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Nonlinear eigenvalue problems for the (p, q)–Laplacian

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ABSTRACT

We consider a parametric (p, q)-equations with sign-changing reaction and Robin boundary condition. We show that for all values of the parameter λ bigger than a certain value which we determine precisely, the problem has at least three nontrivial solutions all with sign information and ordered. For the particular case of $(p, 2)$-equations we produce a second nodal solution, for a total of four nontrivial solutions. Under symmetry conditions, we show the existence of infinitely many nodal solutions. The same results are also valid for the Dirichlet problem.

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1. Introduction

We study the following parametric (p, q)-equation with Robin boundary condition

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{p-2}u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, \ 1 < q < p, \ \lambda > 0. \end{cases} \qquad (P_\lambda)$$

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. For $1 < r < +\infty$ we denote by Δ_r the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}\left(|Du|^{r-2}Du\right) \text{ for all } u \in W^{1,r}(\Omega).$$

In problem (P_λ), in the left-hand side we have the sum of two such operators. So, the differential operator in (P_λ) is not homogeneous. There is also a potential term $\xi(z)|u|^{p-2}u$ with $\xi \geq 0$. The reaction (right-hand side of (P_λ)) is parametric with $\lambda > 0$ being the parameter and $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous).

In contrast to most similar works in the literature, $f(z, \cdot)$ can be sign-changing. In the boundary condition $\frac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative corresponding to the differential operator $u \mapsto -\Delta_p u - \Delta_q u$ (the (p, q)-Laplacian). We interpret this directional derivative using the nonlinear Green's identity (see [21, p. 35]). We know that if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{pq}} = (|Du|^{p-2} + |Du|^{q-2})\frac{\partial u}{\partial n}$$

with $n(\cdot)$ being the outward unit normal.

So, problem (P_λ) is a kind of a nonlinear eigenvalue problem for the Robin (p, q)-Laplacian plus a potential term. We want to find those parameter values for which problem (P_λ) has solutions and provide sign information for all of them. Our work here complements those of Gasiński & Papageorgiou [8], Li & Yang [12], Papageorgiou & Rădulescu [15], Papageorgiou, Rădulescu & Repovš [20]. In these works the reaction $f(z, \cdot)$ is $(p-1)$-superlinear as $x \to \pm\infty$ and they focus only on the existence of positive solutions. In addition, Gasiński & Papageorgiou [8] and Li & Yang [12] deal with equations driven by the Dirichlet p-Laplacian only. Related to our work, is also the last part in the paper of Gasiński & Papageorgiou [6], who consider equations driven by the Dirichlet p-Laplacian and a sign-changing reaction satisfying more restrictive conditions. They prove a bifurcation type result describing the changes in the set of positive solutions as the parameter λ moves on $\overset{\circ}{\mathbb{R}}_+ = (0, +\infty)$. We also mention the recent work of Papageorgiou & Zhang [23], on positive solutions of resonant (p, q)-equations.

Under minimal conditions of $f(z, \cdot)$, we show that for all $\lambda > 0$ problem (P_λ) has constant sign smooth solutions. If the parameter $\lambda > 0$ is restricted to be big enough (we determine the lower bound of the values of λ using the data of the problem), then we can show the existence of a smooth nodal solution. Under a symmetry condition of $f(z,\cdot)$, we show the existence of a sequence of nodal solutions. When $q=2$ (case of $(p,2)$-equations), then we are able to show the existence of a second nodal solution. Our tools are variational from critical point theory, combined with truncation and comparison techniques and critical groups.

The double-phase problem (P_λ) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [1] that appears in electromagnetism:

$$-\mathrm{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right)=h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-1/2}=1+\frac{x}{2}+\frac{3}{2\cdot 2^2}x^2+\frac{5!!}{3!\cdot 2^3}x^3+\cdots+\frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1}+\cdots \text{ for } |x|<1.$$

Taking $x=2|\nabla u|^2$ and adopting the first order approximation, we obtain problem (P_λ) for $p=4$ and $q=2$. Furthermore, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u-\Delta_4 u-\frac{3}{2}\Delta_6 u-\cdots-\frac{(2n-3)!!}{(n-1)!}\Delta_{2n}u.$$

Our work here appears to be the first one on nonlinear eigenvalue problems driven by the (p,q)-Laplacian with Robin boundary condition. Our hypotheses on the reaction are minimal, very general, and they include the case of sign-changing forcing term. Moreover, we provide sign information for all solutions produced.

2. Background material and hypotheses

The main spaces in the analysis of problem (P_λ), are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue spaces $L^s(\partial\Omega)$, $1\le s\le+\infty$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. We have

$$\|u\|=\left(\|u\|_p^p+\|Du\|_p^p\right)^{1/p}.$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+=\left\{u\in C^1(\overline{\Omega}):\ u(z)\ge 0 \text{ for all } z\in\overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : \, u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leq s \leq +\infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \mapsto L^p(\partial\Omega)$, known as the "trace map". We know that if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, then $\gamma_0(u) = u\Big|_{\partial\Omega}$. So, the trace map extends to all Sobolev functions the notion of boundary values. We know that $\gamma_0(\cdot)$ is compact into $L^s(\partial\Omega)$, for all $1 \leq s < \frac{(N-1)p}{N-p}$ if $p < N$ and into $L^s(\partial\Omega)$ for all $1 \leq s < +\infty$ if $N \leq p$. Moreover, we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \left(\frac{1}{p'} + \frac{1}{p} = 1\right) \text{ and } \ker\gamma_0 = W_0^{1,p}(\Omega).$$

In what follows for the sake of notational economy, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of the Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

If we consider the q-Laplace differential operator with Neumann boundary condition, then $\hat{\lambda}_1(q) = 0$ is the first eigenvalue with corresponding eigenspace $\mathbb{R}$ (the constant functions). The positive $L^q(\Omega)$-normalized principal eigenfunction is $\hat{u}_1(q) = \frac{1}{|\Omega|_N}$ with $|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$. By $\hat{\lambda}_2(q)$ we denote the first positive eigenvalue. We have the following variational characterization of $\hat{\lambda}_2(q)$ (see Cuesta, de Figueiredo & Gossez [3] (Dirichlet problems), Mugnai & Papageorgiou [14], Neumann problems with indefinite potential). We set $\partial B_1^{L^q} = \{u \in L^q(\Omega) : \, \|u\|_q = 1\}$, $M = W^{1,p}(\Omega) \cap \partial B_1^{L^q}$ and $\Gamma = \{\gamma \in C([-1,1], M) : \, \gamma(-1) = -\hat{u}_1(q), \gamma(1) = \hat{u}_1(q)\}$.

Proposition 2.1. $\hat{\lambda}_2(q) = \inf\limits_{\gamma \in \Gamma} \max\limits_{-1 \leq t \leq 1} \|D\gamma(t)\|_q^q$.

If $q = 2$, then we know that $-\Delta$ with Neumann boundary condition has a sequence of distinct eigenvalues $\{\hat{\lambda}_m(2)\}_{m \in \mathbb{N}}$ which satisfy $\hat{\lambda}_m(2) \to +\infty$ as $m \to \infty$ and describe completely the spectrum of the operator. Of course $\hat{\lambda}_1(2) = 0$. There is a corresponding sequence $\{\hat{u}_n\}_{n \in \mathbb{N}} \subseteq H_0^1(\Omega)$ of eigenfunctions which are an orthonormal basis for $H^1(\Omega)$. By $E(\hat{\lambda}_m(2))$ we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_m(2)$. These items have the following properties:

(a) $E(\hat{\lambda}_m(2))$ $(m \in \mathbb{N})$ is finite dimensional and $E(\hat{\lambda}_m(2)) \subseteq C^1(\overline{\Omega})$ (see Brezis [2])
(b) Each eigenspace has the so-called "Unique Continuation Property" (UCP for short), which means that if $u \in E(\hat{\lambda}_m(2))$ vanishes on A with $|A|_N > 0$, then $u \equiv 0$.
(c) $H^1(\Omega) = \overline{\mathop{\oplus}\limits_{m \geq 1} E(\hat{\lambda}_m(2))}$ (orthogonal direct sum decomposition) and

$$\hat{\lambda}_1(2) = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : \, u \in H^1(\Omega), u \neq 0\right\} = 0 \tag{1}$$

$$\hat{\lambda}_n(2) = \sup\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \overline{H}_n, u \neq 0\right\} = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \hat{H}_n, u \neq 0\right\} \tag{2}$$

where $\overline{H}_n = \overset{n}{\underset{m=1}{\oplus}} E(\hat{\lambda}_m(2))$, $\hat{H}_n = \overline{\underset{m\geq n}{\oplus} E(\hat{\lambda}_m(2))}$, $n \in \mathbb{N}$ (see Papageorgiou & Rădulescu [18]).

The infimum in (1) is clearly attained on $\mathbb{R}$ (the eigenspace of $\hat{\lambda}_1(2) = 0$), while both the supremum and infimum in (2) are realized on $E(\hat{\lambda}_m(2))$. All eigenvalues $\hat{\lambda}_m(2)$ $(m \geq 2)$ have nodal eigenfunctions.

Using the orthogonality of the eigenspaces, the UCP and (1), (2) we have the following Lemma (see Papageorgiou & Winkert [22]).

Lemma 2.2.

(a) *If $m \in \mathbb{N}$, $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geq \hat{\lambda}_m(2)$ for a.a. $z \in \Omega$ and the inequality is strict on a set A with $|A|_N > 0$, then*

$$C_1\|u\|^2_{H^1(\Omega)} \leq \int_\Omega \vartheta(z)u^2 dz - \|Du\|_2^2$$

for some $C_1 > 0$, all $u \in \overline{H}_m$.

(b) *If $m \in \mathbb{N}$, $\vartheta \in L^\infty(\Omega)$, $\vartheta \leq \hat{\lambda}_m(2)$ for a.a. $z \in \Omega$ and the inequality is strict on a set A with $|A|_N > 0$, then*

$$C_2\|u\|^2_{H^1(\Omega)} \leq \|Du\|_2^2 - \int_\Omega \vartheta(z)u^2 dz$$

for some $C_2 > 0$ all $u \in \hat{H}_m$.

Our hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$ are the following:

$\mathbf{H_0}$: $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $\xi \neq 0$ or $\beta \neq 0$.

If $k_p : W^{1,p}(\Omega) \mapsto \mathbb{R}$ is the C^1-functional defined by

$$k_p(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma,$$

then using the hypotheses $\mathbf{H_0}$, Lemma 4.11 of Mugnai & Papageorgiou [14] and Proposition 2.4 of Gasiński & Papageorgiou [6], we have

$$C_0\|u\|^p \leq k_p(u) \text{ for some } C_0 > 0, \text{ all } W^{1,p}(\Omega). \tag{3}$$

In particular, the nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u + \xi(z)|u|^{p-2}u = \tilde{\lambda}|u|^{p-2}u \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, \end{cases}$$

has a positive smallest eigenvalue $\tilde{\lambda}_1(p)$ which is isolated, simple and

$$\tilde{\lambda}_1(p) = \inf\left\{ \frac{k_p(u)}{\|u\|_p^p} : \, u \in W^{1,p}(\Omega), u \neq 0 \right\} > 0$$

(see Papageorgiou & Rădulescu [18]).

If $u, v : \Omega \mapsto \mathbb{R}$ are measurable functions such that $v(z) \leq u(z)$ for a.a. $z \in \Omega$, then we introduce the following order interval in $W^{1,p}(\Omega)$

$$[v, u] = \left\{ h \in W^{1,p}(\Omega) : v(z) \leq h(z) \leq u(z) \text{ for a.a. } z \in \Omega \right\}.$$

By $\operatorname{int}_{C^1(\overline{\Omega})}[v, u]$ we denote the interior of $[v, u] \cap C^1(\overline{\Omega})$ in $C^1(\overline{\Omega})$.

If $u \in W^{1,p}(\Omega)$, we set $u^{\pm} = \max\{\pm u, 0\}$. We know that $u^{\pm} \in W^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$.

Given $r \in (1, +\infty)$, we denote by $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u, h \in W^{1,r}(\Omega).$$

This operator is continuous, monotone (hence maximal monotone) and of type $(S)_+$, that is,

$$\text{“if } u_n \xrightarrow{w} u \text{ in } W^{1,r}(\Omega) \text{ and } \limsup_{n\to\infty} \langle A_r(u_n), u_n - u\rangle \leq 0,$$
$$\text{then } u_n \to u \text{ in } W^{1,r}(\Omega).\text{”}$$

This property is a consequence of the Kadec-Klee property (also known as the Radon-Riesz property) of uniformly convex spaces. This property says that if X is uniformly convex and $x_n \xrightarrow{w} x$, $\|x_n\| \to \|x\|$, then $x_n \to x$.

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We define

$$K_\varphi = \{u \in X : \, \varphi'(u) = 0\}, \; K_\varphi^c = \{u \in K_\varphi : \, \varphi(u) = c\}, \; \varphi^c = \{u \in X : \, \varphi(u) \leq c\}.$$

We say that $\varphi(\cdot)$ satisfies the “PS-condition”, if:

“Every sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ is bounded}$$
$$\text{and } \varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$
admits a strongly convergent subsequence".

Finally let $Y_2 \subseteq Y_1 \subseteq X$. By $H_k(Y_1, Y_2)$ ($k \in \mathbb{N}_0$), we denote the k^{th}-relative singular homology, group with integer coefficients. If $u \in K_\varphi$ is isolated, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k\left(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}\right)$$

with $c = \varphi(u)$, $k \in \mathbb{N}_0$ and U an open neighborhood of u such that $\varphi^c \cap K_\varphi \cap U = \{u\}$. The excision property of singular homology implies that this definition is independent of the isolating neighborhood U. Suppose that φ satisfies the PS-condition and that K_φ is finite. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c)$$

for all $k \in \mathbb{N}_0$ with $c < \inf \varphi(K_\varphi)$. The Second Deformation Theorem (see [21, p. 386]), implies that this definition is independent of the choice of $c < \inf \varphi(K_\varphi)$. We define

$$M(t, u) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi, u) t^k \text{ for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u\in K_\varphi} M(t, u) = P(t, \infty) + (1 + t)Q(t) \text{ for all } t \in \mathbb{R}$$

with $Q(t) = \sum_{k\geq 0} \beta_k t^k$ a formal series in t with nonnegative integer coefficients.

Next we introduce the hypotheses on $f(z, x)$:

$\mathbf{H_1}$: $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(*i*) $|f(z, x)| \leq a(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$,

$$p < r < p^* = \begin{cases} \dfrac{Np}{N-p}, & \text{if } p < N \\ +\infty, & \text{if } N \leq p \end{cases};$$

(*ii*) $\limsup_{x\to+\infty} \dfrac{f(z, x)}{|x|^{p-2}x} \leq 0$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\vartheta > 0$ such that

$$\vartheta \leq \liminf_{x \to 0} \frac{f(z,x)}{|x|^{q-2}x} \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2.3. Evidently the hypotheses on f are very general and include also functions which may change sign as $x \to \pm\infty$. Note that near zero $f(z,x)x \geq 0$ for a.a. $z \in \Omega$.

Let $F(z,x) = \int_0^x f(z,s)ds$ (the primitive of $f(z,\cdot)$). We introduce the C^1-functionals $\varphi_\lambda, \varphi_\lambda^{\pm} : W^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}k_p(u) + \frac{1}{q}\|Du\|_q^q - \lambda \int_\Omega F(z,u)dz,$$

$$\varphi_\lambda^{\pm}(u) = \frac{1}{p}k_p(u) + \frac{1}{q}\|Du\|_q^q - \lambda \int_\Omega F(z,\pm u^{\pm})dz \text{ for all } u \in W^{1,p}(\Omega).$$

3. Constant sign solutions

First we show that (P_λ) has constant sign solutions for all $\lambda > 0$.

Proposition 3.1. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold, then for every* $\lambda > 0$ *problem* (P_λ) *has at least two constant sign solutions* $u_\lambda \in \operatorname{int} C_+$, $v_\lambda \in -\operatorname{int} C_+$.

Proof. First we show the existence of a positive solution. On account of hypotheses $\mathbf{H_1}(i)$, (ii) given $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$F(z,x) \leq \frac{\varepsilon}{p}|x|^p + C_\varepsilon \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{4}$$

Then for all $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}\varphi_\lambda^+(u) &\geq \frac{1}{p}\left(k_p(u) - \lambda\varepsilon\|u\|_p^p\right) - C_3 \text{ for some } C_3 = C_3(\varepsilon) > 0 \text{ (see (4))} \\ &\geq \frac{1}{p}\left(C_0 - \lambda\varepsilon\right)\|u\|^p - C_3 \text{ (see (3)).}\end{aligned}$$

Choosing $\varepsilon \in \left(0, \frac{C_0}{\lambda}\right)$, we see that

$$\varphi_\lambda^+(\cdot) \text{ is coercive.}$$

The Sobolev embedding theorem and the compactness of the trace map, imply that φ_λ^+ is sequentially weakly lower semicontinuous. Thus, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(u_\lambda) = \inf\left\{\varphi_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{5}$$

Hypothesis $\mathbf{H_1}(iii)$ implies that given $\varepsilon \in (0, \vartheta)$, we can find $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$F(z,x) \geq \frac{1}{q}(\vartheta - \varepsilon)|x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{6}$$

Let $\eta \in (0, \delta)$. Then

$$\begin{aligned}\varphi_\lambda^+(\eta) &\leq \frac{\eta^p}{p}\left(\int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma\right) - \frac{\eta^q}{q}\lambda(\vartheta - \varepsilon) \text{ (see (6))} \\ &= C_4\eta^p - C_5\eta^q \text{ for some } C_4, C_5 > 0. \end{aligned} \tag{7}$$

Since $q < p$, choosing $\eta \in (0, \delta)$ even smaller if necessary we have

$$\begin{aligned}&\varphi_\lambda^+(\eta) < 0 \text{ (see (7))}, \\ \Rightarrow\ &\varphi_\lambda^+(u_\lambda) < 0 = \varphi_\lambda^+(0) \text{ (see (5))}, \\ \Rightarrow\ &u_\lambda \neq 0.\end{aligned}$$

From (5) we have

$$\begin{aligned}&\left(\varphi_\lambda^+\right)'(u_\lambda) = 0, \\ \Rightarrow\ &\langle k_p'(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle = \lambda \int_\Omega f(z, u_\lambda^+)hdz \end{aligned} \tag{8}$$

for all $h \in W^{1,p}(\Omega)$.

In (8) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned}&k_p(u_\lambda^-) \leq 0, \\ \Rightarrow\ &C_0\|u_\lambda^-\|^p \leq 0 \text{ (see (3))}, \\ \Rightarrow\ &u_\lambda \geq 0, u_\lambda \neq 0.\end{aligned}$$

Therefore u_λ is a positive solution of (P_λ). Proposition 2.10 of Papageorgiou & Rădulescu [17], implies that $u_\lambda \in L^\infty(\Omega)$. Then using the nonlinear regularity theory of Lieberman [11], we have $u_\lambda \in C_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_\infty$. Hypotheses $\mathbf{H_1}(i)$, (iii) imply that we can find $\hat{\xi}_\rho > 0$ such that

$$f(z,x)x + \hat{\xi}_\rho|x|^p \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho.$$

We have

$$\Delta_p u_\lambda + \Delta_q u_\lambda \leq \left(\|\xi\|_\infty + \lambda\hat{\xi}_\rho\right) u_\lambda^{p-1} \text{ in } \Omega.$$

Then the maximum principle of Pucci & Serrin [24, pp. 111, 120], implies that $u_\lambda \in \operatorname{int} C_+$.

Similarly working with φ_λ^-, we produce a negative solution $v_\lambda \in -\operatorname{int} C_+$. □

In fact we can show the existence of a smallest positive solution and of a biggest negative solution. We will need these extremal constant sign solutions in order to produce a nodal one (see Section 4).

To produce these extremal constant sign solutions, we need to do some preparatory work. Hypotheses $\mathbf{H_1}(i)$, (iii) imply that given $\varepsilon \in (0, \vartheta)$, we can find $C_6 = C_6(\varepsilon) > 0$ such that

$$f(z,x)x \geq (\vartheta - \varepsilon)\,|x|^q - C_6|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{9}$$

This unilateral growth condition on $f(z, \cdot)$ leads to the following auxiliary Robin problem

$$\begin{cases} -\Delta_p u - \Delta_q u + \xi(z)|u|^{p-2}u = \lambda\left((\vartheta - \varepsilon)|u|^{q-2}u - C_6|u|^{r-2}u\right) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, \lambda > 0, \varepsilon \in (0,\vartheta). \end{cases} \tag{10_λ}$$

Proposition 3.2. *If hypotheses* $\mathbf{H_0}$ *hold, then for every* $\lambda > 0$ *problem* (10_λ) *has a unique positive solution* $\overline{u}_\lambda \in \operatorname{int} C_+$ *and since the equation is odd* $\overline{v}_\lambda = -\overline{u}_\lambda \in -\operatorname{int} C_+$ *is the unique negative solution of problem* (10_λ).

Proof. First we show the existence of a positive solution.

So, we consider the C^1-functional $\psi_\lambda^+ : W^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi_\lambda^+(u) = \frac{1}{p}k_p(u) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda(\vartheta - \varepsilon)}{q}\|u^+\|_q^q + \frac{\lambda C_6}{r}\|u^+\|_r^r$$

for all $u \in W^{1,p}(\Omega)$.

Since $q < p < r$, it is clear that

$$\psi_\lambda^+ \text{ is coercive.}$$

Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(\overline{u}_\lambda) = \inf\left\{\psi_\lambda^+(u) :\ u \in W^{1,p}(\Omega)\right\}. \tag{10}$$

Since $\varepsilon \in (0, \vartheta)$ and $q < p < r$, we see that for $\eta \in (0,1)$ small we have

$$\begin{aligned} &\psi_\lambda^+(\eta) < 0 \\ \Rightarrow\ &\psi_\lambda^+(\overline{u}_\lambda) < 0 = \psi_\lambda^+(0) \text{ (see (10))}, \\ \Rightarrow\ &\overline{u}_\lambda \neq 0. \end{aligned}$$

From (10) we have

$$\begin{aligned} &\left(\psi_\lambda^+\right)'(\overline{u}_\lambda) = 0, \\ \Rightarrow\ &\langle k_p'(\overline{u}_\lambda), h\rangle + \langle A_q(\overline{u}_\lambda), h\rangle = \lambda \int_\Omega \left((\vartheta - \varepsilon)|\overline{u}_\lambda|^{q-2}\overline{u}_\lambda - C_6|\overline{u}_\lambda|^{r-2}\overline{u}_\lambda\right) h dz \qquad (11) \end{aligned}$$

for all $h \in W^{1,p}(\Omega)$.

In (11) we use the test function $h = -\overline{u}_\lambda^- \in W^{1,p}(\Omega)$ and using (3) we obtain that $\overline{u}_\lambda \geq 0$, $\overline{u}_\lambda \neq 0$. This implies that $\overline{u}_\lambda$ is a positive solution of (10_λ). As before the nonlinear regularity theory and the nonlinear maximum principle imply that $\overline{u}_\lambda \in \operatorname{int} C_+$.

In what follows, $\hat{k}_p : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\hat{k}_p(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p dz \text{ for all } u \in W^{1,p}(\Omega).$$

Next, we show the uniqueness of this positive solution. To this end, we introduce the integral functional $j : L^1(\Omega) \mapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \dfrac{1}{p}\hat{k}_p\left(u^{1/q}\right) + \dfrac{1}{q}\|Du^{1/q}\|_q^q, & \text{if } u \geq 0, u^{1/q} \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j = \left\{u \in L^1(\Omega) : \ j(u) < +\infty\right\}$ (the effective domain of $j(\cdot)$). We introduce function $G_0 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by

$$G_0(t) = \frac{1}{p}t^p + \frac{1}{q}t^q \text{ for all } t \geq 0$$

Evidently $G_0(\cdot)$ is increasing and $t \mapsto G_0(t^{1/q})$ is convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Clearly $G(\cdot)$ is convex. So, if $u_1, u_2 \in \operatorname{dom} j$ and $u = (tu_1 + (1-t)u_2)^{1/q}$, $t \in [0,1]$, then from Diaz & Saa [4, Lemma 1], we have

$$\begin{aligned} &|Du| \leq \left(t\left|Du_1^{1/q}\right|^q + (1-t)\left|Du_2^{1/q}\right|^q\right)^{1/q} \\ \Rightarrow\ &G_0(|Du|) \leq G_0\left(\left(t\left|Du_1^{1/q}\right|^q + (1-t)\left|Du_2^{1/q}\right|^q\right)^{1/q}\right) \end{aligned}$$

$$
\begin{aligned}
&\text{(since } G_0(\cdot) \text{ is increasing),}\\
&\le tG_0\left(\left|Du_1^{1/q}\right|\right) + (1-t)G_0\left(\left|Du_2^{1/q}\right|\right)\\
&\text{(since } t \mapsto G_0(t^{1/q}) \text{ is convex),}\\
\Rightarrow\ & G(Du) \le tG\left(Du_1^{1/q}\right) + (1-t)G\left(Du_2^{1/q}\right),\\
\Rightarrow\ & j(\cdot) \text{ is convex (recall that } q<p \text{ and see hypotheses } \mathbf{H_0}).
\end{aligned}
$$

Also, by Fatou's lemma we see that $j(\cdot)$ is lower semicontinuous.

Suppose $\tilde{u}_\lambda$ is another positive solution of problem (10_λ). Again we have $\tilde{u}_\lambda \in \operatorname{int} C_+$. Hence using Proposition 4.1.22 of Papageorgiou, Rădulescu & Repovš [21, p. 274], we have

$$\frac{\overline{u}_\lambda}{\tilde{u}_\lambda} \in L^\infty(\Omega) \text{ and } \frac{\tilde{u}_\lambda}{\overline{u}_\lambda} \in L^\infty(\Omega).$$

Let $h = \overline{u}_\lambda^q - \tilde{u}_\lambda^q$. Then for $|t|<1$ small we have

$$\overline{u}_\lambda^q + th \in \operatorname{dom} j,\ \tilde{u}_\lambda^q + th \in \operatorname{dom} j.$$

Then we can calculate the Gâteaux (directional) derivative of $j(\cdot)$ at $\overline{u}_\lambda^q$ and at $\tilde{u}_\lambda^q$ in the direction h. In fact, using the chain rule and reasoning as in Gasiński and Papageorgiou [7, p. 492], we have

$$
\begin{aligned}
j'(\overline{u}_\lambda^q)(h) &= \frac{1}{q}\left[\left\langle A_p(\overline{u}_\lambda), \frac{h}{\overline{u}_\lambda^{q-1}}\right\rangle + \left\langle A_q(\overline{u}_\lambda), \frac{h}{\overline{u}_\lambda^{q-1}}\right\rangle + \int_\Omega \frac{\xi(z)\overline{u}_\lambda^{p-1}}{\overline{u}_\lambda^{q-1}} h dz\right]\\
&= \frac{1}{q}\int_\Omega \frac{-\Delta_p\overline{u}_\lambda - \Delta_q\overline{u}_\lambda + \xi(z)\overline{u}_\lambda^{p-1}}{\overline{u}_\lambda^{q-1}} h dz\\
&= \frac{1}{q}\int_\Omega \lambda\left((\vartheta-\varepsilon) - C_6\overline{u}_\lambda^{r-q}\right) h dz\\
&\quad \text{(using Green's identity, see [21, p. 35]).}
\end{aligned}
$$

Similarly we have

$$j'(\tilde{u}_\lambda^q)(h) = \frac{1}{q}\int_\Omega \lambda\left((\vartheta-\varepsilon) - C_6\tilde{u}_\lambda^{r-q}\right) hdz.$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$
\begin{aligned}
&0 \le \lambda C_6\int_\Omega \left(\tilde{u}_\lambda^{r-q} - \overline{u}_\lambda^{r-q}\right)(\overline{u}_\lambda^q - \tilde{u}_\lambda^q)dz,\\
\Rightarrow\ & \overline{u}_\lambda = \tilde{u}_\lambda.
\end{aligned}
$$

This proves the uniqueness of the positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$ of problem (10_λ).

Since the equation is odd, then $\overline{v}_\lambda = -\overline{u}_\lambda \in -\operatorname{int} C_+$ is the unique negative solution of problem (10_λ). $\square$

We introduce the following two sets

$$\begin{aligned} S_\lambda^+ &= \text{ set of positive solutions of } (P_\lambda), \\ S_\lambda^- &= \text{ set of negative solutions of } (P_\lambda). \end{aligned}$$

From Proposition 3.1 and its proof, we know that for all $\lambda > 0$, we have

$$\emptyset \neq S_\lambda^+ \subseteq \operatorname{int} C_+ \text{ and } \emptyset \neq S_\lambda^- \subseteq -\operatorname{int} C_+.$$

The solutions of (10_λ) produced in Proposition 3.2 provide bounds for the two solution sets S_λ^+, S_λ^-.

Proposition 3.3. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold and* $\lambda > 0$, *then* $\overline{u}_\lambda \leq u$ *for all* $u \in S_\lambda^+$ *and* $v \leq \overline{v}_\lambda$ *for all* $v \in S_\lambda^-$.

Proof. Let $u \in S_\lambda^+ \subseteq \operatorname{int} C_+$. We consider the Carathéodory function $l(z,x)$ defined by

$$l(z,x) = \begin{cases} (\vartheta - \varepsilon)(x^+)^{q-1} - C_6(x^+)^{r-1}, & \text{if } x \leq u(z) \\ (\vartheta - \varepsilon)u(z)^{q-1} - C_6 u(z)^{r-1}, & \text{if } u(z) < x \end{cases} \tag{12}$$

(recall that $\varepsilon \in (0,\vartheta)$). We set $L(z,x) = \int\limits_0^x l(z,s)ds$ and consider the C^1-functional $\hat{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p} k_p(u) + \frac{1}{q}\|Du\|_q^q - \lambda \int\limits_\Omega L(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3) and (12), it is clear that $\hat{\psi}(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda(\tilde{u}_\lambda) = \inf\left\{\hat{\psi}_\lambda(u) : \ u \in W^{1,p}(\Omega)\right\}. \tag{13}$$

Since $q < p < r$, for $\eta \in (0,1)$ small we will have

$$\begin{aligned} &\hat{\psi}_\lambda(\eta) < 0, \\ \Rightarrow\ &\hat{\psi}_\lambda(\tilde{u}_\lambda) < 0 = \hat{\psi}_\lambda(0), \\ \Rightarrow\ &\tilde{u}_\lambda \neq 0. \end{aligned}$$

From (13) we have

$$
\begin{aligned}
&\hat{\psi}'_\lambda(\tilde{u}_\lambda) = 0, \\
\Rightarrow\ &\langle k'_p(\tilde{u}_\lambda), h\rangle + \langle A_q(\tilde{u}_\lambda), h\rangle = \int_\Omega \lambda l(z,u) h dz
\end{aligned}
\tag{14}
$$

for all $h \in W^{1,p}(\Omega)$.

Using $h = -\tilde{u}_\lambda^-$ we obtain $\tilde{u}_\lambda \geq 0$, $\tilde{u}_\lambda \neq 0$. If we use $h = (\tilde{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$, then

$$
\begin{aligned}
&\langle k'_p(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle + \langle A_q(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle \\
&= \lambda \int_\Omega \left[(\vartheta - \varepsilon)u^{q-1} - C_6 u^{r-1}\right](\tilde{u}_\lambda - u)^+ dz \\
&\leq \lambda \int_\Omega f(z,u)(\tilde{u}_\lambda - u)^+ dz \text{ (see (9))} \\
&= \langle k'_p(u), (\tilde{u}_\lambda - u)^+\rangle + \langle A_q(u), (\tilde{u}_\lambda - u)^+\rangle \text{ (since } u \in S_\lambda^+), \\
\Rightarrow\ &\tilde{u}_\lambda \leq u.
\end{aligned}
$$

So, we have proved that

$$
\tilde{u}_\lambda \in [0, u],\ \tilde{u}_\lambda \neq 0. \tag{15}
$$

From (15), (12) and (14) we see that

$$
\begin{aligned}
&\tilde{u}_\lambda \text{ is a positive solution of problem } (10_\lambda), \\
&\Rightarrow \tilde{u}_\lambda = \overline{u}_\lambda \in \operatorname{int} C_+ \text{ (see Proposition 3.2).}
\end{aligned}
$$

Similarly we show that $v \leq \overline{v}_\lambda$ for all $v \in S_\lambda^- \subseteq -\operatorname{int} C_+$. □

From Papageorgiou, Rădulescu & Repovš [19] (see the proof of Proposition 7), we know that S_λ^+ is downward directed (that is, if $u_1, u_2 \in S_\lambda^+$, then there exists $u \in S_\lambda^+$ such that $u \leq u_1$, $u \leq u_2$) while S_λ^- is upward directed (that is, if $v_1, v_2 \in S_\lambda^-$, then there exists $v \in S_\lambda^-$ such that $v_1 \leq v$, $v_2 \leq v$). In the next proposition we establish the existence of extremal constant sign solutions.

Proposition 3.4. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold and* $\lambda > 0$, *then problem* (P_λ) *has a smallest positive solution* $u_\lambda^* \in S_\lambda^+ \subseteq \operatorname{int} C_+$ *(that is,* $u_\lambda^* \leq u$ *for all* $u \in S_\lambda^+$*) and a biggest negative solution* $v_\lambda^* \in S_\lambda^- \subseteq -\operatorname{int} C_+$ *(that is,* $v \leq v_\lambda^*$ *for all* S_λ^-*).*

Proof. Using Lemma 3.10 of Hu & Papageorgiou [9], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda^+$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf S_\lambda^+.$$

We have

$$\langle k_p'(u_n), h\rangle + \langle A_q(u_n), h\rangle = \lambda \int_\Omega f(z, u_n) h dz \tag{16}$$

$$\text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N},$$

$$\overline{u}_\lambda \le u_n \le u_1 \text{ for all } n \in \mathbb{N} \text{ (see Proposition 3.3)}. \tag{17}$$

In (16) we use the test function $h = u_n \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned}
&C_0\|u_n\|^p \le k_p(u_n) \le C_7 \text{ for some } C_7 > 0, \text{ all } n \in \mathbb{N} \\
&\text{(see (17) and hypothesis } \mathbf{H_1}(i)), \\
\Rightarrow\ &\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \tag{18}$$

In (16) we use $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (18). We obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left(\langle A_p(u_n), u_n - u_\lambda^*\rangle + \langle A_q(u_n), u_n - u_\lambda^*\rangle\right) = 0, \\
\Rightarrow\ &\limsup \left(\langle A_p(u_n), u_n - u_\lambda^*\rangle + \langle A_q(u_\lambda^*), u_n - u_\lambda^*\rangle\right) \le 0 \\
&\text{(since } A_q(\cdot) \text{ is monotone)}, \\
\Rightarrow\ &\limsup_{n\to\infty} \langle A_p(u_n), u_n - u_\lambda^*\rangle \le 0 \text{ (see (18))}, \\
\Rightarrow\ &u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ (by the } (S)_+ \text{ -property of } A_p(\cdot))
\end{aligned} \tag{19}$$

If in (16) we pass to the limit as $n \to \infty$ and use (19), then

$$\begin{aligned}
&\langle k_p'(u_\lambda^*), h\rangle + \langle A_q(u_\lambda^*), h\rangle = \lambda \int_\Omega f(z, u_\lambda^*) h dz \text{ for all } h \in W^{1,p}(\Omega), \\
&\overline{u}_\lambda \le u_\lambda^*.
\end{aligned}$$

It follows that $u_\lambda^* \in S_\lambda^+ \subseteq \operatorname{int} C_+$ and $u_\lambda^* = \inf S_\lambda^+$.

Similarly we produce maximal negative solution $v_\lambda^* \in S_\lambda^- \subseteq -\operatorname{int} C_+$. In this case we can find an increasing sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq S_\lambda^-$ such that $\sup_{n\in\mathbb{N}} v_n = \sup S_\lambda^-$. □

In the next section we use these extremal constant sign solutions in order to produce a nodal one.

4. Nodal solutions

To produce a nodal (sign-changing) solution, we look for nontrivial solutions of problem (P_λ) in the order interval $[v_\lambda^*, u_\lambda^*]$ distinct from u_λ^* and v_λ^*. On account of the extremality of u_λ^* and v_λ^*, any such solution is necessarily nodal. To limit ourselves on the order interval $[v_\lambda^*, u_\lambda^*]$, we use truncations techniques. For this method to lead to the desired nodal solution, we need to restrict the parameter $\lambda > 0$.

Let $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions produced in Proposition 3.4. We introduce the following truncation of $f(z, \cdot)$

$$\hat{f}(z,x) = \begin{cases} f(z, v_\lambda^*(z)), & \text{if } x < v_\lambda^*(z) \\ f(z,x), & \text{if } v_\lambda^*(z) \le x \le u_\lambda^*(z) \\ f(z, u_\lambda^*(z)), & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{20}$$

This is a Carathéodory function. We also consider the positive and negative truncations of $f(z, \cdot)$, namely the Carathéodory functions

$$\hat{f}_\pm(z,x) = \hat{f}(z, \pm x^\pm). \tag{21}$$

We set $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)ds$ and $\hat{F}_\pm(z,x) = \int_0^x \hat{f}_\pm(z,s)ds$ and introduce the C^1-functionals $\hat{\varphi}_\lambda, \hat{\varphi}_\lambda^\pm : W^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}k_p(u) + \frac{1}{q}\|Du\|_q^q - \lambda \int_\Omega \hat{F}(z,u)dz$$

$$\hat{\varphi}_\lambda^\pm(u) = \frac{1}{p}k_p(u) + \frac{1}{q}\|Du\|_q^q - \lambda \int_\Omega \hat{F}_\pm(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (20), (21) and the extremality of u_λ^*, v_λ^*, we obtain easily the following result.

Proposition 4.1. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold and* $\lambda > 0$, *then* $K_{\hat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$, $K_{\hat{\varphi}_\lambda^+} = \{0, u_\lambda^*\}$, $K_{\hat{\varphi}_\lambda^-} = \{0, v_\lambda^*\}$.

Now we are ready to prove the existence of a nodal solution.

Proposition 4.2. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold and* $\lambda > \frac{\hat{\lambda}_2(q)}{\vartheta} + 1$, *then problem* (P_λ) *has a nodal solution*

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}).$$

Proof. First we show that u_λ^* and v_λ^* are local minimizers of $\hat{\varphi}_\lambda(\cdot)$.

From (20) and (21) it is clear that $\hat{\varphi}_\lambda^+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. Hence we can find $\tilde{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda^+(\tilde{u}_\lambda^*) = \inf\left\{\hat{\varphi}_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\} < 0 = \hat{\varphi}_\lambda^+(0)$$
$$\text{(see the proof of Proposition 3.1)},$$
$$\Rightarrow \tilde{u}_\lambda^* \neq 0.$$

It follows that $\tilde{u}_\lambda^* \in K_{\hat{\varphi}_\lambda^+} \setminus \{0\}$ and so using Proposition 4.1 we infer that

$$\tilde{u}_\lambda^* = u_\lambda^* \in \operatorname{int} C_+. \tag{22}$$

From (20) and (21), we see that

$$\hat{\varphi}_\lambda^+\Big|_{C_+} = \hat{\varphi}_\lambda\Big|_{C_+}.$$

But then (22) implies that

$$\begin{aligned} & u_\lambda^* \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \hat{\varphi}_\lambda(\cdot) \\ \Rightarrow\ & u_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\varphi}_\lambda(\cdot). \\ & \text{(see Papageorgiou \& Rădulescu [17, Proposition 2.12]).} \end{aligned} \tag{23}$$

Similarly, using the functional $\hat{\varphi}_\lambda^-$, we show

$$v_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\varphi}_\lambda(\cdot). \tag{24}$$

We may assume that $\hat{\varphi}_\lambda(v_\lambda^*) \leq \hat{\varphi}_\lambda(u_\lambda^*)$. The reasoning is the same if the opposite inequality holds, using (24) instead of (23).

From Proposition 4.1, we see that we may assume that

$$K_{\hat{\varphi}_\lambda} \text{ is finite.} \tag{25}$$

Otherwise we already have a sequence of distinct smooth nodal solutions so we are done.

From (23), (25) and Theorem 5.7.6 of Papageorgiou, Rădulescu & Repovš [21, p. 449], we can find $\rho \in (0,1)$ small such that

$$\hat{\varphi}_\lambda(v_\lambda^*) \leq \hat{\varphi}_\lambda(u_\lambda^*) < \inf\{\hat{\varphi}_\lambda(u) : \|u - u_\lambda^*\| = \rho\} = \hat{m}_\lambda,\ \|v_\lambda^* - u_\lambda^*\| > \rho. \tag{26}$$

From [21] it follows that $\hat{\varphi}_\lambda(\cdot)$ is coercive. Hence by Proposition 5.1.15 of [21, p. 369] we obtain that

$$\hat{\varphi}_\lambda(\cdot) \text{ satisfies the PS-condition.} \tag{27}$$

Then on account of (26) and (27), we see that we can apply the Mountain Pass Theorem and produce $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_k \in K_{\hat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}) \text{ (see Proposition 4.1) and } \hat{m}_\lambda \leq \hat{\varphi}_\lambda(y_\lambda) \text{ (see (26))},$$
$$\Rightarrow y_\lambda \notin \{u_\lambda^*, v_\lambda^*\}.$$

So, if we can show that $y_\lambda \neq 0$, then we can conclude that $y_\lambda \in C^1(\overline{\Omega})$ is a nodal solution of problem (P_λ).

From the Mountain Pass Theorem, we know that

$$\hat{\varphi}_\lambda(y_\lambda) = \inf_{\gamma \in \Gamma} \max_{-1 \leq t \leq 1} \hat{\varphi}_\lambda(\gamma(t)), \tag{28}$$

with $\Gamma = \left\{\gamma \in C\left([-1,1], W^{1,p}(\Omega)\right) : \gamma(-1) = v_\lambda^*, \gamma(1) = u_\lambda^*\right\}$.

Let $\partial B_1^{L^q}$, M be the manifolds from Proposition 2.1 and $M_c = M \cap C^1(\overline{\Omega})$. We introduce the following two sets of paths:

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\},$$
$$\hat{\Gamma}_c = \{\hat{\gamma} \in C([-1,1], M_c) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\}.$$

Claim: $\hat{\Gamma}_c$ is dense in $\hat{\Gamma}$.

Let $\hat{\gamma} \in \hat{\Gamma}$ and $\varepsilon \in (0,1)$. We introduce the multifunction $\hat{H}_\varepsilon : [-1,1] \mapsto 2^{C^1(\overline{\Omega})}$ defined by

$$\hat{H}_\varepsilon(t) = \begin{cases} \{u \in C^1(\overline{\Omega}) : \|u - \hat{\gamma}(t)\| < \varepsilon\}, & \text{if } t \in (-1,1) \\ \{\pm \hat{u}_1(q)\}, & \text{if } t = \pm 1. \end{cases}$$

Evidently $\hat{H}_\varepsilon(\cdot)$ has nonempty convex values. Moreover, for $t \in (-1,1)$, $\hat{H}_\varepsilon(t)$ is open, while $\hat{H}_\varepsilon(\pm 1)$ are singletons. In addition the continuity of $\hat{\gamma}(\cdot)$ implies the lower semicontinuity of the multifunction $\hat{H}_\varepsilon(\cdot)$ (see Proposition 2.6 of Hu & Papageorgiou [9, p. 37]). Therefore we can use Theorem 3.1‴ of Michael [13] and have a continuous map $\hat{\gamma}_\varepsilon : [-1,1] \mapsto C^1(\overline{\Omega})$ such that $\hat{\gamma}_\varepsilon(t) \in \hat{H}_\varepsilon(t)$ for all $t \in [-1,1]$.

Now let $\varepsilon_n = \frac{1}{n}$ and $\hat{\gamma}_n = \hat{\gamma}_{\varepsilon_n}$ $n \in \mathbb{N}$ as above. We have

$$\|\hat{\gamma}_n(t) - \hat{\gamma}(t)\| < \frac{1}{n} \text{ for all } t \in [-1,1]. \tag{29}$$

Recall that $\hat{\gamma}(t) \in \partial B_1^{L^q}$ for all $t \in [-1,1]$. So, from (29) we see that we may assume that $\hat{\gamma}_n(t) \neq 0$ for all $t \in [-1,1]$, all $n \in \mathbb{N}$. We set

$$\tilde{\gamma}_n(t) = \frac{\hat{\gamma}_n(t)}{\|\hat{\gamma}_n(t)\|_q} \text{ for all } t \in [-1,1], \text{ all } n \in \mathbb{N}.$$

We see that $\tilde{\gamma}_n \in C([-1,1], M_c)$, $\tilde{\gamma}_n(\pm 1) = \pm \hat{u}_1(q)$ for all $n \in \mathbb{N}$.

Also we have

$$\begin{aligned}\|\tilde{\gamma}_n(t)-\hat{\gamma}(t)\| &\leq \|\tilde{\gamma}_n(t)-\hat{\gamma}_n(t)\|+\|\hat{\gamma}_n(t)-\hat{\gamma}(t)\| \\ &\leq \frac{|1-\|\hat{\gamma}_n(t)\|_q|}{\|\hat{\gamma}_n(t)\|_q}\|\hat{\gamma}_n(t)\|+\frac{1}{n} \\ &\text{for all } t\in[-1,1], \text{ all } n\in\mathbb{N}.\end{aligned} \tag{30}$$

Note that

$$\begin{aligned}&\max_{-1\leq t\leq 1}|1-\|\hat{\gamma}_n(t)\|_q| \\ &= \max_{-1\leq t\leq 1}|\|\hat{\gamma}(t)\|_q-\|\hat{\gamma}_n(t)\|_q| \ \text{(recall that } \hat{\gamma}\in\hat{\Gamma}) \\ &\leq \max_{-1\leq t\leq 1}\|\hat{\gamma}(t)-\hat{\gamma}_n(t)\|_q \\ &\leq C_8\max_{-1\leq t\leq 1}\|\hat{\gamma}(t)-\hat{\gamma}_n(t)\| \text{ for some } C_8>0 \left(\text{ since } W^{1,p}(\Omega)\hookrightarrow L^q(\Omega)\right). \\ &\leq C_8\frac{1}{n} \text{ for all } n\in\mathbb{N} \text{ (see (29))}.\end{aligned}$$

Let $m^* = \max_{-1\leq t\leq 1}\|\hat{\gamma}(t)\|$ and $m_n^* = \max_{-1\leq t\leq 1}\|\hat{\gamma}_n(t)\|$. We know that

$$\begin{aligned}&\|\hat{\gamma}_n(t)\|\leq\frac{1}{n}+\|\hat{\gamma}(t)\| \\ &\text{for all } t\in[-1,1], \text{ all } n\in\mathbb{N} \text{ (see (29))}, \\ &\Rightarrow m_n^*\leq\frac{1}{n}+m^*, \\ &\Rightarrow \sup_{n\in\mathbb{N}} m_n^*\leq 1+m^*.\end{aligned}$$

Also we have $\|\hat{\gamma}(t)\|_q = 1$ (since $\hat{\gamma}\in\hat{\Gamma}$) and from (29) and since $W^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$, we have

$$\begin{aligned}&\|\hat{\gamma}_n(t)-\hat{\gamma}(t)\|_q\leq\frac{C_9}{n} \text{ for some } C_9>0, \text{ all } n\in\mathbb{N}, \\ &\Rightarrow 1\leq\frac{C_9}{n}+\|\hat{\gamma}_n(t)\|.\end{aligned}$$

So, if $m_*^n = \min_{-1\leq t\leq 1}\|\hat{\gamma}_n(t)\|$, then $1\leq\inf_{n\in\mathbb{N}} m_*^n$. Returning to (30), we have

$$\begin{aligned}&\|\tilde{\gamma}_n(t)-\hat{\gamma}(t)\|\leq\frac{1}{n}\left(C_8(1+m^*)+1\right), \\ &\Rightarrow \hat{\Gamma}_c \text{ is dense in } \Gamma.\end{aligned}$$

This proves the Claim.

Using the Claim and Proposition 2.1, we see that given $\eta \in (0, \vartheta)$, we can find $\hat{\gamma} \in \hat{\Gamma}_c$ such that

$$\|D\hat{\gamma}(t)\|_q^q \leq \hat{\lambda}_2(q) + \eta. \tag{31}$$

Hypothesis $H_1(iii)$ implies that we can find $\delta > 0$ such that

$$F(z, x) \geq \frac{\eta}{q}|x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{32}$$

The set $\hat{\gamma}([-1,1]) \subseteq M_c$ is compact. Recall that $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$. So, using Proposition 4.1.22 of [21, p. 274] we can find $\mu \in (0,1)$ small such that

$$\begin{cases} \mu\hat{\gamma}(t) \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}), & \text{for all } t \in [-1,1], \\ |\mu\hat{\gamma}(t)(z)| \leq \delta, & \text{for all } z \in \overline{\Omega}, \text{ all } t \in [-1,1]. \end{cases} \tag{33}$$

Let $u \in \mu\hat{\gamma}([-1,1])$. We have $u = \mu\hat{u}$ with $\hat{u} \in \gamma([-1,1])$. Then

$$\begin{aligned} \hat{\varphi}_\lambda(u) &\leq \frac{\mu^p}{p}k_p(\hat{u}) + \frac{\mu^q}{q}\left(\|D\hat{u}\|_q^q - \lambda\eta\|\hat{u}\|_q^q\right) \text{ (see (32), (33))} \\ &\leq \frac{\mu^p}{p}k_p(\hat{u}) + \frac{\mu^q}{q}\left(\hat{\lambda}_2(q) + \eta - \lambda\eta\right) \\ &\qquad \text{(see (31) and recall that } \|\hat{u}\|_q = 1). \end{aligned}$$

But $\lambda > \frac{\hat{\lambda}_2(q)}{\vartheta} + 1 \Rightarrow \vartheta(\lambda - 1) > \hat{\lambda_2}(q) \Rightarrow \eta(\lambda - 1) > \hat{\lambda}_2(q)$ for η near ϑ. Therefore we have

$$\hat{\varphi}_\lambda(u) \leq C_{10}\mu^p - C_{11}\mu^q \text{ for some } C_{10}, C_{11} > 0.$$

Since $q < p$, choosing $\mu \in (0,1)$ even smaller if necessary we have

$$\hat{\varphi}_\lambda(u) < 0 \text{ for all } u \in \mu\hat{\gamma}([-1,1]). \tag{34}$$

We set $\gamma_0 = \mu\hat{\gamma}$. Then γ_0 is a continuous path connecting $-\mu\hat{u}_1(q)$ and $\mu\hat{u}_1(q)$ and

$$\hat{\varphi}_\lambda\Big|_{\gamma_0} < 0 \text{ (see (34))}. \tag{35}$$

Next, we produce a continuous path connecting $\mu\hat{u}_1(q)$ and u_λ^* and along this path $\hat{\varphi}_\lambda$ is negative.

So, let $a = \hat{\varphi}_\lambda^+(u_\lambda^*) = \varphi_\lambda(u_\lambda^*)$, $b = 0 = \hat{\varphi}_\lambda^+(0) = \varphi_\lambda(0)$. Recall that $a < 0 = b$. Using Proposition 4.1, we have

$$K^a_{\hat{\varphi}_\lambda^+} = \{u_\lambda^*\},\ K^\circ_{\hat{\varphi}_\lambda^+} = \{0\} \text{ and } \hat{\varphi}_\lambda^+\left(K_{\hat{\varphi}_\lambda^+}\right) \cap (a, 0) = \emptyset.$$

Using the Second Deformation Theorem (see [21], Theorem 5.3.12, p. 386), we produce a deformation $\hat{h} : [0,1] \times \left((\hat{\varphi}_\lambda^+)^\circ \setminus \{0\} \to (\hat{\varphi}_\lambda^+)^\circ\right)$ such that

$$\hat{h}(0,u) = u \text{ for all } u \in (\hat{\varphi}_\lambda^+)^\circ \setminus \{0\} \tag{36}$$

$$\hat{h}(1,u) = u_\lambda^* \text{ for all } u \in (\hat{\varphi}_\lambda^+)^\circ \setminus \{0\} \tag{37}$$

$$\hat{h}(t,u_\lambda^*) = u_\lambda^* \text{ for all } t \in [0,1] \tag{38}$$

$$\hat{\varphi}_\lambda^+(\hat{h}(t,u)) \leq \hat{\varphi}_\lambda^+(\hat{h}(s,u)) \tag{39}$$

$$\text{for all } 0 \leq s \leq t \leq 1, \text{ all } u \in (\hat{\varphi}_\lambda^+)^\circ \setminus \{0\}$$

(recall from Section 2, that $(\hat{\varphi}_\lambda^+)^0 = \{u \in W^{1,p}(\Omega) : \ \hat{\varphi}_\lambda^+(u) \leq 0\}$).

From (36), (37), (38) we see that $K^a_{\hat{\varphi}_\lambda^+} = \{u_\lambda^*\}$ is a strong deformation retract of $(\hat{\varphi}_\lambda^+)^\circ \setminus \{0\} = (\hat{\varphi}_\lambda^+)^\circ \setminus K^\circ_{\hat{\varphi}_\lambda^+}$ and from (39) it follows that the deformation is $\hat{\varphi}_\lambda^+$-decreasing.

We set $\gamma_+(t) = \hat{h}(t, \mu\hat{u}_1(q))^+$ for all $0 \leq t \leq 1$. This is a continuous path in $W^{1,p}(\Omega)$ and since $\mu\hat{u}_1(q) \in (\hat{\varphi}_\lambda^+)^\circ \setminus \{0\}$ (see (35)), we have

$$\begin{aligned}
&\gamma_+(0) = \mu\hat{u}_1(q) \text{ (see (36))}, \ \gamma_+(1) = u_\lambda^* \text{ (see (37))} \\
&\hat{\varphi}_\lambda(\gamma_+(t)) = \hat{\varphi}_\lambda^+(\gamma_+(t)) \leq \hat{\varphi}_\lambda^+(\gamma_+(0)) = \hat{\varphi}_\lambda^+(\mu\hat{u}_1(q)) \\
&= \hat{\varphi}_\lambda(\mu\hat{u}_1(q)) < 0 \text{ for all } t \in [0,1] \text{ (see (35))}, \\
\Rightarrow \quad &\hat{\varphi}_\lambda\Big|_{\gamma_+} < 0.
\end{aligned} \tag{40}$$

In a similar fashion we produce another continuous path $\gamma_-(\cdot)$ in $W^{1,p}(\Omega)$ connecting $-\mu\hat{u}_1(q)$ and v_λ^* such that

$$\hat{\varphi}_\lambda\Big|_{\gamma_-} < 0. \tag{41}$$

We concatenate $\gamma_-, \gamma_0, \gamma_+$ and produce a path $\gamma_* \in \Gamma$ such that

$$\begin{aligned}
&\hat{\varphi}_\lambda\Big|_{\gamma_*} < 0 \text{ (see (35), (40), (41))}, \\
&\Rightarrow \hat{\varphi}_\lambda(y_\lambda) < 0 = \hat{\varphi}_\lambda(0) \text{ (see (28))}, \\
&\Rightarrow y_\lambda \neq 0.
\end{aligned}$$

Therefore $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$ is a nodal solution of problem (P_λ). □

So, summarizing, we can state the following multiplicity theorem for problem (P_λ). Note that we provide sign information for all solutions and the solutions are ordered.

Theorem 4.3. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1}$ *hold, then*

(a) *for all* $\lambda > 0$ *problem* (P_λ) *has constant sign solutions*

$$u_\lambda \in \operatorname{int} C_+ \text{ and } v_\lambda \in -\operatorname{int} C_+;$$

(b) *for all* $\lambda > \frac{\hat{\lambda}_2(q)}{\vartheta} + 1$ *problem* (P_λ) *has at least three nontrivial solutions*

$$u_\lambda \in \operatorname{int} C_+, v_\lambda \in -\operatorname{int} C_+, \ y_\lambda \in [v_\lambda, u_\lambda] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

If we introduce a symmetry hypothesis on $f(z, \cdot)$, we can have a whole sequence of nodal solutions converging to zero in $C^1(\overline{\Omega})$ and the result is valid for every parameter value $\lambda > 0$. We introduce the following stronger version of hypothesis $\mathbf{H_1}$.

$\mathbf{H_1'}$: for a.a. $z \in \Omega$, $f(z, \cdot)$ is odd, hypotheses $\mathbf{H_1}(i)$, (ii) hold and

(iii) $\lim_{x \to 0} \dfrac{f(z, x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$.

Proposition 4.4. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1'}$ *hold and* $\lambda > 0$, *then problem* (P_λ) *has a sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(\overline{\Omega})$ *of nodal solutions such that* $u_n \to 0$ *in* $C^1(\overline{\Omega})$.

Proof. From Proposition 3.4, we know that there exist extremal constant sign solutions

$$u_\lambda^* \in \operatorname{int} C_+ \text{ and } v_\lambda^* \in -\operatorname{int} C_+.$$

The energy functional φ_λ is even (see hypotheses $\mathbf{H_1'}$) and coercive, thus it is bounded below. Hypothesis $\mathbf{H_1'}(iii)$ implies that given any $\eta > 0$, we can find $\delta = \delta(\eta) > 0$ such that

$$F(z, x) \geq \frac{\eta}{q}|x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{42}$$

Let $V \subseteq W^{1,p}(\Omega)$ be a finite dimensional subspace. Then on V all norms are equivalent and so we can find $\rho_V \in (0, 1)$ such that

$$u \in V \text{ and } \|u\| \leq \rho_V \Rightarrow |u(z)| \leq \delta \text{ for a.a. } z \in \Omega. \tag{43}$$

If $u \in V$ with $\|u\| = \rho_V$, then using (42) and (43) we have

$$\begin{aligned} \varphi_\lambda(u) &\leq \frac{1}{p}k_p(u) + \frac{1}{q}\left(\|Du\|_q^q - \eta\|u\|_q^q\right) \\ &\leq C_{10}\rho_V^p + \frac{1}{q}\left(C_{11} - \eta C_V\right)\rho_V^q \end{aligned}$$

for some $C_{10}, C_{11}, C_V > 0$ (since all norms on V are equivalent).

Recall that $\eta > 0$ is arbitrary. So, we choose $\eta > \frac{C_{11}}{C_V}$ and have

$$\varphi_\lambda(u) \le C_{10}\rho_V^p - C_{12}\rho_V^q \text{ for some } C_{12} > 0.$$

Since $q < p$, choosing $\rho_V \in (0,1)$ small we have

$$\sup\{\varphi_\lambda(u) : \ u \in V, \|u\| = \rho_V\} < 0.$$

Then we can apply Theorem 1 of Kajikiya [10] and produce a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq K_{\varphi_\lambda}$ such that

$$\varphi_\lambda(u_n) \le 0 \text{ and } \|u_n\| \to 0. \tag{44}$$

The nonlinear regularity theory (see Lieberman [11]) implies that we can find $\alpha \in (0,1)$ and $C_{13} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le C_{13} \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (44), we have

$$\begin{aligned} & u_n \to 0 \text{ in } C^1(\overline{\Omega}), \\ \Rightarrow\ & u_n \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}) \text{ for all } n \ge n_0, \\ \Rightarrow\ & \{u_n\}_{n\ge n_0} \text{ is a sequence of nodal solutions of problem } (P_\lambda). \end{aligned}$$

This completes the proof. □

Using the same tools we can also treat the Dirichlet problem. So, now the problem under consideration is the following:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = 0, \ 1 < q < p, \ \lambda > 0. \end{cases} \tag{P'_λ}$$

We know that the q-Laplace differential operator with Dirichlet boundary condition, has a smallest eigenvalue $\hat{\lambda}_1(q) > 0$. Then Theorem 4.3 takes the following form.

Theorem 4.5. *If hypotheses* $\mathbf{H_1}$ *hold, then*

(a) for all $\lambda > \hat{\lambda}_1(q)$ *problem* (P'_λ) *has constant sign solutions*

$$u_\lambda \in \operatorname{int} C_+ \text{ and } v_\lambda \in -\operatorname{int} C_+;$$

(b) for all $\lambda > \frac{\hat{\lambda}_2(q)}{\vartheta} + 1$ *problem* (P'_λ) *has at least three nontrivial solutions*

$$u_\lambda \in \operatorname{int} C_+, v_\lambda \in -\operatorname{int} C_+ \text{ and } y_\lambda \in [v_\lambda, u_\lambda] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

Similarly Proposition 4.4 is also valid but with $\lambda > \hat{\lambda}_1(q)$.

Proposition 4.6. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_1'}$ *hold and* $\lambda > \hat{\lambda}_1(q)$, *then problem* (P'_λ) *has a sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(\overline{\Omega})$ *of nodal solutions such that* $u_n \to 0$ *in* $C^1(\overline{\Omega})$.

5. $(p, 2)$–equations

When $q = 2$ (that is, we deal with a $(p, 2)$-equation) and we strengthen the regularity of $f(z, \cdot)$, then we can produce a second nodal solution, for a total of four nontrivial smooth solutions all with sign information.

So, the Robin problem under consideration, is the following

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z)|u(z)|^{p-2}u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_{p2}} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, 1 < 2 < p, \lambda > 0. \end{cases} \tag{Q_λ}$$

Now the hypotheses of the reaction $f(z, x)$ are the following:

$\mathbf{H_2}$: $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \le a(z)\left(1 + |x|^{r-2}\right)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)$ and $p < r < p^*$;

(ii) $\limsup_{x\to\pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \le 0$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $m \in \mathbb{N}$, $m \ge 2$ such that

$$f'_x(z, 0) \in \left[\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right] \text{ for a.a. } z \in \Omega,$$
$$f'_x(\cdot, 0) \not\equiv \hat{\lambda}_m(2),\ f'_x(\cdot, 0) \not\equiv \hat{\lambda}_{m+1}(2).$$
$$f'_x(z, 0) = \lim_{x\to 0} \frac{f(z, x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

We introduce the functional $\hat{\tau}_\lambda : H^1(\Omega) \mapsto \mathbb{R}$ defined by

$$\hat{\tau}_\lambda(u) = \frac{1}{2}\|Du\|_2^2 - \lambda \int_\Omega F(z, u)dz \text{ for all } u \in H^1(\Omega).$$

Note that $\hat{\tau}_\lambda \in C^2(H^1(\Omega))$. We consider the functional

$$\tau_\lambda = \hat{\tau}_\lambda\Big|_{W^{1,p}(\Omega)} \text{ (recall that } 2 < p).$$

Proposition 5.1. *If hypotheses* $\mathbf{H_2}$ *hold, then* $C_k(\tau_\lambda, 0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$, *with* $d_m = \dim \overline{H}_m$.

Proof. As we already mentioned, $\hat{\tau}_k \in C^2(H^1(\Omega))$ and if by $\langle \cdot, \cdot \rangle_{H^1}$ we denote the duality brackets for the pair $\big(H^1(\Omega), H^1(\Omega)^*\big)$, we have

$$\langle \hat{\tau}_\lambda''(u)v, h\rangle_{H^1} = \int_\Omega (Dv, Dh)_{\mathbb{R}^N} dz - \lambda \int_\Omega f_x'(z,u)vhdz \quad \text{for all } u, v, h \in H^1(\Omega). \tag{45}$$

Suppose that $v \in N(\hat{\tau}_\lambda''(0)) = \ker(\hat{\tau}_\lambda''(0))$. We have the unique orthogonal decomposition $v = \overline{v} + \hat{v}$ with $\overline{v} \in \overline{H}_m$ and $\hat{v} \in \hat{H}_{m+1} = \overline{H}_m^{\perp}$. In (45) let $u = 0$, $v \in N(\hat{\tau}_\lambda''(0))$ and choose $h = \hat{v}$. Exploiting the orthogonality of $\overline{H}_m$ and $\hat{H}_{m+1}$ and hypothesis $\mathbf{H_2}(iii)$, we obtain

$$\|D\hat{v}\|_2^2 = \int_\Omega f_x'(z,0)\hat{v}^2 dz \le \hat{\lambda}_{m+1}(2)\|\hat{v}\|_2^2, \tag{46}$$
$$\Rightarrow \hat{v} \in E(\hat{\lambda}_{m+1}(2)) \text{ (see (2))}.$$

If $\hat{v} \neq 0$, then by the UCP (see de Figueiredo & Gossez [5]) we have that $\hat{v}(z) \neq 0$ for a.a. $z \in \Omega$ and so from (46) and hypothesis $\mathbf{H_2}(iii)$, we have

$$\|D\hat{v}\|_2^2 < \hat{\lambda}_{m+1}(2)\|\hat{v}\|_2^2,$$

a contradiction (see (2)). Hence $\hat{v} = 0$. Similarly, we show that $\overline{v} = 0$ and so finally $v = 0$. Therefore $u = 0$ is nondegenerate critical point of $\hat{\tau}_\lambda$ with Morse index $\hat{d}_m$ and so from Proposition 6.2.6 of [21, p. 479], we have

$$C_k(\hat{\tau}_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{47}$$

We know that $W^{1,p}(\Omega) \hookrightarrow H^1(\Omega)$ densely and so by Theorem 6.6.26 of [21, p. 545], we have

$$C_k(\tau_\lambda, 0) = C_k(\hat{\tau}_\lambda, 0) \text{ for all } k \in \mathbb{N}_0,$$
$$\Rightarrow C_k(\tau_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (47))}.$$

The proof is now complete. □

Using this proposition, we can have a second nodal solution.

Proposition 5.2. *If hypotheses $\mathbf{H_0}$, $\mathbf{H_2}$ hold and $\lambda > \frac{\hat{\lambda}_2(2)}{\hat{\lambda}_m(2)} + 1$, then problem (Q_λ) has at least two nodal solutions*

$$y_\lambda,\ \hat{y}_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*].$$

Proof. From Theorem 4.3 we already have a nodal solution

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}).$$

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by

$$a(y) = |y|^{p-2}y + y \text{ for all } y \in \mathbb{R}^N.$$

Since $p > 2$, we see that $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\nabla a(y) = |y|^{p-2}\left[id + (p-2)\frac{y \otimes y}{|y|^2}\right] + id \text{ for all } y \in \mathbb{R}^N \setminus \{0\}.$$

We have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq |\xi|^2 \text{ for all } y \in \mathbb{R}^N \setminus \{0\}, \xi \in \mathbb{R}^N.$$

Since $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$, using the tangency principle of Pucci & Serrin [24, p. 35], we have

$$v_\lambda^*(z) < y_\lambda(z) < u_\lambda^*(z) \text{ for all } z \in \Omega.$$

Consider the following open cone in $C^1(\overline{\Omega})$

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : \, u(z) > 0 \text{ for all } z \in \Omega, \, \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

From Proposition 3.2 of Gasiński & Papageorgiou [8], we have $u_\lambda^* - y_\lambda \in D_+$ and $y_\lambda - v_\lambda^* \in D_+$. Therefore

$$y_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]. \tag{48}$$

Using (48) and the standard homotopy invariance argument, we obtain

$$C_k(\varphi_\lambda, y_\lambda) = C_k(\hat{\varphi}_\lambda, y_\lambda) \text{ for all } k \in \mathbb{N}_0, \tag{49}$$

with $\varphi_\lambda(\cdot)$ and $\hat{\varphi}_\lambda(\cdot)$ as before, only now $q = 2$. Recall that y_λ is a critical point of mountain pass-type for $\hat{\varphi}_\lambda(\cdot)$, hence

$$C_1(\hat{\varphi}_\lambda, y_\lambda) \neq 0 \text{ (see [21, p. 527])}. \tag{50}$$

We assume that $K_{\hat{\varphi}_\lambda}$ is finite or otherwise we already have an infinity of nodal solutions and so we are done. Since now on account of hypotheses $\mathbf{H_2}$, $\varphi_\lambda \in C^2\left(W^{1,p}(\Omega)\right)$, as in Papageorgiou & Rădulescu [16] (p. 414, Claim 3), using (49) and (50), we have

$$\begin{aligned} & C_k(\varphi_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\hat{\varphi}_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (49))}. \end{aligned} \tag{51}$$

The C^1-continuity property of critical groups (see [21, p. 503]) implies that

$$\begin{aligned} & C_k(\varphi_\lambda, 0) = C_k(\tau_\lambda, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \\ & \quad \text{(see Proposition 5.1)}, \\ \Rightarrow\ & C_k(\hat{\varphi}_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (49))}. \end{aligned} \tag{52}$$

From the proof of Proposition 4.2, we know that u^*_λ and v^*_λ are local minimizers of $\hat{\varphi}_\lambda(\cdot)$. Hence

$$C_k(\hat{\varphi}_\lambda, u^*_\lambda) = C_k(\hat{\varphi}_\lambda, v^*_\lambda) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{53}$$

Recall that $\hat{\varphi}_\lambda(\cdot)$ is coercive (see (20)). Therefore

$$C_k(\hat{\varphi}_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see [21, p. 491])}. \tag{54}$$

Suppose $K_{\hat{\varphi}_\lambda} = \{y_\lambda, 0, u^*_\lambda, v^*_\lambda\}$. From (51), (52), (53), (54) and the Morse relation (see Section 2), with $t = -1$, we have

$$\begin{aligned} & (-1)^1 + (-1)^{d_m} + 2(-1)^0 = (-1)^0, \\ \Rightarrow\ & (-1)^{d_m} = 0, \text{ a contradiction.} \end{aligned}$$

So, there exists $\hat{y}_\lambda \in K_{\hat{\varphi}_\lambda}$, $\hat{y}_\lambda \notin \{y_\lambda, 0, u^*_\lambda, v^*_\lambda\}$. We have

$$\begin{aligned} & \hat{y}_\lambda \in [v^*_\lambda, u^*_\lambda] \cap C^1(\overline{\Omega}) \text{ (see Proposition 4.1)}, \\ \Rightarrow\ & \hat{y}_\lambda \in C^1(\overline{\Omega}) \text{ is a nodal solution of problem } (Q_\lambda). \end{aligned}$$

Moreover, as we did for y_λ, we show that

$$\hat{y}_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[v^*_\lambda, u^*_\lambda].$$

This completes the proof. □

So, for the problem (Q_λ) we can state the following multiplicity theorem.

Theorem 5.3. *If hypotheses* $\mathbf{H_0}$, $\mathbf{H_2}$ *hold and* $\lambda > \frac{\hat{\lambda}_2(2)}{\hat{\lambda}_m(2)} + 1$, *then problem* (Q_λ) *has at least four nontrivial solutions*

$$u_\lambda \in \operatorname{int} C_+,\ v_\lambda \in -\operatorname{int} C_+,$$
$$y_\lambda, \hat{y}_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[v_\lambda, u_\lambda] \ \textit{nodal.}$$

The same multiplicity theorem is also true for the Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = 0,\ 2 < p,\ \lambda > 0. \end{cases} \tag{Q'_λ}$$

Theorem 5.4. *If hypotheses* $\mathbf{H_2}$ *hold and* $\lambda > \frac{\hat{\lambda}_2(2)}{\hat{\lambda}_m(2)} + 1$, *then problem* (Q'_λ) *has at least four nontrivial solutions*

$$u_\lambda \in \operatorname{int} C_+,\ v_\lambda \in -\operatorname{int} C_+$$
$$y_\lambda, \hat{y}_\lambda \in \operatorname{int}_{C_0^1}(\overline{\Omega})[v_\lambda, u_\lambda] \ \textit{nodal.}$$

Remark 5.5. Another multiplicity theorem for $(p, 2)$-equations under different hypotheses can be found in [22].

Declaration of competing interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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