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Planar Kirchhoff equations with critical exponential growth and trapping potential

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Abstract
We are concerned with the following Kirchhoff equation:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(u), \quad \text{in } \mathbb{R}^2,\\ u\in H^1(\mathbb{R}^2),\end{cases}$$

where $a,\ b$ are positive constants, $V\in\mathcal{C}(\mathbb{R}^2,(0,\infty))$ is a trapping potential, and f has critical exponential growth of Trudinger–Moser type. By developing some new analytical approaches and techniques, we prove the existence of nontrivial solutions and least energy solutions. Without any monotonicity conditions on f, we also give the mountain pass characterization of the least energy solution by constructing a fine path. In particular, we remove the common restriction on $\liminf_{t\to+\infty}\frac{tf(t)}{e^{\alpha_0 t^2}}$, which is crucial in the literature to overcome the loss of the compactness caused by the critical exponential nonlinearity. Our approach could be extended to other classes of critical exponential growth problems with trapping potentials.

Keywords Kirchhoff equation · Critical exponential growth · Trudinger–Moser inequality · Trapping potential

Mathematics Subject Classification 35J20 · 35J62 · 35Q55

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1 Introduction

This paper is concerned with the following Kirchhoff equation:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(u),\ x\in\mathbb{R}^2; \\ u\in H^1(\mathbb{R}^2), \end{cases} \tag{$\mathcal{K}$}$$

where $a, b > 0$ are two constants, $V \in \mathcal{C}(\mathbb{R}^2, (0,\infty))$ is the Rabinowitz type trapping potential, namely it satisfies

(V1) $0 < V_0 \le V(x) \le \liminf_{|y|\to\infty} V(y) = V_\infty$ for all $x \in \mathbb{R}^2$,

and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies the basic conditions below:

(F1) there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to+\infty}\frac{|f(t)|}{e^{\alpha t^2}}=\begin{cases}0, & \text{for all } \alpha>\alpha_0,\\ +\infty, & \text{for all } \alpha<\alpha_0;\end{cases} \tag{1.1}$$

(F2) $f(t) = o(t)$ as $t \to 0$ and $F(t):=\int_0^t f(s)\mathrm{d}s > 0$ for all $t \in \mathbb{R}\backslash\{0\}$.

As in Adimurthi and Yadava [3] and de Figueiredo et al. [10], we say that $f(t)$ has critical exponential growth at $t = \pm\infty$ if condition (F1) holds. It was shown by Trudinger [28] and Moser [22] that this kind of nonlinearity has the maximal growth that can be treated variationally in $H^1(\mathbb{R}^2)$, which is motivated by the following Trudinger–Moser inequality.

Lemma 1.1 [2, 4, 5] i) *If* $\alpha > 0$ *and* $u \in H^1(\mathbb{R}^2)$, *then*

$$\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)\mathrm{d}x<\infty;$$

ii) *if* $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \le 1$, $\|u\|_2 \le M < \infty$, *and* $\alpha < 4\pi$, *then there exists a constant* $\mathcal{C}(M,\alpha)$, *which depends only on* M *and* α, *such that*

$$\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)\mathrm{d}x\le\mathcal{C}(M,\alpha). \tag{1.2}$$

From (F1) and (F2), it follows that

$$\lim_{t\to0}\frac{F(t)}{t^2}=0 \tag{1.3}$$

and

$$\lim_{t\to+\infty}\frac{t^2F(t)}{e^{\alpha t^2}}=\begin{cases}0, & \text{for all } \alpha>\alpha_0,\\ +\infty, & \text{for all } \alpha<\alpha_0.\end{cases} \tag{1.4}$$

Then for any $\varepsilon > 0$, $\alpha > \alpha_0$ and $q > 0$, there exists $C = C(\varepsilon, \alpha, q) > 0$ such that

$$F(t)\le\varepsilon t^2+C|t|^qe^{\alpha t^2},\quad \forall\, t\in\mathbb{R}. \tag{1.5}$$

Using (1.5), a standard argument can show that the energy functional $\Phi : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left[a|\nabla u|^2+V(x)u^2\right]\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^2}F(u)\mathrm{d}x \tag{1.6}$$

associated with equation $(\mathcal{K})$, is of class $\mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$, and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^2} [a\nabla u \cdot \nabla v + V(x)uv]\,\mathrm{d}x + b\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x \int_{\mathbb{R}^2} \nabla u \cdot \nabla v \mathrm{d}x \\ - \int_{\mathbb{R}^2} f(u)v\mathrm{d}x, \quad \forall\, u, v \in H^1(\mathbb{R}^2). \tag{1.7}$$

Hence, the solutions of $(\mathcal{K})$ are the critical points of the functional Φ.

Problem $(\mathcal{K})$ has a profound physical meaning, which was proposed firstly by Kirchhoff [18] in the case where $\mathbb{R}^2$ is replaced by the bounded domain $\Omega \subset \mathbb{R}$. Nonlocal equations of this type model the vibration of elastic strings by considering the effect of the changes in the length of strings. We also point out that as is customary in quantum mechanics applications, the unknown u is the probability density function of a particle trapped inside a trapping potential well, traditionally modeled by $V(x)$.

After the pioneering contributions of Lions [17] and Pohozaev [25], the following Kirchhoff-type problem

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.8}$$

has been studied intensively by many researchers, where constants $a, b > 0$, $N \geq 2$, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. By variational methods, a number of important results of the existence and multiplicity of solutions for (1.8) were established under various conditions on V and f, especially when $N \geq 3$. As it is known, in the case $N \geq 3$, the nonlinearities are required to have polynomial growth, and the notation of criticality is associated to the sharp Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ with $2^*{:=}2N/(N-2)$. Coming to the case $N = 2$, much faster exponential growth is allowed for the nonlinearity and the Trudinger–Moser inequality replaces the sharp Sobolev inequality used for $N \geq 3$. From now on, we will focus our attention on the dimension $N = 2$ when the nonlinearity exhibits the critical exponential growth, which is more complicated than the case $N \geq 3$. We refer the reader to [1, 6, 8, 11, 12, 14, 20, 21, 24, 30] and the references therein for recent advances on nonlinear problems with exponential growth. Let us describe some of the relevant works on planar Kirchhoff equations with exponential growth below.

To the best of our knowledge, the first result on planar Kirchhoff equation with critical exponential growth is due to Figueiredo and Severo [13]. Precisely, based on the mountain pass theorem, they proved that the following Kirchhoff equation on the bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

has a positive ground state solution where $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies (F1), (F2) and the following assumptions:

(F0) $f(t) = 0$ for all $t \leq 0$;
(F3′) $\lim_{t\to+\infty} \frac{tf(t)}{e^{\alpha_0 t^2}} > \frac{2}{\alpha_0 d^2}\left(a + \frac{4\pi b}{\alpha_0}\right)$, where d is the radius of the largest open ball contained in Ω;
(F4′) $f(t) > 0$ for all $t > 0$, and there exist $\hat{M}_0 > 0$ and $\hat{\beta}_0 > 0$ such that

$$F(t) \leq \hat{M}_0 f(t), \quad \forall\, t \geq \hat{\beta}_0;$$

(F5′) $\frac{f(t)}{t^3}$ is increasing on $(0, \infty)$.

Note that if $\alpha = 4\pi$, the Trudinger–Moser inequality (1.2) gives rise to the possible failure of compactness of the associated functional. In order to restore the compactness property, they proved that the Mountain pass level is less than the threshold $\frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2}$ under which Palais–Smale condition holds with the help of (F3′), following the ideas introduced by de Figueiredo et al. [10] in their pioneering work on the solvability of the elliptic type problem (1.9) with $b = 0$. If the monotonicity condition (F5′) is replaced by the following Ambrosetti–Rabinowitz condition:

(F6′) $f(t)t \geq 4F(t) \geq 0$ for all $t \geq 0$,

Naimen and Tarsi [23] and Chen and Yu [9] obtained the existence of positive solutions for (1.9). It is well-known that the monotonicity condition (F5′) plays an important role in using a Nehari type argument, and the Ambrosetti–Rabinowitz condition (F6′) can help proving the boundedness of (PS) sequences. Recently, the above results were improved and generalized by Chen et al. [7], by weakening (F3′), (F5′) and (F6′) to the following conditions, respectively.

(F3″) $\liminf_{t\to+\infty} \frac{t^2F(t)}{e^{\alpha_0 t^2}} > \frac{1}{e\alpha_0^2 d^2}\left(a + \frac{4\pi b}{\alpha_0}\right)$;

(F5″) $\frac{f(t)-a\lambda_1 t}{t^3}$ is non-decreasing on $(0,\infty)$, where λ_1 denotes the first eigenvalue of $-\Delta$ with a Dirichlet boundary condition;

(F6″) $f(t)t - 4F(t) + \lambda_1 t^2 \geq 0$ for all $t \geq 0$.

Kirchhoff equations with more general nonlocal coefficient were also considered in [9, 13], where $\left(a + b\|\nabla u\|_2^2\right)$ is replaced by more general continuous function $m(\|\nabla u\|_2^2)$ in (1.8). We must point out that (F3″), (F4′) and (F5″) (or (F6″)) are very crucial in these works, for example, (F3″) was used to yield the threshold of the Mountain pass level, and by using (F4′) and (F5″) (or (F6″)) it can be shown the convergence of $(PS)_c$ sequences in $H_0^1(\Omega)$ provided that the mountain pass level lies below the threshold, that is $c < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2}$.

However, the methods used in [7, 9, 13, 23] seem difficult to apply for Kirchhoff equation in $\mathbb{R}^2$ since they depend heavily on the compactness of the embeddings $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q \geq 2$.

To the best of our knowledge, almost all of the works dealing with planar Kirchhoff equations are set in the bounded domain $\Omega \subset \mathbb{R}^2$, and there is no result available on the existence of nontrivial solutions for Kirchhoff equation $(\mathcal{K})$ with the critical exponential growth in $\mathbb{R}^2$, which is the focus of the present paper.

More precisely, we first consider the following Kirchhoff equation with constant potential

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V_\infty u = f(u), \ x \in \mathbb{R}^2; \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{$(\mathcal{K})_\infty$}$$

when f satisfies (F0), (F1), (F2) and the following condition:

(F3) $f(t)t \geq 2F(t)$ for all $t \geq 0$,

which can be derived easily from the conditions (F4′) and (F6′) used in the previous literature. Using a suitable minimization method, completely different from those of [7, 9, 13, 23] relying on the mountain pass theorem, we shall establish the existence of positive *least energy solutions* for $(\mathcal{K})_\infty$. In particular, we also give its mountain pass characterization. For this, we define the functional $\Phi^\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ by

$$\Phi^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^2} \left(a|\nabla u|^2 + V_\infty u^2\right)\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^2} F(u)\mathrm{d}x, \tag{1.10}$$

and denote by c^∞ the mountain pass level of Φ^∞, i.e.

$$c^\infty = \inf_{\gamma\in\Gamma^\infty} \max_{t\in[0,1]} \Phi^\infty(\gamma(t)), \tag{1.11}$$

where

$$\Gamma^\infty := \left\{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0, \Phi^\infty(\gamma(1)) < 0\right\}. \tag{1.12}$$

We recall also that a solution u of problem $(\mathcal{K})_\infty$ is a *least energy solution* if $\Phi^\infty(u) = m^\infty$ with

$$m^\infty := \inf\left\{\Phi^\infty(u) \,\middle|\, u \in H^1(\mathbb{R}^2)\backslash\{0\}, (\Phi^\infty)'(u) = 0\right\}. \tag{1.13}$$

In this direction, we have the following two results.

Theorem 1.2 *Assume that f satisfies* (F0)–(F3). *Then there exists $V^* \in (0, +\infty]$ such that for any $V_\infty \in (0, V^*)$, equation $(\mathcal{K})_\infty$ has a positive least energy solution. Moreover, V^* is equal to the Trudinger–Moser ratio:*

$$C^*_{TM}(F) := \sup\left\{\frac{2}{\|u\|_2^2}\int_{\mathbb{R}^2} F(u)\mathrm{d}x \,\middle|\, u \in H^1(\mathbb{R}^2)\backslash\{0\}, \|\nabla u\|_2^2 \le \frac{4\pi}{\alpha_0}\right\}.$$

In particular, $V^* = +\infty$ is equivalent to $\lim_{t\to+\infty} \frac{t^2F(t)}{e^{\alpha_0 t^2}} = +\infty$.

Theorem 1.3 *Under the assumptions of Theorem 1.2, the least energy level m^∞ is equal to mountain pass value c^∞. Moreover, for any least energy solution w of $(\mathcal{K})_\infty$, there exists a path $\tilde{\gamma} \in \Gamma^\infty$ such that $w \in \tilde{\gamma}([0, 1])$ and*

$$\max_{t\in[0,1]} \Phi^\infty(\tilde{\gamma}(t)) = \Phi^\infty(w).$$

Next, we study the existence of *ground state solutions* for the critical exponential growth Kirchhoff equation $(\mathcal{K})$ with the trapping potential V satisfying (V1), which is introduced by Rabinowitz [26]. Though this kind of potential has been studied in the literature, it seems that there is no paper associated with Kirchhoff equations, dealing with the dimension $N = 2$ when the nonlinearity has critical exponential growth. Some effective methods, treating the dimension $N \ge 3$, do not carry over to our case due to the simultaneous appearance of the nonlocal term and the nonlinear term with critical exponential growth. Before stating our results, we introduce the following assumptions:

(F3′) $f(t)t \ge 2F(t)$ for all $t \in \mathbb{R}$, and

$$\frac{f(t)}{t} \ge V_0 \Rightarrow f(t)t - 2F(t) > 0;$$

(F4) there exist $M_0 > 0$ and $\beta_0 > 0$ such that $F(\beta_0) > 0$ and

$$F(t) \le M_0|f(t)|, \quad \forall\, |t| \ge \beta_0;$$

(F5) $\frac{f(t)-V_0t}{|t|^3}$ is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$.

We say that a solution u of $(\mathcal{K})$ is a *ground state solution* (of Nehari type) if $\Phi(u) = c_{\mathcal{N}}$ with

$$c_{\mathcal{N}} := \inf_{u\in\mathcal{N}} \Phi(u) \tag{1.14}$$

and

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^2)\backslash\{0\} \,|\, \langle\Phi'(u), u\rangle = 0\}. \tag{1.15}$$

Our result is as follows.

Theorem 1.4 *Assume that V satisfies* (V1) *with $V_\infty \in (0, V^*)$, and f satisfies* (F1), (F2), (F3′), (F4) *and* (F5). *Then* ($\mathcal{K}$) *has a ground state solution, where V^* is given by Theorem* 1.2.

Finally, as a by-product of the present paper, we would like to study the existence of nontrivial solutions for ($\mathcal{K}$) when V is radial, namely it satisfies

(V2) $V(x) = V(|x|)$ and $0 < V_0 \leq V(x) \leq V_\infty$ for all $x \in \mathbb{R}^2$.

As we all know, it is nontrivial to show that the weak limit of Cerami sequences is a weak solution because of the fact

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^2) \ \not\Rightarrow\ \|\nabla u_n\|_2^2 \int_{\mathbb{R}^2} \nabla u_n \cdot \nabla \varphi \mathrm{d}x \to \|\nabla u\|_2^2 \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi \mathrm{d}x, \quad \forall\, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

To address this issue, a classical way is to restrict the energy functional in the subspace of radially symmetric functions $H_r^1(\mathbb{R}^2)$ belonging to $H^1(\mathbb{R}^2)$ since the limit

$$\int_{\mathbb{R}^2} [f(u_n) - f(u)](u_n - u)\mathrm{d}x \to 0 \tag{1.16}$$

can be easily deduced from the compactness of the embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for $q > 2$ if f is superlinear at zero and has polynomial growth. However, when f is of critical exponential growth, it is still unknown whether the limit (1.16) holds or not since the embedding of $H_r^1(\mathbb{R}^2)$ into the Orlicz space associated with the function $\varphi(s) = \exp(4\pi s^2) - 1$ is not compact. Thus, a deeper analysis is required for ($\mathcal{K}$) with the radial potential V in this direction, which is the focus in the last part of the present paper. For this purpose, we introduce the following condition:

(F6) $f(t)t - 4F(t) + V_0 t^2 \geq 0$ for all $t \in \mathbb{R}$,

which is weaker than the Ambrosetti–Rabinowitz condition (F6′) and can be derived from the monotonicity condition (F5). Our last result is as follow.

Theorem 1.5 *Assume that V satisfies* (V2) *with $V_\infty \in (0, V^*)$, and f satisfies* (F1), (F2), (F3′), (F4) *and* (F6). *Then* ($\mathcal{K}$) *has a nontrivial radial solution, where V^* is given by Theorem* 1.2.

To complete this section, we sketch our proof as follows.

For the proofs of Theorems 1.2 and 1.3, motivated by the Pohozaev identity for $(\mathcal{K})_\infty$ proved in Lemma 2.1 below, we introduce the auxiliary functional $J^\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$J^\infty(u) = V_\infty \|u\|_2^2 - 2\int_{\mathbb{R}^2} F(u)\mathrm{d}x, \tag{1.17}$$

the set

$$\mathcal{P}_\infty := \left\{ u \in H^1(\mathbb{R}^2)\backslash\{0\} \,\middle|\, J^\infty(u) = 0 \right\} \tag{1.18}$$

and the constrained minimization problem

$$A^\infty := \inf_{u \in \mathcal{P}_\infty} \left(\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 \right) = \inf_{u \in \mathcal{P}_\infty} \Phi^\infty(u). \tag{1.19}$$

Based on a sufficient and necessary condition for compactness of general nonlinear functionals, we will prove that A^∞ can be attained if V^∞ is less than the Trudinger–Moser ratio

$C^*_{\mathrm{TM}}(F)$ depending on the Trudinger–Moser inequality with the exact growth (see Lemmas 2.5 and 2.6 below), and the minimizer, under a suitable change of scale, is a *least energy solution* of $(\mathcal{K})_\infty$ using an analytical method. Different from the dimension $N \geq 3$, the minimum A^∞ has no saddle point structure with respect to the fibres $\{u(\cdot/t) : t > 0\} \subset H^1(\mathbb{R}^2)$, $u \in H^1(\mathbb{R}^2)$ since the Pohozaev functional $J^\infty(u)$ does not have a $\|\nabla u\|_2$-component, thus it is more complicated to establish the relation among the minimum A^∞, the least energy m^∞ and the mountain pass level c^∞ in the dimension $N = 2$. To address this issue, inspired by the idea of Jeanjean and Tanaka [16], we construct a new path belonging to Γ^∞ (see (2.44) below) and derive the mountain pass characterization of the *least energy solution* for $(\mathcal{K})_\infty$ with more subtle analyses, which is the highlight of the proof of Theorem 1.3.

The proofs of Theorems 1.4 and 1.5 are based on Mountain Pass theorem. For this, a standard procedure is to prove the boundedness of Cerami sequences, and verify that the weak limit of Cerami sequences is non-trivial and is also a weak solution. Nevertheless, to do that, compared with the previous works dealing with Kirchhoff-type equation (1.8) involving trapping or radial potential V in $\mathbb{R}^N (N \geq 3)$, some new obstacles arise in the proofs, for example,

i) the lack of the monotonicity condition (F5′) and the Ambrosetti–Rabinowitz type condition (F6′) prevent us from using usual methods to prove the boundedness of Cerami sequences;
ii) it is more difficult to rule out the concentration phenomena and the vanishing phenomena of Cerami sequences;
iii) it does not work that the BL-splitting property for the energy functional along Cerami sequences caused by the appearance of the nonlinear term with critical growth, which is a powerful tool to restore the compactness of Cerami sequences.

To surmount the above obstacles, some new techniques and ideas are expected to be introduced, which is the right issue we intend to address in the proofs of Theorems 1.4 and 1.5.

The paper is organized as follows. In Section 2, we study the existence of least energy solutions for $(\mathcal{K})_\infty$, and establish its mountain pass characterization, where Theorems 1.2 and 1.3 are proved. In Section 3, we investigate the existence of ground state solutions for $(\mathcal{K})$ with the trapping potential and complete the proof of Theorem 1.4. Section 4 is devoted to the study of $(\mathcal{K})$ with the radial potential, where Theorem 1.5 is proved.

Throughout the paper, we make use of the following notations:

- $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall\, u, v \in H^1(\mathbb{R}^2);$$

- $H^1_r(\mathbb{R}^2)$ denotes the space of spherically symmetric functions belonging to $H^1(\mathbb{R}^2)$:

$$H^1_r(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^2\};$$

- $L^s(\mathbb{R}^2) (1 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x\right)^{1/s}$;
- For any $x \in \Omega$ and $r > 0$, $B_r(x) := \{y \in \Omega : |y - x| < r\}$ and $B_r = B_r(0)$;
- $C_1, C_2, \ldots$ denote positive constants possibly different in different places.

2 Least energy solutions for $(\mathcal{K})_\infty$

In this section, we consider the existence of least energy solutions for $(\mathcal{K})_\infty$, and establish its mountain pass characterization, which completes the proofs of Theorems 1.2 and 1.4.

First, using a truncation argument due to Kavian (see [29, Appendix B]), we establish the Pohozaev type identity for $(\mathcal{K})_\infty$ when f has critical exponential growth.

Lemma 2.1 *Assume that f satisfies* (F1)–(F3). *Let $u \in H^1(\mathbb{R}^2)$ be a weak solution of $(\mathcal{K})_\infty$, then we have the following Pohozaev type identity*

$$J^\infty(u) = V_\infty \|u\|_2^2 - 2\int_{\mathbb{R}^2} F(u)\mathrm{d}x = 0.$$

Proof Let $\psi \in \mathcal{C}^\infty([0,+\infty),[0,1])$ such that $\psi(r) = 1$ for $r \in [0,1]$ and $\psi(r) = 0$ for $r \in [2,+\infty)$. Define $\psi_n(x) := \psi(|x|^2/n^2)$ on $\mathbb{R}^2$ for $n \in \mathbb{N}$. Then there exists $C_1 > 0$ such that

$$0 \le \psi_n(x) \le C_1, \quad |x||\nabla\psi_n(x)| \le C_1 \quad \forall\, x \in \mathbb{R}^2. \tag{2.1}$$

By a standard regularity argument, we can show that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$. Let $\bar{\alpha} = (a + b\|\nabla u\|_2^2)$. It follows from $(\mathcal{K})_\infty$ that, for every $n \in \mathbb{N}$,

$$0 = [-\bar{\alpha}\Delta u + V_\infty u - f(u)]\,\psi_n(x\cdot\nabla u). \tag{2.2}$$

It is clear that, for every $n \in \mathbb{N}$,

$$-\psi_n\lambda f(u)(x\cdot\nabla u) = -\mathrm{div}(x\psi_n\lambda F(u)) + 2\psi_n\lambda F(u) + \lambda F(u)(x\cdot\nabla\psi_n), \tag{2.3}$$

$$\begin{aligned} -\psi_n\Delta u(x\cdot\nabla u) &= -\mathrm{div}\left\{\left[\nabla u(x\cdot\nabla u) - x\frac{|\nabla u|^2}{2}\right]\psi_n\right\} \\ &\quad -\frac{|\nabla u|^2}{2}(x\cdot\nabla\psi_n) + (x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u) \end{aligned} \tag{2.4}$$

and

$$\psi_n u(x\cdot\nabla u) = \frac{1}{2}\mathrm{div}\left(u^2\psi_n x\right) - u^2\psi_n - \frac{1}{2}u^2(x\cdot\nabla\psi_n) - \frac{1}{2}u^2\psi_n. \tag{2.5}$$

Hence, for every $n \in \mathbb{N}$, it follows from (2.2), (2.3), (2.4), (2.5) and the divergence theorem that

$$\begin{aligned} &\int_{\partial B_{2n}}\left\{\frac{\bar{\alpha}|x\cdot\nabla u|^2}{2n} - \bar{\alpha}n|\nabla u|^2 - nV_\infty u^2 + 2nF(u)\right\}\psi_n\mathrm{d}\sigma \\ &\quad = -\int_{B_{2n}}\left[V_\infty u^2 - 2F(u)\right]\psi_n\mathrm{d}x - \frac{1}{2}\int_{B_{2n}}\left\{\bar{\alpha}|\nabla u|^2 + V_\infty u^2 - 2F(u)\right\}(x\cdot\nabla\psi_n)\mathrm{d}x \\ &\qquad + \bar{\alpha}\int_{B_{2n}}(x\cdot\nabla u)(\nabla\psi_n\cdot\nabla u)\mathrm{d}x, \end{aligned} \tag{2.6}$$

which together with the fact that $\psi_n|_{\partial B_{2n}} = 0$, implies

$$\begin{aligned}\int_{B_{2n}} \left[V_\infty u^2 - 2F(u)\right] \psi_n \mathrm{d}x &= -\frac{1}{2}\int_{B_{2n}} \left\{\bar{\alpha}|\nabla u|^2 + V_\infty u^2 - 2F(u)\right\} (x \cdot \nabla\psi_n)\mathrm{d}x \\ &\quad + \bar{\alpha}\int_{B_{2n}} (x \cdot \nabla u)(\nabla\psi_n \cdot \nabla u)\mathrm{d}x \\ &= -\frac{1}{2}\int_{B_{\sqrt{2}n}\setminus B_n} \left[\bar{\alpha}|\nabla u|^2 + V_\infty u^2 - 2F(u)\right] (x \cdot \nabla\psi_n)\mathrm{d}x \\ &\quad + \bar{\alpha}\int_{B_{\sqrt{2}n}\setminus B_n} (x \cdot \nabla u)(\nabla\psi_n \cdot \nabla u)\mathrm{d}x. \end{aligned} \tag{2.7}$$

From (2.1) and (2.7), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \left[V_\infty u^2 - 2F(u)\right] \mathrm{d}x\right| &= \left|\lim_{n\to\infty}\int_{B_{2n}} \left[V_\infty u^2 - 2F(u)\right] \psi_n \mathrm{d}x\right| \\ &\leq \frac{1}{2}\lim_{n\to\infty}\int_{B_{\sqrt{2}n}\setminus B_n} \left[3\bar{\alpha}|\nabla u|^2 + V_\infty u^2 + 2F(u)\right] |x||\nabla\psi_n|\mathrm{d}x \\ &\leq \frac{C_1}{2}\lim_{n\to\infty}\int_{B_{\sqrt{2}n}\setminus B_n} \left[3\bar{\alpha}|\nabla u|^2 + V_\infty u^2 + 2F(u)\right] \mathrm{d}x = 0. \end{aligned}$$

The proof is complete. □

In the following, we will solve the constrained minimization problem A^∞, given by (1.19).

Lemma 2.2 *Assume that f satisfies* (F1)–(F3). *Then there exists a minimizing sequence $\{u_n\} \subset \mathcal{P}_\infty$ satisfying $\|u_n\|_2 = 1$ for A^∞. In particular,*

$$A^\infty = \inf_{u\in\mathcal{P}_\infty} \left(\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4\right) = \inf_{u\in\widetilde{\mathcal{P}}_\infty} \left(\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4\right), \tag{2.8}$$

where

$$\widetilde{\mathcal{P}}_\infty = \mathcal{P}_\infty \cap \{u \in H^1(\mathbb{R}^2) : \|u\|_2 = 1\} \tag{2.9}$$

and $\mathcal{P}_\infty$ is given by (1.18).

Proof First, we verify that $\mathcal{P}_\infty \neq \emptyset$. Let $u \in H^1(\mathbb{R}^2)\setminus\{0\}$ be fixed and define a function $\zeta(t){:=}J^\infty(tu)$ on $(0, \infty)$. Using (F1)–(F3), it is easy to check that $\zeta(t) > 0$ for small $t > 0$ and $\zeta(t) < 0$ for large $t > 0$. Then there exists $t_u > 0$ such that $\zeta(t_u) = J^\infty(t_u u) = 0$, and so $\mathcal{P}_\infty \neq \emptyset$. Thus we can assume that there exists a minimizing sequence $\{u_n\} \subset \mathcal{P}_\infty$ satisfying

$$\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 \to A^\infty.$$

Let $\tilde{u}_n = u_n(\|u_n\|_2^{1/2}x)$. Then a simple computation leads to $\tilde{u}_n \in \mathcal{P}_\infty, \|\tilde{u}_n\|_2 = 1$ and $\|\nabla\tilde{u}_n\|_2 = \|\nabla u_n\|_2$. This shows that $\tilde{u}_n \in \widetilde{\mathcal{P}}_\infty$. From this and the fact that $\widetilde{\mathcal{P}}_\infty \subset \mathcal{P}_\infty$, (2.8) follows directly. The proof is complete. □

Lemma 2.3 *Assume that f satisfies* (F1)–(F3).

(i) *If $u \in H^1(\mathbb{R}^2)$ is a critical point of Φ^∞ on the set $\mathcal{P}_\infty$, then it is a nontrivial solution of $(\mathcal{K})_\infty$ under a suitable change of scale;*

(ii) *If the infimum A^∞ is attained, then $A^\infty = m^\infty$.*

Proof (i) Let $u \in H^1(\mathbb{R}^2)$ be a critical point of Φ^∞ on the set $\mathcal{P}_\infty$. Then there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$-(a + b\|\nabla u\|_2^2)\Delta u + V_\infty u - f(u) = 2\lambda[V_\infty u - f(u)], \tag{2.10}$$

namely,

$$-(a + b\|\nabla u\|_2^2)\Delta u = (2\lambda - 1)[V_\infty u - f(u)]. \tag{2.11}$$

Since $u \neq 0$, we deduce from (2.11) that

$$2\lambda - 1 \neq 0 \ \text{ and } \ V_\infty u - f(u) \neq 0. \tag{2.12}$$

For any $T > 0$, by (F1)–(F3), there exist $0 < t_1 < t_2 < T$ such that

$$2F(t) - f(t)t \leq 0, \quad \forall\, t \in [0, T], \quad \text{and} \quad 2F(t) - f(t)t < 0, \quad \forall\, t \in [t_1, t_2]. \tag{2.13}$$

Hence, it follows from (2.13) and the definition of $\mathcal{P}_\infty$ that

$$\int_{\mathbb{R}^2} [V_\infty u - f(u)]u\mathrm{d}x = \int_{\mathbb{R}^2} [2F(u) - f(u)u]\mathrm{d}x < 0. \tag{2.14}$$

This implies that there exists $w \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that

$$\langle (J^\infty)'(u), w\rangle = \int_{\mathbb{R}^2} [V_\infty u - f(u)]w\mathrm{d}x < 0. \tag{2.15}$$

By multiplying (2.11) by w and integrating, we have

$$(a + b\|\nabla u\|_2^2) \int_{\mathbb{R}^2} \nabla u \cdot \nabla w\mathrm{d}x = (2\lambda - 1) \int_{\mathbb{R}^2} [V_\infty u - f(u)]w\mathrm{d}x. \tag{2.16}$$

Recalling that $J^\infty(u) = 0$, we have, it follows from (2.15) that for small enough $\varepsilon > 0$,

$$J^\infty(u + \varepsilon w) < J^\infty(u) = 0. \tag{2.17}$$

Let

$$A(v) := \frac{a}{2}\|\nabla v\|_2^2 + \frac{b}{4}\|\nabla v\|_2^4, \quad \forall\, v \in H^1(\mathbb{R}^2).$$

Noting that $A(u) = A^\infty$, by (2.16) and (2.17), we have

$$\begin{aligned}
A(u + \varepsilon w) = {} & \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \varepsilon(a + b\|\nabla u\|_2^2) \int_{\mathbb{R}^2} \nabla u \cdot \nabla w\mathrm{d}x \\
& + \frac{a\varepsilon^2}{2}\|\nabla w\|_2^2 + \frac{b\varepsilon^4}{4}\|\nabla w\|_2^4 \\
& + \frac{b\varepsilon^2}{2}\|\nabla u\|_2^2\|\nabla w\|_2^2 + b\varepsilon^2\left(\int_{\mathbb{R}^2} \nabla u \cdot \nabla w\mathrm{d}x\right)^2 + b\varepsilon^3\|\nabla w\|_2^2 \int_{\mathbb{R}^2} \nabla u \cdot \nabla w\mathrm{d}x \\
= {} & A^\infty + \varepsilon(2\lambda - 1) \int_{\mathbb{R}^2} [V_\infty u - f(u)]w\mathrm{d}x + O(\varepsilon^2).
\end{aligned} \tag{2.18}$$

We claim that $2\lambda - 1 < 0$. Otherwise, if $2\lambda - 1 > 0$, then there exists $\varepsilon_0 > 0$ small enough such that

$$J^\infty(u + \varepsilon_0 w) < 0 \ \text{ and } \ A(u + \varepsilon_0 w) < A^\infty, \tag{2.19}$$

due to (2.17) and (2.18). Let $u_0 = u + \varepsilon_0 w$. Then $J^\infty(u_0) < 0$ and $J^\infty(su_0) > 0$ for $s > 0$ small enough as a consequence of (F2). Therefore, there exists $s_0 \in (0, 1)$ such that $J^\infty(s_0 u_0) = 0$, moreover, by (2.19), we have

$$A(s_0u_0) = \frac{as_0^2}{2}\left\|\nabla u_0\|_2^2 + \frac{bs_0^4}{4}\|\nabla u_0\right\|_2^4 < s_0^2 A(u_0) < A^\infty. \tag{2.20}$$

This shows that $s_0u_0 \in \mathcal{P}_\infty$ and $\Phi^\infty(s_0u_0) < A^\infty$, which contradicts to the definition of A^∞. Hence, we have $2\lambda - 1 < 0$ as claimed. Thus,

$$\tilde{u}(x){:=}u\left(\frac{x}{(1-2\lambda)^{1/2}}\right) \text{ for a.e. } x \in \mathbb{R}^2 \tag{2.21}$$

is a nontrivial solution of $(\mathcal{K})_\infty$.

(ii) Suppose the infimum A^∞ is attained by $u \in H^1(\mathbb{R}^2)$. From (2.8), we see that $u \in H^1(\mathbb{R}^2)$ is a critical point of Φ^∞ on the set $\mathcal{P}_\infty$. Then (i) of this lemma shows that $\tilde{u} \in H^1(\mathbb{R}^2)$ defined by (2.21) is a nontrivial solution of $(\mathcal{K})_\infty$, and so $(\Phi^\infty)'(\tilde{u}) = 0$ and $A^\infty = \Phi^\infty(\tilde{u}) \geq m^\infty$. To prove $A^\infty = \Phi^\infty(\tilde{u}) = m^\infty$, it is left to show that $A^\infty \leq m^\infty$. Note that Lemma 2.1 shows that if $(\Phi^\infty)'(v) = 0$ for $v \in H^1(\mathbb{R}^2)$, then v satisfies the Pohozaev type identity $J^\infty(v) = 0$, namely,

$$\left\{u \in H^1(\mathbb{R}^2)\backslash\{0\}, \;\middle|\; (\Phi^\infty)'(u) = 0\right\} \subset \mathcal{P}_\infty.$$

This implies that $A^\infty \leq m^\infty$. The proof is complete. □

Before studying the attainability of A^∞, we recall necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals in $H^1(\mathbb{R}^2)$, see Ibrahim et al. [15] and Masmoudi and Sani [19].

Lemma 2.4 *Suppose that $g : \mathbb{R} \to [0, +\infty)$ is a Borel function and define functional G by $G(u){:=}\int_{\mathbb{R}^2} g(u(x))\mathrm{d}x$. Then for any $K > 0$ we have the following* (B) *and* (C)*:*

(B) *Boundedness: The following* (i) *and* (ii) *are equivalent.*

(i) $\limsup_{|t|\to+\infty} e^{-2|t|^2/K}|t|^2 g(t) < \infty$ *and* $\limsup_{|t|\to 0} |t|^{-2} g(t) < \infty$.
(ii) *There exists a constant $C_{g,K} > 0$ such that*

$$u \in H^1(\mathbb{R}^2),\ \|\nabla u\|_2^2 \leq 2\pi K \Rightarrow \int_{\mathbb{R}^2} g(u)\mathrm{d}x \leq C_{g,K}\int_{\mathbb{R}^2} |u|^2\mathrm{d}x.$$

(C) *Compactness: The following* (iii) *and* (iv) *are equivalent.*

(iii) $\limsup_{|t|\to+\infty} e^{-2|t|^2/K}|t|^2 g(t) = 0$ *and* $\lim_{|t|\to 0} |t|^{-2} g(t) = 0$.
(iv) *For any radially symmetric sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} |\nabla u_n|^2\mathrm{d}x \leq 2\pi K$ and weakly converging to some $u \in H^1(\mathbb{R}^2)$, we have $G(u_n) \to G(u)$.*

Now, we establish a relation between the attainability of A^∞ and the Trudinger–Moser inequality with the exact growth:

$$\int_{\mathbb{R}^2} \frac{e^{4\pi|u|^2} - 1}{(1 + |u|)^2}\mathrm{d}x \leq C\|u\|_2^2, \quad \forall\, u \in H^1(\mathbb{R}^2) \text{ with } \|\nabla u\|_2 \leq 1. \tag{2.22}$$

For this purpose, as in [15], we introduce the Trudinger–Moser ratio

$$C_{\mathrm{TM}}^L(F){:=}\sup\left\{\frac{2}{\|u\|_2^2}\int_{\mathbb{R}^2} F(u)\mathrm{d}x \;\middle|\; u \in H^1(\mathbb{R}^2)\backslash\{0\}, \|\nabla u\|_2^2 \leq L\right\}, \tag{2.23}$$

the Trudinger–Moser threshold:

$$R(F) := \sup\left\{L > 0 \,\middle|\, C^{L}_{\mathrm{TM}}(F) < +\infty\right\} \tag{2.24}$$

and we denote by $C^{*}_{\mathrm{TM}}(F)$ the ratio at the threshold, i.e.

$$C^{*}_{\mathrm{TM}}(F) = C^{R(F)}_{\mathrm{TM}}(F). \tag{2.25}$$

Using (1.3) and (1.4), and applying Lemma 2.4, we derive that

$$R(F) = \frac{4\pi}{\alpha_0}. \tag{2.26}$$

If (F0) holds, to apply Schwarz symmetrization, as usual we let

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for all } t > 0, \\ -f(-t), & \text{for all } t \le 0. \end{cases} \tag{2.27}$$

Observe that $\tilde{f}$ satisfies the same conditions as f. Furthermore, by the maximum principle, solutions of $(\mathcal{K})_\infty$ with $\tilde{f}$ are also solutions of $(\mathcal{K})_\infty$ with f. Hence there is no loss in generality in replacing f by $\tilde{f}$, and we will always adopt the convention that f has been replaced by $\tilde{f}$; we keep however the same notation f in the following discussion of this section.

Lemma 2.5 *Assume that f satisfies* (F0)–(F3). *If*

$$A^\infty < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2},$$

then A^∞ is attained and $A^\infty = \Phi^\infty(u)$, where $u \in H^1_r(\mathbb{R}^2)$ is, under a suitable change of scale, a positive least energy solution of equation $(\mathcal{K})_\infty$.

Proof We may always assume that there exists a sequence $\{u_n\} \subset \mathcal{P}_\infty \cap H^1_r(\mathbb{R}^2)$ satisfying

$$\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 \to A^\infty \text{ and } \|u_n\|_2 = 1 \tag{2.28}$$

by Schwarz symmetrization and Lemma 2.2. Then there exists some function $u \in H^1_r(\mathbb{R}^2)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$.

Now, we prove that if

$$A^\infty < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2},$$

then A^∞ is attained. Note that

$$\begin{cases} \frac{at}{2} + \frac{bt^2}{4} < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2} \\ t \ge 0, \end{cases} \quad \Leftrightarrow \quad 0 \le t < R(F) = \frac{4\pi}{\alpha_0}. \tag{2.29}$$

Picking up $\frac{2}{K} > \alpha_0$ satisfying $\lim_{n\to\infty} \|\nabla u_n\|_2^2 \le 2\pi K$, then (1.4) yields

$$\lim_{|t|\to\infty} \frac{|t|^2 F(t)}{e^{2|t|^2/K}} = 0. \tag{2.30}$$

From (1.3), (2.30) and (C) of Lemma 2.4, we derive that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} F(u)\mathrm{d}x. \tag{2.31}$$

Since $J^\infty(u_n) = 0$ and $\|u_n\|_2 = 1$, by (2.31), we have

$$0 < V_\infty = \lim_{n\to\infty} V_\infty\|u_n\|_2^2 = 2\lim_{n\to\infty}\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = 2\int_{\mathbb{R}^2} F(u)\mathrm{d}x, \tag{2.32}$$

which implies that $u \neq 0$ and $J^\infty(u) \leq 0$. By the weak lower semicontinuity of the norm and (2.31), we have

$$J^\infty(u) = V_\infty\|u\|_2^2 - 2\int_{\mathbb{R}^2} F(u)\mathrm{d}x \leq \lim_{n\to\infty}\left(V_\infty\|u_n\|_2^2 - 2\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x\right) = 0 \tag{2.33}$$

and

$$0 < \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 \leq \lim_{n\to\infty}\left(\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4\right) = A^\infty. \tag{2.34}$$

In order to prove that the infimum A^∞ is attained by u, it remains only to show that $u \in \mathcal{P}_\infty$, namely $J^\infty(u) = 0$. Set

$$h(t) = J^\infty(tu) = t^2 V_\infty\|u\|_2^2 - 2\int_{\mathbb{R}^2} F(tu)\mathrm{d}x.$$

Then $h(1) \leq 0$ by (2.33), and from (1.5) one can deduce that $h(t) > 0$ for $t > 0$ small enough. Consequently, there exists $t_0 \in (0, 1]$ such that $J^\infty(t_0 u) = 0$, namely $t_0 u \in \mathcal{P}_\infty$. This together with (2.34) leads to

$$A^\infty \leq \frac{a}{2}\|\nabla(t_0 u)\|_2^2 + \frac{b}{4}\|\nabla(t_0 u)\|_2^4 \leq t_0^2 A^\infty.$$

The above inequality and (2.34) show that $t_0 = 1$ and $\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 = A^\infty > 0$. Combining (2.32) with the fact that $J^\infty(u) = 0$, we have $\|u\|_2 = 1$. Applying Lemma 2.3, we have that the above u is a least energy solution of $(\mathcal{K})_\infty$ under a suitable change of scale. Similarly as in [13, Proof of Theorem 1.3], we can derive that $u > 0$ in $\mathbb{R}^2$. The proof is complete. □

Lemma 2.6 *Assume that f satisfies* (F0)–(F3). *The constrained minimization problem A^∞ associated to the functional Φ^∞ satisfies*

$$A^\infty < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2} \tag{2.35}$$

if $V_\infty < C^*_{\mathrm{TM}}(F)$, where $C^*_{\mathrm{TM}}(F)$ is given by (2.25).

Proof We distinguish two cases: $C^*_{\mathrm{TM}}(F) < +\infty$ and $C^*_{\mathrm{TM}}(F) = +\infty$. In the case $C^*_{\mathrm{TM}}(F) < +\infty$, since $V_\infty < C^*_{\mathrm{TM}}(F)$, then $V_\infty < C^*_{\mathrm{TM}}(F) - \varepsilon_0$ for some $\varepsilon_0 > 0$. By the definition of $C^*_{\mathrm{TM}}(F)$, there exists some $u_0 \in H^1(\mathbb{R}^2)\backslash\{0\}$ such that

$$\|\nabla u_0\|_2^2 \leq R(F) \text{ and } V_\infty < C^*_{\mathrm{TM}}(F) - \varepsilon_0 < \frac{2}{\|u_0\|_2^2}\int_{\mathbb{R}^2} F(u_0)\mathrm{d}x. \tag{2.36}$$

Then

$$J^\infty(u_0) = V_\infty\|u_0\|_2^2 - 2\int_{\mathbb{R}^2} F(u_0)\mathrm{d}x < 0. \tag{2.37}$$

Let $h(t) = J^\infty(tu_0)$ for $t > 0$. Since $h(1) < 0$ by (2.37), and $h(t) > 0$ for $t > 0$ small enough by (1.5), there exists $t_0 \in (0,1)$ such that $h(t_0) = J^\infty(t_0u_0) = 0$, namely $t_0u_0 \in \mathcal{P}^\infty$. Therefore, we have

$$A^\infty \le \frac{a}{2}\|\nabla(t_0u_0)\|_2^2 + \frac{b}{4}\|\nabla(t_0u_0)\|_2^4 < \frac{a}{2}\|\nabla u_0\|_2^2 + \frac{b}{4}\|\nabla u_0\|_2^4 \le \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2},$$

which yields (2.35). In the case $C^*_{\mathrm{TM}}(F) = +\infty$, for any $V_\infty > 0$, there exists some $u_0 \in H^1(\mathbb{R}^2)\backslash\{0\}$ such that

$$\|\nabla u_0\|_2^2 \le R(F) \text{ and } V_\infty\|u_0\|_2^2 < 2\int_{\mathbb{R}^2} F(u_0)\mathrm{d}x.$$

Hence we can repeat the same arguments as above to get the desired conclusion. □

Proof of Theorem 1.2 If $V_\infty < C^*_{\mathrm{TM}}(F)$, then Lemma 2.6 leads to

$$A^\infty < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2}.$$

Hence the assumptions of Lemma 2.5 are fulfilled and we obtain the existence of a positive least energy solution for equation $(\mathcal{K})_\infty$. Moreover, recalling (1.3) and in light of (B) of Lemma 2.4, we can easily derive that $C^*_{\mathrm{TM}}(F) = +\infty$ if and only if $\lim_{t\to+\infty} \frac{t^2F(t)}{e^{\alpha_0 t^2}} = +\infty$. □

Besides the attainability of A^∞, Theorem 1.2 also shows that infimum A^∞ equals to the ground state m^∞. Proceeding to the proof of Theorem 1.3, we next investigate the interesting relation between the infimum A^∞ and the mountain pass level c^∞ defined by (1.11). Before this, we first verify that $\Phi^\infty(u)$ has a mountain pass geometry, in order to show that the mountain pass level c^∞ is well-defined. Indeed it has the following properties:

Lemma 2.7 *Assume that $V_\infty < C^*_{\mathrm{TM}}(F)$ and f satisfies* (F0)–(F3). *Then*

(i) *there exist $\rho_0 > 0$ and $\delta_0 > 0$ such that $\Phi^\infty(u) \ge \delta_0$ for all $\|u\| = \rho_0$;*
(ii) *there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\|u_0\| > \rho_0$ and $\Phi^\infty(u_0) < 0$.*

Proof (i) By the Rellich embedding theorem, for $s \in [2,\infty)$, there exists $\gamma_s > 0$ such that

$$\|u\|_s \le \gamma_s\|u\|, \quad \forall\, u \in H^1(\mathbb{R}^2). \tag{2.38}$$

By (F1) and (F2), one has for some constants $\alpha > 0$ and $C_1 > 0$

$$|F(t)| \le \frac{V_\infty}{4}t^2 + C_1\left(e^{\alpha t^2} - 1\right)|t|^3, \quad \forall\, t \in \mathbb{R}. \tag{2.39}$$

In view of Lemma 1.1 ii), we have

$$\int_{\mathbb{R}^2}\left(e^{2\alpha u^2} - 1\right)\mathrm{d}x = \int_{\mathbb{R}^2}\left(e^{2\alpha\|u\|^2(u/\|u\|)^2} - 1\right)\mathrm{d}x \le \mathcal{C}(2\pi), \quad \forall\, \|u\| \le \sqrt{\pi/\alpha}. \tag{2.40}$$

From (2.39) and (2.40), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} F(u)\mathrm{d}x &\le \frac{V_\infty}{4}\|u\|_2^2 + C_1\int_{\mathbb{R}^2}\left(e^{\alpha u^2} - 1\right)|u|^3\mathrm{d}x \\ &\le \frac{V_\infty}{4}\|u\|_2^2 + C_1\left[\int_{\mathbb{R}^2}\left(e^{2\alpha u^2} - 1\right)\mathrm{d}x\right]^{1/2}\|u\|_6^3 \\ &\le \frac{V_\infty}{4}\|u\|_2^2 + C_2\|u\|^3, \quad \forall\, \|u\| \le \sqrt{\pi/\alpha}.\end{aligned} \tag{2.41}$$

Hence, it follows from (1.10) and (2.41) that

$$\Phi^\infty(u) \geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{V_\infty}{4}\|u\|_2^2 - C_2\|u\|^3, \quad \forall\, \|u\| \leq \sqrt{\pi/\alpha}. \tag{2.42}$$

Therefore, there exist $\delta_0 > 0$ and $0 < \rho_0 < \sqrt{\pi/\alpha}$ such that

$$\Phi^\infty(u) \geq \delta_0, \quad \forall\, u \in S:=\{u \in H^1(\mathbb{R}^2) : \|u\| = \rho_0\}. \tag{2.43}$$

(ii) This conclusion will be done in the proof of next lemma, see (2.54) below. □

Lemma 2.8 *Assume that* $V_\infty < C^*_{\mathrm{TM}}(F)$ *and* f *satisfies* (F0)–(F3). *Then for any least energy solution* $w(x)$ *of* $(\mathcal{K})_\infty$*, there exists a path* $\tilde{\gamma} \in \Gamma^\infty$ *such that* $w(x) \in \tilde{\gamma}([0,1])$ *and*

$$\max_{t\in[0,1]} \Phi^\infty(\tilde{\gamma}(t)) = \Phi^\infty(w).$$

Proof Let w be a given least energy solution of $(\mathcal{K})_\infty$ obtained in Theorem 1.2. We define a curve γ, constituted of the three pieces given by:

$$\gamma(\theta) = \begin{cases} \frac{\theta}{t_1} w_{t_1} & \text{if } \theta \in [0, t_1], \\ w_{[t_3(\theta-t_1)+(t_2-\theta)t_1]/(t_2-t_1)} & \text{if } \theta \in [t_1, t_2], \\ \frac{t_2(\theta-t_2)+t_3-\theta}{t_3-t_2} w_{t_3} & \text{if } \theta \in [t_2, t_3], \end{cases} \tag{2.44}$$

where $w_t(x) = w(x/t)$ and $0 < t_1 < 1 < t_2 < t_3$ are determined later. It is easy to check that $\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^2))$. Since w is a weak solution of $(\mathcal{K})_\infty$, we have $\langle (\Phi^\infty)'(w), w\rangle = 0$, and so

$$\int_{\mathbb{R}^2} [f(w) - V_\infty w]\, w\mathrm{d}x = a\|\nabla w\|_2^2 + b\|\nabla w\|_2^4 > 0.$$

Then we can find $t_2 > 1$ such that

$$\int_{\mathbb{R}^2} [f(\xi w) - V_\infty \xi w]\, w\mathrm{d}x > 0, \quad \forall\, \xi \in [1, t_2]. \tag{2.45}$$

Set

$$\phi(t) = \begin{cases} \frac{f(t)}{t} - V_\infty, & t \neq 0, \\ -V_\infty, & t = 0. \end{cases} \tag{2.46}$$

Then $\phi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ by (F1) and (F2). Moreover, (2.45) and (2.46) give

$$\int_{\mathbb{R}^2} \phi(\xi w) w^2 \mathrm{d}x > 0, \quad \forall\, \xi \in [1, t_2]. \tag{2.47}$$

Note that for any fixed $t > 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}\Phi^\infty(\xi w_t) &= \langle (\Phi^\infty)'(\xi w_t), w_t\rangle \\ &= \xi\left[a\|\nabla w_t\|_2^2 + \xi^2 b\|\nabla w_t\|_2^4 - \int_{\mathbb{R}^2}\phi(\xi w_t) w_t^2 \mathrm{d}x\right] \\ &= \xi\left[a\|\nabla w\|_2^2 + \xi^2 b\|\nabla w\|_2^4 - t^2\int_{\mathbb{R}^2}\phi(\xi w) w^2 \mathrm{d}x\right]. \end{aligned} \tag{2.48}$$

Choosing $t_1 \in (0,1)$, we have

$$\begin{aligned} & a\|\nabla w\|_2^2 + \xi^2 b\|\nabla w\|_2^4 - t_1^2 \int_{\mathbb{R}^2} \phi(\xi w) w^2 \mathrm{d}x \\ & \quad \geq a\|\nabla w\|_2^2 - t_1^2 \int_{\mathbb{R}^2} \phi(\xi w) w^2 \mathrm{d}x > 0, \quad \forall\, \xi \in [0,1]. \end{aligned} \tag{2.49}$$

By (2.47), we can also choose $t_3 > t_2$ such that

$$\begin{aligned} & a\|\nabla w\|_2^2 + \xi^2 b\|\nabla w\|_2^4 - t_3^2 \int_{\mathbb{R}^2} \phi(\xi w) w^2 \mathrm{d}x \\ & \quad \leq a\|\nabla w\|_2^2 + t_2^2 b\|\nabla w\|_2^4 - t_3^2 \int_{\mathbb{R}^2} \phi(\xi w) w^2 \mathrm{d}x \\ & \quad \leq -\frac{2}{t_2^2 - 1}\left(a\|\nabla w\|_2^2 + b\|\nabla w\|_2^4\right), \quad \forall\, \xi \in [1, t_2]. \end{aligned} \tag{2.50}$$

Thus we can see by (2.49) that the function $\Phi^\infty\left(\frac{\theta}{t_1} w_{t_1}\right)$ is increasing on $\theta \in [0, t_1]$ and takes its maximum at $\theta = t_1$, namely

$$\Phi^\infty(\gamma(\theta)) = \Phi^\infty\left(\frac{\theta}{t_1} w_{t_1}\right) \leq \Phi^\infty(w_{t_1}), \quad \forall\, \theta \in [0, t_1]. \tag{2.51}$$

Since $J^\infty(w) = \int_{\mathbb{R}^2}\left[V_\infty w^2 - 2F(w)\right]\mathrm{d}x = 0$ by Pohozaev type identity (see Lemma 2.1), we have

$$\begin{aligned} \Phi^\infty(w_t) &= \frac{a}{2}\|\nabla w\|_2^2 + \frac{b}{4}\|\nabla w\|_2^4 + \frac{t^2}{2}\int_{\mathbb{R}^2}\left[V_\infty w^2 - 2F(w)\right]\mathrm{d}x \\ &= \frac{a}{2}\|\nabla w\|_2^2 + \frac{b}{4}\|\nabla w\|_2^4 \\ &= \Phi^\infty(w) = m^\infty, \quad \forall\, t > 0. \end{aligned} \tag{2.52}$$

By (2.48) and (2.50), we have that $\Phi^\infty(\xi w_{t_3})$ is decreasing on $\xi \in [1, t_2]$. Noting that

$$\frac{t_2(\theta - t_2) + t_3 - \theta}{t_3 - t_2} \in [1, t_2] \;\Leftrightarrow\; \theta \in [t_2, t_3],$$

we then get that $\Phi^\infty\left(\frac{t_2(\theta - t_2) + t_3 - \theta}{t_3 - t_2} w_{t_3}\right)$ is decreasing on $\theta \in [t_2, t_3]$. Therefore,

$$\Phi^\infty(\gamma(\theta)) \leq \Phi^\infty(\gamma(t_2)) = \Phi^\infty(w_{t_3}), \quad \forall\, \theta \in [t_2, t_3]. \tag{2.53}$$

Moreover, (2.50) yields

$$\begin{aligned} \Phi^\infty(\gamma(t_3)) = \Phi^\infty(t_2 w_{t_3}) &= \Phi^\infty(w_{t_3}) + \int_1^{t_2} \frac{\mathrm{d}}{\mathrm{d}\xi}\Phi^\infty(\xi w_{t_3})\mathrm{d}\xi \\ &\leq \frac{a}{2}\|\nabla w\|_2^2 + \frac{b}{4}\|\nabla w\|_2^4 - \int_1^{t_2} \frac{2\xi}{t_2^2 - 1}\left(a\|\nabla w\|_2^2 + b\|\nabla w\|_2^4\right)\mathrm{d}\xi \\ &= -\frac{a}{2}\|\nabla w\|_2^2 - \frac{3b}{4}\|\nabla w\|_2^4 < 0. \end{aligned} \tag{2.54}$$

Combining (2.51), (2.52) and (2.53), we have

$$\Phi^\infty(\gamma(\theta)) \leq \Phi^\infty(w) = m^\infty, \quad \forall\, \theta \in [0, t_3]. \tag{2.55}$$

Let $\tilde{\gamma}(\theta) = \gamma(t_3\theta)$ for all $\theta \in [0, 1]$. Since $\tilde{\gamma} \in \Gamma^\infty$ due to (2.54), then it follows from (2.55) that

$$\max_{t\in[0,t_3]} \Phi^\infty(\gamma(t)) = \Phi^\infty(w) = m^\infty, \tag{2.56}$$

where the definition of Γ^∞ is given by (1.12). The proof is completed. □

From the definition of c^∞, as a corollary to Lemma 2.8, we have the following result.

Corollary 2.9 *Assume that $V_\infty < C^*_{\mathrm{TM}}(F)$ and f satisfies* (F0)–(F3). *Then $c^\infty \le m^\infty$.*

Lemma 2.10 *Assume that $V_\infty < C^*_{\mathrm{TM}}(F)$ and f satisfies* (F0)–(F3). *Then $c^\infty \ge A^\infty$.*

Proof To prove $c^\infty \ge A^\infty = \inf_{u\in\mathcal{P}_\infty} \Phi^\infty(u)$, it suffices to show that

$$\gamma([0, 1]) \cap \mathcal{P}_\infty \ne \emptyset \text{ for all } \gamma \in \Gamma^\infty. \tag{2.57}$$

The proof of (2.57) follows the same line of [16, Lemma 4.1], so we omit it here. □

Proof of Theorem 1.3 In view of Theorem 1.2, we know that $A^\infty = m^\infty$. Therefore, Theorem 1.3 follows directly from Lemmas 2.8 and 2.10. □

3 Ground state solutions for $(\mathcal{K})$ with the trapping potential

In this section, we are concerned with the ground state solutions for $(\mathcal{K})$ with the trapping potential, that is V satisfies (V1).

Arguing as in the proof of Lemma 2.7, we can verify that $\Phi(u)$ has a mountain pass geometry. Applying the mountain pass theorem, we know that Φ possesses a Cerami sequence, reads as follows.

Lemma 3.1 *Assume that $V \in \mathcal{C}(\mathbb{R}^2, [V_0, V_\infty])$ and f satisfies* (F1), (F2) *and* (F4). *Then there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ such that*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{3.1}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) \tag{3.2}$$

and

$$\Gamma = \left\{\gamma \in \mathcal{C}([0, 1], H^1(\mathbb{R}^2)) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\right\}. \tag{3.3}$$

Lemma 3.2 *Assume that $V \in \mathcal{C}(\mathbb{R}^2, [V_0, V_\infty])$ with $V_\infty < C^*_{\mathrm{TM}}(F)$ and f satisfies* (F1), (F2) *and* (F4). *Then*

$$c \le c^\infty < \frac{2\pi a}{\alpha_0} + \frac{4\pi^2 b}{\alpha_0^2},$$

where the definitions of c and c^∞ are given by (3.2) *and* (1.11).

Proof First, we prove that $c \le c^\infty$. Since $\Phi(\gamma(1)) \le \Phi^\infty(\gamma(1)) < 0$ for any $\gamma \in \Gamma^\infty$ due to $V(x) \le V_\infty$ for all $x \in \mathbb{R}^2$, we have $\Gamma^\infty \subset \Gamma$. Then for any $\gamma \in \Gamma^\infty$, we have

$$\max_{t\in[0,1]} \Phi^\infty(\gamma(t)) \ge \max_{t\in[0,1]} \Phi(\gamma(t)) \ge \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) = c,$$

which, together with the arbitrariness of γ, yields

$$c^\infty = \inf_{\gamma\in\Gamma^\infty}\max_{t\in[0,1]}\Phi^\infty(\gamma(t)) \geq c.$$

Next, we prove that $c^\infty < \frac{2\pi a}{\alpha_0} + \frac{4\pi^2 b}{\alpha_0^2}$. Without loss of generality, we just consider the case $C^*_{\mathrm{TM}}(F) < +\infty$. Since $V_\infty < C^*_{\mathrm{TM}}(F)$, then $V_\infty + 2\varepsilon_0 < C^*_{\mathrm{TM}}(F)$ for some $\varepsilon_0 > 0$. In view of the definition of $C^*_{\mathrm{TM}}(F)$, there exists $\hat{u} \in H^1(\mathbb{R}^2)$ with $\|\nabla\hat{u}\|_2^2 \leq \frac{4\pi}{\alpha_0}$ satisfying

$$(V_\infty + \varepsilon_0)\,\|\hat{u}\|_2^2 < 2\int_{\mathbb{R}^2} F(\hat{u})\mathrm{d}x. \tag{3.4}$$

This shows that $J^\infty(\hat{u}) < 0$, where the definition of J^∞ is given by (1.17). Let $h(s){:=}J^\infty(s\hat{u})$ for $s > 0$. Since $h(1) < 0$ and $h(s) > 0$ for $s > 0$ small enough by (F2), then there exists $s_0 \in (0, 1)$ satisfying $h(s_0) = 0$. Therefore, for $\tilde{u}{:=}s_0\hat{u}$, we have

$$(V_\infty + \varepsilon_0)\,\|\tilde{u}\|_2^2 = 2\int_{\mathbb{R}^2} F(\tilde{u})\mathrm{d}x. \tag{3.5}$$

By (1.10) and (3.5), we get

$$\begin{aligned}
\Phi^\infty(\tilde{u}_t) &= \frac{a}{2}\|\nabla\tilde{u}\|_2^2 + \frac{b}{4}\|\nabla\tilde{u}\|_2^4 + \frac{t^2}{2}V_\infty\|\tilde{u}\|_2^2 - t^2\int_{\mathbb{R}^2}F(\tilde{u})\mathrm{d}x\\
&= \frac{a}{2}\|\nabla\tilde{u}\|_2^2 + \frac{b}{4}\|\nabla\tilde{u}\|_2^4 - \frac{\varepsilon_0 t^2}{2}\|\tilde{u}\|_2^2 \qquad (3.6)\\
&\leq \frac{as_0^2}{2}\|\nabla\hat{u}\|_2^2 + \frac{bs_0^4}{2}\|\nabla\hat{u}\|_2^4 < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2}, \quad \forall\, t > 0, \qquad (3.7)
\end{aligned}$$

where the last inequality follows from the fact that $\|\nabla\hat{u}\|_2^2 \leq \frac{4\pi}{\alpha_0}$ and $s_0 \in (0, 1)$. Using (F1), (F2) and Lemma 1.1 i), it is easy to check that there exists a constant $M_1 > 0$ independent of $\xi \in [0, 1]$ such that

$$\int_{\mathbb{R}^2}\left|\frac{f(\xi\tilde{u})}{\xi\tilde{u}}\tilde{u}^2\right|\mathrm{d}x \leq M_1, \quad \forall\,\xi \in [0, 1]. \tag{3.8}$$

Let $\tilde{u}_t(x){:=}\tilde{u}(x/t)$ for $t > 0$. Then it follows from (3.8) that

$$\begin{aligned}
\frac{d}{d\xi}\Phi^\infty(\xi\tilde{u}_t) &= \langle(\Phi^\infty)'(\xi\tilde{u}_t), \tilde{u}_t\rangle\\
&= \xi\left[a\|\nabla\tilde{u}\|_2^2 + \xi^2 b\|\nabla\tilde{u}\|_2^4 + t^2\left(V_\infty\|\tilde{u}\|_2^2 - \int_{\mathbb{R}^2}\frac{f(\xi\tilde{u})}{\xi\tilde{u}}\tilde{u}^2\mathrm{d}x\right)\right]\\
&\geq \xi\left[\|\nabla\tilde{u}\|_2^2 + t^2\left(V_\infty\|\tilde{u}\|_2^2 - M_1\right)\right], \quad \forall\, t > 0,\ \xi \in [0, 1]. \qquad (3.9)
\end{aligned}$$

Since $\tilde{u} \neq 0$, we can choose $t_0 \in (0, 1)$ such that

$$\|\nabla\tilde{u}\|_2^2 + t_0^2\left(V_\infty\|\tilde{u}\|_2^2 - M_1\right) > 0. \tag{3.10}$$

Using (3.6), we know that there exists $T > 1$ such that $\Phi^\infty(\tilde{u}_T) < 0$. Let

$$\gamma^*(t) = \begin{cases} t_0^{-1}t\tilde{u}_{t_0}, & 0 \leq t \leq t_0,\\ \tilde{u}_{t_0+(T-t_0)(t-t_0)/(1-t_0)}, & t_0 \leq t \leq 1. \end{cases} \tag{3.11}$$

Then it is easy to see that $\gamma^* \in \Gamma^\infty$, where Γ^∞ is defined by (1.12). Note that (3.9) and (3.10) show that $\Phi^\infty\left(t_0^{-1}t\tilde{u}_{t_0}\right)$ is increasing on $t \in [0, t_0]$. Hence it follows from (1.10) and (3.7) that

$$\Phi^\infty\left(t_0^{-1}t\tilde{u}_{t_0}\right) \le \Phi^\infty\left(\tilde{u}_{t_0}\right) < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2}, \quad \forall\, 0 \le t \le t_0. \tag{3.12}$$

From the definition of c^∞, (3.7), (3.11) and (3.12), we derive that

$$c^\infty \le \max_{t\in[0,1]} \Phi^\infty(\gamma^*(t)) < \frac{2a\pi}{\alpha_0} + \frac{4b\pi^2}{\alpha_0^2}. \tag{3.13}$$

This completes the proof. □

Note that (F5) implies the following inequality:

$$\begin{aligned}
&\frac{1-t^4}{4} f(u)u + F(tu) - F(u) + \frac{(1-t^2)^2}{4} V_0 u^2 \\
&= \int_t^1 \left[\frac{f(u) - V_0 u}{u^3} - \frac{f(su) - V_0(su)}{(su)^3}\right] s^3 u^4 \mathrm{d}s \\
&\ge 0, \quad \forall\, u \ne 0,\ t \ge 0.
\end{aligned} \tag{3.14}$$

Letting $t = 0$ in (3.14), we have

$$\frac{1}{4} f(u)u - F(u) + \frac{1}{4} V_0 u^2 \ge 0, \quad \forall\, u \in \mathbb{R}. \tag{3.15}$$

Lemma 3.3 *Assume that* (V1), (F1), (F2) *and* (F5) *hold. Then*

$$\Phi(u) \ge \Phi(tu) + \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-t^2)^2 a}{4}\|\nabla u\|_2^2, \quad \forall\, u \in H^1(\mathbb{R}^2),\ t \ge 0. \tag{3.16}$$

Corollary 3.4 *Assume that* (V1), (F1), (F2) *and* (F5) *hold. Then*

$$\Phi(u) \ge \max_{t\ge 0} \Phi(tu), \quad \forall\, u \in \mathcal{N}, \tag{3.17}$$

where $\mathcal{N}$ is defined by (1.15).

Lemma 3.5 *Assume that* (V1), (F1), (F2) *and* (F5) *hold. Then for any $u \in H^1(\mathbb{R}^2)\backslash\{0\}$, there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$.*

Lemma 3.6 *Assume that* (V1), (F1), (F2) *and* (F5) *hold. Then*

$$c = c_N = \inf_{u\in\mathcal{N}} \max_{t\ge 0} \Phi(tu). \tag{3.18}$$

Lemma 3.7 *Assume that* (V1) *and* (F1), (F2), (F4) *and* (F5) *hold. Then any sequence satisfying* (3.1) *is bounded in $H^1(\mathbb{R}^2)$.*

Proof Using (F4), it is easy to check that there exists $R_0 > \beta_0$ such that

$$f(t)t - 8F(t) \ge 0, \quad \forall\, |t| \ge R_0. \tag{3.19}$$

By (1.6), (1.7), (3.1), (3.15) and (3.19), we have

$$
\begin{aligned}
c+o(1) &= \Phi(u_n)+o(1) = \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\
&= \frac{a}{4}\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^2} V(x)u_n^2 \mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right]\mathrm{d}x \qquad (3.20)\\
&\ge \frac{a}{4}\|\nabla u_n\|_2^2 + \frac{1}{8}\int_{|u_n|\ge R_0} f(u_n)u_n \mathrm{d}x. \qquad (3.21)
\end{aligned}
$$

In view of (3.21), to prove the boundedness of $\{\|u_n\|\}$, it suffices to show the boundedness of $\{\|u_n\|_2\}$. To this end, arguing by contradiction, we assume that $\|u_n\|_2 \to \infty$ as $n \to \infty$. Let

$$
t_n = \left[\frac{2(c+1)}{V_0\|u_n\|_2^2}\right]^{1/4}. \qquad (3.22)
$$

Then $t_n \to 0$ as $n \to \infty$. Note that (3.15) implies that

$$
\frac{F(t)-\frac{1}{2}V_0t^2}{t^4} \text{ is nondecreasing on } t \in (-\infty, 0) \text{ and } t \in (0, +\infty). \qquad (3.23)
$$

This gives

$$
F(tu_n) - \frac{1}{2}V_0t^2u_n^2 \le t^4\left[F(u_n) - \frac{1}{2}V_0u_n^2\right], \quad \forall\, t \in (0,1), n \in \mathbb{N}. \qquad (3.24)
$$

Noting that $2t_n^4 \le t_n^2 < 1$ for large enough $n \in \mathbb{N}$, from (F2), (3.20) and (3.24), we deduce that for large enough $n \in \mathbb{N}$,

$$
\begin{aligned}
\int_{\mathbb{R}^2} F(t_nu_n)\mathrm{d}x &= \int_{|u_n|\le R_0} F(t_nu_n)\mathrm{d}x + \int_{|u_n|>R_0} F(t_nu_n)\mathrm{d}x \\
&\le \int_{|u_n|\le R_0} F(t_nu_n)\mathrm{d}x + \frac{V_0}{2}t_n^2(1-t_n^2)\int_{|u_n|>R_0} u_n^2\mathrm{d}x + t_n^4\int_{|u_n|>R_0} F(u_n)\mathrm{d}x \\
&\le \frac{V_0}{4}\int_{|u_n|\le R_0} |t_nu_n|^2\mathrm{d}x + \frac{V_0}{2}t_n^2(1-t_n^2)\int_{|u_n|>R_0} u_n^2\mathrm{d}x \\
&\quad + \frac{t_n^4}{8}\int_{|u_n|>R_0} f(u_n)u_n\mathrm{d}x \\
&\le \frac{V_0}{4}\int_{|u_n|\le R_0} |t_nu_n|^2\mathrm{d}x + \frac{V_0}{2}t_n^2(1-t_n^2)\int_{|u_n|>R_0} u_n^2\mathrm{d}x + (c+1)t_n^4 \\
&= \frac{V_0}{4}t_n^2\int_{|u_n|\le R_0} u_n^2\mathrm{d}x + \frac{V_0}{2}t_n^2\int_{|u_n|>R_0} u_n^2\mathrm{d}x + \frac{V_0}{2}t_n^4\int_{|u_n|\le R_0} u_n^2\mathrm{d}x \\
&\quad - \frac{V_0}{2}t_n^4\|u_n\|_2^2 + (c+1)t_n^4 \\
&\le \frac{V_0}{2}t_n^2\int_{|u_n|\le R_0} u_n^2\mathrm{d}x + \frac{V_0}{2}t_n^2\int_{|u_n|>R_0} u_n^2\mathrm{d}x - \frac{V_0}{2}t_n^4\|u_n\|_2^2 + (c+1)t_n^4 \\
&= \frac{V_0}{2}t_n^2\|u_n\|_2^2 - (c+1) + \frac{2(c+1)^2}{V_0\|u_n\|_2^2}. \qquad (3.25)
\end{aligned}
$$

By (1.6), (1.7), (3.1), (3.16), (3.20) and (3.25), we have

$$
\begin{aligned}
c+o(1) &= \Phi(u_n)+o(1) \geq \Phi(t_n u_n)+o(1) \\
&= \frac{a t_n^2}{2}\|\nabla u_n\|_2^2+\frac{b t_n^4}{4}\|\nabla u_n\|_2^4+\frac{t_n^2}{2}\int_{\mathbb{R}^2} V(x) u_n^2 \mathrm{d}x-\int_{\mathbb{R}^2} F(t_n u_n) \mathrm{d}x \\
&\geq \frac{t_n^2}{2}\int_{\mathbb{R}^2} V(x) u_n^2 \mathrm{d}x-\frac{V_0}{2} t_n^2\|u_n\|_2^2+(c+1)-\frac{2(c+1)^2}{V_0\|u_n\|_2^2}+o(1) \\
&\geq c+1+o(1).
\end{aligned}
$$

This contradiction shows the boundedness of $\{\|u_n\|_2\}$. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. □

As a direct consequence of [10, Lemma 2.1], we can get the following lemma.

Lemma 3.8 *Assume that* (F1) *and* (F2) *hold. Let* $u_n \rightharpoonup \bar{u}$ *in* $H^1(\mathbb{R}^2)$ *and*

$$
\int_{\mathbb{R}^2} |f(u_n) u_n| \mathrm{d}x \leq K_0 \tag{3.26}
$$

for some constant $K_0>0$. *Then for any* $\phi \in \mathcal{C}_0^{\infty}(\mathbb{R}^2)$

$$
\lim_{n \to \infty} \int_{\mathbb{R}^2} f(u_n) \phi \mathrm{d}x=\int_{\mathbb{R}^2} f(\bar{u}) \phi \mathrm{d}x. \tag{3.27}
$$

Similarly to the proof of [27, Lemma 2.13], we can get the following lemma.

Lemma 3.9 *Assume that* (V1), (F1), (F2) *and* (F5) *hold. If* $\bar{u} \in \mathcal{N}$ *and* $\Phi(\bar{u})=c_{\mathcal{N}}$, *then* $\bar{u}$ *is a critical point of* Φ, *namely* $\Phi'(\bar{u})=0$.

Proof of Theorem 1.4 Applying Lemmas 3.1 and 3.7, we deduce that there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying (3.1) and $\|u_n\| \leq C_7$. Then (1.7) and (3.1) give

$$
\int_{\mathbb{R}^2} f(u_n) u_n \mathrm{d}x \leq C_8. \tag{3.28}
$$

We may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^2)$, $u_n \to \bar{u}$ in $L_{\mathrm{loc}}^s(\mathbb{R}^2)$ for $s \in [1, \infty)$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^2$. Now, we distinguish the following two cases: i) $\bar{u} \neq 0$; ii) $\bar{u}=0$.

Case i): $\bar{u} \neq 0$. We assume that $l=\lim_{n \to \infty}\|\nabla u_n\|_2$ up to a subsequence. Then $\|\nabla \bar{u}\|_2 \leq l$. Applying Lemma 3.8, we get

$$
\lim_{n \to \infty} \int_{\mathbb{R}^2} f(u_n) \varphi \mathrm{d}x=\int_{\mathbb{R}^2} f(\bar{u}) \varphi \mathrm{d}x, \quad \forall\, \varphi \in \mathcal{C}_0^{\infty}(\mathbb{R}^2). \tag{3.29}
$$

Note that there exists $\{\varphi_n\} \subset \mathcal{C}_0^{\infty}(\mathbb{R}^2)$ such that $\|\varphi_n-\bar{u}\|=o(1)$ since $C_0^{\infty}(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$. This, together with (3.29), (F1) and Lemma 1.1 i), gives

$$
\begin{aligned}
\left|\int_{\mathbb{R}^2} f(\bar{u})(\varphi_n-\bar{u}) \mathrm{d}x\right| &\leq \int_{\mathbb{R}^2} |f(\bar{u})||\varphi_n-\bar{u}| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^2}\left(|\bar{u}|+C_9 e^{2 \alpha_0 \bar{u}^2}\right)|\varphi_n-\bar{u}| \mathrm{d}x \\
&\leq \|\bar{u}\|_2\|\varphi_n-\bar{u}\|_2+C_9\left(\int_{\mathbb{R}^2} e^{4 \alpha_0 \bar{u}^2} \mathrm{d}x\right)^{\frac{1}{2}}\|\varphi_n-\bar{u}\|_2 \\
&= o(1).
\end{aligned} \tag{3.30}
$$

By (1.7), (3.1) and (3.30), we have

$$\langle \Phi'(\bar{u}), \bar{u}\rangle + b(l^2 - \|\nabla \bar{u}\|_2^2)\int_{\mathbb{R}^2} \nabla \bar{u}\cdot \nabla \bar{u}\mathrm{d}x = \lim_{n\to\infty}\langle \Phi'(u_n), \bar{u}\rangle = 0,$$

which, together with $l^2 - \|\nabla \bar{u}\|_2^2 \geq 0$, implies that $\langle \Phi'(\bar{u}), \bar{u}\rangle \leq 0$.

Note that $\langle \Phi'(t\bar{u}), t\bar{u}\rangle > 0$ for small $t > 0$ by (F1) and (F2). Then there exists $\bar{t} \in (0, 1]$ such that

$$\langle \Phi'(\bar{t}\bar{u}), \bar{t}\bar{u}\rangle = 0 \ \text{ and } \ \Phi(\bar{t}\bar{u}) \geq c_N. \tag{3.31}$$

Using (F5), it is easy to see that

$$f(tu)tu \leq t^4 f(u)u + V_0(1-t^2)(tu)^2, \quad \forall\, 0 \leq t \leq 1,\ u \in \mathbb{R}. \tag{3.32}$$

Note that (3.14) implies

$$F(tu) \geq \frac{t^4-1}{4} f(u)u + F(u) - \frac{1-2t^2+t^4}{4} V_0 u^2, \quad \forall\, t \geq 0,\ u \in \mathbb{R}. \tag{3.33}$$

Combining (3.32) with (3.33), we have

$$\frac{1}{4} f(tu)tu - F(tu) + \frac{V_0}{4}(tu)^2 \leq \frac{1}{4} f(u)u - F(u) + \frac{V_0}{4} u^2, \quad \forall\, 0 \leq t \leq 1,\ u \in \mathbb{R}. \tag{3.34}$$

Since $\bar{t} \in (0, 1]$, from (3.1), (3.31), lemma 3.6 and Fatou's lemma, it follows that

$$\begin{aligned} c_N &\leq \Phi(\bar{t}\bar{u}) - \frac{1}{4}\langle \Phi'(\bar{t}\bar{u}), \bar{t}\bar{u}\rangle \\ &= \frac{\bar{t}^2}{4}\int_{\mathbb{R}^2}\left[a|\nabla \bar{u}|^2 + (V(x)-V_0)\bar{u}^2\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{4} f(\bar{t}\bar{u})\bar{t}\bar{u} - F(\bar{t}\bar{u}) + \frac{V_0}{4}(\bar{t}\bar{u})^2\right]\mathrm{d}x \\ &\leq \frac{1}{4}\int_{\mathbb{R}^2}\left[a|\nabla \bar{u}|^2 + (V(x)-V_0)\bar{u}^2\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{4} f(\bar{u})\bar{u} - F(\bar{u}) + \frac{V_0}{4}\bar{u}^2\right]\mathrm{d}x \\ &= \Phi(\bar{u}) - \frac{1}{4}\langle \Phi'(\bar{u}), \bar{u}\rangle \\ &\leq \lim_{n\to\infty}\left[\Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle\right] = c = c_N. \end{aligned}$$

This implies that $\bar{t} = 1$ and $u_n \to \bar{u}$ in $H^1(\mathbb{R}^2)$. Hence, we have $\Phi'(\bar{u}) = 0$ and $\Phi(\bar{u}) = c_N = c$.

Case ii): $\bar{u} = 0$. Then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$. Moreover, by a standard argument, we have

$$\Phi^\infty(u_n) \to c, \quad \|(\Phi^\infty)'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.35}$$

We claim that

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_1(y)} |u_n|^2\mathrm{d}x > 0. \tag{3.36}$$

Otherwise, if $\delta = 0$, then by Lion's concentration compactness principle [29, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s > 2$. Arguing as in the proof of [8, (4.20)] (or [6, (4.56)]), and using (F2) and (F4), we can get

$$\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = o(1). \tag{3.37}$$

In view of Theorem 1.3, Lemma 3.2 and (2.35), we have

$$c \leq c^{\infty} = A^{\infty} < \frac{2\pi a}{\alpha_0} + \frac{4\pi^2 b}{\alpha_0^2}. \tag{3.38}$$

From (3.1), (3.37) and (3.38), it follows that

$$\begin{aligned} \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u_n^2 \mathrm{d}x + \frac{b}{4}\|\nabla u_n\|_2^4 + o(1) = c \leq c^{\infty} = A^{\infty} \\ = \frac{2a\pi}{\alpha_0}(1-4\bar{\varepsilon}) + \frac{4b\pi^2}{\alpha_0^2}(1-4\bar{\varepsilon})^2, \end{aligned} \tag{3.39}$$

where $0 < \bar{\varepsilon} < \frac{1}{4}$. Using (3.39), it is easy to check that

$$\|\nabla u_n\|_2^2 \leq \frac{4\pi}{\alpha_0}(1-4\bar{\varepsilon}). \tag{3.40}$$

Choosing $q \in (1, 2)$ satisfying

$$\frac{(1+\bar{\varepsilon})(1-3\bar{\varepsilon})q}{1-\bar{\varepsilon}} < 1, \tag{3.41}$$

by (F1), we have

$$|f(t)|^q \leq C_8\left[e^{\alpha_0(1+\bar{\varepsilon})qt^2} - 1\right], \quad \forall\, |t| \geq 1. \tag{3.42}$$

From (3.40), (3.41), (3.42) and Lemma 1.1 ii), we deduce that

$$\begin{aligned} \int_{|u_n|\geq 1} |f(u_n)|^q \mathrm{d}x &\leq C_8 \int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})qu_n^2} - 1\right]\mathrm{d}x \\ &= C_8 \int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})q(\|\nabla u_n\|_2^2 + 4\pi\bar{\varepsilon}/\alpha_0)\frac{u_n^2}{(\|\nabla u_n\|_2^2+4\pi\bar{\varepsilon}/\alpha_0)}} - 1\right]\mathrm{d}x \leq C_9. \end{aligned} \tag{3.43}$$

Let $q' := q/(q-1) > 2$. Then (3.43) and the Hölder inequality yield

$$\int_{|u_n|\geq 1} f(u_n)u_n \mathrm{d}x \leq \left[\int_{|u_n|\geq 1} |f(u_n)|^q \mathrm{d}x\right]^{1/q} \|u_n\|_{q'} = o(1). \tag{3.44}$$

From (1.6), (1.7), (3.1), (3.37) and (3.44), it follows that

$$\begin{aligned} c + o(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \\ &= -\frac{b}{4}\|\nabla u_n\|_2^4 + \int_{\mathbb{R}^2}\left[\frac{1}{2}f(u_n)u_n - F(u_n)\right]\mathrm{d}x \\ &\leq o(1). \end{aligned} \tag{3.45}$$

This contradiction shows that the claim (3.36) holds, namely $\delta > 0$. Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^2$ such that

$$\int_{B_2(k_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2} > 0.$$

Let $v_n(x) = u_n(x + k_n)$. Then

$$\int_{B_2(0)} |v_n|^2 \mathrm{d}x > \frac{\delta}{2} > 0. \tag{3.46}$$

Moreover, (3.35) gives

$$\Phi^\infty(v_n) \to c, \quad \|(\Phi^\infty)'(v_n)\|(1 + \|v_n\|) \to 0. \tag{3.47}$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^2)$, $v_n \to \bar{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for $s > 2$ and $v_n \to \bar{v}$ a.e. in $\mathbb{R}^2$. Then (3.46) yields $\bar{v} \neq 0$. In view of Theorem 1.2 and Theorem 1.3, one can prove easily

$$c^\infty = m^\infty = \inf\left\{\Phi^\infty(u) \;\middle|\; u \in H^1(\mathbb{R}^2)\backslash\{0\},\ \langle(\Phi^\infty)'(u), u\rangle = 0\right\}. \tag{3.48}$$

As in the proof of (3.31), using (3.48), we can derive that there exists $\tilde{t} \in (0, 1]$ such that

$$\langle(\Phi^\infty)'(\tilde{t}\bar{v}), \tilde{t}\bar{v}\rangle = 0 \;\text{ and }\; \Phi^\infty(\tilde{t}\bar{v}) \geq c^\infty. \tag{3.49}$$

Since $\tilde{t} \in (0, 1]$, from (4.1), (3.49), lemma 3.2 and Fatou's lemma, we deduce

$$\begin{aligned}
c^\infty &\leq \Phi^\infty(\tilde{t}\bar{v}) - \frac{1}{4}\langle(\Phi^\infty)'(\tilde{t}\bar{v}), \tilde{t}\bar{v}\rangle \\
&= \frac{\tilde{t}^2}{4}\int_{\mathbb{R}^2}\left[a|\nabla\bar{v}|^2 + (V_\infty - V_0)\bar{v}^2\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{4}f(\tilde{t}\bar{v})\tilde{t}\bar{v} - F(\tilde{t}\bar{v}) + \frac{V_0}{4}(\tilde{t}\bar{v})^2\right]\mathrm{d}x \\
&\leq \frac{1}{4}\int_{\mathbb{R}^2}\left[a|\nabla\bar{u}|^2 + (V_\infty - V_0)\bar{u}^2\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{4}f(\bar{v})\bar{v} - F(\bar{v}) + \frac{V_0}{4}\bar{v}^2\right]\mathrm{d}x \\
&= \Phi^\infty(\bar{v}) - \frac{1}{4}\langle(\Phi^\infty)'(\bar{v}), \bar{v}\rangle \\
&\leq \lim_{n\to\infty}\left[\Phi^\infty(v_n) - \frac{1}{4}\langle(\Phi^\infty)'(v_n), v_n\rangle\right] = c = c_N \leq c^\infty,
\end{aligned}$$

which implies that $\tilde{t} = 1$, $v_n \to \bar{v}$ in $H^1(\mathbb{R}^2)$, $\langle(\Phi^\infty)'(\bar{v}), \bar{v}\rangle = 0$ and $\Phi^\infty(\bar{v}) = c = c_N$. As in Corollary 3.4, we have $c = c_N = \Phi^\infty(\bar{v}) \geq \max_{t\geq 0}\Phi^\infty(t\bar{v})$. Since $\bar{v} \neq 0$, by Lemma 3.5, there exists $\hat{t} > 0$ such that $\hat{t}\bar{v} \in \mathcal{N}$, and so $\Phi(\hat{t}\bar{v}) \geq c_N = c$. Therefore, it follows that

$$\begin{aligned}
c = \Phi^\infty(\bar{v}) &\geq \Phi^\infty(\hat{t}\bar{v}) = \Phi(\hat{t}\bar{v}) + \frac{\hat{t}^2}{2}\int_{\mathbb{R}^2}[V_\infty - V(x)]\bar{v}^2\mathrm{d}x \\
&\geq \Phi(\hat{t}\bar{v}) \geq c_N = c,
\end{aligned}$$

which yields $\Phi(\hat{t}\bar{v}) = c_N = c$. Applying Lemma 3.9, we get $\Phi'(\hat{t}\bar{v}) = 0$. This completes the proof of Theorem 1.4. □

4 Nontrivial solutions for $(\mathcal{K})$ with the radial potential

In this section, we study the existence of nontrivial solutions for $(\mathcal{K})$ with radial potential V satisfying (V2), by restricting the working space in $H^1_r(\mathbb{R}^2)$. It is well-known that the embedding $H^1_r(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ is compact for any $s > 2$. In view of [29, Theorem 1.28], if u is a critical point of Φ restricted to $H^1_r(\mathbb{R}^2)$, then u is a critical point of Φ on $H^1(\mathbb{R}^2)$.

Similarly as in Lemma 3.1, we can deduce that there exists a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ satisfying

$$\Phi(u_n) \to c_r < \frac{2\pi a}{\alpha_0} + \frac{4\pi^2 b}{\alpha_0^2}, \quad \|\Phi'(u_n)\|_{(H_r^1(\mathbb{R}^2))^*}(1 + \|u_n\|) \to 0, \tag{4.1}$$

$$c_r = \inf_{\gamma \in \Gamma_r} \max_{t \in [0,1]} \Phi(\gamma(t)), \tag{4.2}$$

where

$$\Gamma_r = \left\{\gamma \in \mathcal{C}([0, 1], H_r^1(\mathbb{R}^2)) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\right\}. \tag{4.3}$$

Lemma 4.1 *Assume that* (V2) *and* (F1), (F2), (F3′), (F4) *and* (F6) *hold. Then any sequence satisfying* (4.1) *is bounded in* $H^1(\mathbb{R}^2)$.

Proof Arguing as in (3.19)–(3.21), we know that

$$\|\nabla u_n\|_2^2 \le C_1, \;\; 8\int_{|u_n| \ge R_0} F(u_n)\mathrm{d}x \le \int_{|u_n| \ge R_0} f(u_n)u_n\mathrm{d}x \le C_2. \tag{4.4}$$

In view of (4.4), to prove the boundedness of $\{\|u_n\|\}$, it remains to show the boundedness of $\{\|u_n\|_2\}$. To this end, arguing by contradiction, suppose that $\|u_n\|_2 \to +\infty$. Let $v_n := \frac{u_n}{\|u_n\|_2}$,

$$D_1 := \left\{t \in [-R_0, R_0] : \frac{f(t)}{t} < V_0\right\}, \quad D_2 := \left\{t \in [-R_0, R_0] : \frac{f(t)}{t} \ge V_0\right\} \tag{4.5}$$

and

$$\mathcal{F}(t) := \frac{1}{2} f(t)t - F(t), \quad \forall\, t \in \mathbb{R}. \tag{4.6}$$

Using (F3′), it is easy to get

$$\mathcal{F}(t) \ge C_4, \quad \forall\, t \in D_2. \tag{4.7}$$

From (F1), (F2) and (4.7), we have

$$|f(t)|^2 \le C_5|t| \le C_6|t|\mathcal{F}(t), \quad \forall\, t \in D_2. \tag{4.8}$$

By (4.1), we have

$$\begin{aligned} c + o(1) = \Phi(u_n) + o(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \\ &\ge -\frac{b}{4}\|\nabla u_n\|_2^4 + \int_{|u_n| \le R_0} \mathcal{F}(u_n)\mathrm{d}x, \end{aligned}$$

which, together with (4.4), yields

$$\int_{|u_n| \le R_0} \mathcal{F}(u_n)\mathrm{d}x \le C_7. \tag{4.9}$$

Note that $\{\|v_n\|\}$ is bounded due to (4.4). Thus it follows from (1.7), (4.1) and (4.9) that

$$\begin{aligned}
V_0 &\le \lim_{n\to\infty}\left[\frac{1}{\|u_n\|_2^2}\int_{\mathbb{R}^2}\left[a|\nabla u_n|^2+V(x)u_n^2\right]\mathrm{d}x+\frac{b\|\nabla u_n\|_2^4}{\|u_n\|_2^2}\right]\\
&=\lim_{n\to\infty}\left[\frac{1}{\|u_n\|_2^2}\int_{|u_n|\le R_0} f(u_n)u_n\mathrm{d}x+\frac{1}{\|u_n\|_2^2}\int_{|u_n|>R_0} f(u_n)u_n\mathrm{d}x\right]\\
&\le \limsup_{n\to\infty}\int_{D_1}\frac{f(u_n)}{u_n}v_n^2\mathrm{d}x+\limsup_{n\to\infty}\frac{1}{\|u_n\|_2^{1/2}}\int_{D_2}\frac{|f(u_n)|}{|u_n|^{1/2}}|v_n|^{3/2}\mathrm{d}x\\
&< V_0+\limsup_{n\to\infty}\frac{\sqrt{C_6}}{\|u_n\|_2^{1/2}}\left[\int_{|u_n|\le R_0}\mathcal{F}(u_n)\mathrm{d}x\right]^{1/2}\|v_n\|_3^{3/2}\\
&=V_0.
\end{aligned}$$

This contradiction shows the boundedness of $\{\|u_n\|_2\}$. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. □

Proof of Theorem 1.4 In view of Lemma 4.1, there exists a sequence $\{u_n\}\subset H_r^1(\mathbb{R}^2)$ satisfying (4.1) and $\|u_n\|\le C_8$. We may thus assume, passing to a subsequence if necessary, that $u_n\rightharpoonup\bar{u}$ in $H_r^1(\mathbb{R}^2)$, $u_n\to\bar{u}$ in $L^s(\mathbb{R}^2)$ for $s>2$ and $u_n\to\bar{u}$ a.e. in $\mathbb{R}^2$. Arguing as in the proof of [8, (4.27)], we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}F(u_n)\mathrm{d}x=\int_{\mathbb{R}^2}F(\bar{u})\mathrm{d}x. \tag{4.10}$$

Step 1. First, we prove that up to a subsequence,

$$L:=\lim_{n\to\infty}\|\nabla u_n\|_2^2<\frac{4\pi}{\alpha_0}+\|\nabla\bar{u}\|_2^2. \tag{4.11}$$

Arguing by contradiction, we assume that $L\ge\frac{4\pi}{\alpha_0}+\|\nabla\bar{u}\|_2^2$. By (1.7) and (3.1), we get

$$0=\lim_{n\to\infty}\langle\Phi'(u_n),u_n\rangle=aL+bL^2+\lim_{n\to\infty}\int_{\mathbb{R}^2}\left[V(x)u_n^2-f(u_n)u_n\right]\mathrm{d}x \tag{4.12}$$

and

$$\begin{aligned}
0&=\lim_{n\to\infty}\langle\Phi'(u_n),\varphi\rangle\\
&=\lim_{n\to\infty}\left\{\int_{\mathbb{R}^2}[a\nabla u_n\cdot\nabla\varphi+V(x)u_n\varphi]\,\mathrm{d}x+bL\int_{\mathbb{R}^2}\nabla u_n\cdot\nabla\varphi\mathrm{d}x-\int_{\mathbb{R}^2}f(u_n)\varphi\mathrm{d}x\right\}\\
&=\int_{\mathbb{R}^2}[a\nabla\bar{u}\cdot\nabla\varphi+V(x)\bar{u}\varphi]\,\mathrm{d}x+bL\int_{\mathbb{R}^2}\nabla\bar{u}\cdot\nabla\varphi\mathrm{d}x-\int_{\mathbb{R}^2}f(\bar{u})\varphi\mathrm{d}x,\quad\forall\,\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2),
\end{aligned}$$

which, together with (3.30), the definition of L and the density of $\mathcal{C}_0^\infty(\mathbb{R}^2)$ in $H_r^1(\mathbb{R}^2)$, implies

$$0=\lim_{n\to\infty}\langle\Phi'(u_n),\bar{u}\rangle=a\|\nabla\bar{u}\|_2^2+\int_{\mathbb{R}^2}V(x)\bar{u}^2\mathrm{d}x+bL\|\nabla\bar{u}\|_2^2-\int_{\mathbb{R}^2}f(\bar{u})\bar{u}\mathrm{d}x. \tag{4.13}$$

If $L \geq \frac{4\pi}{\alpha_0} + \|\nabla \bar{u}\|_2^2$, from (3.1), (3.38), (4.12), (4.13) and Fatou's lemma, we then deduce

$$
\begin{aligned}
\frac{2\pi a}{\alpha_0} + \frac{4\pi^2 b}{\alpha_0^2} > c^\infty \geq c &= \lim_{n\to\infty}\left[\Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), \bar{u}\rangle\right] \\
&= \frac{a}{2}L + \frac{b}{4}L^2 - \int_{\mathbb{R}^2} F(\bar{u})\mathrm{d}x + \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)u_n^2\mathrm{d}x \\
&\quad - \frac{a}{4}\|\nabla \bar{u}\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^2} V(x)\bar{u}^2\mathrm{d}x + \frac{b}{4}L\|\nabla \bar{u}\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^2} f(\bar{u})\bar{u}\mathrm{d}x \\
&= \frac{a}{4}L + \frac{a+bL}{4}(L - \|\nabla \bar{u}\|_2^2) + \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)\left(u_n^2 - \bar{u}^2\right)\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^2}[f(\bar{u})\bar{u} - 4F(\bar{u}) + V(x)\bar{u}^2]\mathrm{d}x \\
&\geq \frac{2\pi a}{\alpha_0} + \frac{4\pi^2 b}{\alpha_0^2} + \left(a + \frac{4\pi b}{\alpha_0^2}\right)\frac{\|\nabla \bar{u}\|_2^2}{4},
\end{aligned}
$$

which is a contradiction. Hence, we complete the proof of Step 1.

Step 2. We prove that $\Phi(\bar{u}) = b$ and $\Phi'(\bar{u}) = 0$.

From Step 1, we know that there exists $\hat{\varepsilon} > 0$ satisfying

$$
\lim_{n\to\infty}\left[\|\nabla(u_n - \bar{u})\|_2^2\right] = \lim_{n\to\infty}\left[\|\nabla u_n\|_2^2 - \|\nabla \bar{u}\|_2^2\right] = \frac{4\pi(1-3\hat{\varepsilon})}{\alpha_0}. \tag{4.14}
$$

Choosing $q \in (1, 2)$ satisfying

$$
\frac{(1+\hat{\varepsilon})^2(1-3\hat{\varepsilon})q^2}{1-\hat{\varepsilon}} < 1, \tag{4.15}
$$

by (4.14), the Young's inequality and Lemma 1.1, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2}\left(e^{\alpha_0(1+\hat{\varepsilon})qu_n^2} - 1\right)\mathrm{d}x &\leq \int_{\mathbb{R}^2}\left(e^{\alpha_0(1+\hat{\varepsilon})^2\hat{\varepsilon}^{-1}q\bar{u}^2} e^{\alpha_0(1+\hat{\varepsilon})^2 q(u_n-\bar{u})^2} - 1\right)\mathrm{d}x \\
&\leq \frac{(q-1)}{q}\int_{\mathbb{R}^2}\left(e^{\alpha_0(1+\hat{\varepsilon})^2\hat{\varepsilon}^{-1}q^2(q-1)^{-1}\bar{u}^2} - 1\right)\mathrm{d}x \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^2}\left(e^{\alpha_0(1+\hat{\varepsilon})^2 q^2(u_n-\bar{u})^2} - 1\right)\mathrm{d}x \\
&\leq C_9.
\end{aligned} \tag{4.16}
$$

Using (F1) and (F2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
|f(t)| \leq \varepsilon|t| + C_\varepsilon\left(e^{\alpha_0(1+\hat{\varepsilon})t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{4.17}
$$

Let $q' := q/(q-1) > 2$. Since $u_n \to u$ in $L^{q'}(\mathbb{R}^2)$, from (4.16), (4.17) and the Hölder inequality, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2} f(u_n)(u_n-\bar{u})\mathrm{d}x &\le \varepsilon\|u_n-\bar{u}\|_2^2 + C_9\int_{\mathbb{R}^2}\left(e^{\alpha u_n^2}-1\right)|u_n-\bar{u}|\mathrm{d}x\\ &\le \varepsilon\left(C_1+\|\bar{u}\|_2^2\right)+C_9\left[\int_{\mathbb{R}^2}\left(e^{\alpha_0(1+\hat{\varepsilon})u_n^2}-1\right)^q\mathrm{d}x\right]^{1/q}\|u_n-\bar{u}\|_{q'}\\ &\le \varepsilon\left(C_1+\|\bar{u}\|_2^2\right)+C_{10}\left[\int_{\mathbb{R}^2}\left(e^{\alpha_0(1+\hat{\varepsilon})qu_n^2}-1\right)\mathrm{d}x\right]^{1/q}\|u_n-\bar{u}\|_{q'}\\ &\le \varepsilon\left(C_1+\|\bar{u}\|_2^2\right)+C_{11}\|u_n-\bar{u}\|_{q'}\\ &\le \varepsilon\left(C_1+\|\bar{u}\|_2^2\right)+o(1),\end{aligned}\tag{4.18}$$

which, together with the arbitrariness of ε, yields

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f(u_n)(u_n-\bar{u})\mathrm{d}x = 0.\tag{4.19}$$

Therefore, it follows from (1.7), (3.1), (4.19) that

$$\begin{aligned}o(1) &= \langle\Phi'(u_n), u_n-\bar{u}\rangle\\ &= \int_{\mathbb{R}^2}\left[a\nabla u_n\cdot\nabla(u_n-\bar{u})+V(x)u_n(u_n-\bar{u})\right]\mathrm{d}x\\ &\quad + b\|\nabla u_n\|_2^2\int_{\mathbb{R}^2}\nabla u_n\cdot\nabla(u_n-\bar{u})\mathrm{d}x-\int_{\mathbb{R}^2} f(u_n)(u_n-\bar{u})\mathrm{d}x\\ &\ge a\|\nabla(u_n-\bar{u})\|_2^2+V_0\|u_n-\bar{u}\|_2^2+b\|\nabla u_n\|_2^2\|\nabla(u_n-\bar{u})\|_2^2+o(1)\\ &\ge \min\{a, V_0\}\|u_n-\bar{u}\|^2+o(1),\end{aligned}$$

which implies that $u_n \to \bar{u}$ in $H_r^1(\mathbb{R}^2)$. Then the continuity of Φ and Φ' leads to $\Phi(\bar{u}) = c$ and $\Phi'(\bar{u})\big|_{H_r^1(\mathbb{R}^2)} = 0$. From [29, Theorem 1.28], we conclude that u is a critical point of Φ on $H^1(\mathbb{R}^2)$. □

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