Ground state solutions of non-linear singular Schrödinger equations with lack of compactness

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SUMMARY

We study a class of time-independent non-linear Schrödinger-type equations on the whole space with a repulsive singular potential in the divergence operator and we establish the existence of non-trivial standing wave solutions for this problem in an appropriate weighted Sobolev space. Such equations have been derived as models of several physical phenomena. Our proofs rely essentially on critical point theory tools combined with the Caffarelli–Kohn–Nirenberg inequality. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: non-linear Schrödinger equation; Caffarelli–Kohn–Nirenberg inequality; singular potential; entire weak solution

1. INTRODUCTION

This paper is motivated by several works on non-linear Schrödinger equations. Problems of this type appear in the study of several physical phenomena: self-channelling of a high-power ultra-short laser in matter [1–4], in the theory of Heisenberg ferromagnets and magnons [5–9] in dissipative quantum mechanics [10] in condensed matter theory [11], in plasma physics (e.g. the Kurihara superfluid film equation) [12–15], etc.

Consider the model problem

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbf{R}^N \tag{1}$$

where $p < 2^*$. In the study of this equation Oh [16] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some a) introduced in Kato [17]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov–Schmidt-type reduction, Oh proved the existence of a standing wave solution of (1), i.e. a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x) \tag{2}$$

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Note that substituting the ansatz (2) into (1) leads to

$$-\frac{\hbar^2}{2}\Delta u + (V(x) - E)u = |u|^{p-1}u$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2(V_\hbar(x) - E)u = |u|^{p-1}u \tag{3}$$

where $V_\hbar(x) = V(\hbar x)$.

Our goal is to show how variational methods can be used to find existence results for stationary non-linear Schrödinger equations. In particular, some classes of highly oscillatory potentials in the class (V_a) are allowed. Our approach is based on the fact that many non-linear problems such as those that naturally arise in the study of geodesics, minimal surfaces, harmonic maps, conformal metrics with prescribed curvature, subharmonics of Hamiltonian systems, solutions of boundary value problems and Yang–Mills fields can all be characterized as critical points u of some energy functional I on an appropriate manifold X, i.e. $I'(u) = 0$.

We study in this paper non-linear Schrödinger equations of form (3), but with a degenerate potential under the divergence operator. We point out that the study of degenerate elliptic boundary value problems was initiated in Mikhlin [18,19] and many papers were devoted in the past decades to the study of several questions related to these problems. We refer only to Murthy–Stampacchia [20], Baouendi–Goulaouic [21], Stredulinsky [22] Caldiroli–Musina [28] and the references therein.

2. THE MAIN RESULT

We are concerned with a problem on the existence of critical points and how they relate to the (weak) solutions they represent for the corresponding Euler–Lagrange equations. More precisely, we study the existence of non-trivial solutions to degenerate elliptic equations of the type

$$-\mathrm{div}(A(x)\nabla u) = g(x, u), \quad x \in \mathbf{R}^N$$

where the weight A is a non-negative measurable function that is allowed to have 'essential' zeroes at some points or even to be unbounded. Problems of this type come from the consideration of standing waves in anisotropic Schrödinger equations. A model equation that we consider in this paper is

$$-\mathrm{div}(|x|^\alpha \nabla u) = f(x, u) - b(x)u, \quad x \in \mathbf{R}^N$$

The main interest of this equation is the presence of the singular potential $|x|^\alpha$ in the divergence operator. Problems of this type arise in many areas of applied physics, including nuclear physics, field theory, solid waves, and problems of false vacuum. These problems are introduced as models for several physical phenomena related to equilibrium of continuous media which may somewhere be '*perfect*' *insulators*, cf. [23, p. 79]. These equations are reduced to elliptic equations with Hardy singular potential. For example, the solutions of the model problem

$$-\mathrm{div}(|x|^\alpha \nabla u) = |x|^{\beta p}u^{p-1}, \quad u \geqslant 0 \quad \text{in } \mathbf{R}^N$$

are in one-to-one correspondence to solutions of the Schrödinger-type equation with Hardy potential

$$-\mathrm{div}(|x|^\gamma \nabla v) + \frac{\lambda}{|x|^{2+\gamma}}\, v = |x|^{\delta p} v^{p-1}, \quad u \geqslant 0 \quad \text{in } \mathbf{R}^N$$

where

$$\alpha = \sqrt{(N-2+\gamma)^2 + 4\lambda} - N + 2$$

and

$$\beta = \delta - \frac{N-2+\gamma}{2} + \sqrt{\left(\frac{N-2+\gamma}{2}\right)^2 + \lambda}$$

A straightforward computation shows that this correspondence is given by

$$u(x) = |x|^{(N-2+\gamma)/2 - \sqrt{((N-2+\gamma)/2)^2 + \lambda}} v(x)$$

The starting point of the variational approach to problems of this type is the following inequality which can be obtained essentially 'interpolating' between Sobolev's and Hardy's inequalities [24].

Lemma 1 (*Caffarelli–Kohn–Nirenberg*)
Let $N \geqslant 2$, $\alpha \in (0,2)$ and denote $2^*_\alpha = 2N/(N-2+\alpha)$. Then there exists $C_\alpha > 0$ such that

$$\left(\int_{\mathbf{R}^N} |\varphi|^{2^*_\alpha}\, \mathrm{d}x\right)^{2/2^*_\alpha} \leqslant C_\alpha \int_{\mathbf{R}^N} |x|^\alpha |\nabla \varphi|^2\, \mathrm{d}x$$

for every $\varphi \in C_0^\infty(\mathbf{R}^N)$.

Consider the problem

$$-\mathrm{div}(|x|^\alpha \nabla u) + b(x)u = f(x,u) \quad \text{in } \mathbf{R}^N \tag{4}$$

where $N \geqslant 3$, $0 < \alpha < 2$. Suppose that b and f satisfy the hypotheses:

(b1) $b \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N \setminus \{0\})$ and there exists $b_0 > 0$ such that $b(x) \geqslant b_0$, for any $x \in \mathbf{R}^N$;
(b2) $\lim_{|x| \to \infty} b(x) = \lim_{|x| \to 0} b(x) = \infty$;
(f1) $f \in C^1(\mathbf{R}^N \times \mathbf{R})$, $f = f(x,z)$, with $f(x,0) = 0 = f_z(x,0)$ for all $x \in \mathbf{R}^N$;
(f2) there exist $a_1, a_2 > 0$ and $s \in (1, (N+2-\alpha)/(N-2+\alpha))$ such that

$$|f_z(x,z)| \leqslant a_1 + a_2|z|^{s-1} \quad \forall x \in \mathbf{R}^N \ \forall z \in \mathbf{R}$$

(f3) there exists $\mu > 2$ such that

$$0 < \mu F(x,z) := \mu \int_0^z f(x,t)\,\mathrm{d}t \leqslant z f(x,z) \quad \forall x \in \mathbf{R}^N \ \forall z \in \mathbf{R} \setminus \{0\}$$

The function $b(x) = e^{|x|}/|x|$ satisfies hypotheses (bl)–(b2), while the mapping $f(x,z) = R(x) z^s$ satisfies assumptions (f1)–(f3) for s given in (f2) and $R \in C^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ is a positive function.

We point out that problem (4) (for $\alpha = 0$) was studied in References [26,29] (see Reference [25] for a non-smooth treatment of this problem). More precisely, under similar assumptions on b and f, Rabinowitz shows that the problem

$$-\Delta u + b(x)u = f(x,u) \quad x \in \mathbf{R}^N \tag{5}$$

has a non-trivial solution $u \in H^1(\mathbf{R}^N)$.

Let E be the space defined as the completion of $C_0^\infty(\mathbf{R}^N \setminus \{0\})$ with respect to the norm

$$\|u\|^2 := \int_{\mathbf{R}^N} (|x|^\alpha |\nabla u|^2 + b(x)u^2) \, dx$$

We denote by E^* the dual space of E. We are seeking solutions in $\mathscr{D}_\alpha^{1,2}(\mathbf{R}^N)$, which is defined as the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the inner product

$$\langle u, v \rangle_\alpha := \int_{\mathbf{R}^N} |x|^\alpha \nabla u \nabla v \, dx$$

Recall that $\mathscr{D}_\alpha^{1,2}(\mathbf{R}^N)$ is a Hilbert space with respect to the norm

$$\|u\|_\alpha^2 := \int_{\mathbf{R}^N} |x|^\alpha |\nabla u|^2 \, dx$$

We say that $u \in \mathscr{D}_\alpha^{1,2}(\mathbf{R}^N)$ is a *weak solution* of (4) if

$$\int_{\mathbf{R}^N} (|x|^\alpha \nabla u \nabla v + b(x)uv) \, dx - \int_{\mathbf{R}^N} f(x,u)v \, dx = 0$$

for all $v \in C_0^\infty(\mathbf{R}^N)$.

Remark 1
Since $\mathscr{D}_\alpha^{1,2}(\mathbf{R}^N) = \overline{C_0^\infty(\mathbf{R}^N \setminus \{0\})}^{\|\cdot\|_\alpha}$ (see Reference [27]) we deduce that $E \subset \mathscr{D}_\alpha^{1,2}(\mathbf{R}^N)$.

Remark 2
If Ω is a bounded domain in $\mathbf{R}^N$ and $0 \notin \bar{\Omega}$ then the embedding $\mathscr{D}_\alpha^{1,2}(\Omega) \hookrightarrow L^{2_\alpha^*}(\Omega)$ is compact for $\alpha \in (0,2)$.

We prove

Theorem 1
Assume conditions (b1)–(b2) and (f1)–(f3) are fulfilled. Then (4) has a non-trivial weak solution.

3. PROOF OF THEOREM 1

We first observe that the weak solutions of (4) correspond to the critical points of the energy functional

$$I(u) := \frac{1}{2} \int_{\mathbf{R}^N} (|x|^\alpha |\nabla u|^2 + b(x)|u|^2) \, dx - \int_{\mathbf{R}^N} F(x,u) \, dx$$

where $u \in E \subset \mathscr{D}_\alpha^{1,2}(\mathbf{R}^N)$. A simple calculation based on Lemma 1, Remark 2 and the conditions (f1)–(f3) on f shows that I is well defined on E and $I \in C^1(E, \mathbf{R})$ with

$$\langle I'(u), v \rangle = \int_{\mathbf{R}^N} (|x|^\alpha \nabla u \nabla v + b(x)uv) \, \mathrm{d}x - \int_{\mathbf{R}^N} f(x, u)v \, \mathrm{d}x$$

for all $u, v \in E$. We have denoted by $\langle , \rangle$ the duality pairing between E and E^*.

Lemma 2
If (b1),(b2), (f1)–(f3) hold then there exist $\varrho > 0$ and $a > 0$ such that for all $u \in E$ with $\|u\| = \varrho$,

$$I(u) \geqslant a > 0$$

Proof
Using (f1) we have

$$\lim_{z \to 0} \frac{F(x, z)}{z^2} = \lim_{z \to 0} \frac{f(x, z)}{2z} = \lim_{z \to 0} \frac{1}{2} f_z(x, z) = 0 \tag{6}$$

for all $x \in \mathbf{R}^N$. From (f2) and (f3) we obtain

$$0 \leqslant F(x, z) \leqslant A_1 |z|^2 + A_2 |z|^{s+1} \tag{7}$$

where A_1, A_2 are positive constants. We conclude that

$$\lim_{z \to \infty} \frac{F(x, z)}{z^{2_\alpha^*}} = 0 \tag{8}$$

Using (6), (8), we deduce that for every $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that

$$F(x, z) < \varepsilon z^2 \quad \text{for all} \quad z \text{ with } |z| < \delta_1$$

$$F(x, z) < \varepsilon z^{2_\alpha^*} \quad \text{for all} \quad z \text{ with } |z| > \delta_2$$

Relation (7) implies that there exists a constant $C > 0$ such that

$$F(x, z) \leqslant C \quad \text{for all } z \text{ with } |z| \in [\delta_1, \delta_2]$$

We conclude that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(x, z) \leqslant \varepsilon |z|^2 + C_\varepsilon |z|^{2_\alpha^*} \tag{9}$$

Using (9) and Lemma 1 we deduce that

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\mathbf{R}^N} F(x, u) \, \mathrm{d}x \\ &\geqslant \frac{1}{2} \|u\|^2 - \varepsilon \int_{\mathbf{R}^N} |u|^2 \, \mathrm{d}x - C_\varepsilon \int_{\mathbf{R}^N} |u|^{2_\alpha^*} \, \mathrm{d}x \end{aligned}$$

$$\geqslant \frac{1}{2}\,\|u\|^2 - \frac{\varepsilon}{b_0}\int_{\mathbf{R}^N} b(x)|u|^2\,\mathrm{d}x - C_\varepsilon \int_{\mathbf{R}^N}|u|^{2^*_\alpha}\,\mathrm{d}x$$

$$\geqslant \|u\|^2\left[\left(\frac{1}{2}-\frac{\varepsilon}{b_0}\right)-C_\varepsilon\|u\|^{2^*_\alpha-2}\right]\geqslant a>0$$

for some fixed $\varepsilon\in(0,\tfrac{1}{2}b_0)$, where a and $\|u\|$ are sufficiently small.   □

Lemma 3
Assume conditions (b1), (b2) and (f1)–(f3) hold true.
   Then there exists $e\in E$ with $\|e\|>\varrho$ (ϱ given in Lemma 2) such that

$$I(e)<0$$

Proof
Using (f3) we deduce that $F(x,z)\geqslant A_3|z|^\mu$, for $|z|$ large enough, where $A_3>0$ is a constant.
Let $u\in E$ be fixed. Then, since $\mu>2$, we have

$$I(tu) \leqslant \frac{t^2}{2}\,\|u\|^2 - \int_{\{x;|tu|\leqslant\eta\}} F(x,tu)\,\mathrm{d}x - A_3|t|^\mu\int_{\{x;|tu|\geqslant\eta\}}|u|^\mu\,\mathrm{d}x$$

$$\leqslant \frac{t^2}{2}\,\|u\|^2 - A_3|t|^\mu\int_{\{x;|tu|\geqslant\eta\}}|u|^\mu\,\mathrm{d}x$$

Hence $I(tu)\to-\infty$ as $t\to\infty$ which concludes our lemma.   □

Lemma 4
Suppose that the hypotheses of Lemmas 2 and 3 are fulfilled. Set

$$\Gamma:=\{\gamma\in C([0,1],E);\ \gamma(0)=0,\ \gamma(1)=e\}$$

where e is given in Lemma 3 and $c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))$.
   Then $c>0$.

Proof
It is obvious that $c\geqslant 0$ because $c\geqslant\inf_{\gamma\in\Gamma}\max_{t\in\{0,1\}}I(\gamma(t))$ and

$$\gamma(0)=0\Rightarrow I(\gamma(0))=I(0)=0$$

$$\gamma(1)=e\Rightarrow I(\gamma(1))=I(1)=e<0$$

By contradiction, assume that $c=0$. Then $0=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))$. It follows that

   (1) $\max_{t\in[0,1]}I(\gamma(t))\geqslant 0,\ \forall\gamma\in\Gamma$;
   (2) for all $\varepsilon>0$ there exists $\gamma_\varepsilon\in\Gamma$ such that $\max_{t\in[0,1]}I(\gamma_\varepsilon(t))<\varepsilon$.

   Using a given by Lemma 2 we fix $0<\varepsilon<a$. We have $\gamma_\varepsilon(0)=0$, $\gamma_\varepsilon(1)=e$. Hence $\|\gamma_\varepsilon(0)\|=0,\|\gamma_\varepsilon(1)\|=\|e\|>\varrho$ (where ϱ is given by Lemma 2). But the application $t\mapsto\|\gamma_\varepsilon(t)\|$ is continuous and thus we conclude that there exists $t_\varepsilon\in[0,1]$ such that

$\|\gamma_\varepsilon(t_\varepsilon)\| = \varrho$. Then $I(\gamma_\varepsilon(t_\varepsilon)) \geqslant a > \varepsilon$ and we thus obtain a contradiction with 2). We conclude that $c > 0$ and our result follows. $\qquad\square$

Proof of Theorem 1
Using Lemma 4 and the Mountain Pass Theorem we find $\{u_n\} \subset E$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0 \tag{10}$$

We claim that $\{u_n\}$ is bounded in E. Arguing by contradiction and passing eventually to a subsequence, we have $\|u_n\| \to \infty$. Using (f3) it follows that for n large enough,

$$
\begin{aligned}
c + 1 + \|u_n\| &\geqslant I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + \int_{\mathbf{R}^N}\left(\frac{1}{\mu}u_n f(x, u_n) - F(x, u_n)\right)\mathrm{d}x \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2
\end{aligned}
$$

Now dividing by $\|u_n\|$ and passing to limit we obtain a contradiction. Hence $\{u_n\}$ is bounded in E, say by M. So, up to a subsequence, $\{u_n\}$ converges weakly in E to some $u \in E$, and strongly in $L^{2_\alpha^*}(\Omega)$, for all Ω bounded domains in $\mathbf{R}^N$ with $0 \notin \bar{\Omega}$ (see Remark 2). If we prove that

$$\langle I'(u_n), \varphi\rangle \to \langle I'(u), \varphi\rangle \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$$

then, by (10), u is a weak solution of (4). To do this, let $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$ be fixed. We set $\Omega = \mathrm{supp}(\varphi)$ $(0 \notin \bar{\Omega})$. Since $u_n \to u$ in E it follows that

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}(|x|^\alpha\nabla u_n\nabla\varphi + b(x)u_n\varphi)\,\mathrm{d}x = \int_{\mathbf{R}^N}(|x|^\alpha\nabla u\nabla\varphi + b(x)u\varphi)\,\mathrm{d}x$$

Furthermore, by (f2) and the Hölder inequality,

$$
\begin{aligned}
\left|\int_\Omega(f(x, u_n) - f(x, u))\varphi(x)\,\mathrm{d}x\right| &\leqslant \int_\Omega|f(x, u_n) - f(x, u)| \cdot |\varphi(x)|\,\mathrm{d}x \\
&\leqslant \|\varphi\|_{L^\infty(\Omega)}\int_\Omega|f_z(x, v_n)| \cdot |u_n - u|\mathrm{d}x \\
&\leqslant \|\varphi\|_{L^\infty(\Omega)}\int_\Omega[a_1 + a_2|v_n(x)|^{s-1}] \cdot |u_n(x) - u(x)\,\mathrm{d}x \\
&\leqslant \|\varphi\|_{L^\infty(\Omega)}[a_1\|u_n - u\|_{L^1(\Omega)} + a_2\|v_n\|_{L^s(\Omega)}^{s-1} \cdot \|u_n - u\|_{L^s(\Omega)}]
\end{aligned}
$$

where $v_n(x) \in [u_n(x), u(x)]$, for all $x \in \Omega$ and for all $n \geqslant 1$. Taking into account that $u_n \to u$ strongly in $L^i(\Omega)$, for all $i \in [1, 2_\alpha^*]$ and remarking that for all $x \in \Omega$ and for all $n \geqslant 1$ there exists $\lambda_n(x) \in [0, 1]$ such that $v_n(x) = \lambda_n(x)u_n(x) + [1 - \lambda_n(x)]u(x)$ we deduce

$$\int_\Omega|v_n - u|^s\,\mathrm{d}x = \int_\Omega|\lambda_n(x)|^s|u_n - u|^s\,\mathrm{d}x \leqslant \int_\Omega|u_n - u|^s\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty$$

It follows that

$$\int_\Omega |v_n|^s \, \mathrm{d}x \to \int_\Omega |u|^s \, \mathrm{d}x \quad \text{as } n \to \infty$$

From the above considerations we obtain

$$\left| \int_\Omega (f(x, u_n) - f(x, u))\varphi(x) \, \mathrm{d}x \right| \to 0 \quad \text{as } n \to \infty$$

and we conclude that

$$\langle I'(u_n), \varphi \rangle \to \langle I'(u), \varphi \rangle$$

for all $\varphi \in C_0^\infty(\mathbf{R}^N \backslash \{0\})$. To end the proof of Theorem 1 it remains to show that $u \not\equiv 0$. For n large enough, using (10), we have

$$\frac{c}{2} \leqslant I(u_n) - \frac{1}{2}\langle I'(u_n), u_n \rangle = \int_{\mathbf{R}^N} \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n) \right] \mathrm{d}x \qquad (11)$$

By (f2), it follows that

$$|f(x, z)| \leqslant B_1|z| + B_2|z|^s$$

for some constants $B_1, B_2 > 0$. Hence

$$\lim_{z \to \infty} \frac{|f(x, z)|}{|z|^{2_\alpha^* - 1}} = 0$$

Furthermore, by (f1),

$$\lim_{z \to 0} \frac{f(x, z)}{z} = \lim_{z \to 0} f_z(x, z) = f_z(x, 0) = 0$$

As in the proof of Lemma 2 we may state that for all $\varepsilon > 0$, there exists some $D_\varepsilon > 0$ such that

$$|f(x, z)| \leqslant \varepsilon|z|^{2_\alpha^* - 1} + D_\varepsilon|z| \qquad (12)$$

From (11) and (12) and Lemma 1 we obtain

$$\frac{c}{2} \leqslant \int_{\mathbf{R}^N} \left[\frac{\varepsilon}{2}|u_n|^{2_\alpha^*} + D_\varepsilon|u_n|^2 \right] \mathrm{d}x \leqslant \frac{\varepsilon}{2} C_\alpha \|u_n\|^{2_\alpha^*} + D_\varepsilon \int_{\mathbf{R}^N} |u_n|^2 \, \mathrm{d}x$$

Choose ε such that

$$\frac{\varepsilon}{2} C_\alpha M^{2_\alpha^*} \leqslant \frac{c}{4}$$

Then

$$\frac{c}{4} \leqslant A \|u_n\|_{L^2(\mathbf{R}^N)}^2$$

where $A > 0$ is a constant. We suppose, by contradiction, that $u \equiv 0$. From $u_n \to 0$ in $L^{2_z^*}(\Omega)$ for all bounded domain $\Omega \subset \mathbf{R}^N$ with $0 \notin \bar{\Omega}$, it follows that $u_n \to 0$ in $L^2(\Omega)$. If $0 < r < R$ we set $\Omega := B_R(0) \setminus \bar{B}_r(0)$. Then there exists $m_0 = m_0(r, R)$ such that for all $n \geqslant m_0$ we have

$$A\|u_n\|_{L^2(\Omega)}^2 \leqslant \frac{c}{8}$$

Therefore,

$$\frac{c}{8} \leqslant A\|u_n\|_{L^2(\mathbf{R}^N \setminus \Omega)}^2$$

$$\leqslant \frac{A}{\inf_{|x| \geqslant R} b(x)} \left(\int_{|x| \geqslant R} b(x)|u_n|^2 \, \mathrm{d}x \right) + \frac{A}{\inf_{|x| \leqslant r} b(x)} \left(\int_{|x| \leqslant r} b(x)|u_n|^2 \, \mathrm{d}x \right)$$

$$\leqslant AM \left[\frac{1}{\inf_{|x| \geqslant R} b(x)} + \frac{1}{\inf_{|x| \leqslant r} b(x)} \right]$$

Now using (b2) we remark that R can be made so large and r can be taken so small so that the right hand side of the last inequality becomes less than $c/8$, a contradiction. This concludes our proof. $\qquad\square$

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