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Critical singular problems on infinite cones

Vincenţiu Rădulescu[a], Didier Smets[b,*]

[a] Department of Mathematics, University of Craiova, 1100 Craiova, Romania
[b] Laboratoire Jacques-Louis Lions, Université de Paris 6, 175 rue du Chevaleret, 75013 Paris, France

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Abstract

We prove existence results for non autonomous perturbations of critical singular elliptic boundary value problems. The non singular case was treated by Tarantello (Ann. Inst. H. Poincaré, Analyse Non-linéaire 9 (1992) 281) for bounded domains; here the singular weight allows for unbounded domains as cones and give rise to a different non compactness picture (as was first remarked by Caldiroli and Musina (Calc. Variations PDE 8 (1999) 365)).

Keywords: Singular weights; Critical exponent; Unbounded domains; Caffarelli–Kohn–Nirenberg inequalities

1. Introduction

Let Ω be an open set in $\mathbb{R}^N$, $N \geqslant 2$ and let $\alpha \in (0,2)$. For any $\zeta \in C_c^\infty(\Omega)$, define

$$\|\zeta\|_\alpha = \left(\int_\Omega |x|^\alpha |\nabla \zeta|^2 \, \mathrm{d}x \right)^{1/2}.$$

Let $H_0^1(\Omega; |x|^\alpha)$ be the closure of $C_c^\infty(\Omega)$ with respect to the $\|\cdot\|_\alpha$-norm. It turns out that $H_0^1(\Omega; |x|^\alpha)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_\alpha = \int_\Omega |x|^\alpha \nabla u \cdot \nabla v \, \mathrm{d}x, \qquad \forall u, v \in H_0^1(\Omega; |x|^\alpha).$$

If $\Omega = \mathbb{R}^N$ we set $H^1(\mathbb{R}^N; |x|^\alpha) = H_0^1(\mathbb{R}^N; |x|^\alpha)$. We remark that if Ω_1 and Ω_2 are arbitrary open sets in $\mathbb{R}^N$ such that $\Omega_1 \subset \Omega_2$ then $H_0^1(\Omega; |x|^\alpha) \hookrightarrow H_0^1(\Omega_2; |x|^\alpha)$, with continuous embedding. We also point out that since we allow the cases $0 \in \bar{\Omega}$ or Ω

* Corresponding author.
E-mail addresses: radules@ann.jussieu.fr (V. Rădulescu), smets@ann.jussieu.fr (D. Smets).

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unbounded then there is no inclusion relationship between $H_0^1(\Omega;|x|^\alpha)$ and the standard Sobolev space $H_0^1(\Omega)$. However, the Caffarelli–Kohn–Nirenberg inequality [4] (see also [6]) asserts that $H_0^1(\Omega;|x|^\alpha)$ is continuously embedded in $L^{2_\alpha^*}(\Omega)$, where $2_\alpha^* = 2N/(N-2+\alpha)$. More precisely, there exists $C_\alpha > 0$ such that

$$\left(\int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}x\right)^{1/2_\alpha^*} \leqslant C_\alpha \left(\int_\Omega |x|^\alpha |\nabla u|^2 \mathrm{d}x\right)^{1/2},$$

for any $u \in H_0^1(\Omega;|x|^\alpha)$.

Consider the problem

$$\begin{cases} -\mathrm{div}(|x|^\alpha \nabla u) = |u|^{2_\alpha^*-2}u & \text{in } \Omega, \\ u \geqslant 0,\ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1)$$

We observe that degeneracy occurs in (1) if $0 \in \bar{\Omega}$ or if Ω is unbounded. We also point out that if 2_α^* in problem (1) is replaced by a subcritical exponent $p \in [2, 2_\alpha^*)$ then the corresponding equation is characterized by local compactness, and existence results are carried out in an easier way.

Consider the quotient

$$S_\alpha(u;\Omega) = \frac{\int_\Omega |x|^\alpha |\nabla u|^2 \mathrm{d}x}{\left(\int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}x\right)^{2/2_\alpha^*}},$$

and denote

$$S_\alpha(\Omega) = \inf_{u \in H_0^1(\Omega;|x|^\alpha)\setminus\{0\}} S_\alpha(u;\Omega). \qquad (2)$$

It is obvious that if $u \in H_0^1(\Omega;|x|^\alpha)$ satisfies

$$\int_\Omega |x|^\alpha |\nabla u|^2 \mathrm{d}x = S_\alpha(\Omega) \quad \text{and} \quad \int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}x = 1,$$

then the function $U(x) = [S_\alpha(\Omega)]^{1/(2_\alpha^*-2)}u(x)$ is a solution of (1).

Caldiroli and Musina [5] studied the critical case and they showed that some concentration phenomena may occur in (1), due to the action of the non compact group of dilations in $\mathbb{R}^N$. They proved in [5] that if $\alpha \in (0,2)$ then, in certain cases, $S_\alpha(\Omega)$ is attained in $H_0^1(\Omega;|x|^\alpha)$ by a positive function, so problem (1) has a solution. We point out (see [10, Theorem III.1.2]) that $S_\alpha(\Omega)$ is never attained in $H_0^1(\Omega)$ in the limiting case $\alpha = 0$ and if $\Omega \neq \mathbb{R}^N$. For the study of further Critical Singular problems we also refer to [7,9].

Let $H^{-1}(\Omega;|x|^\alpha)$ be the dual space of $H_0^1(\Omega;|x|^\alpha)$ and denote by $\|\cdot\|_{-1}$ the norm in $H^{-1}(\Omega;|x|^\alpha)$. For any $f \in H^{-1}(\Omega;|x|^\alpha)$, consider the perturbed problem

$$\begin{cases} -\mathrm{div}(|x|^\alpha \nabla u) = |u|^{2_\alpha^*-2}u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (3)$$

We say that a function $u \in H_0^1(\Omega; |x|^\alpha)$ is a solution of problem (3) if u is a critical point of the energy functional

$$J(u) = \frac{1}{2} \int_\Omega |x|^\alpha |\nabla u|^2 \mathrm{d}x - \frac{1}{2_\alpha^*} \int_\Omega |u|^{2_\alpha^*} \mathrm{d}x - \int_\Omega f u \, \mathrm{d}x.$$

We observe that the Caffarelli–Kohn–Nirenberg inequality ensures that J is well defined on the space $H_0^1(\Omega; |x|^\alpha)$. Moreover, by the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{2_\alpha^*}(\Omega)$, the functional J is Fréchet differentiable on $H_0^1(\Omega; |x|^\alpha)$.

Perturbations of critical semilinear boundary value problems on bounded domains were initially studied by Tarantello in [11]. Our purpose is to prove a corresponding multiplicity result for the degenerate problem (3). Notice that in our case, Ω will be unbounded. We first need some preliminaries. Set

$$s_\alpha^0(\Omega) = \lim_{r \to 0} S_\alpha(\Omega \cap B_r)$$

and

$$s_\alpha^\infty(\Omega) = \lim_{r \to \infty} S_\alpha(\Omega \setminus B_r).$$

These limits are well defined because the mappings $r \mapsto S_\alpha(\Omega \cap B_r)$ and $r \mapsto S_\alpha(\Omega \setminus B_r)$ are easily seen to be, respectively, nonincreasing and nondecreasing.

Condition $\mathscr{C}$. *We say that $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ satisfies Condition $\mathscr{C}$ provided that Ω is a cone in $\mathbb{R}^N$, or $\Omega = \mathbb{R}^N$, or*

$$S_\alpha(\Omega) < \min\{s_\alpha^0(\Omega), s_\alpha^\infty(\Omega)\}. \tag{4}$$

We recall that $\Omega \subset \mathbb{R}^N$ is a cone if Ω has Lipschitz boundary and if $\lambda x \in \Omega$ for every $\lambda > 0$ and $x \in \Omega$. If Ω is a cone then

$$S_\alpha(\Omega) = s_\alpha^0(\Omega) = s_\alpha^\infty(\Omega),$$

so equality holds in (4) (see [5, Leemma 3.9]). We also point out (see Caldiroli–Musina [5]) the following situations in which property (4) is fulfilled:

(i) $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 is a cone and Ω_1 is an open bounded set such that $0 \notin \overline{\Omega_1}$;
(ii) $\Omega = I \times \mathbb{R}^{N-1}$, where $I = \mathbb{R}$, or $I = (0, +\infty)$, or $I = (-\infty, 0)$, or I is bounded and $0 \notin \bar{I}$.

Denote by E_+ the positive cone of $E = H^{-1}(\Omega; |x|^\alpha)$. This means that $f \in E_+$ if and only if $f \neq 0$ and

$$\int_\Omega f u \, \mathrm{d}x \geqslant 0,$$

for any $u \in H_0^1(\Omega; |x|^\alpha)$ such that $u \geqslant 0$ a.e. in Ω.

Our main result is the following

Theorem 1.1. *Assume that* $\alpha\in(0,2)$ *and* Ω *satisfies Condition* $\mathscr{C}$. *Then, for each* $g\in E_+$, *there exists* $\varepsilon_0>0$ *such that for all* $0<\varepsilon\leqslant\varepsilon_0$, *problem* (3) *with* $f=\varepsilon g$ *has at least two positive solutions.*

Remark 1.2. (a) In the previous theorem, ε_0 can be chosen uniformly for g in a compact subset of E_+.

(b) The existence of at least two solutions (not necessarily positive) when g belongs to E instead of E_+ is less clear. The sign condition can easily be weakened, but we think the general case should require some additional assumption.

2. The first solution

We first recall that if c is a real number, X is a Banach space and $F: X\to\mathbb{R}$ is a C^1-functional then F satisfies condition $(PS)_c$ if any sequence (u_n) in X such that $F(u_n)\to c$ and $\|F'(u_n)\|_{X*}\to 0$ as $n\to\infty$, is relatively compact. It is obvious that if a Palais–Smale, sequence converges strongly, then its limit is a critical point. Our first result shows that if a $(PS)_c$ sequence of J is weakly convergent then its limit is a solution of problem (3).

Lemma 2.1. *Let* $(u_n)\subset H_0^1(\Omega;|x|^\alpha)$ *be a* $(PS)_c$ *sequence of* J, *for some* $c\in\mathbb{R}$. *Assume that* (u_n) *converges weakly to some* u_0. *Then* u_0 *is a solution of problem* (3).

Proof. Consider an arbitrary function $\zeta\in C_0^\infty(\Omega)$ and set $\omega=\operatorname{supp}(\zeta)$. Obviously $J'(u_n)\to 0$ in $H_0^1(\Omega;|x|^\alpha)$ implies $\langle J'(u_n),\zeta\rangle\to 0$ as $n\to\infty$, that is

$$\lim_{n\to\infty}\left(\int_\omega |x|^\alpha\nabla u_n\cdot\nabla\zeta\,\mathrm{d}x-\int_\omega |u_n|^{2_\alpha^*-2}u_n\zeta\,\mathrm{d}x-\int_\omega f\zeta\,\mathrm{d}x\right)=0. \tag{5}$$

Since $u_n\rightharpoonup u_0$ in $H_0^1(\Omega;|x|^\alpha)$ it follows that

$$\lim_{n\to\infty}\int_\omega |x|^\alpha\nabla u_n\cdot\nabla\zeta\,\mathrm{d}x=\int_\omega |x|^\alpha\nabla u_0\cdot\nabla\zeta\,\mathrm{d}x. \tag{6}$$

The boundedness of (u_n) in $H_0^1(\Omega;|x|^\alpha)$ and the Caffarelli–Kohn–Nirenberg inequality imply that $|u_n|^{2_\alpha^*-2}u_n$ is bounded in $L^{2_\alpha^*/(2_\alpha^*-1)}(\Omega;|x|^\alpha)$. Combining this with the convergence (up to be a sequence)

$$|u_n|^{2_\alpha^*-2}u_n\to|u_0|^{2_\alpha^*-2}u_0\quad\text{a.e. in }\Omega$$

we deduce (see [1]) that $|u_0|^{2_\alpha^*-2}u_0$ is the weak limit of the sequence $|u_n|^{2_\alpha^*-2}u_n$ in the space $L^{2_\alpha^*/(2_\alpha^*-1)}(\Omega;|x|^\alpha)$. So

$$\lim_{n\to\infty}\int_\omega |u_n|^{2_\alpha^*-2}u_n\zeta\,\mathrm{d}x=\int_\omega |u_0|^{2_\alpha^*-2}u_0\zeta\,\mathrm{d}x. \tag{7}$$

From (5)–(7) we deduce that

$$\int_{\omega} |x|^{\alpha}\nabla u_0 \cdot \nabla\zeta \, dx - \int_{\omega} |u_0|^{2^*_\alpha-2}u_0\zeta \, dx - \int_{\omega} f\zeta \, dx = 0.$$

By density, this equality holds for any $\zeta \in H^1_0(\Omega; |x|^{\alpha})$ which means that $J'(u_0)=0$. □

Lemma 2.2. *There exists $\varepsilon_1 > 0$ such that problem (3) has at least one solution u_0 provided that $f \neq 0$ and $\|f\|_{-1} < \varepsilon_1$. Moreover, u_0 is positive if $f \in E_+$.*

Proof. The idea is to show that there exist $c_0 < 0$ and $R > 0$ such that J has the $(PS)_{c_0}$ property, where

$$c_0 = \inf\{J(u); u \in H^1_0(\Omega; |x|^{\alpha}) \text{ and } \|u\| \leqslant R\}. \tag{8}$$

Then we prove that c_0 is achieved by some $u_0 \in H^1_0(\Omega; |x|^{\alpha})$ and, furthermore, $J'(u_0)=0$.

Applying the Caffarelli–Kohn–Nirenberg inequality we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*_\alpha}\int_{\Omega}|u|^{2^*_\alpha}\, dx - \int_{\Omega} fu\, dx \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{1}{2^*_\alpha}\int_{\Omega}|u|^{2^*_\alpha}\, dx - \|f\|_{-1}\cdot\|u\| \\ &\geqslant \left(\frac{1}{2} - \frac{\varepsilon^2}{2}\right)\|u\|^2 - C\|u\|^{2^*_\alpha} - C_\varepsilon\|f\|^2_{-1}. \end{aligned}$$

Fixing $\varepsilon \in (0,1)$ we find $R > 0$, $\varepsilon_1 > 0$ and $\delta > 0$ such that $J(u) \geqslant \delta$ if $\|u\| = R$ and $\|f\|_{-1} < \varepsilon_1$.

Let c_0 be defined in (8). Since $f \neq 0$, $c_0 < J(0) = 0$. The set

$$\bar{B}_R := \{u \in H^1_0(\Omega; |x|^{\alpha}); \|u\| \leqslant R\}$$

becomes a complete metric space with respect to the distance

$$\operatorname{dist}(u,v) = \|u - v\| \quad \text{for any } u, v \in \bar{B}_R.$$

On the other hand, J is lower semi-continuous and bounded from below on $\bar{B}_R$. So, by Ekeland's variational principle [8, Theorem 1.1], for any positive integer n there exists u_n such that

$$c_0 \leqslant J(u_n) \leqslant c_0 + \frac{1}{n}, \tag{9}$$

and

$$J(w) \geqslant J(u_n) - \frac{1}{n}\|u_n - w\| \quad \text{for all } w \in \bar{B}_R. \tag{10}$$

We claim that $\|u_n\| < R$ for n large enough. Indeed, if $\|u_n\| = R$ for infinitely many n, we may assume, without loss of generality, that $\|u_n\| = R$ for all $n \geqslant 1$. It follows that $J(u_n) \geqslant \delta > 0$. Combining this with (9) and letting $n \to \infty$, we have $0 \geqslant c_0 \geqslant \delta > 0$ which is a contradiction.

We now prove that $\|J'(u_n)\|_{-1} \to 0$. Indeed, for any $u \in H_0^1(\Omega; |x|^\alpha)$ with $\|u\| = 1$, let $w_n = u_n + t_u$. For a fixed n, we have $\|w_n\| \leqslant \|u_n\| + t < R$, where $t > 0$ is small enough. Using (10) we obtain

$$J(u_n + tu) \geqslant J(u_n) - \frac{t}{n}\|u\|$$

that is

$$\frac{J(u_n + tu) - J(u_n)}{t} \geqslant -\frac{1}{n}\|u\| = -\frac{1}{n}.$$

Letting $t \searrow 0$, we deduce that $\langle J'(u_n), u\rangle \geqslant -1/n$ and a similar argument for $t \nearrow 0$ produces $|\langle J'(u_n), u\rangle| \leqslant 1/n$ for any $u \in H_0^1(\Omega; |x|^\alpha)$ with $\|u\| = 1$. So,

$$\|J'(u_n)\|_{-1} = \sup_{\|u\|=1} |\langle J'(u_n), u\rangle| \leqslant \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

We have obtained the existence of a $(PS)_{c_0}$ sequence, i.e. a sequence $(u_n) \subset H_0^1(\Omega; |x|^\alpha)$ with

$$J(u_n) \to c_0 \quad \text{and} \quad \|J'(u_n)\|_{-1} \to 0. \tag{11}$$

But $\|u_n\| \leqslant R$ shows that (u_n) converges weakly in $H_0^1(\Omega; |x|^\alpha)$, up to a subsequence. Therefore, by (11) and Lemma 2.1 we find that for some $u_0 \in H_0^1(\Omega; |x|^\alpha)$,

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega; |x|^\alpha), \quad u_n \to u_0 \text{ a.e in } \mathbb{R}^N \tag{12}$$

and

$$J'(u_0) = 0. \tag{13}$$

We now prove that $J(u_0) = c_0$. By (11) and (12) we have

$$o(1) = \langle J'(u_n), u_n\rangle = \int_\Omega |x|^\alpha |\nabla u_n|^2 \,\mathrm{d}x - \int_\Omega |u_n|^{2_\alpha^*} \,\mathrm{d}x - \int_\Omega f u_n \,\mathrm{d}x.$$

Therefore

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \int_\Omega |u_n|^{2_\alpha^*} \,\mathrm{d}x - \left(1 - \frac{1}{2_\alpha^*}\right) \int_\Omega f u_n \,\mathrm{d}x + o(1).$$

By (11)–(13) and Fatou's lemma we have

$$c_0 = \liminf_{n\to\infty} J(u_n) \geqslant \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \int_\Omega |x|^\alpha |u_0|^{2_\alpha^*} \,\mathrm{d}x - \left(1 - \frac{1}{2_\alpha^*}\right) \int_\Omega f u_0 \,\mathrm{d}x = J(u_0).$$

Since $u_0 \in \bar{B}_R$, it follows that $J(u_0) = c_0$. If $f \in E_+$, u_0 can be replaced by $|u_0|$, and the proof is complete. $\square$

3. A priori estimates for the second solution

Set

$$I(u)=\frac{1}{2}\int_\Omega |x|^\alpha|\nabla u|^2\,\mathrm{d}x-\frac{1}{2^*_\alpha}\int_\Omega |u|^{2^*_\alpha}\,\mathrm{d}x$$

and denote

$$S=\{u\in H^1_0(\Omega;|x|^\alpha)\setminus\{0\};\langle I'(u),u\rangle=0\}.$$

We first justify that $S\neq\emptyset$. Indeed, fix $u_0\in H^1_0(\Omega;|x|^\alpha)\backslash\{0\}$ and set, for any $\lambda>0$,

$$\Psi(\lambda)=\langle I'(\lambda u_0),\lambda u_0\rangle=\lambda^2\int_\Omega |x|^\alpha|\nabla u_0|^2\,\mathrm{d}x-\lambda^{2^*_\alpha}\int_\Omega |u_0|^{2^*_\alpha}\,\mathrm{d}x.$$

Since $2^*_\alpha>2$, it follows that $\Psi(\lambda)<0$ for λ large enough and $\Psi(\lambda)>0$ for λ sufficiently close to zero.

Hence there exists $\lambda_0\in(0,\infty)$ such that $\Psi(\lambda_0)=0$. This means that $\lambda_0u_0\in S$.

Lemma 3.1. *Let $I_\infty=\inf\{I(u);u\in S\}$. Then there exists $\bar u\in H^1_0(\Omega;|x|^\alpha)$ such that*

$$I_\infty=I(\bar u)=\sup_{t\geqslant 0}I(t\bar u). \tag{14}$$

Proof. We first claim that

$$I_\infty(u)=\sup_{t\geqslant 0}I(tu)\quad \forall u\in S. \tag{15}$$

Indeed, for some fixed $\varphi\in H^1_0(\Omega;|x|^\alpha)\setminus\{0\}$, denote

$$f(t)=I(t\varphi)=\frac{t^2}{2}\int_\Omega |x|^\alpha|\nabla u|^2\,\mathrm{d}x-\frac{t^{2^*_\alpha}}{2^*_\alpha}\int_\Omega |\varphi|^{2^*_\alpha}\,\mathrm{d}x.$$

We have

$$f'(t)=t\int_\Omega |x|^\alpha|\nabla u|^2\,\mathrm{d}x-t^{2^*_\alpha-1}\int_\Omega |\varphi|^{2^*_\alpha}\,\mathrm{d}x,$$

which vanishes for

$$t_0=t_0(\varphi)=\left\{\frac{\int_\Omega |x|^\alpha|\nabla u|^2\,\mathrm{d}x}{\int_\Omega |\varphi|^{2^*_\alpha}|\,\mathrm{d}x}\right\}^{1/(2^*_\alpha-2)}.$$

Hence

$$f(t_0)=I(t_0\varphi)=\sup_{t\geqslant 0}I(t\varphi)=\frac{2-\alpha}{2N}\left\{\frac{\int_\Omega |x|^\alpha|\nabla u|^2\,\mathrm{d}x}{\left(\int_\Omega |\varphi|^{2^*_\alpha}\,\mathrm{d}x\right)^{(N-2+\alpha)/N}}\right\}^{N/(2-\alpha)}.$$

It follows that

$$\inf_{\varphi\in H^1_0(\Omega;|x|^\alpha)\setminus\{0\}}\ \sup_{t\geqslant 0}I(t\varphi)=\frac{2-\alpha}{2N}\,[S_\alpha(\Omega)]^{N/(2-\alpha)}. \tag{16}$$

We now easily observe that for every $u \in S$ we have $t_0(u) = 1$. So, by (16), we find (15).

By Caldiroli–Musina [5, Theorems 2.2 and 3.1] the minimum is achieved in (2) by some function $U \in H_0^1(\Omega; |x|^\alpha)$. We prove in what follows that the function $\bar{u} := [S_\alpha(\Omega)]^{1/(2_\alpha^* - 2)} U$ satisfies (14). We first observe that $\bar{u} \in S$ and

$$I(\bar{u}) = \frac{2-\alpha}{2N} [S_\alpha(\Omega)]^{N/(2-\alpha)}. \tag{17}$$

So, by (15) and (17),

$$I_\infty = \inf_{u \in S} I(u) = \inf_{u \in S} \sup_{t \geqslant 0} I(tu) \geqslant \inf_{u \in H_0^1(\Omega;|x|^\alpha)\setminus\{0\}} \sup_{t \geqslant 0} I(tu)$$

$$= \frac{2-\alpha}{2N} [S_\alpha(\Omega)]^{N/(2-\alpha)} = I(\bar{u}),$$

which concludes our proof. □

Lemma 3.2. *Assume (u_n) is a $(PS)_c$ sequence of J that converges weakly to u_0 in $H_0^1(\Omega; |x|^\alpha)$. Then either (u_n) converges strongly in $H_0^1(\Omega; |x|^\alpha)$, or $c \geqslant J(u_0) + I_\infty$.*

Proof. Since (u_n) is a $(PS)_c$ sequence and $u_n \rightharpoonup u_0$ in $H_0^1(\Omega; |x|^\alpha)$ we have

$$J(u_n) = c + o(1) \quad \text{and} \quad \langle J'(u_n), u_n \rangle = o(1). \tag{18}$$

Set $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in $H_0^1(\Omega; |x|^\alpha)$ which implies

$$\int_\Omega |x|^\alpha \nabla v_n \cdot \nabla u_0 \, dx \to 0 \quad \text{as } n \to \infty,$$

$$\int_\Omega f v_n \, dx \to 0 \quad \text{as } n \to \infty.$$

We rewrite the above relations as

$$\|u_n\|^2 = \|u_0\|^2 + \|v_n\|^2 + o(1),$$

$$J(v_n) = I(v_n) + o(1). \tag{19}$$

The Brezis–Lieb Lemma (see [2]) combined with the Caffarelli–Kohn–Nirenberg Inequality yield

$$\int_\Omega (|u_n|^{2_\alpha^*} - |v_n|^{2_\alpha^*}) \, dx = \int_\Omega |u_0|^{2_\alpha^*} \, dx + o(1). \tag{20}$$

From (18)–(20) and Lemma 2.1 we find

$$o(1) + c = J(u_n) = J(u_0) + J(v_n) + o(1) = J(u_0) + I(v_n) + o(1),$$

$$o(1) = \langle J'(u_n), u_n \rangle = \langle J'(u_0), u_0 \rangle + \langle J'(v_n), v_n \rangle + o(1)$$

$$= \langle I'(v_n), v_n \rangle + o(1). \tag{21}$$

If $v_n \to 0$ in $H_0^1(\Omega;|x|^\alpha)$, then $u_n \to u_0$ in $H_0^1(\Omega;|x|^\alpha)$ and $J(u_0)=\lim_{n\to\infty} J(u_n)=c$.

If $v_n \nrightarrow 0$ in $H_0^1(\Omega;|x|^\alpha)$, then combining this with the fact that $v_n \rightharpoonup 0$ in $H_0^1(\Omega;|x|^\alpha)$ we may assume that $\|v_n\| \to l > 0$. Then, by (21),

$$c = J(u_0) + I(v_n) + o(1) \tag{22}$$

$$\mu_n = \langle I'(v_n), v_n\rangle = \int_\Omega |x|^\alpha |\nabla v_n|^2\,dx - \int_\Omega |v_n|^{2_\alpha^*}\,dx = \alpha_n - \beta_n, \tag{23}$$

where $\lim_{n\to\infty}\mu_n=0$, $\alpha_n=\int_\Omega |x|^\alpha|\nabla v_n|^2\,dx \geqslant \|v_n\|^2$ and $\beta_n=\int_\Omega |v_n|^{2_\alpha^*}\,dx \geqslant 0$. In virtue of (22), it remains to show that $I(v_n) \geqslant I_\infty + o(1)$. For $t>0$, we have

$$\langle I'(tv_n), tv_n\rangle = t^2\int_\Omega |x|^\alpha|\nabla v_n|^2\,dx - t^{2_\alpha^*}\int_\Omega |v_n|^{2_\alpha^*}\,dx.$$

If we prove the existence of a sequence (t_n) with $t_n \to 1$ and $\langle I'(t_nv_n), t_nv_n\rangle = 0$, then

$$I(v_n) = I(t_nv_n) + \frac{1-t_n^2}{2}\alpha_n - \frac{1-t_n^{2_\alpha^*}}{2_\alpha^*}\|v_n\|_{L^{2_\alpha^*}}^{2_\alpha^*} = I(t_nv_n) + o(1) \geqslant I_\infty + o(1)$$

and the conclusion follows. To do this, let $t=1+\delta$ with $\delta>0$ small enough and using (23) we obtain

$$\begin{aligned}\langle I'(tv_n), tv_n\rangle &= (1+\delta)^2\alpha_n - (1+\delta)^{2_\alpha^*}\beta_n = (1+\delta)^2\alpha_n - (1+\delta)^{2_\alpha^*}(\alpha_n-\mu_n)\\ &= \alpha_n(2\delta - 2_\alpha^*\delta + o(\delta)) + (1+\delta)^{2_\alpha^*}\mu_n = \alpha_n(2-2_\alpha^*)\delta + \alpha_n o(\delta)\\ &\quad + (1+\delta)^{2_\alpha^*}\mu_n.\end{aligned}$$

Since $\alpha_n \to \bar{l} \geqslant l^2 > 0$, $\lim_{n\to\infty}\mu_n = 0$ and $2_\alpha^* > 2$ then, for n large enough, we can define the sequence $\delta_n = 2|\mu_n|/\alpha_n(2_\alpha^*-2) > 0$ and $\delta_n \to 0$. Then

$$\langle I'((1+\delta_n)v_n), (1+\delta_n)v_n\rangle < 0 \quad \langle I'((1-\delta_n)v_n), (1-\delta_n)v_n\rangle > 0. \tag{24}$$

From (24) we deduce the existence of $t_n \in (1-\delta_n, 1+\delta_n)$ such that

$$t_n \to 1 \quad \text{and} \quad \langle I'(t_nv_n), t_n, v_n\rangle = 0.$$

This concludes our proof. □

Fix $\bar{u} \in H_0^1(\Omega;|x|^\alpha)$ such that (14) holds. Since $2 < 2_\alpha^*$, there exists $t_0 > 0$ such that

$$I(t\bar{u}) < 0 \quad \text{if } t \geqslant t_0$$

$$J(t\bar{u}) < 0 \quad \text{if } t \geqslant t_0.$$

Set

$$\mathscr{P} = \{\gamma \in C([0,1], H_0^1(\Omega;|x|^\alpha));\ \gamma(0)=0, \gamma(1)=t_0\bar{u}\} \tag{25}$$

$$c_1 = \inf_{\gamma\in\mathscr{P}}\sup_{u\in\gamma} J(u). \tag{26}$$

In the next result c_0, resp. c_1, are those defined in (8), resp. (26).

Lemma 3.3. *Given $g \in E_+$, $\|g\|_{-1} = 1$, there exist $R > 0$ and $\varepsilon_2 = \varepsilon_2(R) > 0$ such that $c_1 < c_0 + I_\infty$, for all $f = \varepsilon g$ with $\varepsilon \leqslant \varepsilon_2$.*

Proof. We first remark that

$$I_\infty + c_0 > 0, \tag{27}$$

provided that ε_1 and R given in the proof of Lemma 2.2 are sufficiently small. Indeed, let u_0 be the solution obtained in Lemma 2.2. Then, by Cauchy–Schwarz,

$$\begin{aligned} c_0 &= \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \int_\Omega |x|^\alpha |\nabla u_0|^2 \, dx - \left(1 - \frac{1}{2_\alpha^*}\right) \int_\Omega f u_0 \, dx \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \int_\Omega |x|^\alpha |\nabla u_0|^2 \, dx - \left(1 - \frac{1}{2_\alpha^*}\right) \|f\|_{-1} \cdot \|u_0\|. \end{aligned} \tag{28}$$

Applying the inequality

$$\alpha\beta \leqslant \frac{\alpha^2}{2} + \frac{\beta^2}{2} \quad \forall \alpha, \beta > 0$$

We find

$$\left(1 - \frac{1}{2_\alpha^*}\right) \|f\|_{-1} \cdot \|u_0\| \leqslant \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u_0\|^2 + \frac{(N - \alpha + 2)^2}{16N(2 - \alpha)} \|f\|_{-1}^2. \tag{29}$$

So, by (28) and (29),

$$c_0 \geqslant -\frac{(N - \alpha + 2)^2}{16N(2 - \alpha)} \|f\|_{-1}^2. \tag{30}$$

It follows that the negative number c_0 is close enough to 0 if $\|f\|_{-1}$ is small. But, by Lemma 3.1,

$$I_\infty = \frac{2 - \alpha}{2N} [S_\alpha(\Omega)]^{N/(2-\alpha)} > 0,$$

so (27) follows obviously.

In order to conclude the proof we observe, by the definition of c_1, that if suffices to show that

$$\sup_{t \geqslant 0} J(t\bar{u}) < c_0 + I_\infty, \tag{31}$$

if $\|f\|_{-1}$ is sufficiently small.

Next, using (27), the continuity of J and $J(0) = 0$, we obtain some $T_0 > 0$ which is uniform with respect to all f satisfying $0 < \|f\|_{-1} < \varepsilon_1$ such that, for some $\varepsilon' < \varepsilon_1$,

$$c_0 + I_\infty > \sup_{t \in [0, T_0]} J(t\bar{u}),$$

if $\|f\|_{-1} < \varepsilon'$. So, in order to prove (31), it sufficies to show that if $\|f\|_{-1}$ is small then

$$c_0 + I_\infty > \sup_{t \geqslant T_0} J(t\bar{u}). \tag{32}$$

But

$$J(t\bar{u}) = \frac{t^2}{2} \int_\Omega |x|^\alpha |\nabla \bar{u}|^2 \,\mathrm{d}x - \frac{t^{2^*_\alpha}}{2^*_\alpha} \int_\Omega |\bar{u}|^{2^*_\alpha} \,\mathrm{d}x - t \int_\Omega f \bar{u} \,\mathrm{d}x$$
$$\leqslant \frac{t^2}{2} \int_\Omega |x|^\alpha |\nabla \bar{u}|^2 \,\mathrm{d}x - \frac{t^{2^*_\alpha}}{2^*_\alpha} \int_\Omega |\bar{u}|^{2^*_\alpha} \,\mathrm{d}x - T_0 \int_\Omega f \bar{u} \,\mathrm{d}x,$$

for any $t \geqslant T_0$. But, by Lemma 3.1,

$$I(\bar{u}) = \frac{2-\alpha}{2N} [S_\alpha(\Omega)]^{N/(2-\alpha)}.$$

Hence, using an argument similar to that used for proving (28), we find

$$\sup_{t \geqslant T_0} J(t\bar{u}) \leqslant \sup_{t \geqslant T_0} \left(\frac{t^2}{2} \int_\Omega |x|^\alpha |\nabla \bar{u}|^2 \,\mathrm{d}x - \frac{t^{2^*_\alpha}}{2^*_\alpha} \int_\Omega |\bar{u}|^{2^*_\alpha} \,\mathrm{d}x \right) - T_0 \int_\Omega f \bar{u} \,\mathrm{d}x$$
$$\leqslant I_\infty - T_0 \int_\Omega f \bar{u} \,\mathrm{d}x < I_\infty + c_0,$$

if $f = \varepsilon g$ with $\varepsilon \leqslant \varepsilon''$. Indeed, it follows (30) that c_0 is quadratic in ε while $\int f \bar{u}$ is linear. Letting $\varepsilon_2 = \min\{\varepsilon', \varepsilon''\}$, we conclude the proof. □

4. Proof of Theorem 1.1 concluded

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Hence, by Lemma 2.2, we obtain the existence of a positive solution $u_0 \in H^1_0(\Omega; |x|^\alpha)$ of (3) such that $J(u_0) = c_0$.

On the other hand, since $J(|u|) \leqslant J(u)$ when $f \in E_+$, it follows from the Mountain Pass Theorem without the Palais–Smale condition [3, Theorem 2.2] that there exists a positive $(PS)_{c_1}$ sequence (u_n) of J, that is

$$J(u_n) = c_1 + o(1) \quad \text{and } \|J'(u_n)\|_{-1} \to 0.$$

This implies

$$c_1 + \frac{1}{2^*_\alpha} \|J'(u_n)\|_{-1} \cdot \|u_n\| + o(1) \geqslant J(u_n) - \frac{1}{2^*_\alpha} \langle J'(u_n), u_n \rangle$$
$$\geqslant \left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \|u_n\|^2$$
$$- \left(1 - \frac{1}{2^*_\alpha} \right) \|f\|_{-1} \cdot \|u_n\|. \tag{33}$$

Hence $\{u_n\}$ is a bounded sequence $H^1_0(\Omega; |x|^\alpha)$. So, up to a subsequence, we may assume that $u_n \rightharpoonup u_1 \geqslant 0$ in $H^1_0(\Omega; |x|^\alpha)$. Lemma 2.1 implies that u_1 is a solution of (3).

We prove in what follows that $u_0 \neq u_1$. For this aim we shall prove that $J(u_0) \neq J(u_1)$. Indeed, by Lemma 3.2, either $u_n \to u_1$ in $H_0^1(\Omega; |x|^\alpha)$ which gives

$$J(u_1) = \lim_{n\to\infty} J(u_n) = c_1 > 0 > c_0 = J(u_0)$$

and the conclusion follows, or

$$c_1 = \lim_{n\to\infty} J(u_n) \geqslant J(u_1) + I_\infty.$$

If we suppose that $J(u_1) = J(u_0) = c_0$, then $c_1 \geqslant c_0 + I_\infty$ which contradicts Lemma 3.3. This concludes our proof. □

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