ON A SPECTRAL VARIATIONAL PROBLEM ARISING IN THE STUDY OF EARTHQUAKES

Multiplicity and perturbation from symmetry

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Abstract We consider an eigenvalue variational inequality problem arising in the earthquake initiation. Our purpose is twofold. Firstly, in the symmetric case, we establish the existence of infinitely many distinct solutions. Next, in the case where the problem is affected by a non-symmetric perturbation, we prove that the number of solutions of the perturbed problem becomes larger and larger if the perturbation "tends" to zero with respect to a suitable topology. Since the canonical energy functionals are included neither in the theory of monotone operators, nor in their Lipschitz perturbations, the proofs of the main results rely on non-smooth critical point theories in the sense of De Giorgi and Degiovanni combined with methods from algebraic topology.

Keywords: Earthquake initiation, variational inequality, non-smooth critical point theory, perturbation.

Introduction

The purpose of this paper is twofold: first, we establish a multiplicity result for a nonlinear symmetric variational inequality; next, we study the effect of an arbitrary perturbation. For related results and further comments we refer to our recent papers [11] (for an appropriate variational inequality) and [2, 5, 14] for the hemivariational framework.

We have been inspired in this work by the following simple phenomenon which occurs in elementary mathematics. Usually, an equation with a certain symmetry admits infinitely many solutions. For instance, the equation $\sin x = 1/2$ ($x \in \mathbf{R}$) has an infinite number of solutions. In this case, the "symmetry" is given by periodicity. If the above equation is affected in an arbitrary non-symmetric way by a certain perturbation, then the number of solutions of the new equation becomes larger and larger if the perturbation "tends to zero" in a suitable sense. For instance, the equation

$$\sin x = \frac{1}{2} + \varepsilon\, x^2, \qquad x \in \mathbf{R} \tag{1}$$

has finitely many solutions, for any $\varepsilon > 0$. However, the number of solutions of (1) tends to infinity if ε tends to 0.

Our purpose in this work is to illustrate the above elementary phenomenon to the study of a nonlinear eigenvalue variational inequality arising in earthquake initiation. More precisely, using a multiplicity result of Lusternik-Schnirelmann type combined with the fact that an adequate function space is infinite dimensional, we first establish the existence of infinitely many solutions. Next, we are concerned with the study of the effect of a small non-symmetric perturbation and we prove that the number of solutions of the perturbed tends to infinity if the perturbation tends to zero with respect to an appropriate topology. The main novelty in our framework is the presence of the convex cone of functions with non-negative jump across an internal boundary which is composed of a finite number of bounded connected arcs.

1. Main Results and Physical Motivation

Let $\Omega \subset \mathbf{R}^N$ be a domain, not necessarily bounded, containing a finite number of cuts. Its boundary $\partial\Omega$ is supposed to be smooth and divided into two disjoint parts: the exterior boundary $\Gamma_d = \partial\bar{\Omega}$ and the internal one Γ composed by N_f bounded connected arcs $\Gamma_f^i, i = 1, .., N_f$, called cracks or faults. On Γ we denote by $[\ \ ]$ the jump across Γ, (i.e. $[w] = w^+ - w^-$) and by $\partial_n = \nabla \cdot n$ the corresponding normal derivative with the unit normal n outwards the positive side.

Set $V = \{v \in H^1(\Omega);\ v = 0 \ \text{ on } \ \Gamma_d\}$. Denote by $\|\cdot\|$ the norm in the space V, and by $\Lambda_0 : L^2(\Omega) \to L^2(\Omega)^*$ and $\Lambda_1 : V \to V^*$ the duality isomorphisms $\Lambda_0 u(v) = \int_\Omega uv dx$, for any $u, v \in L^2(\Omega)$, and $\Lambda_1 u(v) = \int_\Omega \nabla u \cdot \nabla v dx$, for any $u, v \in V$. Consider the Lipschitz map $\gamma = i \circ \eta : V \to L^2(\Gamma)$, where $\eta : V \to H^{1/2}(\Gamma)$ is the trace operator, $\eta(v) = [v]$ on Γ and $i : H^{1/2}(\Gamma) \to L^2(\Gamma)$ is the embedding operator. Let K be the convex closed cone defined by $K = \{v \in V;\ [v] \geq 0 \text{ on } \Gamma\}$.

Consider the nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} \text{find } u \in K \text{ and } \lambda \in \mathbf{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla (v-u) dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \displaystyle\lambda \int_\Omega u(v-u) dx \geq 0, \quad \forall v \in K, \end{array} \right. \tag{2}$$

where $j(t) = -\frac{\beta}{2}t^2$, $t \in \mathbf{R}$, for some real constant β.

If $N = 2$, then any solution of problem (2) can be viewed as a stationary solution of an appropriate evolution variational inequality that describes the slip-dependent friction law introduced in the geophysical context of earthquake modelling (see [1], [3]). We look for $w : \mathbf{R}_+ \times \Omega \to \mathbf{R}$ solution of the wave equation

$$\partial_{tt} w(t) = c^2 \Delta w(t) \quad \text{in} \quad \Omega,$$

with the boundary condition

$$w(t) = 0 \ \text{ on } \ \Gamma_d.$$

On the contact zone Γ we have $[\partial_n w] = 0$ and the following slip dependent friction law (introduced in the geophysical context of earthquakes modelling) is assumed (see [1], [3])

$$G\partial_n w(t) = -\mu(|[w(t)]|)S\text{sign}([\partial_t w(t)]) - q, \ \text{ if } \ [\partial_t w(t)] \neq 0,$$

$$|G\partial_n w(t) + q| \leq \mu(|[w(t)]|)S \quad \text{if} \quad \partial_t [w(t)] = 0,$$

where G denotes the elastic medium shear rigidity, ρ is the density, and $c = \sqrt{G/\rho}$ is the shear velocity. The non-vanishing shear stress components are $\sigma_{zx} = \tau_x^\infty + G\partial_x w$, $\sigma_{zy} = \tau_y^\infty + G\partial_y w$, $\sigma_{xx} = \sigma_{yy} = -S$ ($S > 0$ is the normal stress on the fault plane), and $q = \tau_x^\infty n_x + \tau_y^\infty n_y$. The initial conditions are

$$w(0) = w_0, \quad \partial_t w(0) = w_1 \quad \text{in} \quad \Omega.$$

Any solution of the above problem satisfies the following nonlinear eigenvalue problem: find $w : [0, T] \to V$ such that

$$\int_\Omega \frac{1}{c^2} \partial_{tt} w(t)(v - \partial_t w(t) \, dx \; + \int_\Omega \nabla w(t) \cdot \nabla (v - \partial_t w(t)) \, dx \; + \tag{3}$$
$$\int_\Gamma \frac{S}{G} \mu(|[w(t)]|)(|[v]| - |[\partial_t w(t)]|) \, d\sigma \; \geq \int_\Gamma \frac{1}{G} q([v] - [\partial_t w(t)]) \, d\sigma,$$

for all $v \in V$.

The main difficulty in the study of problem (3) is the non-monotone dependence of μ with respect to the slip $|[w]|$. However, in modelling unstable phenomena, as earthquakes, we have to expect "bad" mathematical properties of the operators involved in the abstract problem. The existence of a solution $w \in W^{1,\infty}(0, T, V) \cap W^{2,\infty}(0, T, L^2(\Omega))$ (if $N = 2$) was recently proved by Ionescu et al. [10]. The uniqueness was obtained only in the one-dimensional case. Since our intention is to study the evolution of the elastic system near an unstable equilibrium position, we shall suppose that $q = \mu(0)S$. We remark that $w \equiv 0$ is an equilibrium solution of (3), and w_0, w_1 may be considered as small perturbations of it. For simplicity, let us assume in the following that the friction law is homogeneous on the fault plane having the form of a piecewise linear function (see [13]) :

$$\mu(x,u) = \mu_s - \frac{\mu_s - \mu_d}{2D_c} u \ \text{ if } \ u \leq 2D_c, \ \ \mu(x,u) = \mu_d \ \text{ if } \ u > 2D_c, \tag{4}$$

where u is the relative slip, μ_s and μ_d ($\mu_s > \mu_d$) are the static and dynamic friction coefficients, and D_c is the critical slip. Since the initial perturbation (w_0, w_1) of the equilibrium ($w \equiv 0$) is small we have $[w(t,x))] \leq 2D_c$ for $t \in [0, T_c]$ for all $x \in \Gamma$, where T_c is a critical time for which the slip on the fault reaches the critical value $2D_c$ at least at one point. Hence for a first period $[0, T_c]$, called the *initiation phase*, we deal with a linear function μ. Our aim is to analyze the evolution of the perturbation during this initial phase. That is why we are interested in the existence of solutions of the type

$$w(t,x) = \sinh(|\lambda|ct)u(x), \quad w(t,x) = \sin(|\lambda|ct)u(x) \tag{5}$$

during the initiation phase $t \in [0, T_c]$. If we put the above expression in (3) and we have in mind that from (4) we have $\mu(s) = \mu_s - (\mu_s - \mu_d)/(2D_c)s$ then we deduce that (u, λ) is solution of the problem (2), where $\beta = (\mu_s - \mu_d)S/(2D_cG) > 0$. The first type of solution described by (5) has an exponential growth in time and corresponds to $\lambda > 0$. The second one has the same amplitude during the initiation phase and corresponds to $\lambda < 0$.

Returning to problem (2), we observe that, due to its homogeneity, we can reformulate this problem in terms of a constrained inequality problem as follows. For any fixed $r > 0$, consider the smooth manifold $M = \left\{u \in V;\ \int_\Omega u^2 dx = r^2\right\}$. We shall study the problem

$$\left\{ \begin{array}{l} \text{find } u \in K \cap M \text{ and } \lambda \in \mathbf{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Gamma j'\left(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\right) d\sigma + \\ \displaystyle\lambda \int_\Omega u(v-u)dx \geq 0, \quad \forall v \in K. \end{array} \right. \tag{6}$$

The multiplicity of solutions to problem (6) is described in

THEOREM 1 *Problem (6) has infinitely many solutions (u, λ) and the set of eigenvalues $\{\lambda\}$ is bounded from above and its infimum equals to $-\infty$. Let $\lambda_0 = \sup\{\lambda\}$. Then there exists u_0 such that (u_0, λ_0) is a* *solution of (6). Moreover the function $\beta \longmapsto \lambda_0(\beta)$ is convex and the following inequality holds*

$$\int_\Omega |\nabla v|^2 \, dx + \lambda_0(\beta) \int_\Omega v^2 \, dx \geq \beta \int_\Gamma [v]^2 \, d\sigma, \qquad \forall v \in K. \tag{7}$$

Our next purpose is to describe the effect of an arbitrary perturbation in problem (2). More precisely, we consider the problem

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in K \text{ and } \lambda_\varepsilon \in \mathbf{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) dx + \\ \displaystyle\int_\Gamma (j' + \varepsilon g') \left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x))\right) d\sigma + \\ \displaystyle\lambda_\varepsilon \int_\Omega u_\varepsilon (v - u_\varepsilon) dx \geq 0, \quad \forall v \in K, \end{array} \right. \tag{8}$$

where $\varepsilon > 0$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function with no symmetry hypothesis, but satisfying the growth assumption

$$\begin{array}{ll} \exists\, a > 0, \ \exists\, 2 \leq p \leq \dfrac{2(N-1)}{N-2} : \ |g(t)| \leq a(1 + |t|^p) & , \text{ if } N \geq 3; \\ \exists\, a > 0, \ \exists\, 2 \leq p < +\infty : \ |g(t)| \leq a(1 + |t|^p) & , \text{ if } N = 2. \end{array} \tag{9}$$

This hypothesis is motivated by the following embedding inequality of Ionescu [9] that will be used in an essential manner in the proof.

Lemma 2 *(Lemma 5.1 in [9]). Let $2 \leq \alpha \leq 2(N-1)/(N-2)$ if $N \geq 3$ and $2 \leq \alpha < +\infty$ if $N = 2$. Then for $\beta = [(\alpha-2)N + 2]/(2\alpha)$ if $N \geq 3$ or if $N = 2$ and $\alpha = 2$ and for all $(\alpha - 1)/\alpha < \beta < 1$ if $N = 2$ and $\alpha > 2$, there exists $C = C(\beta)$ such that*

$$\left(\int_\Gamma |[u]|^\alpha d\sigma \right)^{1/\alpha} \leq C \left(\int_\Omega u^2 dx \right)^{(1-\beta)/2} \left(\int_\Omega |\nabla u|^2 dx \right)^{\beta/2}, \tag{10}$$

for any $u \in V$.

Our perturbation result is

Theorem 3 *For every positive integer n, there exists $\varepsilon_n > 0$ such that problem (8) has at least n distinct solutions $(u_\varepsilon, \lambda_\varepsilon)$ if $\varepsilon < \varepsilon_n$. There exists and is finite $\lambda_{0\varepsilon} = \sup\{\lambda_\varepsilon\}$ and there exists $u_{0\varepsilon}$ such that $(u_{0\varepsilon}, \lambda_{0\varepsilon})$ is a solution of (8). Moreover, $\lambda_{0\varepsilon}$ converges to λ_0 as ε tends to 0, where λ_0 was defined in Theorem 1.*

2. Proofs

We first recall some of the notions used in the proofs of Theorems 1 and 3. An important role in our arguments in order to locate the solution of problem (6) is played by the indicator function of M, that is,

$$I_M(u) = \begin{cases} 0 & , \quad \text{if } u \in M \\ +\infty & , \quad \text{if } u \in V \setminus M. \end{cases}$$

Then I_M is lower semicontinuous. However, since the natural energy functional associated to problem (6) is neither smooth nor convex, it is necessary to introduce a more general concept of gradient. We shall employ the following notion of lower subdifferential which is due to De Giorgi, Marino and Tosques [8].

Definition 4 *Let X be a Banach space and let $f : X \to \mathbf{R} \cup \{+\infty\}$ be an arbitrary proper functional. Let $x \in D(f)$. The Fréchet (regular) subdifferential of f at x is the (possibly empty) set*

$$\partial^- f(x) = \left\{ \xi \in X^*;\ \liminf_{y \to x} \frac{f(y) - f(x) - \xi(y-x)}{\|y - x\|} \geq 0 \right\}.$$

An element $\xi \in \partial^- f(x)$ is called a lower subgradient of f at x.

Accordingly, we say that $x \in D(f)$ is a critical (lower stationary) point of f if $0 \in \partial^- f(x)$.

Then $\partial^- f(x)$ is a convex set. If $\partial^- f(x) \neq \emptyset$ we denote by $\text{grad}^- f(x)$ the element of minimal norm of $\partial^- f(x)$, that is,

$$\text{grad}^- f(x) = \min\{\|\xi\|_{X^*};\ \xi \in \partial^- f(x)\}.$$

This notion plays a central role in the statement of the following basic compactness condition.

Definition 5 *Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be an arbitrary functional. We say that $(x_n) \subset D(f)$ is a Palais-Smale sequence if $\sup_n |f(x_n)| < +\infty$ and $\lim_{n \to \infty} \text{grad}^- f(x_n) = 0$. The functional f is said to satisfy the Palais-Smale condition provided that any Palais-Smale sequence is relatively compact.*

Definition 4 implies that if $g : X \to \mathbf{R}$ is Fréchet differentiable and $f : X \to \mathbf{R} \cup \{+\infty\}$ is an arbitrary proper function then

$$\partial^-(f+g)(x) = \{\xi + g'(x);\ \xi \in \partial^- f(x)\},$$

for any $x \in D(f)$.

We also point out that in [4] it is proved the formula

$$\partial^- I_M(u) = \{\lambda\Lambda_0 u;\ \lambda \in \mathbf{R}\} \subset L^2(\Omega)^* \subset V^*, \tag{11}$$

for any $u \in M$, where $\Lambda_0 : L^2(\Omega) \to L^2(\Omega)^*$ denotes the canonical duality isomorphism.

2.1 Proof of Theorem 1

Define $E = F + G : V \to \mathbf{R} \cup \{+\infty\}$, where

$$F(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx & , \quad \text{if } u \in K \\ +\infty & , \quad \text{if } u \notin K \end{cases}$$

and

$$G(u) = -\frac{\beta}{2}\int_\Gamma [\gamma(u(x))]^2 d\sigma.$$

Then $E+I_M$ is lower semicontinuous. Moreover, $E+I_M$ is the canonical energy functional associated to problem (6). This assertion is refined in the following auxiliary result.

Lemma 6 *Let (u, λ) be an arbitrary solution of problem (6). Then $0 \in \partial^-(E + I_M)(u)$. Conversely, let u be a critical point of $E + I_M$ and denote $\lambda = -2E(u)r^{-2}$. Then (u, λ) is a solution of problem (6).*

Proof of Lemma 6. If (u, λ) is a solution of problem (6) then, by the definition of the lower subdifferential,

$$-\lambda u \in \partial^- E(u). \tag{12}$$

On the other hand,

$$\partial^-(E + I_M)(u) = \partial^- E(u) + \partial^- I_M(u), \qquad \text{for any } u \in K \cap M. \tag{13}$$

So, by (11) and (12), $0 \in \partial^-(E + I_M)(u)$.

Conversely, let $0 \in \partial^-(E + I_M)(u)$. Thus, by (11) and (13), there exists $\lambda \in \mathbf{R}$ such that (u, λ) is a solution of problem (6). If we put $v = 0$ in (6) then we deduce $\lambda r^2 \le -2E(u)$ and for $v = 2u$ we get $\lambda r^2 \ge -2E(u)$, that is $\lambda = -2E(u)r^{-2}$.

The next step in our proof consists in showing that the functional $E+I_M$ satisfies the Palais-Smale condition. This is done by using standard arguments, but applied in the framework of the non-smooth critical point theory in the sense of De Giorgi, Marino and Tosques.

Due to the symmetry of problem (6), we can extend our study to the symmetric cone $(-K)$. More precisely, if (u,λ) is a solution of (6) then $u_0 := -u \in (-K)\cap M$ satisfies

$$\int_\Omega \nabla u_0 \cdot \nabla(v-u_0)dx + \int_\Gamma j'\left(\gamma(u_0(x));\gamma(v(x))-\gamma(u_0(x))\right)d\sigma +$$
$$\lambda\int_\Omega u_0(v-u_0)dx \geq 0, \qquad \text{for all } v\in(-K).$$

This means that we can extend the energy functional associated to problem (6) to the symmetric set $\widetilde{K} := K\cup(-K)$. We put, by definition,

$$\widetilde{E}(u) = \begin{cases} E(u) & , \quad \text{if } u\in K \\ E(-u) & , \quad \text{if } u\in(-K) \\ +\infty & , \quad \text{otherwise.}\end{cases}$$

We are interested in finding the lower stationary points of the extended energy functional $J := \widetilde{E} + I_M$.

We endow the set $\widetilde{K}\cap M$ with the graph metric of $\widetilde{E}$ defined by

$$d(u,v) = \|u-v\| + |\widetilde{E}(u)-\widetilde{E}(v)|, \qquad \text{for any } u,v\in\widetilde{K}\cap M.$$

Denote by $\mathcal{X}$ the metric space $(\widetilde{K}\cap M, d)$.

The next step in the proof of Theorem 1 consists in showing

Lemma 7 *We have* $\mathrm{Cat}_{\mathcal{X}}(\widetilde{K}\cap M) = +\infty$.

The proof is straightforward and is accomplished by using adequate tools from Algebraic Topology.

The above results enable us to apply the Lusternik-Schnirelmann theorem in the sense established by Marino and Scolozzi [12]. This implies that problem (6) admits infinitely many solutions (u,λ). Next, we observe that the set of eigenvalues is bounded from above. Indeed, if (u,λ) is a solution of our problem then choosing $v=0$ in (6) and using (10), it follows that $\lambda r^2 \leq -2\|u\|^2 + \frac{\beta}{2}\,\|u\|^2_{L^2(\Gamma)} \leq C$, where C does not depend on u.

It remains to prove that

$$\inf\{\lambda;\ \lambda \text{ is an eigenvalue of problem (6)}\} = -\infty.$$

For this purpose, it is sufficient to show that

$$\sup\{J(u);\ u\in\widetilde{K}\cap M\} = +\infty.$$

But this follows directly from (10) after observing that

$$\sup_{u\in\widetilde{K}\cap M}\int_\Omega |\nabla u|^2dx = +\infty.$$

In order to prove the last part of the theorem we remark that $-\lambda_0$, as a function of β, is the upper bound of a family of affine functions

$$-\lambda_0(\beta) = \inf_{v \in K \cap M} \frac{1}{r^2} \left\{ \int_\Omega |\nabla v|^2 \, dx - \beta \int_\Gamma [v]^2 \, d\sigma \right\}, \tag{14}$$

hence it is a concave function. Thus $\beta \longmapsto \lambda_0(\beta)$ is convex and (7) yields. This completes the proof of Theorem 1.

2.2 Proof of Theorem 3

The main idea is to establish the multiplicity result with respect to a prescribed level of energy. More precisely, let us fix $r > 0$. Consider the manifold $N = \left\{ u \in V;\ \int_\Gamma [u]^p d\sigma = r^p \right\}$, where p is as in (9).

We reformulate problem (8) as follows:

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in K \cap N \text{ and } \lambda_\varepsilon \in \mathbf{R} \text{ such that} \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla (v - u_\varepsilon) dx + \\ \displaystyle\int_\Gamma (j' + \varepsilon g') \left(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x)) \right) d\sigma + \\ \displaystyle\lambda_\varepsilon \int_\Omega u_\varepsilon (v - u_\varepsilon) dx \geq 0, \quad \forall v \in K. \end{array} \right. \tag{15}$$

We start with the preliminary result

LEMMA 8 *There exists a sequence (b_n) of essential values of E such that $b_n \to +\infty$ as $n \to \infty$.*

Proof of Lemma 8. For any $n \geq 1$, set $a_n = \inf_{S \in \Gamma_n} \sup_{u \in S} E(u)$, where Γ_n is the family of compact subsets of $K \cap N$ of the form $\phi(S^{n-1})$, with $\phi : S^{n-1} \to K \cap N$ continuous and odd. The function E restricted to $K \cap N$ is continuous, even and bounded from below. So, by Theorem 2.12 in [7], it is sufficient to prove that $a_n \to +\infty$ as $n \to \infty$. But, as in the preceding section, the functional E restricted to $K \cap N$ satisfies the Palais-Smale condition. So, taking into account Theorem 3.5 in [6] and Theorem 3.9 in [7], we deduce that the set E^c has finite genus for any $c \in \mathbf{R}$. Using now the definition of the genus combined with the fact that $K \cap N$ is a weakly locally contractible metric space, we deduce that $a_n \to +\infty$. This completes our proof.

The canonical energy associated to problem (15) is the functional J restricted to $K \cap N$, where $J = E + \Phi$ and Φ is defined by

$$\Phi(u) = \varepsilon \int_\Gamma g(\gamma(u(x))) d\sigma.$$

A straightforward computation with the same arguments as in the proof of Lemma 6 shows that if u is a lower stationary point of J then there exists $\lambda \in \mathbf{R}$ such that (u, λ) is a solution of problem (15). In virtue of this result, it is sufficient for concluding the proof of Theorem 3 to show that the functional J has at least n distinct critical values, provided that $\varepsilon > 0$ is sufficiently small. We first prove that J is a small perturbation of E. More precisely, we have

LEMMA 9 *For every $\eta > 0$, there exists $\delta = \delta_\eta > 0$ such that for any $\varepsilon \leq \delta$, $\sup_{u \in K \cap N} |J(u) - E(u)| \leq \eta$.*

Proof of Lemma 9. We have

$$|J(u) - E(u)| = |\Phi(u)| \leq \varepsilon \int_\Gamma |g(\gamma(u(x)))| \, d\sigma.$$

So, by (9) and Lemma 2,

$$|J(u) - E(u)| \leq \varepsilon \, a \int_\Gamma (1 + [u(x)]^p) \, d\sigma \leq C\varepsilon \leq \eta,$$

if ε is sufficiently small.

By Lemma 8, there exists a sequence (b_n) of essential values of $E_{|K\cap N}$ such that $b_n \to +\infty$. Without loss of generality we can assume that $b_i < b_j$ if $i < j$. Fix an integer $n \geq 1$ and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 < 1/2 \min_{2 \leq i \leq n}(b_i - b_{i-1})$. Applying now [7, Theorem 2.6], we obtain that for any $1 \leq j \leq n$, there exists $\eta_j > 0$ such that if $\sup_{K\cap N} |J(u) - E(u)| < \eta_j$ then $J_{|K\cap N}$ has an essential value $c_j \in (b_j - \varepsilon_0, b_j + \varepsilon_0)$. So, by Lemma 9 applied for $\eta = \min\{\eta_1, \ldots, \eta_n\}$, there exists $\delta_n > 0$ such that $\sup_{K\cap N} |J(u) - E(u)| < \eta$, provided that $\varepsilon \leq \delta_n$. This shows that the energy functional J has at least n distinct essential values $c_1, \ldots, c_n$ in $(b_1 - \varepsilon_0, b_n + \varepsilon_0)$.

The next step consists in showing that $c_1, \ldots, c_n$ are critical values of $J_{|K\cap N}$. Arguing by contradiction, let us suppose that c_j is not a critical value of $J_{|K\cap N}$. We show in what follows that

(A_1) There exists $\bar{\delta} > 0$ such that $J_{|K\cap N}$ has no critical value in $(c_j - \bar{\delta}, c_j + \bar{\delta})$.

(A_2) For every $a, b \in (c_j - \bar{\delta}, c_j + \bar{\delta})$ with $a < b$, the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial.

Suppose, by contradiction, that (A_1) is no valid. Then there exists a sequence (d_k) of critical values of $J_{|K\cap N}$ with $d_k \to c_j$ as $k \to \infty$. Since d_k is a critical value, it follows that there exists $u_k \in K \cap N$ such that $J(u_k) = d_k$ and $0 \in \partial^- J(u_k)$. Using now the fact that J satisfies the Palais-Smale condition at the level c_j, it follows that, up to a subsequence, (u_k) converges to some $u \in K \cap N$ as $k \to \infty$. So, by the continuity of J and the lower semicontinuity of grad $J(\,\cdot\,)$, we obtain $J(u) = c_j$ and $0 \in \partial^- J(u)$, which contradicts the initial assumption on c_j.

Let us now prove assertion (A_2). For this purpose we apply the Noncritical Point Theorem (see [6], Theorem 2.15]). So, there exists a continuous map $\chi : (K \cap N) \times [0,1] \to K \cap N$ such that

$$\begin{aligned} &\chi(u,0) = u, \quad J(\chi(u,t)) \le J(u), \\ &J(u) \le b \Rightarrow J(\chi(u,1)) \le a, \quad J(u) \le a \Rightarrow \chi(u,t) = u. \end{aligned} \tag{16}$$

Define the map $\rho : J^b_{|K\cap N} \to J^a_{|K\cap N}$ by $\rho(u) = \chi(u,1)$. From (16) we obtain that ρ is well defined and it is a retraction. Set

$$\mathcal{J} : J^b_{|K\cap N} \times [0,1] \to J^b_{|K\cap N}, \qquad \mathcal{J}(u,t) = \chi(u,t).$$

The definition of $\mathcal{J}$ implies that, for every $u \in J^b_{|K\cap N}$,

$$\mathcal{J}(u,0) = u \qquad \text{and} \qquad \mathcal{J}(u,1) = \rho(u) \tag{17}$$

and, for any $(u,t) \in J^a_{|K\cap N} \times [0,1]$,

$$\mathcal{J}(u,t) = \mathcal{J}(u,0). \tag{18}$$

From (17) and (18) it follows that $\mathcal{J}$ is $J^a_{|K\cap N}$-homotopic to the identity of $J^a_{|K\cap N}$, that is, $\mathcal{J}$ is a strong deformation retraction, so the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial. Assertions (A_1), and (A_2) show that c_j is not an essential value of $J_{|K\cap N}$. This contradiction concludes the proof of Theorem 3.

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