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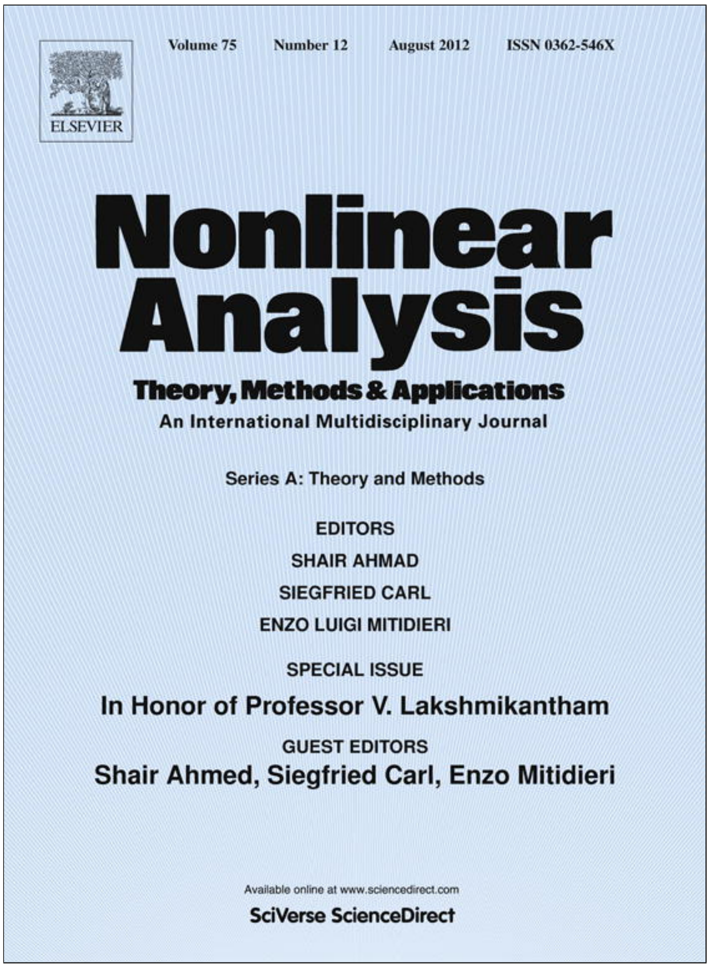

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Nonlinear Analysis 75 (2012) 4471–4482

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Anisotropic Neumann problems in Sobolev spaces with variable exponent

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ARTICLE INFO

Communicated by Enzo Mitidieri

This paper is dedicated to Professor V. Lakshmikantham on the occasion of his retirement as Editor of Nonlinear Analysis: Theory, Methods and Applications

MSC:
35J25
35J62
35D30
46E35
35J20

Keywords:
Anisotropic Sobolev space
Variable exponent
Neumann elliptic problem
Weak solutions
Existence
Uniqueness
Trace theorem

ABSTRACT

We prove a trace theorem that allows the treatment of Neumann problems with nonlinearities on the boundary in anisotropic spaces with variable exponent. Then we proceed to the study of such a problem that involves general operators of the $\vec{p}(\cdot)$-Laplace type. We deduce the existence of solutions and we direct attention to the situation where the solution is unique.

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1. Introduction

In the last few decades, one of the topics from the field of partial differential equations that has continuously attracted interest is that concerning the Sobolev space with variable exponents, $W^{1,p(\cdot)}$ (where p is a function depending on x); see for example the monograph [1] and the references therein. Naturally, problems involving the $p(\cdot)$-Laplace operator

$$\Delta_{p(x)}u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) \tag{1}$$

were intensively studied. At the same time, due to the development of the theory regarding the anisotropic Sobolev space, $W^{1,\vec{p}}$ (where $\vec{p}$ is a constant vector, $\vec{p} = (p_1, \ldots, p_N)$), there are also many authors discussing problems involving the $\vec{p}$-Laplace operator,

$$\Delta_{\vec{p}}(u) = \sum_{i=1}^{N} \partial_{x_i}\left(\left|\partial_{x_i}u\right|^{p_i-2}\partial_{x_i}u\right); \tag{2}$$

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doi:10.1016/j.na.2011.09.033

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see for example [2,3] and the references therein. Furthermore, a new theory captured attention when it introduced the anisotropic space with variable exponent, $W^{1,\vec{p}(\cdot)}$ (where $\vec{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ is a vector with variable components); see [4–6]. Consequently, a new operator takes its place in the mathematical literature, namely

$$\Delta_{\vec{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}\left(\left|\partial_{x_i}u\right|^{p_i(x)-2}\partial_{x_i}u\right); \tag{3}$$

see also [7,8]. This operator will be referred to as the anisotropic variable exponent $\vec{p}(\cdot)$-Laplace operator and it is closely related to (1) and (2). To be more specific, when choosing $p_1(\cdot) = \cdots = p_N(\cdot) = p(\cdot)$ we obtain an operator with similar properties to the variable exponent $p(\cdot)$-Laplace operator (1), while when choosing $p_1, \ldots, p_N$ to be constant functions we arrive at the anisotropic $\vec{p}$-Laplace operator (2).

In the present paper we consider a problem involving a more general type of operator, that is,

$$\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i}u\right). \tag{4}$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. The applications $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling the following hypotheses for all $i \in \{1, \ldots, N\}$:

(A1) There exists a positive constant $\bar{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x, s)| \le \bar{c}_i(d_i(x) + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$, where $d_i \in L^{p_i'(\cdot)}(\Omega)$ (with $1/p_i(x) + 1/p_i'(x) = 1$) is a nonnegative function.

(A2) If we define $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$A_i(x, s) = \int_0^s a_i(x, t)dt,$$

then the following inequalities hold:

$$|s|^{p_i(x)} \le a_i(x, s)s \le p_i(x)A_i(x, s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(A3) a_i fulfills

$$(a_i(x, s) - a_i(x, t))(s - t) > 0,$$

for all $x \in \Omega$ and $s, t \in \mathbb{R}$ with $s \ne t$.

The operator presented above is a $\vec{p}(\cdot)$-Laplace type operator because when we take

$$a_i(x, s) = |s|^{p_i(x)-2}s \quad \text{for all } i \in \{1, \ldots, N\},$$

we have $A_i(x, s) = \frac{1}{p_i(x)}|s|^{p_i(x)}$ for all $i \in \{1, \ldots, N\}$ and we get the $\vec{p}(\cdot)$-Laplace operator (3). Obviously, there are many other operators deriving from (4). Indeed, to give another interesting example, if we take

$$a_i(x, s) = (1 + |s|^2)^{(p_i(x)-2)/2}s \quad \text{for all } i \in \{1, \ldots, N\},$$

we have $A_i(x, s) = \frac{1}{p_i(x)}[(1 + |s|^2)^{p_i(x)/2} - 1]$ for all $i \in \{1, \ldots, N\}$, and we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i}\left[\left(1 + \left|\partial_{x_i}u\right|^2\right)^{(p_i(x)-2)/2} \partial_{x_i}u\right].$$

Notice that the general operator given by (4) can admit degenerate and singular points. So, since there are a number of features favoring this $\vec{p}(\cdot)$-Laplace type operator described by (4) and (A1)–(A3), it is no surprise to find that there are already papers treating problems with this kind of operator. To give some examples, we refer the reader to [9–11] where the authors were concerned with Dirichlet problems. We, on the other hand, are interested in a Neumann problem. One of the novelties of our work is that we consider a problem with a nonlinear term on the boundary, for whose study it is necessary to introduce and prove a trace theorem. More exactly, we analyze the problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i}u\right) + b(x)|u|^{p_M(x)-2}u = f(x, u) & \text{in } \Omega, \\ u \ge 0 & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i\left(x, \partial_{x_i}u\right)\nu_i = g(x, u) & \text{on } \partial\Omega. \end{cases} \tag{5}$$

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Here $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, are the Carathéodory functions characterized by (A1)–(A3), $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary and ν_i, $i \in \{1, \dots, N\}$, are the components of the outer normal unit vector. As for the rest of the functions involved in (5), we will enumerate their properties after we give some notation.

For any $\Omega \subset \mathbb{R}^N$ we set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \inf_{x \in \Omega} h(x) > 1\}$$

and we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

Moreover,

$$h^\partial(x) = \begin{cases} (N-1)h(x)/[N-h(x)] & \text{if } h(x) < N, \\ \infty & \text{if } h(x) \geq N. \end{cases}$$

In our case, $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$, $\overrightarrow{p}\,(x) = (p_1(x), p_2(x), \dots, p_N(x))$ with $p_i \in C_+(\overline{\Omega})$, $i \in \{1, \dots, N\}$, and for all $x \in \overline{\Omega}$ we put

$$p_M(x) = \max\{p_1(x), \dots, p_N(x)\} \quad \text{and} \quad p_m(x) = \min\{p_1(x), \dots, p_N(x)\}.$$

In addition, for the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, we consider the antiderivatives $F : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$F(x, s) = \int_0^s f(x, t)dt,$$

and $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$,

$$G(x, s) = \int_0^s g(x, t)dt,$$

respectively. With the previous notation, the functions $\overrightarrow{p}$, b, f and g satisfy the conditions:

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

(F) There exist a positive constant k_1 and $q \in L^\infty_+(\Omega)$ with $q^+ < p_m^-$ such that

$$|f(x, s)| \leq k_1 \left(1 + |s|^{q(x)-1}\right),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(G) There exist a positive constant k_2 and $r \in C(\overline{\Omega})$ with $r^+ < \min_{x \in \partial\Omega}\{p_1^\partial(x), \dots, p_N^\partial(x)\}$ and $r^+ < p_m^-$ such that

$$|g(x, s)| \leq k_2 \left(1 + |s|^{r(x)-1}\right),$$

for all $x \in \partial\Omega$ and $s \in \mathbb{R}$.

Note that by adding the following assumptions:

(F0) f fulfills the monotonicity condition

$$(f(x, s) - f(x, t))(s - t) < 0,$$

for all $x \in \Omega$ and $s, t \in \mathbb{R}$ with $s \neq t$,

(G0) g fulfills the monotonicity condition

$$(g(x, s) - g(x, t))(s - t) < 0,$$

for all $x \in \partial\Omega$ and $s, t \in \mathbb{R}$ with $s \neq t$,

we can deduce the uniqueness of the solution.

The variable exponent spaces have various applications like those in electrorheological fluids [12–16], thermorheological fluids [17], elastic mechanics [18] and image restoration [19]. However, there are some nonhomogeneous materials that have different behaviors in different space directions; hence the need for anisotropic spaces with variable exponent. The next section is dedicated to a brief presentation of the variable exponent spaces, shedding light on the properties that will help us to establish our main results.

2. A brief framework and preliminary results

In this section we will use the notation Ω_0 for a generic set from $\mathbb{R}^N$ that will be used to recall definitions and properties. For any measurable subset $\Omega_0 \subset \mathbb{R}^N$, $N \geq 2$, with $0 < |\Omega_0| < \infty$ (where $|\Omega_0|$ represents the Lebesgue measure of the set

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Ω_0), we consider $p \in C_+(\overline{\Omega_0})$. The isotropic Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\Omega_0) = \left\{ u : u \text{ is a measurable real-valued function such that } \int_{\Omega_0} |u(x)|^{p(x)}\, dx < \infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega_0)} = \inf\left\{ \mu > 0 : \int_{\Omega_0} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}. \tag{6}$$

Notice that the norm of the classical Lebesgue space $L^p(\Omega_0)$, that is,

$$\|u\|_{L^p(\Omega_0)} = \left(\int_{\Omega_0} |u(x)|^p \right)^{1/p},$$

is a particular case of (6) for p constant. Let us recall some of the most important properties of the space $\left(L^{p(\cdot)}(\Omega_0), \|\cdot\|_{L^{p(\cdot)}(\Omega_0)}\right)$. This space is a separable and reflexive Banach space (see [20, Theorem 2.5, Corollary 2.7]) and the inclusion between spaces generalizes naturally.

Theorem 1 *([20, Theorem 2.8]). If $0 < |\Omega_0| < \infty$ and $p_1, p_2 \in C(\overline{\Omega_0}; \mathbb{R})$, $1 < p_i^- \le p_i^+ < \infty$ $(i = 1, 2)$, are such that $p_1 \le p_2$ in Ω_0, then the embedding $L^{p_2(\cdot)}(\Omega_0) \hookrightarrow L^{p_1(\cdot)}(\Omega_0)$ is continuous.*

Furthermore, the following Hölder type inequality:

$$\left| \int_{\Omega_0} u(x)v(x)\, dx \right| \le 2\|u\|_{L^{p(\cdot)}(\Omega_0)} \|v\|_{L^{p'(\cdot)}(\Omega_0)} \tag{7}$$

holds true for all $u \in L^{p(\cdot)}(\Omega_0)$ and $v \in L^{p'(\cdot)}(\Omega_0)$ (see [20, Theorem 2.1]), where we denoted by $L^{p'(\cdot)}(\Omega_0)$ the conjugate space of $L^{p(\cdot)}(\Omega_0)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$ (see [20, Corollary 2.7]).

Moreover, the application $\rho_{\Omega_0,p(\cdot)} : L^{p(\cdot)}(\Omega_0) \to \mathbb{R}$,

$$\rho_{\Omega_0,p(\cdot)}(u) = \int_{\Omega_0} |u(x)|^{p(x)}\, dx,$$

called the $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega_0)$ space, is very useful in handling these Lebesgue spaces with variable exponent. Indeed, we have the following properties (see for example [21, Theorems 1.3 and 1.4]). If $u \in L^{p(\cdot)}(\Omega_0)$ and $p < \infty$, then

$$\|u\|_{L^{p(\cdot)}(\Omega_0)} < 1 (=1; > 1) \Leftrightarrow \rho_{\Omega_0,p(\cdot)}(u) < 1 (=1; > 1) \tag{8}$$

$$\|u\|_{L^{p(\cdot)}(\Omega_0)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega_0)} \le \rho_{\Omega_0,p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega_0)} \tag{9}$$

$$\|u\|_{L^{p(\cdot)}(\Omega_0)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega_0)} \le \rho_{\Omega_0,p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega_0)} \tag{10}$$

$$\|u\|_{L^{p(\cdot)}(\Omega_0)} \to 0 (\to \infty) \Leftrightarrow \rho_{\Omega_0,p(\cdot)}(u) \to 0\ (\to \infty). \tag{11}$$

If, in addition, $(u_n)_n \subset L^{p(\cdot)}(\Omega_0)$, then

$$\begin{aligned}\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega_0)} = 0 &\Leftrightarrow \lim_{n\to\infty} \rho_{\Omega_0,p(\cdot)}(u_n - u) = 0 \\ &\Leftrightarrow (u_n)_n \text{ converges to } u \text{ in measure and } \lim_{n\to\infty} \rho_{\Omega_0,p(\cdot)}(u_n) = \rho_{\Omega_0,p(\cdot)}(u).\end{aligned}$$

Let us introduce now the definition of the isotropic Sobolev space with variable exponent, $W^{1,p(\cdot)}(\Omega_0)$. We set

$$W^{1,p(\cdot)}(\Omega_0) = \left\{ u \in L^{p(\cdot)}(\Omega_0) : |\nabla u| \in L^{p(\cdot)}(\Omega_0) \right\}$$

endowed with the norm

$$\|u\| = \|u\|_{L^{p(\cdot)}(\Omega_0)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega_0)},$$

where by $\|\nabla u\|_{L^{p(\cdot)}(\Omega_0)}$ we understand $\|\, |\nabla u|\, \|_{L^{p(\cdot)}(\Omega_0)}$. The space $\left(W^{1,p(\cdot)}(\Omega_0), \|\cdot\|\right)$ is a separable and reflexive Banach space (see [20, Theorem 1.3]) and we have the following trace theorem.

Theorem 2 *([22, Corollary 2.4]). Let $\Omega_0 \subset \mathbb{R}^N$, $N \ge 2$, be a bounded open set with smooth boundary. Suppose that $p \in C_+(\overline{\Omega_0})$ and $r \in C(\overline{\Omega_0})$ satisfy the condition*

$$1 \le r(x) < p^\partial(x), \quad \forall x \in \partial\Omega_0.$$

Then there is a compact boundary trace embedding $W^{1,p(\cdot)}(\Omega_0) \hookrightarrow L^{r(\cdot)}(\partial\Omega_0)$.

We refer the reader to [22] for more details regarding the extension of the classical trace to Lebesgue–Sobolev spaces with variable exponent. Everywhere below, when we refer to the trace of u we will write u instead of $u|_{\partial\Omega_0}$ or γu.

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Next, we will introduce the space $W^{1,\overrightarrow{p}(\cdot)}(\Omega_0)$, where $\overrightarrow{p}: \overline{\Omega}_0 \to \mathbb{R}^N$ is the vectorial function

$$\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$$

and $p_i \in C_+(\overline{\Omega}_0)$ for all $i \in \{1, \ldots, N\}$. The anisotropic variable exponent Sobolev space is defined by

$$\begin{aligned} W^{1,\overrightarrow{p}(\cdot)}(\Omega_0) &= \{u \in L^{p_M(\cdot)}(\Omega_0) : \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega_0) \text{ for all } i \in \{1, \ldots, N\}\} \\ &= \{u \in L^1_{\mathrm{loc}}(\Omega_0) : u \in L^{p_i(\cdot)}(\Omega_0), \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega_0) \text{ for all } i \in \{1, \ldots, N\}\} \end{aligned}$$

endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega_0)} = \|u\|_{L^{p_M(\cdot)}(\Omega_0)} + \sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i(\cdot)}(\Omega_0)}.$$

The space $\left(W^{1,\overrightarrow{p}(\cdot)}(\Omega_0), \|\cdot\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega_0)}\right)$ is a reflexive Banach space (see [6, Theorems 2.1 and 2.2]) and the following embedding theorem applies.

Theorem 3 *([6, Corollary 2.1]). Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded open set and for all $i \in \{1, \ldots, N\}$, $p_i \in L^\infty(\Omega_0)$, let $p_i(x) \geq 1$ a.e. in Ω_0. Then for any $q \in L^\infty(\Omega_0)$ with $q(x) \geq 1$ a.e. in Ω_0 such that*

$$\operatorname{ess}\inf_{x \in \Omega_0} (p_M(x) - q(x)) > 0$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_0) \hookrightarrow L^{q(\cdot)}(\Omega_0).$$

As we announced in the introduction, a trace theorem is also needed and we will provide one in the following section. Keep in mind that, since $p_i^- > 1$,

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_0) \hookrightarrow W^{1,1}(\Omega_0) \quad \text{continously}$$

and by the Gagliardo trace theorem,

$$W^{1,1}(\Omega_0) \hookrightarrow L^1(\partial\Omega_0) \quad \text{compactly,}$$

with $\Omega_0 \subset \mathbb{R}^N$ being a bounded open set with smooth boundary. Hence for $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega_0)$ the trace has definite meaning.

Meanwhile, let us recall two other important results that are necessary for the proofs from Section 3.

Theorem 4 *([23, Theorem 6.2.1.]). Let X be a reflexive Banach space, and let $f : M \subseteq X \to \mathbb{R}$ be Gâteaux differentiable over the closed, convex set M. Then the following conditions are equivalent:*

(i) *f is convex over M.*
(ii) *We have*

$$f(u) - f(v) \geq \langle f'(v), u - v\rangle_{X^\star \times X} \quad \forall u, v \in M,$$

where $X^\star$ denotes the dual of the space X.
(iii) *The first Gâteaux derivative is monotone, that is,*

$$\langle f'(u) - f'(v), u - v\rangle_{X^\star \times X} \geq 0, \quad \forall u, v \in M.$$

(iv) *The second Gâteaux derivative of f exists and it is positive, that is,*

$$\langle f''(u) \circ v, v\rangle_{X^\star \times X} \geq 0, \quad \forall v \in M.$$

Theorem 5 *([24, Theorem 1.2]). Suppose X is a reflexive Banach space with norm $\|\cdot\|_X$ and let $M \subset X$ be a weakly closed subset of X. Suppose $\Phi : M \to \mathbb{R} \cup \{\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to X, that is, suppose the following conditions are fulfilled:*

(i) *$\Phi(u) \to \infty$ as $\|u\|_X \to \infty$, $u \in M$.*
(ii) *For any $u \in M$, and any subsequence $(u_m)_m$ in M such that $u_m \rightharpoonup u$ weakly in X, it holds that*

$$\Phi(u) \leq \liminf_{m \to \infty} \Phi(u_m).$$

Then Φ is bounded from below on M and attains its infimum in M.

3. The main results

Before analyzing the existence of solutions to problem (5) we introduce a trace theorem.

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Theorem 6. *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded open set with smooth boundary and let $\overrightarrow{p} \in \left(C_+(\overline{\Omega})\right)^N$, $r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x \in \partial\Omega}\{p_1^\partial(x), \ldots, p_N^\partial(x)\}, \quad \forall x \in \partial\Omega.$$

Then there is a compact boundary trace embedding

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega).$$

Proof. For clarity, for any subset $M \subset \overline{\Omega}$ and any function h, we will use the notation

$$h^-(M) = \inf_{x \in M} h(x) \quad \text{and} \quad h^+(M) = \sup_{x \in M} h(x).$$

We know that $r(x) < \min_{x \in \partial\Omega}\{p_1^\partial(x), \ldots, p_N^\partial(x)\}$, for all $x \in \partial\Omega$. In particular, for any $x \in \partial\Omega$, we have that $r(x) < p_i^\partial(x)$, for all $i \in \{1, \ldots, N\}$. Then for an arbitrarily fixed i and for any given $x \in \partial\Omega$, there exists a relatively open neighborhood Ω_x of x in $\overline{\Omega}$ such that

$$r^+(\partial\Omega \cap \Omega_x) < p_i^\partial(x).$$

Thus, by Theorem 2 there exists $\tilde{c}_i > 0$ such that

$$\|u\|_{L^{r^+(\partial\Omega\cap\Omega_x)}(\partial\Omega\cap\Omega_x)} \leq \tilde{c}_i \|u\|_{W^{1,p_i(\cdot)}(\Omega_x)},$$

that is,

$$\|u\|_{L^{r^+(\partial\Omega\cap\Omega_x)}(\partial\Omega\cap\Omega_x)} \leq \tilde{c}_i \left(\|u\|_{L^{p_i(\cdot)}(\Omega_x)} + \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega_x)}\right). \tag{12}$$

Due to the fact that $p_i(x) \leq p_M(x)$ for all $x \in \overline{\Omega}$, by Theorem 1 there exists $\tilde{c}_0 > 0$ such that

$$\|u\|_{L^{p_i(\cdot)}(\Omega_x)} \leq \tilde{c}_0 \|u\|_{L^{p_M(\cdot)}(\Omega_x)}. \tag{13}$$

Using (12) and (13) we get

$$\|u\|_{L^{r^+(\partial\Omega\cap\Omega_x)}(\partial\Omega\cap\Omega_x)} \leq \tilde{c} \|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega_x)},$$

where $\tilde{c} = \frac{\max\{\tilde{c}_0, 1\}}{N} \sum_{i=1}^N c_i > 0$. Hence

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_x) \hookrightarrow L^{r^+(\partial\Omega\cap\Omega_x)}(\partial\Omega \cap \Omega_x) \quad \text{continuously.} \tag{14}$$

On the other hand, the fact that $p_i(x) > 1$ for all $x \in \overline{\Omega}$ implies that

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_x) \hookrightarrow W^{1,1}(\Omega_x) \quad \text{continuously.} \tag{15}$$

Moreover, by the Gagliardo trace theorem,

$$W^{1,1}(\Omega_x) \hookrightarrow L^1(\partial\Omega \cap \Omega_x) \quad \text{compactly.} \tag{16}$$

Combining (15) and (16) we obtain that

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_x) \hookrightarrow L^1(\partial\Omega \cap \Omega_x) \quad \text{compactly.} \tag{17}$$

By (14) and (17) and by interpolation between $L^1(\partial\Omega \cap \Omega_x)$ and $L^{r^+(\partial\Omega\cap\Omega_x)}(\partial\Omega \cap \Omega_x)$ we arrive at

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_x) \hookrightarrow L^{r^+(\partial\Omega\cap\Omega_x)}(\partial\Omega \cap \Omega_x) \quad \text{compactly.}$$

Since $r(x) < r^+(\partial\Omega \cap \Omega_x)$ we use again Theorem 1 and by the above relation

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega_x) \hookrightarrow L^{r(\cdot)}(\partial\Omega \cap \Omega_x) \quad \text{compactly.}$$

Taking into account the finite covering theorem for the compact set $\partial\Omega$ we can deduce that

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega) \quad \text{compactly.} \quad \square$$

We now have all the necessary tools for proving the existence of a weak solution to (5). Before starting the discussion, we point out that everywhere below we will work under the following hypothesis.

Hypothesis (H). We consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, to be a bounded open set with smooth boundary and $\overrightarrow{p} \in \left(C_+(\overline{\Omega})\right)^N$, and we assume that for all $i \in \{1, \ldots, N\}$, the applications $a_i, f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying (A1)–(A3), (F), and (G), respectively, and $b : \Omega \to \mathbb{R}$ satisfies (B).

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For simplicity, we define

$$W^{1,\vec{p}(\cdot)}(\Omega) = E \quad \text{and} \quad \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|\cdot\|.$$

Now let us define the notion of a weak solution.

Definition 1. By a weak solution to problem (5) we mean a function $u \in E$ such that

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi\, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u\varphi\, dx - \int_\Omega f(x,u)\varphi\, dx - \int_{\partial\Omega} g(x,u)\varphi\, dS = 0, \tag{18}$$

for all $\varphi \in E$.

With problem (5) we associate the energy functional $I : E \to \mathbb{R}$, defined by

$$I(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right)\, dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)}\, dx - \int_\Omega F(x, u_+)\, dx - \int_{\partial\Omega} G(x, u_+)\, dx,$$

where $u_+(x) = \max\{u(x), 0\}$.

For simplicity, we denote by $\Lambda, J : E \to \mathbb{R}$ the functionals

$$\Lambda(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx$$

and

$$J(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right)\, dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)}\, dx = \Lambda(u) + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)}\, dx.$$

We recall the following result.

Lemma 1 *(See [10, Lemma 3.4]). The functional Λ is well-defined on E. In addition, Λ is of class $C^1(E, \mathbb{R})$ and*

$$\langle \Lambda'(u), \varphi\rangle = \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi\, dx,$$

for all $u, \varphi \in E$.

Remark 1. The study in [10] is conducted for $E = W_0^{1,\vec{p}(\cdot)}(\Omega)$, that is, the closure of $C_0^\infty(\Omega)$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} = \sum_{i=1}^N \left\|\partial_{x_i} u\right\|_{L^{p_i(\cdot)}(\Omega)}.$$

Since we want to avoid repeating the same arguments and the calculus is almost identical, we omit the proof.

Due to Lemma 1, standard calculus leads to the fact that I is well-defined on E and $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), \varphi\rangle = \int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi\, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u\varphi\, dx - \int_\Omega f(x,u)\varphi\, dx - \int_{\partial\Omega} g(x,u)\varphi\, dS,$$

for all $u, \varphi \in E$. Obviously, the critical points of I are weak solutions to (5), so, by means of Theorem 5, we intend to establish the existence of critical points in order to deduce the existence of weak solutions. Our second main result is the following.

Theorem 7. *If Hypothesis (H) is fulfilled, then there exists a weak solution to problem (5).*

As suggested above, for the proof of Theorem 7 we show that the energy functional I fulfills the hypotheses of Theorem 5. To this end, we proceed with the following lemma.

Lemma 2. *If Hypothesis (H) is fulfilled, then the functional I is coercive.*

Proof. Let $u \in E$ be such that $\|u\| \to \infty$. Using (A2) we deduce

$$\Lambda(u) \geq \frac{1}{p_M^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}. \tag{19}$$

We define the following notation:

$$\mathcal{I}_1 = \{i \in \{1, \dots, N\} : \|\partial_{x_i} u\|_{L^{p_i(\cdot)}} \leq 1\}$$

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and

$$\mathcal{I}_2 = \{i \in \{1, \ldots, N\} : \|\partial_{x_i} u\|_{L^{p_i(\cdot)}} > 1\}.$$

By (8)–(10) and (19) we find that

$$\begin{aligned} \Lambda(u) &\geq \frac{1}{p_M^+}\left(\sum_{i\in\mathcal{I}_1} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}}^{p_M^+} + \sum_{i\in\mathcal{I}_2} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}}^{p_m^-}\right) \\ &\geq \frac{1}{p_M^+}\left(\sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}}^{p_m^-} - \sum_{i\in\mathcal{I}_1} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}}^{p_m^-}\right) \\ &\geq \frac{1}{p_M^+}\left(\sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}}^{p_m^-} - N\right). \end{aligned}$$

By the generalized mean inequality or the Jensen inequality applied to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+, a(t) = t^{p_m^-}, p_m^- > 1$ we get

$$\Lambda(u) \geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i(\cdot)}}\right)^{p_m^-} - N\right]. \tag{20}$$

We analyze now the two cases corresponding to the values of $\|u\|_{L^{p_M(\cdot)}}$.

Case 1: $\|u\|_{L^{p_M(\cdot)}} \geq 1$.

By (20) and (B) we have

$$J(u) \geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i(\cdot)}}\right)^{p_m^-} - N\right] + \frac{b_0}{p_M^+}\|u\|_{L^{p_M(\cdot)}}^{p_m^-},$$

and thus

$$J(u) \geq \frac{1}{2^{p_m^-} p_M^+} \min\left\{\frac{1}{N^{p_m^- -1}}, b_0\right\} \|u\|^{p_m^-} - \frac{1}{p_M^+ N^{p_m^- -2}}. \tag{21}$$

Case 2: $\|u\|_{L^{p_M(\cdot)}} < 1$. Then,

$$\begin{aligned} J(u) &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i(\cdot)}}\right)^{p_m^-} - N\right] \\ &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i(\cdot)}}\right)^{p_m^-} + \|u\|_{L^{p_M(\cdot)}}^{p_m^-} - N - 1\right]. \end{aligned}$$

We obtain

$$J(u) \geq \frac{1}{2^{p_m^-} p_M^+} \min\left\{\frac{1}{N^{p_m^- -1}}, 1\right\} \|u\|^{p_m^-} - \frac{N+1}{p_M^+ N^{p_m^- -1}}. \tag{22}$$

In conclusion, by (21) and (22), we deduce that there exists $\tilde{k}_0, \tilde{k}_3 > 0$ such that

$$J(u) \geq \tilde{k}_0 \|u\|^{p_m^-} - \tilde{k}_3. \tag{23}$$

Let us evaluate now the other two terms from the formula of I. By (F),

$$\int_\Omega F(x, u_+)\, dx \leq k_1\|u\|_{L^1(\Omega)} + \frac{k_1}{q^-}\int_\Omega u_+^{q(x)} dx.$$

Since

$$(u_+(x))^{q(x)} \leq |u(x)|^{q^-} + |u(x)|^{q^+} \quad \forall x \in \overline{\Omega}$$

and

$$L^{q^+}(\Omega) \hookrightarrow L^{q^-}(\Omega) \hookrightarrow L^1(\Omega) \quad \text{continuously},$$

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we infer that there exists $\bar{k}_1 > 0$ such that

$$\int_\Omega F(x, u_+)dx \le \bar{k}_1 \left(\|u\|^{q^-}_{L^{q^+}(\Omega)} + \|u\|^{q^+}_{L^{q^+}(\Omega)} \right). \tag{24}$$

Theorem 3 yields the continuous embedding

$$E \hookrightarrow L^{q^+}(\Omega)$$

because $q^+ < p_m^-$. Thus, using (24) and the fact that $\|u\| \ge 1$, we deduce that there exists $\tilde{k}_1 > 0$ such that

$$\int_\Omega F(x, u_+)\, dx \le \tilde{k}_1 \|u\|^{q^+}. \tag{25}$$

Using condition (G) instead of (F) and Theorem 6 instead of Theorem 3, by similar arguments, we obtain the existence of a positive constant $\tilde{k}_2$ such that

$$\int_\Omega G(x, u_+)\, dx \le \tilde{k}_2 \|u\|^{r^+}, \tag{26}$$

where r is given by condition (G). Putting together relations (23), (25) and (26), we arrive at

$$I(u) \ge \tilde{k}_0 \|u\|^{p_m^-} - \tilde{k}_1 \|u\|^{q^+} - \tilde{k}_2 \|u\|^{r^+} - \tilde{k}_3.$$

Keeping in mind the fact that $q^+, r^+ < p_m^-$, we find that $I(u) \to \infty$ when $\|u\| \to \infty$, and hence I is coercive. □

Lemma 3. *If Hypothesis (H) is fulfilled, then the functional I is weakly lower semi-continuous.*

Proof. We start by showing that J is weakly lower semi-continuous. By [25, Section 1.4], it is enough to prove that J is lower semi-continuous. To this end, fix $u \in E$ and $\epsilon > 0$. By (A3) and Theorem 4 we deduce that for any $v \in E$, the following inequality holds:

$$J(v) \ge J(u) + \sum_{i=1}^N \int_\Omega a_i(x, \partial_{x_i} u)(\partial_{x_i} v - \partial_{x_i} u)\, dx + \int_\Omega b(x)|u|^{p_M(x)-2} u(v-u)\, dx.$$

Using (A1), (B) and the Hölder type inequality (7) we infer

$$\begin{aligned} J(v) \ge\ & J(u) - \max\{\bar{c}_1, \dots, \bar{c}_N\} \sum_{i=1}^N \int_\Omega d_i(x)|\partial_{x_i} v - \partial_{x_i} u|\, dx \\ & - \max\{\bar{c}_1, \dots, \bar{c}_N\} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)-1} \left| \frac{\partial}{\partial x_i}(v-u) \right| dx - \|b\|_{L^\infty(\Omega)} \int_\Omega \left| |u|^{p_M(x)-2} u \right| |v-u|\, dx \\ \ge\ & J(u) - 2\max\{\bar{c}_1, \dots, \bar{c}_N\} \sum_{i=1}^N \|d_i\|_{L^{p_i'(\cdot)}(\Omega)} \|\partial_{x_i} v - \partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)} \\ & - 2\max\{\bar{c}_1, \dots, \bar{c}_N\} \sum_{i=1}^N \| |\partial_{x_i} u|^{p_i(x)-1} \|_{L^{p_i'(\cdot)}(\Omega)} \|\partial_{x_i} v - \partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)} \\ & - \|b\|_{L^\infty(\Omega)} \| |u|^{p_M(x)-1} \|_{L^{p_M'(\cdot)}(\Omega)} \|v-u\|_{L^{p_M(\cdot)}(\Omega)}. \end{aligned}$$

The above inequality and relation (11) imply that there exists $C > 0$ such that

$$J(v) \ge J(u) - C\|v-u\| \ge J(u) - \epsilon,$$

for all $v \in E$ with $\|v-u\| < \delta = \epsilon/C$. Therefore J is weakly lower semi-continuous.

Next, we define

$$h_1(u) = \int_\Omega F(x, u)\, dx \quad \text{and} \quad h_2(u) = \int_{\partial\Omega} G(x, u)\, dx.$$

Then $h_1', h_2' : E \to E^\star$ are completely continuous, that is, if $u_n \rightharpoonup u$, then $h_1'(u_n) \to h_1'(u)$ and $h_2'(u_n) \to h_2'(u)$. Hence the functionals h_1', h_2' are weakly continuous and, since J is weakly lower semi-continuous, we conclude that I is weakly lower semi-continuous. □

Proof of Theorem 7. The proof follows directly from Lemmas 2 and 3 and Theorem 5. □

Since we have an existence result, we are concerned with the uniqueness of the solution.

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Theorem 8. *If, in addition to Hypothesis* (H)*, the conditions* (F0), (G0) *are fulfilled, then the weak solution to problem* (5) *is unique.*

Proof. We suppose that there exist two weak solutions to problem (5), that is, u_1 and u_2. We replace the solution u by u_1 in (18) and we choose $\varphi = u_1 - u_2$. Then

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u_1\right) \partial_{x_i}(u_1 - u_2)\, dx + \int_\Omega b(x)|u_1|^{p_M(x)-2} u_1 (u_1 - u_2)\, dx \\ - \int_\Omega f(x, u_1)(u_1 - u_2)\, dx - \int_{\partial\Omega} g(x, u_1)(u_1 - u_2)\, dx = 0.$$

Next, we replace the solution u by u_2 in (18) and we choose $\varphi = u_2 - u_1$. We have

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u_2\right) \partial_{x_i}(u_2 - u_1)\, dx + \int_\Omega b(x)|u_2|^{p_M(x)-2} u_2 (u_2 - u_1)\, dx \\ - \int_\Omega f(x, u_2)(u_2 - u_1)\, dx - \int_{\partial\Omega} g(x, u_2)(u_2 - u_1)\, dx = 0.$$

By combining the previous two equalities we obtain

$$\int_\Omega \left\{ \sum_{i=1}^N \left[a_i\left(x, \partial_{x_i} u_1\right) - a_i\left(x, \partial_{x_i} u_2\right)\right] (\partial_{x_i} u_1 - \partial_{x_i} u_2) \right\} dx + \int_\Omega b(x)\left[|u_1|^{p_M(x)-2} u_1 - |u_2|^{p_M(x)-2} u_2\right] (u_1 - u_2)\, dx \\ - \int_\Omega [f(x, u_1) - f(x, u_2)]\,(u_1 - u_2)\, dx - \int_{\partial\Omega} [g(x, u_1) - g(x, u_2)]\,(u_1 - u_2)\, dx = 0.$$

By (A3), (F0) and (G0), all the terms in the above equality are positive unless $u_1 = u_2$, and this yields the uniqueness of the solution. □

4. Additional comments

One of the questions that the reader might ask is the following: what happens if we change the order of the exponents q^+ and p_m^-, or r^+ and p_m^-? Does problem (5) still have solution? Two of the most well known variational tools are the mountain pass theorem of Ambrosetti and Rabinowitz and the Ekeland variational principle (see for example [26]). Are they suitable for our problem? We consider the following assumptions.

(F1) There exist a positive constant k_1 and $q \in L_+^\infty(\Omega)$ with $1 < p_M^+ < q^-$, such that

$$|f(x, s)| \le k_1\left(1 + |s|^{q(x)-1}\right),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(F2) f verifies an Ambrosetti–Rabinowitz type condition: there exists a constant $\alpha_1 > p_M^+$ such that for every $x \in \Omega$

$$0 < \alpha_1 F(x, s) \le s f(x, s), \quad \forall s > 0.$$

(G1) There exist a positive constant k_2 and $r \in C(\overline{\Omega})$ with $1 < p_M^+ < r^-$ and $r(x) < \min\{p_1^\partial(x), \ldots, p_N^\partial(x)\}$ for all $x \in \partial\Omega$ such that

$$|g(x, s)| \le k_2\left(1 + |s|^{r(x)-1}\right),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(G2) g verifies an Ambrosetti–Rabinowitz type condition: there exists a constant $\alpha_2 > p_M^+$ such that for every $x \in \partial\Omega$

$$0 < \alpha_1 G(x, s) \le s g(x, s), \quad \forall s > 0.$$

If we replace conditions (F) and (G) by (F1) and (G1), respectively, and we add conditions (F2), (G2) to Hypothesis (H), we can expect to obtain the existence of solutions to problem (5) by using the mountain pass theorem instead of Theorem 5. However, a first remark is that we need a better embedding theorem because when $q^- > p_M^+$ we cannot apply Theorem 3. For $h, p_i \in C_+(\overline{\Omega}),\ i \in \{1, \ldots, N\}$, we define the notation

$$h^\star(x) = \begin{cases} Nh(x)/[N - h(x)] & \text{if} \quad h(x) < N, \\ \infty & \text{if} \quad h(x) \ge N, \end{cases}$$

and

$$\bar{p}(x) = \frac{N}{\sum_{i=1}^N 1/p_i(x)}.$$

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Furthermore, we say that a set $\Omega \subset \mathbb{R}^N$ is a rectangular-like domain if it is a union of finitely many rectangular domains with edges parallel to the coordinate axes in $\mathbb{R}^N$. Recently, the following result was proved.

Theorem 9 (*See* [6, *Theorem 2.5*]). *Let $\Omega \subset \mathbb{R}^N$ be a rectangular-like domain and $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$. If $q \in C_+(\overline{\Omega})$ and*

$$q(x) < \max\{\bar{p}^\star(x), p_M(x)\} \quad \text{for all } x \in \overline{\Omega},$$

then we have the compact embedding

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

Notice that, in the statement of Theorem 9, Ω is not a set with smooth boundary; hence we cannot define the outer normal. Moreover, since $W^{1,\vec{p}(\cdot)}(\Omega)$ can be considered a generalization of the space $W^{1,\vec{p}}(\Omega)$, with the exponent $\vec{p}$ being a constant vector, there are reasons for believing that Theorem 9 fails for domains that are not of the rectangular type; see [27]. Therefore, at least for the time being, we must remain faithful to Theorem 3 and we cannot apply the mountain pass theorem.

The next question arises: do we have an existence result if in Hypothesis (H) we replace (G) by (G1), we keep (F) as it is and we add (F2), (G2)? Unfortunately, there is another impediment in our way: we do not know whether Λ is of type (S+). We recall that Λ is said to be of type (S+) if any sequence $(u_n)_n \subset E$ that is weakly convergent to $u \in E$ such that

$$\limsup_{n\to\infty} \langle J'(u_n), u_n - u\rangle \leq 0$$

converges strongly to u in E. Due to [9, Lemma 2], we know that Λ is of type (S+) when $E = W_0^{1,\vec{p}(\cdot)}(\Omega)$. To follow the idea from the proof of [9, Lemma 2] and to apply it to the case $E = W^{1,\vec{p}(\cdot)}(\Omega)$ we need the following embedding:

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{p_M(\cdot)}(\Omega) \quad \text{compactly}. \tag{27}$$

This embedding is not provided by Theorem 3, and, as we previously discussed, we do not have an improved embedding theorem for open bounded sets with smooth boundary. Adding a uniform convexity assumption to (A1)–(A3), the authors of [11] obtained a result similar to [9, Lemma 2]. However, the problem persists: the proof can be adapted to the case of $W^{1,\vec{p}(\cdot)}(\Omega)$ from the case $W_0^{1,\vec{p}(\cdot)}(\Omega)$ if (27) holds. Therefore, until this difficulty is overcome, we can use neither the mountain pass theorem nor the Ekeland principle.

And finally, we make our last comment. There exists a weak solution to the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) + b_1(x)|u|^{p_M(x)-2}u = f(x,u) + b_2(x) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \nu_i = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where $\Omega, a_i, b_1, \vec{p}, f, g$ satisfy Hypothesis (H) and $b_2 \in L^\infty(\Omega)$, $b_2 \not\equiv 0$. Obviously the proof will follow the same steps as were presented in Section 3 and the last term will not raise any problems when showing that the functional attached to problem (28) is coercive and weakly lower semi-continuous. We direct attention to problem (28) only because, on adding the function b_2, the function $u \equiv 0$ cannot be a solution, independently of the forms of f and g.

Acknowledgment

The authors acknowledge the support through Grant CNCSIS–PNII-ID-PCE-2011-3-0195.

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