslant \int_{\mathbb{R}^3} g(\varepsilon_n\, x, u_n)u_n \,\mathrm{d}x \leqslant \frac{V_0}{K} \int_{\mathbb{R}^3} |u_n|^p \,\mathrm{d}x$$

which yields $\|u_n\|_{\varepsilon_n} = 0$, and this is an absurd. Accordingly, (3.23) holds true.

Taking into account (3.22) and (3.23) we can deduce that the maximum points $p_n \in \mathbb{R}^3$ of u_n belong to $\mathcal{B}_R(\tilde{y}_n)$. Therefore, $p_n = \tilde{y}_n + q_n$ for some $q_n \in \mathcal{B}_R$. Hence, $\eta_n := \varepsilon_n\, \tilde{y}_n + \varepsilon_n\, q_n$ is the maximum point of $\hat{u}_n(x) = u_n(x/\varepsilon_n)$. Since $|q_n| < R$ for any $n \in \mathbb{N}$ and $\varepsilon_n\, \tilde{y}_n \to y_0 \in M$ (in view of lemma 3.4), by the continuity of V we can infer that

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0,$$

which ends the proof of theorem. □

4. Critical and supercritical fractional Kirchhoff problems

This section is devoted to the existence of positive solutions to

$$\begin{cases} \left(\varepsilon^{sp}\, a + \varepsilon^{2sp-3}\, b\, [u]^p_{s,p}\right)(-\Delta)^s_p u + V(x)u^{p-1} = u^{q-1} + \gamma u^{r-1} & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3. \end{cases}$$

After rescaling, we study the following Kirchhoff problem

$$\begin{cases} \left(a + b\, [u]^p_{s,p}\right)(-\Delta)^s_p u + V(\varepsilon\, x)u^{p-1} = u^{q-1} + \gamma u^{r-1} & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.1}$$

where $\gamma > 0$ and the powers q and r are such that $2p < q < p^*_s \leqslant r$. In what follows, we truncate the nonlinearity $\phi(u) := u^{q-1} + \gamma u^{r-1}$ in a suitable way.

Let $K > 0$ be a real number, whose value will be fixed later, and we set

$$\phi_\gamma(t) := \begin{cases} 0 & \text{if } t < 0, \\ t^{q-1} + \gamma t^{r-1} & \text{if } 0 \leqslant t < K, \\ (1 + \gamma K^{r-q}) t^{q-1} & \text{if } t \geqslant K. \end{cases}$$

Let us note that ϕ_γ satisfies the following properties:

(ϕ_1) $\lim_{t \to 0} \phi_\gamma(t)/t^{2p-1} = 0$;

(ϕ_2) $\lim_{t \to \infty} \phi_\gamma(t)/t^{\nu-1} = 0$ for some $\nu \in (q, p_s^*)$;

(ϕ_3) $0 < q\Phi_\gamma(t) \leqslant t\phi_\gamma(t)$ for all $t > 0$, where $\Phi_\gamma(t) = \int_0^t \phi_\gamma(\tau)\, \mathrm{d}\tau$;

(ϕ_4) $t \mapsto \phi_\gamma(t)/t^{2p-1}$ is increasing in $(0, \infty)$.

Moreover

$$\phi_\gamma(t) \leqslant (1 + \gamma K^{r-q}) t^{q-1} \text{ for all } t \geqslant 0. \tag{4.2}$$

Therefore, we consider the following truncated problem

$$\begin{cases} \left(a + b\,[u]_{s,p}^p\right)(-\Delta)_p^s u + V(\varepsilon\, x) u^{p-1} = \phi_\gamma(u) & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.3}$$

It is easy to see that weak solutions of (4.3) are critical points of the energy functional $\mathcal{I}_{\varepsilon,\gamma} : \mathcal{H}_\varepsilon \to \mathbb{R}$ defined by

$$\mathcal{I}_{\varepsilon,\gamma}(u) = \frac{1}{p}\|u\|_\varepsilon^p + \frac{b}{2p}[u]_{s,p}^{2p} - \int_{\mathbb{R}^3} \Phi_\gamma(u)\, \mathrm{d}x.$$

We also consider the autonomous functional

$$\mathcal{E}_{V_0,\gamma}(u) = \frac{1}{p}\|u\|_{V_0}^p + \frac{b}{2p}[u]_{s,p}^{2p} - \int_{\mathbb{R}^3} \Phi_\gamma(u)\, \mathrm{d}x.$$

Using theorem 1.1, we know that for any $\gamma \geqslant 0$ there exists $\bar{\varepsilon}(\gamma) > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}(\gamma))$, problem (4.3) admits a positive solution $u_{\varepsilon,\gamma}$. Now, we prove that it is possible to estimate the $\mathcal{H}_\varepsilon$-norm of these solutions uniformly with respect to γ. More precisely:

LEMMA 4.1. *There exists $\bar{C} > 0$ such that $\|u_{\varepsilon,\gamma}\|_\varepsilon \leqslant \bar{C}$ for any $\varepsilon > 0$ sufficiently small and uniformly in γ.*

Proof. A simple inspection of the proof of theorem 1.1 shows that any solution $u_{\varepsilon,\gamma}$ of (4.3) satisfies the following inequality

$$\mathcal{I}_{\varepsilon,\gamma}(u_{\varepsilon,\gamma}) \leqslant d_{V_0,\gamma} + h_\gamma(\varepsilon),$$

where $d_{V_0,\gamma}$ is the mountain pass level related to the functional $\mathcal{E}_{V_0,\gamma}$, and $h_\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, decreasing $\bar{\varepsilon}(\gamma)$ if necessary, we may suppose that

$$\mathcal{I}_{\varepsilon,\gamma}(u_{\varepsilon,\gamma}) \leqslant d_{V_0,\gamma} + 1 \quad \text{for any } \varepsilon \in (0, \bar{\varepsilon}(\gamma)). \tag{4.4}$$

Using the fact that $d_{V_0,\gamma} \leqslant d_{V_0,0}$ for any $\gamma \geqslant 0$, we deduce that

$$\mathcal{I}_{\varepsilon,\gamma}(u_{\varepsilon,\gamma}) \leqslant d_{V_0,0} + 1 \quad \text{for any } \varepsilon \in (0, \bar{\varepsilon}(\gamma)). \tag{4.5}$$

From (ϕ_3) and $q > 2p$ we infer that

$$\begin{aligned}
\mathcal{I}_{\varepsilon,\gamma}(u_{\varepsilon,\gamma}) &= \mathcal{I}_{\varepsilon,\gamma}(u_{\varepsilon,\gamma}) - \frac{1}{q}\langle \mathcal{I}'_{\varepsilon,\gamma}(u_{\varepsilon,\gamma}), u_{\varepsilon,\gamma}\rangle \\
&= \left(\frac{1}{p} - \frac{1}{q}\right) \|u_{\varepsilon,\gamma}\|_\varepsilon^p \\
&\quad + \left(\frac{1}{2p} - \frac{1}{q}\right) [u_{\varepsilon,\gamma}]_{s,p}^{2p} + \int_{\mathbb{R}^3} \frac{1}{q}\phi_\gamma(u_{\varepsilon,\gamma})u_{\varepsilon,\gamma} - \Phi_\gamma(u_{\varepsilon,\gamma})\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{p} - \frac{1}{q}\right) \|u_{\varepsilon,\gamma}\|_\varepsilon^p.
\end{aligned} \tag{4.6}$$

Putting together (4.5) and (4.6), we have

$$\|u_{\varepsilon,\gamma}\|_\varepsilon \leqslant \left[\left(\frac{p\,q}{q-p}\right)(d_{V_0,0}+1)\right]^{1/p} \qquad \text{for any } \varepsilon \in (0, \bar{\varepsilon}(\gamma)).$$

□

Now, our claim is to prove that $u_{\varepsilon,\gamma}$ is a solution of the original problem (4.1). To do this, we will show that we can find $K_0 > 0$ such that for any $K \geqslant K_0$, there exists $\gamma_0 = \gamma_0(K) > 0$ such that

$$|u_{\varepsilon,\gamma}|_\infty \leqslant K \quad \text{for all } \gamma \in [0, \gamma_0]. \tag{4.7}$$

In order to achieve our purpose, we make use of a variant of the Moser iteration technique [**45**]. For simplicity, we set $u := u_{\varepsilon,\gamma}$. For any $L > 0$, we define $u_L := \min\{u, L\} \geqslant 0$, and $w_L := u u_L^{\sigma-1}$, where $\sigma > 1$ will be chosen later. Taking $u_L^{p(\sigma-1)}u$ in (4.3), we can see that

$$\begin{aligned}
&a \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u(x)u_L^{p(\sigma-1)}(x) - u(y)u_L^{p(\sigma-1)}(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + b[u]_{s,p}^p \iint_{\mathbb{R}^6}
\end{aligned}$$

$$\times \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u(x)u_L^{p(\sigma-1)}(x)-u(y)u_L^{p(\sigma-1)}(y))}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^3}\phi_\gamma(u)u_L^{p(\sigma-1)}u\,\mathrm{d}x - \int_{\mathbb{R}^3}V(\varepsilon x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \tag{4.8}$$

Using (2.30) and (2.31) with $\tilde{u}_n$ and $\tilde{u}_{L,n}$ replaced by u and u_L respectively, we can note that

$$\begin{aligned} a\frac{C_*^{-1}}{\sigma^p}|w_L|_{p_s^*}^p &\leqslant a[\mathcal{L}(u)]_{s,p}^p \\ &\leqslant a\iint_{\mathbb{R}^6}\frac{\begin{array}{c}|u(x)-u(y)|^{p-2}(u(x)-u(y))(u(x)u_L^{p(\sigma-1)}(x)\\ -u(y)u_L^{p(\sigma-1)}(y))\end{array}}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + b[u]_{s,p}^p\iint_{\mathbb{R}^6}\frac{\begin{array}{c}|u(x)-u(y)|^{p-2}(u(x)-u(y))(u(x)u_L^{p(\sigma-1)}(x)\\ -u(y)u_L^{p(\sigma-1)}(y))\end{array}}{|x-y|^{3+sp}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.9}$$

On the other hand, by (4.2) and (V_1) we can see that

$$\begin{aligned} &\int_{\mathbb{R}^3}\phi_\gamma(u)u_L^{p(\sigma-1)}u\,\mathrm{d}x - \int_{\mathbb{R}^3}V(\varepsilon x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \\ &\qquad \leqslant (1+\gamma K^{r-q})\int_{\mathbb{R}^3}u^q u_L^{p(\sigma-1)}\,\mathrm{d}x. \end{aligned} \tag{4.10}$$

Putting together (4.8), (4.9) and (4.10) we get

$$|w_L|_{p_s^*}^p \leqslant \frac{\sigma^p}{a}C_*(1+\gamma K^{r-q})\int_{\mathbb{R}^3}u^q u_L^{p(\sigma-1)}\,\mathrm{d}x. \tag{4.11}$$

Now, by Hölder's inequality, we have

$$\int_{\mathbb{R}^3}u^q u_L^{p(\sigma-1)}\,\mathrm{d}x \leqslant |u|_{p_s^*}^{q-p}\,|w_L|_{(pp_s^*/p_s^*-(q-p))}^p$$

which together with (4.11) yields

$$|w_L|_{p_s^*}^p \leqslant \frac{\sigma^p}{a}C_*(1+\gamma K^{r-q})|u|_{p_s^*}^{q-p}\,|w_L|_{(pp_s^*/p_s^*-(q-p))}^p. \tag{4.12}$$

Set

$$C_{\gamma,K} := \frac{C_*}{a}(1+\gamma K^{r-q}) \quad \text{and} \quad \alpha^* := \alpha^*(s,p,q) = \frac{pp_s^*}{p_s^*-(q-p)}$$

so that (4.12) becomes

$$|w_L|_{p_s^*}^p \leqslant \sigma^p C_{\gamma,K}|u|_{p_s^*}^{q-p}\,|w_L|_{\alpha^*}^p. \tag{4.13}$$

On the other hand, by theorem 2.1 and lemma 4.1 we know that

$$|u|_{p_s^*}^p \leqslant C_* \|u\|_\varepsilon^p \leqslant C_* \bar{C}^p$$

which together with (4.13) gives

$$|w_L|_{p_s^*}^p \leqslant \sigma^p C_{\gamma,K} M_1 |w_L|_{\alpha^*}^p \tag{4.14}$$

where $M_1 := (C_* \bar{C}^p)^{(q-p)/p}$. Now, we observe that if $u^\sigma \in L^{\alpha^*}(\mathbb{R}^3)$, by the definition of w_L, $u_L \leqslant u$, and (4.14), it follows that

$$|w_L|_{p_s^*}^p \leqslant \sigma^p C_{\gamma,K} M_1 |u|_{\sigma\alpha^*}^{p\sigma} < \infty. \tag{4.15}$$

Passing to the limit as $L \to +\infty$ in (4.15) and using Fatou's Lemma we have

$$|u|_{p_s^* \sigma} \leqslant (C_{\gamma,K} M_1)^{1/p\sigma} \sigma^{1/\sigma} |u|_{\sigma\alpha^*} \tag{4.16}$$

whenever $u^{\sigma\alpha^*} \in L^1(\mathbb{R}^3)$.

Now, we set $\sigma := p_s^*/\alpha^* > 1$, and we observe that, being $u \in L^{p_s^*}(\mathbb{R}^3)$, the above inequality holds for this choice of σ. Then, using the fact that $\sigma^2\alpha^* = p_s^*\sigma$, it follows that (4.16) holds with σ replaced by σ^2. Therefore, we can see that

$$|u|_{p_s^*\sigma^2} \leqslant (C_{\gamma,K} M_1)^{1/p\sigma^2} \sigma^{2/\sigma^2} |u|_{\sigma^2\alpha^*} \leqslant (C_{\gamma,K} M_1)^{1/p(1/\sigma + 1/\sigma^2)} \sigma^{1/\sigma} + \frac{2}{\sigma^2} |u|_{\sigma\alpha^*}.$$

Iterating this process and recalling that $\sigma\alpha^* := p_s^*$, we can infer that for every $m \in \mathbb{N}$

$$|u|_{p_s^*\sigma^m} \leqslant (C_{\gamma,K} M_1)^{\sum_{j=1}^m 1/p\sigma^j} \sigma^{\sum_{j=1}^m j\sigma^{-j}} |u|_{p_s^*}. \tag{4.17}$$

Taking the limit in (4.17) as $m \to +\infty$ and using lemma 4.1, we get

$$|u|_\infty \leqslant (C_{\gamma,K} M_1)^{\delta_1} \sigma^{\delta_2} M_2 \tag{4.18}$$

where $M_2 := C_*^{1/p} \bar{C}$ and

$$\delta_1 := \frac{1}{p} \sum_{j=1}^\infty \frac{1}{\sigma^j} < \infty \quad \text{and} \quad \delta_2 := \sum_{j=1}^\infty \frac{j}{\sigma^j} < \infty.$$

Next, we will find some suitable values of K and γ such that the following inequality holds

$$(C_{\gamma,K} M_1)^{\delta_1} \sigma^{\delta_2} M_2 \leqslant K,$$

or equivalently

$$1 + \gamma K^{r-q} \leqslant (K M_1^{-1})^{1/\delta_1} \left(\frac{C_* M_1}{a} \right)^{-1} \sigma^{-(\delta_2/\delta_1)}.$$

Take $K > 0$ such that

$$(K M_1^{-1})^{1/\delta_1} \left(\frac{C_* M_1}{a} \right)^{-1} \sigma^{-(\delta_2/\delta_1)} - 1 > 0,$$

and fix $\gamma_0 > 0$ such that

$$\gamma \leqslant \gamma_0 \leqslant \left[(KM_1^{-1})^{1/\delta_1} \left(\frac{C_* M_1}{a} \right)^{-1} \sigma^{-\delta_2/\delta_1} - 1 \right] \frac{1}{K^{r-q}}.$$

Therefore, thanks to (4.18), we can infer that

$$|u|_\infty \leqslant K \text{for all } \gamma \in [0, \gamma_0],$$

that is $u = u_{\varepsilon,\gamma}$ is a solution of (4.1). This ends the proof of theorem 1.3.

REMARK 4.1. We point out that, by assuming $q > 2p$ (since we aim to use theorem 1.1), the combined effect of concave–convex type growth $1 < q < 2p$, $r \geqslant p_s^*$ has been excluded. Anyway, when $\varepsilon = 1$, $s \in (0, 1)$ and $p \in (1, \infty)$ are such that $sp < 3$, the potential V is constant, and $p < q < p_s^* \leqslant r$, we suspect that it is possible to obtain an existence result to (4.1) with $b > 0$ sufficiently small. Indeed, one can truncate the nonlinearity $\phi(u)$ as before, and taking into account remark 2.1, we can deduce an existence result for (4.3) provided that $b > 0$ is small enough. Combining this fact with a Moser iteration argument, the desired existence result for (4.1) follows.

References

1 C. O. Alves, F. J. S. A. Corrêa and G. M. Figueiredo. On a class of nonlocal elliptic problems with critical growth. *Differ. Equ. Appl.* **2** (2010), 409–417.

2 C. O. Alves, F. J. S. A. Corrêa and T. F. Ma. Positive solutions for a quasilinear elliptic equation of Kirchhoff type. *Comput. Math. Appl.* **49** (2005), 85–93.

3 C. O. Alves and O. H. Miyagaki. Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method. *Calc. Var. Partial Differ. Equ.* **55** (2016), 47. 19 pp.

4 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14** (1973), 349–381.

5 V. Ambrosio. Multiplicity of positive solutions for a class of fractional Schrödinger equations via penalization method. *Ann. Mat. Pura Appl. (4)* **196** (2017), 2043–2062.

6 V. Ambrosio. An existence result for a fractional Kirchhoff-Schrödinger-Poisson system. *Z. Angew. Math. Phys.* **69** (2018), 30. 13 pp.

7 V. Ambrosio. Concentrating solutions for a class of nonlinear fractional Schrödinger equations in $\mathbb{R}^N$. *Rev. Mat. Iberoam.* **35** (2019), 1367–1414.

8 V. Ambrosio and T. Isernia. A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^N$ with a general nonlinearity. *Commun. Contemp. Math.* **20** (2018), 1750054. 17 pp.

9 V. Ambrosio and T. Isernia. Concentration phenomena for a fractional Schrödinger-Kirchhoff type problem. *Math. Methods Appl. Sci.* **41** (2018), 615–645.

10 V. Ambrosio and T. Isernia. Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional p-Laplacian. *Discrete Contin. Dyn. Syst.* **38** (2018), 5835–5881.

11 G. Autuori, A. Fiscella and P. Pucci. Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity. *Nonlinear Anal.* **125** (2015), 699–714.

12 V. Benci and G. Cerami. Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. *Calc. Var. Partial Differ. Equ.* **2** (1994), 29–48.

13 S. Bernstein. Sur une classe d'équations fonctionnelles aux dérivées partielles. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **4** (1940), 17–26.

14 H. Brézis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Am. Math. Soc.* **88** (1983), 486–490.

15 L. A. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Commun. Partial Differ. Equ.* **32** (2007), 1245–1260.

16 J. Chabrowski and J. Yang. Existence theorems for elliptic equations involving supercritical Sobolev exponent. *Adv. Differ. Equ.* **2** (1997), 231–256.

17 M. Chipot and B. Lovat. Some remarks on nonlocal elliptic and parabolic problems. *Nonlinear Anal.* **30** (1997), 4619–4627.

18 J. Dávila, M. del Pino, S. Dipierro and E. Valdinoci. Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum. *Anal. PDE* **8** (2015), 1165–1235.

19 J. Dávila, M. del Pino and J. Wei. Concentrating standing waves for the fractional nonlinear Schrödinger equation. *J. Differ. Equ.* **256** (2014), 858–892.

20 L. M. Del Pezzo and A. Quaas. A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian. *J. Differ. Equ.* **263** (2017), 765–778.

21 L. M. Del Pezzo and A. Quaas. Spectrum of the fractional p-Laplacian in $\mathbb{R}^N$ and decay estimate for positive solutions of a Schrödinger equation. *Nonlinear Anal.* (in press) (2019).

22 M. Del Pino and P. L. Felmer. Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. Partial Differ. Equ.* **4** (1996), 121–137.

23 A. Di Castro, T. Kuusi and G. Palatucci. Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), 1279–1299.

24 E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. math.* **136** (2012), 521–573.

25 S. Dipierro, M. Medina and E. Valdinoci. *Fractional elliptic problems with critical growth in the whole of* $\mathbb{R}^n$. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 15, pp. viii+152 (Pisa: Edizioni della Normale, 2017).

26 I. Ekeland. On the variational principle. *J. Math. Anal. Appl.* **47** (1974), 324–353.

27 P. Felmer, A. Quaas and J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. R. Soc. Edinburgh Sect. A* **142** (2012), 1237–1262.

28 G. M. Figueiredo and M. Furtado. Positive solutions for some quasilinear equations with critical and supercritical growth. *Nonlinear Anal.* **66** (2007), 1600–1616.

29 G. M. Figueiredo, G. Molica Bisci and R. Servadei. On a fractional Kirchhoff-type equation via Krasnoselskii's genus. *Asymptot. Anal.* **94** (2015), 347–361.

30 G. M. Figueiredo and J. R. Santos. Multiplicity and concentration behavior of positive solutions for a Schrödinger-Kirchhoff type problem via penalization method. *ESAIM Control Optim. Calc. Var.* **20** (2014), 389–415.

31 G. M. Figueiredo and G. Siciliano. A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in $\mathbb{R}N$. *NoDEA Nonlinear Differ. Equ. Appl.* **23** (2016), 12. 22 pp.

32 A. Fiscella and P. Pucci. Kirchhoff-Hardy fractional problems with lack of compactness. *Adv. Nonlinear Stud.* **17** (2017), 429–456.

33 A. Fiscella and E. Valdinoci. A critical Kirchhoff type problem involving a nonlocal operator. *Nonlinear Anal.* **94** (2014), 156–170.

34 G. Franzina and G. Palatucci. Fractional p-eigenvalues. *Riv. Math. Univ. Parma (N.S.)* **5** (2014), 373–386.

35 Y. He, G. Li and S. Peng. Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. *Adv. Nonlinear Stud.* **14** (2014), 483–510.

36 X. He and W. Zou. Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. *J. Differ. Equ.* **252** (2012), 1813–1834.

37 A. Iannizzotto, S. Mosconi and M. Squassina. Global Hölder regularity for the fractional p-Laplacian. *Rev. Mat. Iberoam.* **32** (2016), 1353–1392.

38 G. Kirchhoff. *Mechanik.* (Leipzig: Teubner, 1883).

39 N. Laskin. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A* **268** (2000), 298–305.

40 S. Liang, G. Molica Bisci and B. Zhang. Multiple solutions for a noncooperative Kirchhoff-type system involving the fractional p-Laplacian and critical exponents. *Math. Nachr.* **291** (2018), 1533–1546.

41 J. L. Lions. *On some questions in boundary value problems of mathematical physics, Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, North-Holland Math. Stud., vol. 30, pp. 284–346 (Amsterdam-New York: North-Holland, 1978).

42 X. Mingqi, G. Molica Bisci, G. Tian and B. Zhang. Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian. *Nonlinearity* **29** (2016), 357–374.

43 X. Mingqi, V. Rădulescu and B. Zhang. Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian. *Nonlinearity* **29** (2016), 3186–3205.

44 G. Molica Bisci, V. Rădulescu and R. Servadei. *Variational methods for nonlocal fractional problems, with a foreword by Jean Mawhin*. Encyclopedia of Mathematics and its Applications, vol. 162 (Cambridge: Cambridge University Press, 2016).

45 J. Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Commun. Pure Appl. Math.* **13** (1960), 457–468.

46 N. Nyamoradi. Existence of three solutions for Kirchhoff nonlocal operators of elliptic type. *Math. Commun.* **18** (2013), 489–502.

47 N. Nyamoradi and L. I. Zaidan. Existence and multiplicity of solutions for fractional p-Laplacian Schrödinger–Kirchhoff type equations. *Complex Var. Elliptic Equ.* **63** (2018), 346–359.

48 G. Palatucci. The Dirichlet problem for the p-fractional Laplace equation. *Nonlinear Anal.* **177** (2018), 699–732.

49 K. Perera and Z. Zhang. Nontrivial solutions of Kirchhoff-type problems via the Yang index. *J. Differ. Equ.* **221** (2006), 246–255.

50 S. I. Pohožaev. A certain class of quasilinear hyperbolic equations. *Mat. Sb.* **96** (1975), 152–166.

51 P. Pucci, M. Xiang and B. Zhang. Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. *Calc. Var. Partial Differ. Equ.* **54** (2015), 2785–2806.

52 P. H. Rabinowitz. Variational methods for nonlinear elliptic eigenvalue problems. *Indiana Univ. Math. J.* **23** (1973/74), 729–754.

53 P. Rabinowitz. On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* **43** (1992), 270–291.

54 J. Simon. *Régularité de la solution d'une équation non linéaire dans* $\mathbb{R}^N$. Lectures Notes in Math. No. 665 (Berlin: Springer, 1978).

55 A. Szulkin and T. Weth. The method of Nehari manifold. In *Handbook of nonconvex analysis and applications* (eds. D. Y. Gao and D. Montreanu), pp. 597–632 (Boston: International Press, 2010).

56 J. Wang, L. Tian, J. Xu and F. Zhang. Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. *J. Differ. Equ.* **253** (2012), 2314–2351.

57 M. Willem, Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24. *Birkhäuser Boston, Inc., Boston, MA*, 1996. x+162 pp.