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NORMALIZED SOLUTIONS OF THE AUTONOMOUS KIRCHHOFF EQUATION WITH SOBOLEV CRITICAL EXPONENT: SUB- AND SUPER-CRITICAL CASES

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Abstract. In the present paper, we investigate the existence and multiplicity properties of the normalized solutions to the following Kirchhoff-type equation with Sobolev critical growth

$$(P)\qquad \begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+\lambda u=\mu|u|^{p-2}u+|u|^4u, & \text{in } \mathbb{R}^3,\\ u>0,\ \int_{\mathbb{R}^3}|u|^2dx=c^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $a,\ b,\ c,\ \mu>0$ and $4<p<6$. We consider both the L^2-subcritical and the L^2-supercritical cases. Precisely, in the L^2-subcritical case, by combining the truncation method, the concentration-compactness principle and genus theory, we obtain the multiplicity of the normalized solutions for problem (P). In the L^2-supercritical case, by using a fiber map and the concentration-compactness principle, we obtain a couple of normalized solutions for problem (P), as well as their asymptotic behavior. These results extend and complement the existing results from Sobolev subcritical growth to the critical Sobolev setting.

1. Introduction and main results

In the past years, the following nonlinear Kirchhoff-type equations

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+\lambda u=f(u),\quad x\ \in\mathbb{R}^3 \tag{1.1}$$

have attracted considerable attention, where $a,\ b>0$ and $\lambda\in\mathbb{R}$. The appearance of the nonlocal term $\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u$ causes several mathematical difficulties that make the study of (1.1) particularly interesting. Furthermore, the interest

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of studying Kirchhoff-type equations comes from the physical background of equations. Usually, u denotes the displacement, $f(u)$ is the external force, b is the initial tension, while a is related to the intrinsic properties of the string, such as Young' modulus.

The Kirchhoff equation was introduced by Kirchhoff [11] in 1883 in the one-dimensional case. This first model is without forcing term and with Dirichlet boundary conditions and it describes the transversal free vibrations of a clamped string in which the dependence of the tension on the deformation cannot be neglected. This is a quasilinear partial differential equation, namely the nonlinear part of the equation contains as many derivatives as the linear differential operator. The Kirchhoff equation is an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. We refer to Arosio and Panizzi [3] and D'Ancona and Spagnolo [6] for more details.

At present, there are two substantially different view points in terms of the frequency λ in problem (1.1). One is to regard the frequency λ as a given constant. In this situation, solutions of Eq. (1.1) are critical points of the corresponding action functional on the working space. We point out that the existence, multiplicity and concentration of solutions for (1.1) involving subcritical, critical and supercritical exponents have been extensively studied under different assumptions about the nonlinearity f, see [9] and their references therein. The other one is to regard the frequency λ as an unknown quantity to the problem (1.1). In this situation, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier. Nowadays, some physicists are very interested in the solutions satisfying

$$\int_{\mathbb{R}^3} |u|^2 dx = c^2 > 0$$

for a priori given c, since the mass admits a clear physical meaning. For example, from a physical point of view, the mass $\|u\|_2^2$ may represent the number of particles of each component in Bose-Einstein condensates or the power supply in the nonlinear optics framework. In addition, such solutions can give a better insight of the dynamical properties, like orbital stability or instability, and can describe attractive Bose-Einstein condensates. This type of solutions is usually called prescribed L^2-norm solutions or normalized solutions in mathematics, and the above condition is called normalized condition. In order to study the solution of Eq. (1.1) satisfying the normalized condition $\int_{\mathbb{R}^3} |u|^2 dx = c^2$, it suffices to consider the critical point of the functional

$$E(u) = \frac{1}{2} a \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^2 - \int_{\mathbb{R}^3} F(u) dx$$

on the constrained manifold

$$S(c) := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = c^2 \right\}.$$

In recent years, Guo *et al.* [8] studied the existence and blow-up behavior of the normalized solutions for the equation

$$-\Big(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\Big) \Delta u + V(x) u = \mu |u|^{p-2} u + \lambda u, \tag{1.2}$$

where $1 \leq N \leq 4$ and $2 < p < 2^*$. Ye [17] pointed out that $2 + \frac{8}{N}$ is the L^2-critical exponent for (1.2), and she studied in [18] the existence of normalized solutions to (1.2) with $p = 2 + \frac{8}{N}$ and $V \equiv 0$. Zeng and Zhang [20] obtained the existence and uniqueness of normalized solutions to the equation (1.2) with $V \equiv 0$, $\mu = 1$ and $2 < p < 2^*$. Li and Ye [15] studied the existence and concentration phenomenon of the normalized solutions to (1.2), and they also assumed that $2 < p < 2^*$. Li, Hao and Shi [13] also obtained the existence of normalized solutions to (1.2) with $N = 4$ by fine calculations. By using a minimax procedure, Luo and Wang [14] studied the multiplicity of normalized solutions to (1.2) with $V \equiv 0$, $\mu = 1$ and $\frac{14}{3} < p < 6$. Chen *et al.* [5] established the existence of the normalized solutions for Kirchhoff-type equation

$$-\big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\big)\Delta u - \lambda u = K(x)f(u),$$

where $f \in C(\mathbb{R}, \mathbb{R})$ satisfies general L^2-supercritical or L^2-subcritical condition.

From the commentaries above, the existing work is mainly focused on the existence of normalized solutions for Kirchhoff-type equations with Sobolev subcritical growth. But there is very few result about Sobolev critical case. Motivated by the works aforementioned and [1, 2], where they all considered the local semilinear equations, we study the normalized solutions to the nonlocal and Sobolev critical problem (P).

Set

$$H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} [a\nabla u \cdot \nabla v + uv]dx$$

and the norm

$$\|u\| = \langle u, u\rangle^{1/2}.$$

We are now in a position to state the first two main results of this paper. Essentially, these results establish the existence of normalized solutions for high perturbations of the forcing term. Our analysis covers both the L^2-subcritical and the L^2-supercritical cases. In the first setting, following some arguments in [7], we obtain the multiplicity of the normalized solutions, while in the L^2-supercritical case we find a couple of normalized solutions.

Theorem 1.1. *If $4 < p < \frac{14}{3}$, for given $k \in \mathbb{N}$, there exist $\alpha > 0$ independent of k and $\mu_k := \mu(k)$ such that problem (P) possesses at least k couples $(u_j, \lambda_j) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions for $\mu \geq \mu_k$ and $c \in \big(0, (\frac{\alpha}{\mu})^{\frac{2}{6-p}}\big]$ with $\int_{\mathbb{R}^3} |u_j|^2 dx = c^2$, $\lambda_j > 0$ for all $j \in [1, k]$.*

Theorem 1.2. *If $\frac{14}{3} < p < 6$, there exists $\mu^* = \mu^*(c) > 0$ such that as $\mu \geq \mu^*$, problem (P) possesses a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions with $\int_{\mathbb{R}^3} |u_c|^2 dx = c^2$ and $\lambda_c > 0$.*

Remark 1.3. It is well known that problem (P) on the whole space $\mathbb{R}^3$ is invariant under translations, which leads to the lack of compactness. In order to overcome it, we can take the space $H^1_{rad}(\mathbb{R}^3)$ of radial functions as the working space, where

$$H^1_{rad}(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : \ u \text{ is radial}\},$$

and by Proposition 1.7.1 in [4] we know that $H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^t(\mathbb{R}^3)$ is compact for any $2 < t < 6$. To be precise, we will consider the functional $I : H^1_{rad}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}a\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2 - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx,$$

restricted to the following sphere in $L^2(\mathbb{R}^3)$

$$S(c) := \{u \in H^1_{rad}(\mathbb{R}^3) : \|u\|_2 = c\}.$$

Moreover, we give the asymptotic behavior of solution u_c as $c \to +\infty$. Our main result in this aspect is Theorem 1.4.

Theorem 1.4. *If u_c is the solution obtained in Theorem 1.2, then* $\lim_{c\to+\infty} I(u_c) = 0$.

Remark 1.5. The Sobolev critical exponent also leads to the lack of compactness. Even the embedding of the radially symmetric space of $H^1_{rad}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$ is not compact. Furthermore, $H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is also not compact. Then, the weak limit of Palais-Smale sequences could leave the constrained manifold $S(c)$. Hence, we need to estimate finely the Lagrange multiplier, which is vital in obtaining compactness. With the aid of the concentration-compactness principle, we overcome the difficulty.

Remark 1.6. No matter $4 < p < \frac{14}{3}$ or $\frac{14}{3} < p < 6$, $I(u)$ on the constrained manifold $S(c)$ is all unbounded from below. Hence, it is unlikely to obtain a solution to problem (P) by minimizing method. We adopt some ideas from [1, 2] to overcome the difficulty.

Remark 1.7. To the best of our knowledge, the main results in this paper are new. They extend the main results in the above mentioned references except [19] from Sobolev subcritical growth to Sobolev critical growth. When $\frac{14}{3} < p < 6$, Zhang and Han [19] considered the existence of normalized solutions for problem (P) by calculating the threshold of the mountain pass level. Our approach is different and easier than their approach. Alternatively, we also consider the L^2-subcritical case. We point out that $p > 4$ only ensures the corresponding Lagrange multiplier is negative, see the proof of Lemma 2.1(iii).

2. The subcritical case

We first recall the definition of genus. Let X be a Banach space and A be a subset of X. The set A is said to be symmetric if $u \in A$ implies that $-u \in A$. Denote by Σ the family of closed symmetric subsets A of X such that $0 \notin A$, i.e.,

$$\Sigma = \{A \subset X\backslash\{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, define

$$\gamma(A) = \begin{cases} 0, & \text{if } A = \varnothing, \\ \inf\Big\{k \in \mathbb{N} : \exists \text{ an odd } \varphi \in C\big(A, \mathbb{R}^k\backslash\{0\}\big)\Big\}, \\ +\infty, & \text{if no such an odd map,} \end{cases}$$

and $\Sigma_k = \{A \in \Sigma : \gamma(A) \geq k\}$.

For $u \in S(c)$, by the Gagliardo-Nirenberg inequality and the Sobolev embedding theorem we have

$$\begin{aligned} I(u) =& \frac{1}{2}a\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^pdx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx \\ \geq& \frac{1}{2}a\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{\mu}{p}C_pc^{\frac{6-p}{2}}\|\nabla u\|_2^{\frac{3(p-2)}{2}} - \frac{1}{6S^3}\|\nabla u\|_2^6 \\ :=& M(\|\nabla u\|_2), \end{aligned}$$

where

$$M(t) = \frac{1}{2}at^2 + \frac{b}{4}t^4 - \frac{\mu}{p}C_pc^{\frac{6-p}{2}}t^{\frac{3(p-2)}{2}} - \frac{1}{6S^3}t^6.$$

Since $4 < p < \frac{14}{3}$, we obtain $\frac{3(p-2)}{2} < 4 < 6$, and there exists $\alpha > 0$ such that as $\mu c^{\frac{6-p}{2}} \leq \alpha$, the function $M(\cdot)$ attains its positive local maximum. More precisely, there exist two constants $0 < R_1 < R_2 < +\infty$ such that $M(\cdot) < 0$ in the intervals $(0, R_1)$ and $(R_2, +\infty)$, and $M(\cdot) > 0$ in the interval (R_1, R_2). Let $\tau(\cdot) \in C^\infty(\mathbb{R}^+, [0,1])$ be a nonincreasing function such that $\tau(t) = 1$ for $t \leq R_1$ and $\tau(t) = 0$ for $t \geq R_2$.

2.1. **Proof of Theorem 1.1.** Define the truncated functional

$$I_\tau(u) = \frac{1}{2}a\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^pdx - \frac{\tau(\|\nabla u\|_2)}{6}\int_{\mathbb{R}^3}|u|^6dx.$$

For $u \in S(c)$, again by the Gagliardo-Nirenberg inequality and the Sobolev embedding theorem one has

$$\begin{aligned} I_\tau(u) \geq& \frac{1}{2}a\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{\mu}{p}C_pc^{\frac{6-p}{2}}\|\nabla u\|_2^{\frac{3(p-2)}{2}} \\ &- \frac{\tau(\|\nabla u\|_2)}{6S^3}\|\nabla u\|_2^6 \\ :=& \widetilde{M}(\|\nabla u\|_2), \end{aligned}$$

where

$$\widetilde{M}(t) = \frac{1}{2}at^2 + \frac{b}{4}t^4 - \frac{\mu}{p}C_pc^{\frac{6-p}{2}}t^{\frac{3(p-2)}{2}} - \frac{\tau(t)}{6S^3}t^6.$$

Then the definition of $\tau(\cdot)$ implies that when $c \in \Big(0, (\frac{\alpha}{\mu})^{\frac{2}{6-p}}\Big]$, $\widetilde{M}(\cdot) < 0$ in the interval $(0, R_1)$ and $\widetilde{M}(\cdot) > 0$ in the interval $(R_1, +\infty)$. In the following, we always assume $c \in \Big(0, (\frac{\alpha}{\mu})^{\frac{2}{6-p}}\Big]$. Without loss of generality, we may assume that

$$\frac{1}{2}ar^2 + \frac{b}{4}r^4 - \frac{1}{6S^3}r^6 \geq 0 \text{ for } r \in [0, R_1] \text{ and } R_1^2 < a^{\frac{1}{2}}S^{\frac{3}{2}}. \tag{2.1}$$

Lemma 2.1.

(i) $I_\tau \in C^1(H^1_{rad}(\mathbb{R}^3), \mathbb{R})$.

(ii) *I_τ is coercive and bounded from below on $S(c)$. Moreover, if $I_\tau \leq 0$, then $\|\nabla u\|_2 \leq R_1$ and $I_\tau(u) = I(u)$.*

(iii) *$I_\tau|_{S(c)}$ satisfies the $(PS)_d$ condition for all $d < 0$.*

Proof. The proofs of (i) and (ii) are easy. For (iii), let $\{u_n\}$ be a $(PS)_d$ sequence of $I_\tau|_{S(c)}$ with $d < 0$, i.e., $I_\tau(u_n) \to d < 0$ and $\|I_\tau|'_{S(c)}(u_n)\| \to 0$ as $n \to \infty$. By (ii), $\|\nabla u_n\|_2 \leq R_1$ for large n, and $\{u_n\}$ is also a $(PS)_d$ sequence of $I|_{S(c)}$ with $d < 0$. Then, $\{u_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^3)$. Hence, up to a subsequence, there exists $u \in H^1_{rad}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1_{rad}(\mathbb{R}^3)$ and $u_n \to u$ in $L^t(\mathbb{R}^3)$ for $2 < t < 6$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$. Since $4 < p < \frac{14}{3}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^p dx = \int_{\mathbb{R}^3} |u|^p dx.$$

We assert that $u \not\equiv 0$. Otherwise, $\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^p dx = 0$, and whence by (2.1) we have

$$\begin{aligned} 0 > d &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty} \Big[\frac{1}{2}a \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{b}{4}\big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\big)^2 - \frac{\mu}{p}\int_{\mathbb{R}^3} |u_n|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx\Big] \\ &\geq \lim_{n\to\infty} \Big[\frac{1}{2}a\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{\mu}{p}\int_{\mathbb{R}^3} |u_n|^p dx - \frac{1}{6S^3}\|\nabla u_n\|_2^6\Big] \\ &\geq -\frac{\mu}{p}\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^p dx = 0, \end{aligned}$$

a contradiction. On the other hand, let $\Phi(v) := \frac{1}{2}\int_{\mathbb{R}^3} |v|^2 dx$, $\forall v \in H^1(\mathbb{R}^3)$, then $S(c) = \Phi^{-1}(\{\frac{c^2}{2}\})$. By Proposition 5.12 in [16], there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that $\|I'(u_n) - \lambda_n \Phi'(u_n)\| \to 0$ as $n \to \infty$, which means that

$$-\big(a+b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\big)\Delta u_n - \mu|u_n|^{p-2}u_n - |u_n|^4 u_n = \lambda_n u_n + o(1) \text{ in } H^{-1}_{rad}(\mathbb{R}^3). \tag{2.2}$$

Therefore, for $\varphi \in H^1_{rad}(\mathbb{R}^3)$,

$$\begin{aligned} &\big(a + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\big)\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla\varphi dx - \mu\int_{\mathbb{R}^3} |u_n|^{p-2}u_n\varphi dx - \int_{\mathbb{R}^3} |u_n|^4 u_n \varphi dx \\ =&\lambda_n \int_{\mathbb{R}^3} u_n\varphi dx + o(1)\|\varphi\|. \end{aligned} \tag{2.3}$$

Especially,

$$\big(a + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\big)\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \mu\int_{\mathbb{R}^3} |u_n|^p dx - \int_{\mathbb{R}^3} |u_n|^6 dx = \lambda_n c^2 + o(1).$$

The boundedness of $\{\|u_n\|\}$ yields that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Then, up to a subsequence, there exists $\lambda_c \in \mathbb{R}$ such that $\lambda_n \to \lambda_c$ as $n \to \infty$. We claim that $\lambda_c < 0$. Indeed, by the above identity and $I(u_n) \to d < 0$ as $n \to \infty$ and $p > 4$ we know that

$$\lambda_n c^2 + o(1) < 4d + \frac{4-p}{p}\mu\int_{\mathbb{R}^3} |u_n|^p dx - \frac{1}{3}\int_{\mathbb{R}^3} |u_n|^6 dx + o(1) \leq 4d + o(1).$$

Let $n \to \infty$, we have $\lambda_c < 0$.

In the sequel, we shall prove $u_n \to u$ in $L^6(\mathbb{R}^3)$ by using the concentration-compactness principle due to Lions [12]. In fact, there exist two positive measures $\mu,\ \nu \in \mathcal{M}(\mathbb{R}^3)$ such that $|\nabla u_n|^2 \rightharpoonup \mu$ and $|u_n|^6 \rightharpoonup \nu$ in $\mathcal{M}(\mathbb{R}^3)$ as $n \to \infty$, and for an at most countable index set J, we have

$$\begin{cases} \nu = |u|^6 + \sum_{j\in J} \nu_j \delta_{x_j}, & \nu_j > 0, \\ \mu \geq |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, & \mu_j > 0, \\ S\nu_j^{\frac{1}{3}} \leq \mu_j, & \forall j \in J. \end{cases}$$

Suppose that J is nonempty but finite. Let $\varphi_\rho \in C^\infty(\mathbb{R}^3)$ be a cut-off function with $\varphi_\rho(x) = 1$ for $|x - x_j| \leq \frac{1}{2}\rho$, $\varphi_\rho(x) = 0$ for $|x - x_j| \geq \rho$ and $|\nabla\varphi_\rho| \leq \frac{1}{\sqrt{\rho}}$. By the boundedness of $\{u_n\}$ in $H^1_{rad}(\mathbb{R}^3)$ we know that $\{\varphi_\rho u_n\}$ is also bounded in $H^1_{rad}(\mathbb{R}^3)$. Therefore,

$$\begin{aligned}
o(1) =&\langle I'(u_n), \varphi_\rho u_n\rangle \\
=&\big(a + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\big) \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(\varphi_\rho u_n)dx - \mu\int_{\mathbb{R}^3} \varphi_\rho|u_n|^p dx - \int_{\mathbb{R}^3} \varphi_\rho|u_n|^6 dx.
\end{aligned}$$

Clearly,

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(\varphi_\rho u_n)dx = \int_{\mathbb{R}^3} |\nabla u_n|^2\varphi_\rho dx + \int_{\mathbb{R}^3} u_n\nabla u_n \cdot \nabla\varphi_\rho dx.$$

By Hölder inequality we derive that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} u_n\nabla u_n \cdot \nabla\varphi_\rho dx\Big| \\
\leq&\Big(\int_{\mathbb{R}^3} u_n^2|\nabla\varphi_\rho|^2 dx\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^{\frac{1}{2}} \\
\leq&C\Big(\int_{\mathbb{R}^3} u_n^2|\nabla\varphi_\rho|^2 dx\Big)^{\frac{1}{2}} = C\Big(\int_{B_\rho(x_j)\setminus B_{\frac{\rho}{2}}(x_j)} u_n^2|\nabla\varphi_\rho|^2 dx\Big)^{\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

Noting that

$$\begin{aligned}
&\Big|\int_{B_\rho(x_j)\setminus B_{\frac{\rho}{2}}(x_j)} u_n^2|\nabla\varphi_\rho|^2 dx - \int_{B_\rho(x_j)\setminus B_{\frac{\rho}{2}}(x_j)} u^2|\nabla\varphi_\rho|^2 dx\Big| \\
\leq&\frac{1}{\rho}\int_{B_\rho(x_j)\setminus B_{\frac{\rho}{2}}(x_j)} |u_n + u||u_n - u|dx \\
\leq&\frac{1}{\rho}\Big(\int_{\mathbb{R}^3} |u_n - u|^p dx\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^3} |u_n + u|^6 dx\Big)^{\frac{1}{6}}\Big(\int_{B_\rho(x_j)} dx\Big)^{\frac{5}{6}-\frac{1}{p}} \\
\leq&C\rho^{3(\frac{5}{6}-\frac{1}{p})-1}\Big(\int_{\mathbb{R}^3} |u_n - u|^p dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

By the absolute continuity of the Lebesgue integral, it implies that

$$\begin{aligned}
\lim_{\rho\to 0}\lim_{n\to\infty}\int_{B_\rho(x_j)\setminus B_{\frac{\rho}{2}}(x_j)} u_n^2|\nabla\varphi_\rho|^2 dx =& \lim_{\rho\to 0}\int_{B_\rho(x_j)\setminus B_{\frac{\rho}{2}}(x_j)} u^2|\nabla\varphi_\rho|^2 dx \\
\leq& \lim_{\rho\to 0}\Big[\frac{1}{\rho}\big(\int_{B_\rho(x_j)} |u|^6 dx\big)^{\frac{1}{3}}\big(\int_{B_\rho(x_j)} dx\big)^{\frac{2}{3}}\Big] = 0.
\end{aligned}$$

By (2.4), $\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} u_n\nabla u_n \cdot \nabla\varphi_\rho dx = 0$. Again by the absolute continuity of the Lebesgue integral we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n|^p\varphi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3} |u|^p\varphi_\rho dx = 0.$$

All together with the above estimates, we get that

$$\begin{aligned}
&\lim_{\rho\to 0}\Big[\int_{\mathbb{R}^3}|u|^6\varphi_\rho dx+\sum_{j\in J}\int_{\mathbb{R}^3}\nu_j\delta_{x_j}\varphi_\rho dx\Big]\\
=&\lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\nu=\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6\varphi_\rho dx\\
=&\lim_{\rho\to 0}\lim_{n\to\infty}\Big(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho dx\\
\geq& a\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho dx=a\lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\mu\\
\geq& a\lim_{\rho\to 0}\Big[\int_{\mathbb{R}^3}|\nabla u|^2\varphi_\rho dx+\sum_{j\in J}\int_{\mathbb{R}^3}\mu_j\delta_{x_j}\varphi_\rho dx\Big].
\end{aligned}$$

Hence, $\nu_j\geq a\mu_j$. Then, $\mu_j\geq S\nu_j^{\frac{1}{3}}\geq S(a\mu_j)^{\frac{1}{3}}$, i.e., $\mu_j\geq a^{\frac{1}{2}}S^{\frac{3}{2}}$, so

$$\begin{aligned}
R_1^2\geq&\limsup_{n\to\infty}\|\nabla u_n\|_2^2\geq\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho dx=\int_{\mathbb{R}^3}\varphi_\rho d\mu\\
\geq&\int_{\mathbb{R}^3}|\nabla u|^2\varphi_\rho dx+\sum_{j\in J}\int_{\mathbb{R}^3}\mu_j\delta_{x_j}\varphi_\rho dx\geq\mu_j\geq a^{\frac{1}{2}}S^{\frac{3}{2}},
\end{aligned}$$

a contradiction with (2.1). Hence, $J=\emptyset$. Then,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6\varphi_\rho dx=\int_{\mathbb{R}^3}\varphi_\rho d\nu=\int_{\mathbb{R}^3}|u|^6\varphi_\rho dx,$$

so $u_n\to u$ in $L^6_{loc}(\mathbb{R}^3)$.

On the other hand, since $u_n\in H^1_{rad}(\mathbb{R}^3)$ and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, $|u_n(x)|\leq\frac{\|u_n\|}{|x|}\leq\frac{C}{|x|}$ for every $|x|\geq 1$. So $|u_n(x)|^6\leq\frac{C}{|x|^6}$ for every $|x|\geq 1$. Since $\frac{1}{|\cdot|^6}\in L^1(\mathbb{R}^3\backslash B_R(0)$ and $u_n(x)\to u(x)$ a.e. on $\mathbb{R}^3\backslash B_R(0)$, by the Lebesgue dominated convergence theorem, $u_n\to u$ in $L^6(\mathbb{R}^3\backslash B_R(0))$.

Consequently, $u_n\to u$ in $L^6(\mathbb{R}^3)$. Along a subsequence if necessary, set $B:=\lim_{n\to\infty}\|\nabla u_n\|_2^2\geq 0$. Then $0<\|\nabla u\|_2^2\leq B$. By (2.3),

$$\Big(a+bB\Big)\int_{\mathbb{R}^3}\nabla u\cdot\nabla\varphi dx-\mu\int_{\mathbb{R}^3}|u|^{p-2}u\varphi dx-\int_{\mathbb{R}^3}|u|^4u\varphi dx=\lambda_c\int_{\mathbb{R}^3}u\varphi dx.$$

Consequently,

$$\begin{aligned}
&\lim_{n\to\infty}\Big[\Big(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}|\nabla u_n|^2dx-\lambda_n\|u_n\|_2^2\Big]\\
=&\lim_{n\to\infty}\Big[\mu\|u_n\|_p^p+\|u_n\|_6^6\Big]\\
=&\mu\|u\|_p^p+\|u\|_6^6=\Big(a+bB\Big)\int_{\mathbb{R}^3}|\nabla u|^2dx-\lambda_c\|u\|_2^2.
\end{aligned}$$

By $\lambda_c<0$ we deduce that

$$\lim_{n\to\infty}-\lambda_c\|u_n\|_2^2=-\lambda_c\|u\|_2^2,\quad a\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2dx=a\int_{\mathbb{R}^3}|\nabla u|^2dx.$$

Hence, $u_n\to u$ in $H^1_{rad}(\mathbb{R}^3)$ and $\|u\|_2=c$. □

For $\varepsilon > 0$, set

$$I_\tau^{-\varepsilon} = \{u \in H^1_{rad}(\mathbb{R}^3) \cap S(c) : I_\tau(u) \leq -\varepsilon\} \subset H^1_{rad}(\mathbb{R}^3).$$

By the fact that I_τ is even and continuous on $H^1_{rad}(\mathbb{R}^3)$, $I_\tau^{-\varepsilon}$ is closed and symmetric. Then, we have Lemma 2.2, whose proof is similar to the Lemma 3.2 in [1].

Lemma 2.2. *Given $n \in \mathbb{N}$, there exist $\varepsilon_n := \varepsilon(n) > 0$ and $\mu_n := \mu(n) > 0$ such that $\gamma(I_\tau^{-\varepsilon}) \geq n$ for $0 < \varepsilon \leq \varepsilon_n$ and $\mu \geq \mu_n$.*

Set

$$\Sigma_k := \{D \subset H^1_{rad}(\mathbb{R}^3) \cap S(c) : D \text{ is closed and symmetric, } \gamma(D) \geq k\},$$

and $d_k := \inf_{D \in \Sigma_k} \sup_{u \in D} I_\tau(u) > -\infty$ for all $k \in \mathbb{N}$ by Lemma 2.1 (ii) and

$$K_d := \{u \in H^1_{rad}(\mathbb{R}^3) \cap S(c) : I'_\tau(u) = 0, I_\tau(u) = d\}.$$

Then, the following lemmas holds.

Lemma 2.3. *If $d = d_k = d_{k+1} = \cdots = d_{k+r}$, then $\gamma(K_d) \geq r + 1$. In particular, I_τ possesses at least $r + 1$ nontrivial critical points.*

Proof. For $\varepsilon > 0$, it is easy to see that $I_\tau^{-\varepsilon} \in \Sigma$. For any $k \in \mathbb{N}$, by the previous lemma there exist $\varepsilon_k = \varepsilon(k) > 0$ and $\mu_k = \mu(k) > 0$ such that if $0 < \varepsilon \leq \varepsilon_k$ and $\mu \geq \mu_k$, we get $\gamma(I_\tau^{-\varepsilon}) \geq k$. Then $I_\tau^{-\varepsilon_k} \in \Sigma_k$, and $d_k \leq \sup_{u \in I_\tau^{-\varepsilon_k}} I_\tau(u) = -\varepsilon_k < 0$. Suppose that $0 > d = d_k = d_{k+1} = \cdots = d_{k+r}$, then Lemma 2.1 (iii) implies that I_τ satisfies the $(PS)_d$ condition. Hence, K_d is a compact set. By [1, Theorem 2.1], $I_\tau|_{S(c)}$ possesses at least $r + 1$ critical points. □

Proof of Theorem 1.1. By Lemma 2.1 (ii), the critical points of I_τ founded in Lemma 2.3 are the critical points of I. So Theorem 1.1 is proved. □

3. The supercritical case

In this section, we consider the case $\frac{14}{3} < p < 6$. At this time, $\frac{3(p-2)}{2} > 4$. It follows that the truncated functional I_τ in Section 2 is still unbounded from below on $S(c)$. Therefore, we cannot use the truncation technique from Section 2 to study problem (P) when $\frac{14}{3} < p < 6$. Inspired by [2], for convenience, we set $f(t) = \mu|t|^{p-2}t + |t|^4 t$ for all $t \in \mathbb{R}$, and introduce the following auxiliary functional

$$\widetilde{I} : S(c) \times \mathbb{R} \to \mathbb{R}, \quad (u, \tau) \mapsto I(\tau * u),$$

where $(\tau * u)(x) := e^{\frac{3}{2}\tau} u(e^\tau x)$. Then

$$\int_{\mathbb{R}^3} |\nabla(\tau * u)|^2 dx = e^{2\tau} \int_{\mathbb{R}^3} |\nabla u|^2 dx$$

and

$$\int_{\mathbb{R}^3} |\tau * u|^q dx = e^{\frac{q-2}{2} 3\tau} \int_{\mathbb{R}^3} |u|^q dx, \quad \forall q \in [2, 6].$$

Then

$$
\begin{aligned}
\widetilde{I}(u,\tau) =& I(\tau * u) = I\big(e^{\frac{3}{2}\tau}u(e^{\tau}x)\big)\\
=&\frac{1}{2}a\int_{\mathbb{R}^3}\big|\nabla(\tau * u)\big|^2dx+\frac{b}{4}\Big(\int_{\mathbb{R}^3}\big|\nabla(\tau * u)\big|^2dx\Big)^2-\int_{\mathbb{R}^3}F(\tau * u)dx\\
=&\frac{1}{2}ae^{2\tau}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{b}{4}e^{4\tau}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2-e^{-3\tau}\int_{\mathbb{R}^3}F\Big(e^{\frac{3\tau}{2}}u(x)\Big)dx\\
=&\frac{1}{2}ae^{2\tau}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{b}{4}e^{4\tau}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2-\frac{\mu}{p}\cdot e^{\frac{3(p-2)}{2}\tau}\int_{\mathbb{R}^3}|u|^pdx\\
&-\frac{1}{6}\cdot e^{6\tau}\int_{\mathbb{R}^3}|u|^6dx.
\end{aligned}
$$

Clearly, the above estimates imply the Lemma 3.1.

Lemma 3.1 ([19, Lemma 3.1])**.** *Let $u\in S(c)$ be arbitrary but fixed. Then*

(i) $\int_{\mathbb{R}^3}\big|\nabla(\tau * u)\big|^2dx\to 0$ *and* $\widetilde{I}(u,\tau)\to 0$ *as* $\tau\to-\infty$.

(ii) $\int_{\mathbb{R}^3}\big|\nabla(\tau * u)\big|^2dx\to +\infty$ *and* $\widetilde{I}(u,\tau)\to -\infty$ *as* $\tau\to+\infty$.

With the aid of the Gagliardo-Nirenberg inequality we can obtain the next lemma.

Lemma 3.2 ([19, Lemma 3.2])**.** *There exists $K(c)>0$ sufficiently small such that*

$$
I(u)>0 \quad \textit{for} \quad u\in A \quad \textit{and} \quad 0<\sup_{u\in A}I(u)<\inf_{u\in B}I(u),
$$

where

$$
A:=\{u\in S(c):\int_{\mathbb{R}^3}|\nabla u|^2dx\le K(c)\}
$$

and

$$
B:=\{u\in S(c):\int_{\mathbb{R}^3}|\nabla u|^2dx=2K(c)\}.
$$

As a consequence of Lemmas 3.1 and 3.2, we see that for fixed $u_0\in S(c)$, there exist two constants τ_1, τ_2 satisfying $\tau_1<0<\tau_2$ such that

$$
\int_{\mathbb{R}^3}|\nabla u_1|^2dx<\frac{K(c)}{2},\ \int_{\mathbb{R}^3}|\nabla u_2|^2dx>2K(c)
$$

and

$$
I(u_1)>0,\ I(u_2)<0,
$$

where $u_1:=\tau_1 * u_0\in S(c)$ and $u_2:=\tau_2 * u_0\in S(c)$. In the following, denote the mountain pass level $\gamma_\mu(c)$ by

$$
\gamma_\mu(c):=\inf_{g\in\Gamma}\max_{t\in[0,1]}I(g(t)),
$$

where

$$
\Gamma:=\{g\in C([0,1],S(c)):g(0)=u_1,g(1)=u_2\}.
$$

Then for any $g\in\Gamma$,

$$
\max_{t\in[0,1]}I(g(t))>\max\{I(u_1),I(u_2)\}.
$$

It yields that $\gamma_\mu(c)>0$. About $\gamma_\mu(c)$, Lemma 3.3 holds.

Lemma 3.3. $\lim_{\mu\to+\infty}\gamma_\mu(c)=0$.

Proof. Taking $g_0(t) := \big[(1-t)\tau_1 + t\tau_2\big] * u_0 \in \Gamma$, then

$$
\begin{aligned}
0 <&\gamma_\mu(c) \le \max_{t\in[0,1]} I(g_0(t)) = \max_{t\in[0,1]} I\Big(\big[(1-t)\tau_1 + t\tau_2\big] * u_0\Big)\\
=& \max_{t\in[0,1]} \Big\{\frac{1}{2}ae^{2\big[(1-t)\tau_1+t\tau_2\big]}\int_{\mathbb{R}^3}|\nabla u_0|^2dx + \frac{b}{4}e^{4\big[(1-t)\tau_1+t\tau_2\big]}\big(\int_{\mathbb{R}^3}|\nabla u_0|^2dx\big)^2\\
&-\frac{\mu}{p}\cdot e^{\frac{3(p-2)}{2}\big[(1-t)\tau_1+t\tau_2\big]}\int_{\mathbb{R}^3}|u_0|^pdx - \frac{1}{6}\cdot e^{6\big[(1-t)\tau_1+t\tau_2\big]}\int_{\mathbb{R}^3}|u_0|^6dx\Big\}\\
\le& \max_{r\ge 0}\Big\{\frac{1}{2}ar^2\int_{\mathbb{R}^3}|\nabla u_0|^2dx + \frac{b}{4}r^4\big(\int_{\mathbb{R}^3}|\nabla u_0|^2dx\big)^2 - \frac{\mu}{p}\cdot r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\\
&-\frac{1}{6}r^6\int_{\mathbb{R}^3}|u_0|^6dx\Big\}\\
\le& \max_{r\ge 0}\Big\{\frac{1}{2}ar^2\int_{\mathbb{R}^3}|\nabla u_0|^2dx + \frac{b}{4}r^4\big(\int_{\mathbb{R}^3}|\nabla u_0|^2dx\big)^2 - \frac{\mu}{p}\cdot r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\}\\
:=&A.
\end{aligned}
$$

If $r^2\int_{\mathbb{R}^3}|\nabla u_0|^2dx \ge 1$, then

$$
\begin{aligned}
A \le& \max_{r\ge 0}\Big\{\frac{1}{2}ar^4\big(\int_{\mathbb{R}^3}|\nabla u_0|^2dx\big)^2 + \frac{b}{4}r^4\big(\int_{\mathbb{R}^3}|\nabla u_0|^2dx\big)^2 - \frac{\mu}{p}\cdot r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\}\\
\le& C\big(\frac{1}{\mu}\big)^{\frac{8}{3p-14}} \to 0 \ (\mu\to+\infty).
\end{aligned}
$$

If $r^2\int_{\mathbb{R}^3}|\nabla u_0|^2dx < 1$, then

$$
\begin{aligned}
A \le& \max_{r\ge 0}\Big\{\frac{1}{2}ar^2\int_{\mathbb{R}^3}|\nabla u_0|^2dx + \frac{b}{4}r^2\int_{\mathbb{R}^3}|\nabla u_0|^2dx - \frac{\mu}{p}\cdot r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\}\\
\le& C\big(\frac{1}{\mu}\big)^{\frac{4}{3p-10}} \to 0 \ (\mu\to+\infty).
\end{aligned}
$$

□

By Proposition 2.2 in [10] or Proposition 3.5 in [19], there exists a sequence $\{u_n\}\subset S(c)$ satisfying

$$I(u_n)\to\gamma_\mu(c) \quad\text{and}\quad \|I'|_{S(c)}(u_n)\|\to 0 \quad\text{and}\quad Q(u_n)\to 0$$

as $n\to\infty$, where

$$Q(u_n) = a\int_{\mathbb{R}^3}|\nabla u_n|^2dx + b\big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)^2 + 3\int_{\mathbb{R}^3}F(u_n)dx - \frac{3}{2}\int_{\mathbb{R}^3}f(u_n)u_ndx.$$

Arguing as the proof of Lemmas 2.3-2.4 of [10], we know that $\{u_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^3)$. Set $\Phi(v) := \frac{1}{2}\int_{\mathbb{R}^3}|v|^2dx$, $\forall v\in H^1(\mathbb{R}^3)$, then $S(c) = \Phi^{-1}\big(\{\frac{c^2}{2}\}\big)$. By Proposition 5.12 in [16], there exists a sequence $\{\lambda_n\}\subset\mathbb{R}$ such that

$$\|I'(u_n) - \lambda_n\Phi'(u_n)\|\to 0$$

as $n\to\infty$, which implies that

$$-\big(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)\Delta u_n - f(u_n) = \lambda_n u_n + o(1) \text{ in } H^{-1}_{rad}(\mathbb{R}^3). \tag{3.1}$$

Therefore, for $\varphi\in H^1_{rad}(\mathbb{R}^3)$,

(3.2)

$$\big(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\varphi dx - \int_{\mathbb{R}^3}f(u_n)\varphi dx = \lambda_n\int_{\mathbb{R}^3}u_n\varphi dx + o(1)\|\varphi\|.$$

We can estimate λ_n as follows.

Lemma 3.4. *$\{\lambda_n\}$ is bounded in $\mathbb{R}$ and $\lambda_n = -\frac{1}{c^2}\cdot\frac{6-p}{2p}\mu\int_{\mathbb{R}^3}|u_n|^p dx + o(1)$.*

Proof. By (3.2) and $u_n \in S(c)$,

$$(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx)\int_{\mathbb{R}^3}|\nabla u_n|^2dx-\int_{\mathbb{R}^3}f(u_n)u_ndx=\lambda_n\int_{\mathbb{R}^3}|u_n|^2dx+o(1)=\lambda_n c^2+o(1),$$

which indicates that

$$\lambda_n=\frac{1}{c^2}\Big[\big(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)\int_{\mathbb{R}^3}|\nabla u_n|^2dx-\int_{\mathbb{R}^3}f(u_n)u_ndx\Big]+o(1).$$

By the boundedness of $\{u_n\}$ in $H^1_{rad}(\mathbb{R}^3)$, we know that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Moreover, combining with $Q(u_n)\to 0$ as $n\to\infty$ we see that

$$\begin{aligned}\lambda_n=&\frac{1}{c^2}\Big[\frac{3}{2}\int_{\mathbb{R}^3}f(u_n)u_ndx-3\int_{\mathbb{R}^3}F(u_n)dx-\int_{\mathbb{R}^3}f(u_n)u_ndx\Big]+o(1)\\=&\frac{1}{c^2}\Big[\frac{1}{2}\int_{\mathbb{R}^3}f(u_n)u_ndx-3\int_{\mathbb{R}^3}F(u_n)dx\Big]+o(1)\\=&\frac{1}{c^2}\Big[\frac{1}{2}\int_{\mathbb{R}^3}(\mu|u_n|^p+|u_n|^6)dx-3\int_{\mathbb{R}^3}(\frac{\mu}{p}|u_n|^p+\frac{1}{6}|u_n|^6)dx\Big]+o(1)\\=&-\frac{1}{c^2}\cdot\frac{6-p}{2p}\cdot\mu\int_{\mathbb{R}^3}|u_n|^pdx+o(1).\end{aligned}$$

□

From the boundedness of $\{u_n\}$ in $H^1_{rad}(\mathbb{R}^3)$, up to a subsequence, there exists $u\in H^1_{rad}(\mathbb{R}^3)$ such that $u_n\rightharpoonup u$ in $H^1_{rad}(\mathbb{R}^3)$ and $u_n\to u$ in $L^t(\mathbb{R}^3)$ for $2<t<6$ and $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^3$. Since $\frac{14}{3}<p<6$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^pdx=\int_{\mathbb{R}^3}|u|^pdx. \tag{3.3}$$

Lemma 3.5. *There exists $\mu^*=\mu^*(c)>0$ such that $u\neq 0$ for all $\mu\geq\mu^*$.*

Proof. Suppose that $u=0$. Then taking into account of (3.3) and Lemma 3.4 one has $\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^pdx=0$ and $\lim_{n\to\infty}\lambda_n=0$. Combining with (3.2), we get that

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+b\big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)^2-\|u_n\|_6^6=o(1).$$

Up to a subsequence,

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+b\big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)^2\to l\geq 0\quad\text{and}\quad\|u_n\|_6^6\to l$$

as $n\to\infty$. If $l=0$, we can deduce from the expression of $I(u_n)$ that $\gamma_\mu(c)=0$. It is a contradiction. Hence, $l>0$. By the definition of S, we have

$$S\leq\frac{\int_{\mathbb{R}^3}|\nabla u_n|^2dx}{\|u_n\|_6^2}\leq\frac{1}{a}\cdot\frac{a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+b\big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\big)^2}{\|u_n\|_6^2}\to\frac{1}{a}\cdot l^{\frac{2}{3}}$$

as $n \to \infty$. It follows that $l \geq a^{\frac{3}{2}} S^{\frac{3}{2}}$. Consequently, by (3.3) we have

$$\begin{aligned}
\gamma_\mu(c) &= \lim_{n\to\infty} I(u_n) \\
&= \lim_{n\to\infty} \Big\{ \frac{1}{2} a \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{b}{4} \big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \big)^2 - \frac{\mu}{p} \int_{\mathbb{R}^3} |u_n|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} |u_n|^6 dx \Big\} \\
&\geq \frac{1}{12} l \geq \frac{1}{12} a^{\frac{3}{2}} S^{\frac{3}{2}},
\end{aligned}$$

a contradiction. □

Subsequently, by virtue of the concentration-compactness principle due to Lions [12], we can obtain Lemma 3.6.

Lemma 3.6. *$u_n \to u$ in $L^6(\mathbb{R}^3)$ for $\mu \geq \mu^*$.*

Proof. We follow the proof of Lemma 2.1 (iii). Indeed, it suffices to prove that $J = \emptyset$, where J appears in Lemma 2.1 (iii). We argue by contradiction, assume that J is nonempty but finite. From the proof of Lemma 2.1 (iii) we can infer that $\limsup_{n\to\infty} \|\nabla u_n\|_2^2 \geq a^{\frac{1}{2}} S^{\frac{3}{2}}$. As a result,

$$\begin{aligned}
&\gamma_\mu(c) + o(1) = I(u_n) - \frac{2}{3p-6} Q(u_n) \\
&= \Big(\frac{1}{2} - \frac{2}{3p-6}\Big) a \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \\
&\quad + \Big(\frac{1}{4} - \frac{2}{3p-6}\Big) b \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \Big)^2 + \Big(\frac{2}{3p-6} - \frac{1}{6}\Big) \int_{\mathbb{R}^3} |u_n|^6 dx,
\end{aligned}$$

which implies that

$$\gamma_\mu(c) \geq \Big(\frac{1}{2} - \frac{2}{3p-6}\Big) a \limsup_{n\to\infty} \|\nabla u_n\|_2^2 \geq \Big(\frac{1}{2} - \frac{2}{3p-6}\Big) a^{\frac{3}{2}} S^{\frac{3}{2}}.$$

Evidently, from Lemma 3.3 we can see that this is impossible. Consequently, $J = \emptyset$. □

3.1. **Proof of Theorem 1.2.** Fix $\mu \geq \mu^*$. By Lemma 3.4, we may assume that $\lambda_n \to \lambda_c$ as $n \to \infty$. Combining with Lemma 3.5 and (3.3), it is easy to see that

$$\lim_{n\to\infty} \lambda_n = -\frac{1}{c^2} \cdot \frac{6-p}{2p} \mu \lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^p dx = -\frac{1}{c^2} \cdot \frac{6-p}{2p} \mu \int_{\mathbb{R}^3} |u|^p dx < 0,$$

and so $\lambda_c < 0$. By (3.2) and arguments from Section 2, together with $\lambda_c < 0$ we can derive that

$$\lim_{n\to\infty} a \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = a \int_{\mathbb{R}^3} |\nabla u|^2 dx \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_2^2 = \|u\|_2^2.$$

Hence, $u_n \to u$ in $H^1_{rad}(\mathbb{R}^3)$ and $\|u\|_2 = c$. □

4. Asymptotic analysis in the supercritical case

In this section, we study the asymptotic behavior when c tends infinity under the case $\frac{14}{3} < p < 6$. By Theorem 1.2, for any $c > 0$, there exists $\mu^* = \mu^*(c) > 0$ such that as $\mu \geq \mu^*$, problem (P) possesses a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions with $\int_{\mathbb{R}^3} |u_c|^2 dx = c^2$, $\lambda_c > 0$, and by its proof we know $I(u_c) = \gamma_\mu(c)$.

4.1. **Proof of Theorem 1.4.** We firstly give some lemmas.

Lemma 4.1. *For any* $u \in H^1(\mathbb{R}^3)\backslash\{0\}$,

(i) *there exists a unique* $\tau(u) \in \mathbb{R}$ *such that* $Q(\tau(u) * u) = 0$.
(ii) $I(\tau(u) * u) > I(\tau * u)$ *for any* $\tau \neq \tau(u)$.

Proof. (i) It is easy to see that

$$\begin{aligned}\frac{d}{d\tau}I(\tau * u) =& ae^{2\tau}\int_{\mathbb{R}^3}|\nabla u|^2dx + be^{4\tau}(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2 - e^{6\tau}\int_{\mathbb{R}^3}|u|^6dx\\ &-\frac{3\mu(p-2)}{2p}e^{\frac{3(p-2)}{2}\tau}\int_{\mathbb{R}^3}|u|^pdx\\ =& Q(\tau * u).\end{aligned}$$

It follows from Lemma 3.1 that $I(\tau * u)$ reaches the global maximum at some $\tau(u) \in \mathbb{R}$, and so

$$Q(\tau(u) * u) = \frac{d}{d\tau}I(\tau(u) * u) = 0.$$

In the sequel, we prove that such $\tau(u)$ is unique. Argument by indirection. Suppose that there exist $\tau_1(u) < \tau_2(u)$ such that $Q(\tau_i(u) * u) = 0$, where $i = 1,\ 2$. Then a simple calculation yields that

$$\begin{aligned}&a[\frac{1}{e^{2\tau_1(u)}} - \frac{1}{e^{2\tau_2(u)}}]\int_{\mathbb{R}^3}|\nabla u|^2dx\\ =&\frac{3\mu(p-2)}{2p}[e^{\frac{3(p-2)}{2}\tau_1(u)-4\tau_1(u)} - e^{\frac{3(p-2)}{2}\tau_2(u)-4\tau_2(u)}]\int_{\mathbb{R}^3}|u|^pdx\\ &+[e^{2\tau_1(u)} - e^{2\tau_2(u)}]\int_{\mathbb{R}^3}|u|^6dx,\end{aligned}$$

which makes no sense in view of the fact that $\frac{3(p-2)}{2} > 4$.

(ii) This is a direct consequence of (i). □

Set

$$\mathcal{Q}(c) = \{u \in S(c) : Q(u) = 0\}.$$

Then we have

Lemma 4.2. $\gamma_\mu(c) \leq \inf_{u\in\mathcal{Q}(c)} I(u)$.

Proof. For any $u \in \mathcal{Q}(c)$, then $u \in S(c)$ and $Q(u) = 0$. The above argument between Lemmas 3.2 and 3.3 implies that there exist $\tau_1(u) < 0 < \tau_2(u)$ such that

$$\tau_1(u) * u \in S(c) \text{ and } \tau_2(u) * u \in S(c).$$

Taking

$$g(t) = [(1-t)\tau_1(u) + t\tau_2(u)] * u \in \Gamma,\ \forall t \in [0,1].$$

Then, by Lemma 4.1 we get that

$$\gamma_\mu(c) \leq \max_{t\in[0,1]} I(g(t)) = \max_{t\in[0,1]} I([(1-t)\tau_1(u) + t\tau_2(u)] * u) = I(u),$$

which implies that the conclusion holds. □

Lemma 4.3. $\gamma_\mu(c) \to 0$ *as* $c \to +\infty$.

Proof. Take $u \in S_1 \cap L^\infty(\mathbb{R}^N)$ and set $u_c := cu \in S(c)$ for any $c > 1$. By virtue of Lemma 4.1 (i), there exists a unique $\tau_1(c) \in \mathbb{R}$ such that $\tau_1(c) * u_c \in \mathcal{Q}(c)$. As a consequence, by Lemma 4.2 one has

$$\begin{aligned} 0 <&\gamma_\mu(c) \leq \inf_{u\in\mathcal{Q}(c)} I(u) \leq I(\tau_1(c) * u_c) \\ \leq& \frac{ac^2}{2} e^{2\tau_1(c)} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} c^4 e^{4\tau_1(c)} (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2. \end{aligned}$$

In what follows, it suffices to prove

$$\lim_{c\to+\infty} c^2 e^{2\tau_1(c)} = 0.$$

As a matter of fact, by $Q(\tau_1(c) * u_c) = 0$ we know that

$$\begin{aligned} &ae^{2\tau_1(c)} c^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + be^{4\tau_1(c)} c^4 (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 \\ =&\frac{3\mu(p-2)}{2p} e^{\frac{3(p-2)}{2}\tau_1(c)} c^p \int_{\mathbb{R}^3} |u|^p dx + e^{6\tau_1(c)} c^6 \int_{\mathbb{R}^3} |u|^6 dx, \end{aligned}$$

namely

$$\begin{aligned} &a \int_{\mathbb{R}^3} |\nabla u|^2 dx + be^{2\tau_1(c)} c^2 (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 \\ =&\frac{3\mu(p-2)}{2p} e^{[\frac{3(p-2)}{2}-2]\tau_1(c)} c^{p-2} \int_{\mathbb{R}^3} |u|^p dx + e^{4\tau_1(c)} c^4 \int_{\mathbb{R}^3} |u|^6 dx \\ =&\frac{3\mu(p-2)}{2p} [c^2 e^{2\tau_1(c)}]^{\frac{3p-10}{4}} c^{\frac{6-p}{2}} \int_{\mathbb{R}^3} |u|^p dx + e^{4\tau_1(c)} c^4 \int_{\mathbb{R}^3} |u|^6 dx. \end{aligned}$$

Since $\lim_{c\to+\infty} c^{\frac{6-p}{2}} = +\infty$, we can conclude that $\lim_{c\to+\infty} c^2 e^{2\tau_1(c)} = 0$. Thereby, the conclusion holds. □

Proof of Theorem 1.4. This is a direct consequence of Lemma 4.3. □

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