Multiplicity of solutions for a class of non-symmetric eigenvalue hemivariational inequalities

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SUMMARY

The aim of this paper is to establish the influence of a non-symmetric perturbation for a symmetric hemivariational eigenvalue inequality with constraints. The original problem was studied by Goeleven *et al.* (Math. Methods Appl. Sci. 1997; **20**:548) who deduced the existence of infinitely many solutions for the symmetric case. In this paper it is shown that the number of solutions of the perturbed problem becomes larger and larger if the perturbation tends to zero with respect to a natural topology. The approach relies on topological methods in non-smooth critical point theory leading to this new multiplicity information. Copyright © 2003 John Wiley & Sons, Ltd.

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1. INTRODUCTION AND THE MAIN RESULT

The study of variational inequality problems began around 1965 with the pioneering works of Fichera [1], Lions and Stampacchia [2]. The connection of the theory of variational inequalities with the notion of subdifferentiability of convex analysis was achieved by Moreau (see Reference [3]) who introduced the notion of convex superpotential which permitted the formulation and the solving of a wide ranging class of complicated problems in mechanics and engineering which could not until then be treated correctly by the methods of classical bilateral mechanics. All the inequality problems studied to the middle of the ninth decade were related to convex energy functions and therefore were firmly linked with the notion of monotonicity; for instance, only monotone, possibly multivalued, boundary conditions and stress–strain laws could be studied. In order to overcome this limitation, Panagiotopoulos introduced in [4–6] the notion of non-convex superpotential by using the generalized gradi-

ent of Clarke [7]. Due to the lack of convexity new types of variational expressions were obtained. These are the so-called *hemivariational inequalities* and they are no longer connected with monotonicity. Generally speaking, mechanical problems involving non-monotone, possibly multivalued stress–strain laws or boundary conditions derived by non-convex super-potentials lead to hemivariational inequalities (see References [4–6,8–10]). Moreover, while in the convex case the static variational inequalities generally give rise to minimisation problems for the potential or the complementary energy, in the non-convex case the problem of substationarity of the potential or the complementary energy at an equilibrium position emerges.

For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs Panagiotopoulos [8,9], Motreanu-Panagiotopoulos [11] and Naniewicz-Panagiotopoulos [12].

Throughout this paper V will denote a real Hilbert space which is densely and compactly imbedded in $L^p(\Omega; \mathbb{R}^N)$, for some $1 < p < \infty$ and $N \geqslant 1$, where Ω is a bounded domain in $\mathbb{R}^m$, $m \geqslant 1$. Denote by $\|\cdot\|$ the norm on V and by $(\cdot, \cdot)$ the corresponding inner product. Let $a : V \times V \to \mathbb{R}$ be a continuous, symmetric and bilinear form, not necessarily coercive. We denote by $A : V \to V$ the self-adjoint bounded linear operator corresponding to a, i.e.

$$(Au, v) = a(u, v) \quad \text{for all } u, v \in V$$

Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^N$, while the duality pairing between V^* and V (resp., between $(\mathbb{R}^N)^*$ and $\mathbb{R}^N$) will be denoted by $\langle \cdot, \cdot \rangle_V$ (resp., $\langle \cdot, \cdot \rangle$). For $r > 0$, let S_r denote the sphere of radius r in V centred at the origin, i.e.

$$S_r = \{u \in V; \|u\| = r\}$$

Let $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $j(\cdot, 0) \in L^1(\Omega)$. Thus, we can define the generalized directional derivative in the sense of Clarke (see Reference [7]):

$$j^0(x, \xi; \eta) = \limsup_{(h, \lambda) \to (0, 0^+)} \frac{j(x, \xi + h + \lambda\eta) - j(x, \xi + h)}{\lambda} \quad \text{for } \xi, \eta \in \mathbb{R}^N$$

and the Clarke generalized gradient

$$\partial_y j(x, y) = \{w \in (\mathbb{R}^N)^*, \quad \langle w, \eta \rangle \leqslant j^0(x, y; \eta), \quad \forall \eta \in \mathbb{R}^N\}, \quad (x, y) \in \Omega \times \mathbb{R}^N$$

Let G be a finite subgroup of the group of linear isometries of V. Assume further that the following conditions are satisfied:

(A$_1$) a and j are G-invariant in the sense that

$$a(gu, gv) = a(u, v) \quad \forall u, v \in V, \quad \forall g \in G$$

and

$$j(x, (gu)(x)) = j(x, u(x)) \quad \forall u \in V, \quad \forall g \in G \text{ and for a.e. } x \in \Omega$$

(A_2) there exist $a_1 \in L^{p/(p-1)}(\Omega)$ and $b \in \mathbb{R}_+$ such that

$$|w| \leqslant a_1(x) + b|y|^{p-1} \quad \text{for a.e. } (x, y) \in \Omega \times \mathbb{R}^N \text{ and all } w \in \partial_y j(x, y)$$

and there exists $v \in L^p(\Omega; \mathbb{R}^N)$ with $\int_\Omega j^0(x, 0; v(x)) \, dx < 0$

Consider $\Lambda: V \to V^*$ the duality isomorphism

$$\langle \Lambda u, v \rangle_V = (u, v) \quad \text{for all } u, v \in V$$

Suppose also that the following assumption holds:

(A_3) For every sequence $(u_n) \subset V$ with $\|u_n\| = r$, for every number $\alpha \in [-r^2\|A\|, r^2\|A\|]$ and for every measurable map $w: \Omega \to (\mathbb{R}^N)^*$ such that $u_n \to u$ strongly in $L^p(\Omega; \mathbb{R}^N)$ for some $u \in V \setminus \{0\}$, $w(x) \in \partial_y j(x, u(x))$ for a.e. $x \in \Omega$ and $a(u_n, u_n) \to \alpha$ one has

$$\inf_{\|\tau\|=1} \{a(\tau, \tau)\} - r^{-2}\left(\alpha + \int_\Omega \langle w(x), u(x) \rangle \, dx\right) > 0$$

Consider the following eigenvalue problem:

$$(\mathrm{P}_1) \quad \begin{cases} (u, \lambda) \in V \times \mathbb{R} \\ a(u, v) + \displaystyle\int_\Omega j^0(x, u(x); v(x)) \, dx \geqslant \lambda(u, v) \quad \text{for all } v \in V \\ \|u\| = r \end{cases}$$

Under Assumptions (A_1)–(A_3), Goeleven, Motreanu and Panagiotopoulos proved in Reference [13], Theorem 5.1 that this problem admits infinitely many distinct pairs $(Gu_n, \lambda_n)_{n \geqslant 1} \subset S_r \times \mathbb{R}$ such that $\lambda = \lambda_n$ and every $u \in Gu_n$ solves (P_1).

We remark that in Reference [13] it was assumed $a_1 = \text{const.}$ in (A_2), so the statement therein is formulated under a slightly less general hypothesis. We observe that in order to show that the arguments of Reference [13] hold in our case, it is sufficient to verify that the energy functional

$$F(u) = \tfrac{1}{2} a(u, u) + J(u), \quad u \in V \tag{1}$$

is bounded from below on S_r, where the locally Lipschitz function $J: L^p(\Omega; \mathbb{R}^N) \to \mathbb{R}$ is defined by

$$J(u) = \int_\Omega j(x, u(x)) \, dx, \quad u \in L^p(\Omega; \mathbb{R}^N)$$

Indeed, using Lebourg's mean value theorem for locally Lipschitz functions (see Reference [7, p. 41]) we obtain

$$|j(x, y)| \leqslant |j(x, 0)| + |j(x, y) - j(x, 0)|$$

$$\leqslant |j(x, 0)| + \sup\{|w|; w \in \partial_y j(x, Y), Y \in [0, y]\} \cdot |y|$$

$$\leqslant |j(x, 0)| + a_1(x)|y| + b|y|^p \tag{2}$$

Therefore,

$$|J(u)| \leqslant \|j(\cdot, 0)\|_{L^1} + \|a_1\|_{L^{p/(p-1)}} \|u\|_{L^p} + b\|u\|_{L^p}^p \tag{3}$$

The continuity of the imbedding $V \subset L^p(\Omega; \mathbb{R}^N)$ ensures the existence of a positive constant $C_p(\Omega)$ such that

$$\|u\|_{L^p} \leqslant C_p(\Omega)\|u\|_V \quad \text{for all } u \in V$$

From (1) and (3) it follows that

$$F_{|S_r}(u) \geqslant -\tfrac{1}{2}\|A\|r^2 - \|j(\cdot, 0)\|_{L^1} - C_p(\Omega)r\|a_1\|_{L^{p/(p-1)}} - bC_p^p(\Omega)r^p, \quad u \in S_r$$

From now on the proof follows in the same way as in Reference [13].

Let us now consider the following non-symmetric perturbed hemivariational inequality:

$$(\mathrm{P}_2) \quad \begin{cases} (u, \lambda) \in V \times \mathbb{R} \\[6pt] a(u, v) + \displaystyle\int_\Omega (j^0(x, u(x); v(x)) + h^0(x, u(x); v(x)))\,\mathrm{d}x + \langle \varphi, v \rangle_V \geqslant \lambda(u, v), \quad \forall v \in V \\[6pt] \|u\| = r \end{cases}$$

where $\varphi \in V^*$ and $h : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $h(\cdot, 0) \in L^1(\Omega)$. We do not make any symmetry assumption on h, but we require only the natural growth condition

(A$_4$) $|z| \leqslant a_2(x) + c|y|^{p-1}$, for a.e. $(x, y) \in \Omega \times \mathbb{R}^N$ and for all $z \in \partial_y h(x, y)$ where $a_2 \in L^{p/(p-1)}(\Omega)$ and $c > 0$, and if $\varphi \in L^{p/(p-1)}(\Omega; \mathbb{R}^N)$ there exists $v \in L^p(\Omega; \mathbb{R}^N)$ with $\int_\Omega (\varphi(x)v(x) + j^0(x, 0; v(x)) + h^0(x, 0; v(x)))\,\mathrm{d}x < 0$.

The corresponding variant of the compactness condition (A$_3$) is

(A$_5$) For every sequence $(u_n) \subset V$ with $\|u_n\| = r$, for every number $\alpha \in [-r^2\|A\|, r^2\|A\|]$, and for every measurable maps $z, w : \Omega \to (\mathbb{R}^N)^*$ such that $u_n \to u$ strongly in $L^p(\Omega; \mathbb{R}^N)$ for some $u \in V \setminus \{0\}$, $w(x) \in \partial_y j(x, u(x))$, $z(x) \in \partial_y h(x, u(x))$ for a.e. $x \in \Omega$ and $a(u_n, u_n) \to \alpha$ one has

$$\inf_{\|\tau\|=1} \{a(\tau, \tau)\} - r^{-2}\left(\alpha + \langle \varphi, u \rangle_V + \int_\Omega \langle w(x) + z(x), u(x) \rangle\,\mathrm{d}x \right) > 0 \tag{4}$$

The present paper deals with the study of (possibly non-symmetric) perturbed hemivariational inequality (P_2). For different perturbation results and their applications we refer to [14–18]. Our main result asserts that the number of solutions of (P_2) goes to infinity as the perturbation becomes smaller and smaller.

Theorem 1
Suppose that assumptions (A$_1$)–(A$_5$) hold. Then, for every $n \geqslant 1$, there exists $\delta_n > 0$ such that problem (P_2) admits at least n distinct solutions, provided that $\|h(\cdot, 0)\|_{L^1} \leqslant \delta_n$, $\|a_2\|_{L^{p/(p-1)}} \leqslant \delta_n$, $c \leqslant \delta_n$ and $\|\varphi\|_{V^*} \leqslant \delta_n$.

The rest of the paper is organized as follows. Section 2 deals with some auxiliary results. Section 3 is devoted to a topological approach in non-smooth critical point theory leading to the basic properties stated in Propositions 1 and 2. Section 4 contains the proof of Proposition 2. In Section 5 the proof of our main result given in Theorem 1 is presented.

2. AUXILIARY RESULTS

We define the energy functional $W : V \to \mathbb{R}$ associated to the hemivariational problem (P$_2$) by

$$W(u) = \tfrac{1}{2} a(u,u) + J(u) + H(u) + \langle \varphi, u \rangle_V \quad u \in V$$

where $H(u) = \int_\Omega h(x, u(x)) \, \mathrm{d}x$. We first prove that W can be viewed as a small perturbation of the functional F in (1) whenever the data h and φ are sufficiently small in a suitable sense.

Lemma 1
For every number $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\sup_{u \in S_r} |F(u) - W(u)| < \varepsilon$$

provided $\|h(\cdot, 0)\|_{L^1} \leqslant \delta_\varepsilon$, $\|a_2\|_{L^{p/(p-1)}} \leqslant \delta_\varepsilon$, $c \leqslant \delta_\varepsilon$ and $\|\varphi\|_{V^*} \leqslant \delta_\varepsilon$.

Proof
Proceeding in the same manner as we did for proving (2) we obtain

$$|h(x, y)| \leqslant |h(x, 0)| + a_2(x)|y| + c|y|^p$$

Thus, for all $u \in S_r$ we have

$$
\begin{aligned}
|F(u) - W(u)| &\leqslant |H(u)| + |\langle \varphi, u \rangle_V| \leqslant |H(u)| + r\|\varphi\|_{V^*} \\
&\leqslant \|h(\cdot, 0)\|_{L^1} + \|a_2\|_{L^{p/(p-1)}} C_p(\Omega) r + c C_p^p(\Omega) r^p + r\|\varphi\|_{V^*} < \varepsilon
\end{aligned}
$$

for small $h(\cdot, 0)$, a_2, c and φ. $\qquad \square$

Our next result shows that $W_{|S_r}$ satisfies the Palais–Smale condition in the sense of Chang [19].

Lemma 2
The functional W satisfies the Palais–Smale condition (in short, (PS) condition) on S_r.

Proof
Let (u_n) be a sequence in S_r such that

$$\sup_n |W(u_n)| < \infty \tag{5}$$

and

$$\lambda_{W_{|S_r}}(u_n) \to 0 \quad \text{as } n \to \infty \tag{6}$$

where $\lambda_{W_{|S_r}}(u) = \min\{\|\theta\|; \theta \in \partial(W_{|S_r})(u)\}$. The functional $\lambda_{W_{|S_r}}$ is well defined and lower semi-continuous (see Reference [19]). The expression of the generalized gradient of W on S_r is given by

$$\partial(W_{|S_r})(u) = \{\xi - r^{-2}\langle \xi, u \rangle_V \Lambda u; \ \xi \in \partial W(u)\} \tag{7}$$

(see Reference [19]). Notice that (5) is automatically fulfilled due to the growth conditions in (A_2) and (A_4). From (6) and (7) we deduce the existence of a sequence $(\xi_n) \subset V^*$ such that

$$\xi_n \in \partial W(u_n) \tag{8}$$

and

$$\xi_n - r^{-2} \langle \xi_n, u_n \rangle_V \Lambda u_n \to 0 \quad \text{strongly in } V^* \tag{9}$$

For every $u \in V$, the generalized gradient $\partial W(u) \subset V^*$ satisfies

$$\partial W(u) \subset \Lambda A u + \partial(J_{|V})(u) + \partial(H_{|V})(u) + \varphi \tag{10}$$

From (8)–(10) it follows that there exist

$$w_n \in \partial(J_{|V})(u_n) \quad \text{and} \quad z_n \in \partial(H_{|V})(u_n)$$

such that

$$\Lambda A u_n + w_n + z_n + \varphi - r^{-2} \langle \Lambda A u_n + w_n + z_n + \varphi, u_n \rangle_V \Lambda u_n \to 0 \quad \text{strongly in } V^* \tag{11}$$

The density of V in $L^p(\Omega; \mathbb{R}^N)$ implies (see Reference [19, Theorem 2.2])

$$\partial(J_{|V})(u) \subset \partial J(u) \quad \text{and} \quad \partial(H_{|V})(u) \subset \partial H(u), \quad u \in V \tag{12}$$

Hence, from (12), one sees that

$$w_n \in \partial J(u_n) \quad \text{and} \quad z_n \in \partial H(u_n) \tag{13}$$

Since V is a reflexive space and $\|u_n\| = r$, we can extract a subsequence, denoted again by (u_n), such that

$$u_n \rightharpoonup u \quad \text{weakly in } V \text{ as } n \to \infty \tag{14}$$

The compactness of the imbedding $V \subset L^p(\Omega; \mathbb{R}^N)$ implies that, up to a subsequence,

$$u_n \to u \quad \text{strongly in } L^p(\Omega; \mathbb{R}^N) \text{ as } n \to \infty \tag{15}$$

Using (13), (15) and the fact that the functionals J and H are locally Lipschitz on $L^p(\Omega; \mathbb{R}^N)$ we deduce that the sequences (w_n) and (z_n) are bounded in $L^{p/(p-1)}(\Omega; \mathbb{R}^N)$. Thus, passing eventually to subsequences, we have

$$w_n \rightharpoonup w \quad \text{weakly in } L^{p/(p-1)}(\Omega; \mathbb{R}^N) \text{ as } n \to \infty \tag{16}$$

$$z_n \rightharpoonup z \quad \text{weakly in } L^{p/(p-1)}(\Omega; \mathbb{R}^N) \text{ as } n \to \infty \tag{17}$$

Since the imbedding $L^{p/(p-1)}(\Omega; \mathbb{R}^N) \subset V^*$ is compact, relations (16) and (17) imply (up to subsequences)

$$w_n \to w \quad \text{strongly in } V^* \text{ as } n \to \infty \tag{18}$$

$$z_n \to z \quad \text{strongly in } V^* \text{ as } n \to \infty \tag{19}$$

Combining (14), (18) and (19) we obtain that

$$\langle w_n + z_n, u_n \rangle_V \to \langle w + z, u \rangle_V \quad \text{as } n \to \infty \tag{20}$$

By virtue of the boundedness of the sequence (u_n) in V and the continuity of the bilinear form a we may suppose that, along a subsequence, we have

$$a(u_n, u_n) \to \alpha \quad \text{as } n \to \infty \text{ for some } \alpha \in [-r^2\|A\|, r^2\|A\|]$$

Taking into account (18)–(20) we see that (11) implies that

$$Au_n - r^{-2}(\alpha + \langle \varphi, u \rangle_V + \langle w + z, u \rangle_V)u_n \quad \text{converges strongly in } V \text{ as } n \to \infty \tag{21}$$

Using (13), (15)–(17) and the fact that the Clarke's generalized gradient is a weak*-closed multifunction (see Reference [7, Proposition 2.1.5]) we deduce

$$w \in \partial J(v) \tag{22}$$

$$z \in \partial H(u) \tag{23}$$

Our hypotheses (A_2) and (A_4) allow to apply Theorem 2.7.5 in Reference [7], and from relations (22) and (23) we get the existence of two measurable mappings $w, z : \Omega \to (\mathbb{R}^N)^*$ such that

$$w(x) \in \partial_y j(x, u(x)) \quad \text{for a.e. } x \in \Omega \tag{24}$$

$$z(x) \in \partial_y h(x, u(x)) \quad \text{for a.e. } x \in \Omega \tag{25}$$

$$\langle w, u \rangle_V = \langle w, u \rangle_{L^p(\Omega; \mathbb{R}^N)} = \int_\Omega \langle w(x), u(x) \rangle \, \mathrm{d}x \tag{26}$$

$$\langle z, u \rangle_V = \langle z, u \rangle_{L^p(\Omega; \mathbb{R}^N)} = \int_\Omega \langle z(x), u(x) \rangle \, \mathrm{d}x \tag{27}$$

Remark that, due to the first part of (A_2), (24) and $u \in L^p(\Omega; \mathbb{R}^N)$, we have that $\langle w(\cdot), u(\cdot) \rangle \in L^1(\Omega; \mathbb{R})$ since

$$\int_\Omega |\langle w(x), u(x) \rangle| \, \mathrm{d}x \leqslant \int_\Omega (a_1(x) + b|u(x)|^{p-1})|u(x)| \, \mathrm{d}x \leqslant \|a_1\|_{L^{p/(p-1)}} \|u\|_{L^p} + b\|u\|_{L^p}^p$$

In the same way, using the first part of (A_4), (25) and $u \in L^p(\Omega; \mathbb{R}^N)$, we obtain that $\langle z(\cdot), u(\cdot) \rangle \in L^1(\Omega; \mathbb{R})$. Replacing (26) and (27) in (21) one gets that

$$Au_n - r^{-2}\left(\alpha + \langle \varphi, u \rangle_V + \int_\Omega \langle w(x) + z(x), u(x) \rangle \, \mathrm{d}x \right)u_n \quad \text{converges in } V \text{ as } n \to \infty \tag{28}$$

with w and z satisfying (24) and (25), respectively.

We note that $u \neq 0$. Indeed, if $u = 0$, then (11) yields that $w_n + z_n \rightharpoonup -\varphi$ weakly in $L^{p/(p-1)}(\Omega; \mathbb{R}^N)$. Hence $-\varphi \in \partial J(0) + \partial H(0)$ which contradicts the final part of assumption

(A$_4$). Consequently, in view of (15), (24), (25), we are in a position to use assumption (A$_5$) and therefore inequality (4) is valid. For all n, k we have

$$\left(\inf_{\|\tau\|=1} \{a(\tau, \tau)\} - r^{-2} \left(\alpha + \langle \varphi, u \rangle_V + \int_{\Omega} \langle (w + z)(x), u(x) \rangle \, \mathrm{d}x \right) \right) \|u_n - u_k\|^2$$

$$\leqslant a(u_n - u_k, u_n - u_k) - r^{-2} \left(\left(\alpha + \langle \varphi, u \rangle_V + \int_{\Omega} \langle (w + z)(x), u(x) \rangle \, \mathrm{d}x \right) (u_n - u_k), u_n - u_k \right)$$

$$= \left(A(u_n - u_k) - r^{-2} \left(\alpha + \langle \varphi, u \rangle_V + \int_{\Omega} \langle (w + z)(x), u(x) \rangle \, \mathrm{d}x \right) (u_n - u_k), u_n - u_k \right)$$

$$\leqslant \left\| A(u_n - u_k) - r^{-2} \left(\alpha + \langle \varphi, u \rangle_V + \int_{\Omega} \langle (w + z)(x), u(x) \rangle \, \mathrm{d}x \right) (u_n - u_k) \right\| \|u_n - u_k\|$$

The convergence in (28), the above estimate and (4) show that (u_n) contains a Cauchy subsequence in V. Hence (u_n) converges strongly along a subsequence in V to u. This completes the proof of lemma.   □

The next result shows that W plays indeed the role of energy functional for the perturbed problem (P$_2$).

Lemma 3
If $u \in S_r$ is a critical point of $W_{|S_r}$ then there exists $\lambda \in \mathbb{R}$ such that (u, λ) is a solution of problem (P$_2$).

Proof
Since $0 \in \partial(W_{|S_r})(u)$ it follows from (7), (10), (12) that there exist

$$w \in \partial(J_{|V})(u) \subset \partial J(u) \quad \text{and} \quad z \in \partial(H_{|V})(u) \subset \partial H(u) \tag{29}$$

such that u is a solution of

$$\Lambda Au + w + z + \varphi = r^{-2} \langle \Lambda Au + w + z + \varphi, u \rangle_V \Lambda u \tag{30}$$

By Theorem 2.7.3 in Reference [7] we have that for every $u \in L^p(\Omega; \mathbb{R}^N)$,

$$\partial J(u) \subset \int_{\Omega} \partial_y j(x, u(x)) \, \mathrm{d}x \quad \text{and} \quad \partial H(u) \subset \int_{\Omega} \partial_y h(x, u(x)) \, \mathrm{d}x$$

Thus, by (29), the mappings $w, z : \Omega \to (\mathbb{R}^N)^*$ satisfy

$$w(x) \in \partial_y j(x, u(x)) \quad \text{for a.e. } x \in \Omega \tag{31}$$

$$z(x) \in \partial_y h(x, u(x)) \quad \text{for a.e. } x \in \Omega \tag{32}$$

and, for all $u \in V$,

$$\langle w, v \rangle_V = \int_\Omega \langle w(x), v(x) \rangle \, \mathrm{d}x \tag{33}$$

$$\langle z, v \rangle_V = \int_\Omega \langle z(x), v(x) \rangle \, \mathrm{d}x \tag{34}$$

Let us take

$$\lambda = r^{-2} \left(\langle \Lambda A u + \varphi, u \rangle_V + \int_\Omega \langle w(x) + z(x), u(x) \rangle \, \mathrm{d}x \right) \tag{35}$$

From (30) to (35) it follows that, for every $v \in V$,

$$\lambda(u, v) - a(u, v) - \langle \varphi, v \rangle_V = \int_\Omega \langle w(x) + z(x), v(x) \rangle \, \mathrm{d}x$$

$$\leqslant \int_\Omega \max\{\langle \mu_1, v(x) \rangle; \, \mu_1 \in \partial_y j(x, u(x))\} \, \mathrm{d}x$$

$$+ \int_\Omega \max\{\langle \mu_2, v(x) \rangle; \, \mu_2 \in \partial_y h(x, u(x))\} \, \mathrm{d}x$$

$$= \int_\Omega j^0(x, u(x); v(x)) \, \mathrm{d}x + \int_\Omega h^0(x, u(x); v(x)) \, \mathrm{d}x$$

In order to write the last equality we have used Proposition 2.1.2 in Reference [7]. The proof of lemma is complete. □

3. TRIVIAL PAIRS AND ESSENTIAL VALUES

Throughout this section, X denotes a metric space, A is a subset of X and i stands for the inclusion map of A into X. For the topological notions mentioned here we refer to References [16,20,21].

Definition 1
A map $r : X \to A$ is said to be a retraction if it is continuous and $r_{|A} = \mathrm{id}_A$.

Definition 2
A retraction r is called a strong deformation retraction provided that there exists a homotopy $\zeta : X \times [0,1] \to X$ of $i \circ r$ and id_X which satisfies $\zeta(x, t) = \zeta(x, 0)$ for all $(x, t) \in A \times [0, 1]$.

Definition 3
The metric space X is said to be weakly locally contractible if for every $u \in X$ there exists a neighbourhood U of u contractible in X.

Given a continuous function $f : X \to \mathbb{R}$, for every $a \in \mathbb{R}$ we denote

$$f^a = \{u \in X : \, f(u) \leqslant a\}$$

Definition 4

Let $a, b \in \mathbb{R}$ with $a \leqslant b$. The pair (f^b, f^a) is said to be trivial provided that, for all neighbourhoods $[a', a'']$ of a and $[b', b'']$ of b, there exist closed sets A and B such that $f^{a'} \subseteq A \subseteq f^{a''}$, $f^{b'} \subseteq B \subseteq f^{b''}$ and there is a strong deformation retraction of B onto A.

Definition 5

A real number c is called an essential value of f if for every $\varepsilon > 0$ there exist $a, b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$ such that the pair (f^b, f^a) is not trivial.

The following property of essential values is due to Degiovanni–Lancelotti (see Reference [16, Theorem 2.6]).

Proposition 1

Let c be an essential value of f. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous function $g : X \to \mathbb{R}$ with

$$\sup\{|g(u) - f(u)| : u \in X\} < \delta$$

admits an essential value in $(c - \varepsilon, c + \varepsilon)$.

We turn now to the use of the notions above in the setting of problem (P_1). For every $n \geqslant 1$, we introduce

$$\mathscr{A}_n = \{A \subset S_r : A \text{ is compact and } \mathrm{cat}_{\pi(S_\tau)} \pi(A) \geqslant n\}$$

where $\pi : V \to V/G$ is the canonical projection and $\mathrm{cat}_Y A$ is the smallest $k \in \mathbb{N} \cup \{+\infty\}$ such that A can be covered by k closed and contractible sets in Y (see Reference [13]). Goeleven, Motreanu and Panagiotopoulos proved in Reference [13] that the corresponding min–max values of F in (1) over $\mathscr{A}_n$

$$c_n = \inf_{A \in \mathscr{A}_n} \max_{u \in A} F(u), \quad n \geqslant 1$$

are critical values of $F_{|S_r}$.

The result below is useful in proving Theorem 1.

Proposition 2

Under assumptions $(A_1)–(A_3)$ there exists an increasing sequence (b_n) of essential values of $F_{|S_r}$ converging to $\sup_{j \geqslant 1} c_j$.

The proof of Proposition 2 is inspired from an argument in Degiovanni–Lancelotti [16] and some constructions in Rabinowitz [18] and will be given in the next section.

4. PROOF OF PROPOSITION 2

In order to prove Proposition 2 we follow the steps:

(a) $c_1 = \inf_{u \in S_r} F(u)$;
(b) $c_n < \bar{c} := \sup_{j \geqslant 1} c_j$, $\forall n \geqslant 1$, and $c_n \to \bar{c}$ as $n \to +\infty$;

(c) there exists the sequence (b_n) as required in the statement of Proposition 2.

(a) We have that

$$c_1 = \inf_{A \in \mathscr{A}_1} \max_{u \in A} F(u)$$

where $\mathscr{A}_1 = \{A \subset S_r \,|\, A \text{ compact and } \mathrm{cat}_{\pi(S_r)} \pi(A) \geqslant 1\}$, with $\pi : S_r \to S_r/G$, $\pi(x) = Gx$. Consider any $x \in S_r$ and let $A_0 := \{x\}$ which is a compact set. Since S_r is weakly locally contractible, there exists a neighbourhood U_x of x (in S_r) contractible in S_r. We denote $T_x := \bigcup_{g \in G} g U_x = \pi(U_x)$. The contractibility of U_x implies that there exists a homotopy $\mathscr{H}_x : \bar{U}_x \times [0,1] \to S_r$ and a point $z_x \in S_r$ such that $\mathscr{H}_x(y,0) = y$ and $\mathscr{H}_x(y,1) = z_x$ for all $y \in \bar{U}_x$. We can also suppose that π is a homeomorphism on $\bar{U}_x$. Let us define $\mathscr{K}_x : \bar{T}_x \times [0,1] \to \pi(S_r)$ by $\mathscr{K}_x(y,t) = \pi \circ \mathscr{H}_x(\pi^{-1}(y),t)$. We have that

$$\mathscr{K}_x(y,0) = (\pi \circ \pi^{-1})(y) = y \quad \text{and} \quad \mathscr{K}_x(y,1) = \pi(z_x) \ \forall y \in \bar{T}_x = \pi(\bar{U}_x)$$

Therefore, $\bar{T}_x$ is contractible in $\pi(S_r)$ and $\mathrm{cat}_{\pi(S_r)} \pi(A_0) = 1$. This is a consequence of the fact that $\pi(\bar{U}_x)$ is a closed and contractible subset of $\pi(S_r)$ which contains $\pi(A_0)$. We obtain that

$$c_1 = \inf_{A \in \mathscr{A}_1} \max_{u \in A} F(u) \leqslant \max_{u \in A_0} F(u) = F(x)$$

and therefore $c_1 \leqslant \inf_{u \in S_r} F(u)$. The converse inequality is obvious.

(b) Since $\mathscr{A}_1 \supseteq \mathscr{A}_2 \supseteq \cdots \supseteq \mathscr{A}_n \supseteq \cdots$ it follows that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n \leqslant \cdots$. Taking into account that $c_n \leqslant \sup_{u \in S_r} F(u)$ for all n and that F is bounded on S_r (cf. (A_2)), we deduce that the sequence (c_n) converges to $\bar{c}$. This establishes the second part of assertion (b).

For proving the first part of property (b) we argue by contradiction. Let us admit that there is some j with $c_j = \bar{c}$. By the monotonicity of the sequence (c_n), one has necessarily that $c_n = \bar{c}$ for all $n \geqslant j$. As shown in the proof of Theorem 3.7 in Reference [13] we have that $c := c_j = c_{j+i} = \cdots = c_{j+p}$ yields $\mathrm{cat}_{\pi(S_r)} \pi(K_c) \geqslant p+1$ for every p, where K_c stands for the set of critical points of $F_{|S_r}$ at level c. This ensures that $\mathrm{cat}_{\pi(S_r)} \pi(K_{\bar{c}}) = +\infty$. Since F satisfies the (PS) condition (see Reference [13]), the set $K_{\bar{c}}$ is compact, which in turn implies that $\mathrm{cat}_{\pi(S_r)} \pi(K_{\bar{c}}) < +\infty$. The obtained contradiction proves the claim, so the first part of assertion (b) is justified.

(c) Proceeding inductively, first we construct the essential value b_1. Let us assume by contradiction that there are no essential values in the open interval $(c_1, \bar{c})$. By Theorem 2.5 in Reference [16] the pair $(F^{\bar{c}}, F^{c_1})$ is trivial. Choose $\alpha', \alpha'' \in \mathbb{R}$ and the least positive integer m such that $\alpha' < c_1 < \alpha'' < c_m$. This is possible because property (b) holds. Then we fix $\beta', \beta'' \in \mathbb{R}$ such that $c_m < \beta' < \bar{c} < \beta''$. Since the pair $(F^{\bar{c}}, F^{c_1})$ is trivial, we can find two closed subsets A, B of S_r and a strong deformation retraction $r : B \to A$ such that $A \subseteq F^{\alpha''}$, $F^{\beta'} \subseteq B$ and, with a homotopy $\eta : B \times [0,1] \to B$,

$$\eta(x,0) = x \quad \forall x \in B$$

$$\eta(x,1) = r(x) \quad \forall x \in B$$

$$\eta(x,0) = \eta(x,t) \quad \forall (x,t) \in A \times [0,1]$$

The inequality $c_m < \beta'$ ensures that there exists $C \in \mathscr{A}_m$ such that $C \subseteq F^{\beta'}$, while the inequality $c_m > \alpha''$ enables us to deduce that for every set $D \in \mathscr{A}_m$ there is a point $u \in D$ satisfying $\alpha'' < F(u)$.

The inclusions $C \subseteq F^{\beta'} \subseteq B$ insure that $\eta(C, 1) \subseteq \eta(B, 1) = r(B) = A \subseteq F^{\alpha''}$. We show that $\eta(C, 1) \in \mathscr{A}_m$. To this end we observe that for the set C one can find a subset $\tilde{C} \subseteq C$ such that $\pi(\tilde{C}) = \pi(C)$ and π is a homeomorphism on $\tilde{C}$. We note that a homotopy $\tilde{\eta} : \pi(\tilde{C}) \times [0, 1] \to \pi(S_r)$ can be defined by the relation $\tilde{\eta}(\pi(x), t) = \pi(\eta(x, t))$, $\forall x \in \tilde{C}$, $\forall t \in [0, 1]$. Using Proposition 2.3 in Reference [13] we derive that

$$\mathrm{cat}_{\pi(S_r)} \pi(\eta(C, 1)) \geqslant \mathrm{cat}_{\pi(S_r)} \pi(\eta(\tilde{C}, 1)) = \mathrm{cat}_{\pi(S_r)} \tilde{\eta}(\pi(\tilde{C}), 1)$$

$$\geqslant \mathrm{cat}_{\pi(S_r)} \pi(\tilde{C}) = \mathrm{cat}_{\pi(S_r)} \pi(C) \geqslant m$$

which expresses that $\eta(C, 1) \in \mathscr{A}_m$. This leads to a contradiction between $\eta(C, 1) \subseteq F^{\alpha''}$ and the property of $D = \eta(C, 1)$ to contain a point $u \in D$ with $\alpha'' < F(u)$. The achieved contradiction allows to conclude that there exists an essential value b_1 of $F_{|S_r}$ satisfying $c_1 < b_1 < \bar{c}$.

Suppose now inductively that there exist essential values $b_1, \ldots, b_{n-1} \in \mathbb{R}$ with $b_1 < \cdots < b_{n-1} < \bar{c}$. Assertion (b) guarantees the existence of some c_p, with p depending on n, which satisfies $b_{n-1} < c_p < \bar{c}$. Repeating the reasoning used for constructing b_1, with c_1 replaced by c_p, we find an essential value b_n belonging to the open interval $(c_p, \bar{c})$. This completes the inductive process. In view of property (b) one obtains that the sequence (b_n) converges to $\bar{c}$. The proof of Proposition 2 is thus complete. $\square$

5. PROOF OF THEOREM 1

Fix any $n \geqslant 1$. To prove Theorem 1 we see from Lemmas 1 and 3 that it is sufficient to establish the existence of some $\delta_n > 0$ such that the functional $W_{|S_r}$ has at least n distinct critical values, provided that $\|h(\cdot, 0)\|_{L^1} \leqslant \delta_n$, $\|a_2\|_{L^{p/(p-1)}} \leqslant \delta_n$, $c \leqslant \delta_n$ and $\|\varphi\|_{V^*} \leqslant \delta_n$. By Proposition 2 we can find an increasing sequence (b_k) of essential values of $F_{|S_r}$ which converges to $\bar{c} = \sup c_k$. Let $\varepsilon_0 > 0$ be chosen such that $\varepsilon_0 < \frac{1}{2} \min_{2 \leqslant i \leqslant n} (b_i - b_{i-1})$. Applying Proposition 1 to $F_{|S_r}$ and $W_{|S_r}$, for every $1 \leqslant j \leqslant n$, there exists $\eta_j > 0$ such that if

$$\sup_{u \in S_r} |F(u) - W(u)| < \eta_j$$

there exists an essential value e_j of $W_{|S_r}$ in $(b_j - \varepsilon_0, b_j + \varepsilon_0)$. Then, from Lemma 1 for $\eta = \min\{\eta_1, \ldots, \eta_n\}$, we get the existence of some $\delta_n > 0$ such that

$$\sup_{u \in S_r} |F(u) - W(u)| < \eta$$

provided that $\|h(\cdot, 0)\|_{L^1} \leqslant \delta_n$, $\|a_2\|_{L^{p/(p-1)}} \leqslant \delta_n$, $c \leqslant \delta_n$ and $\|\varphi\|_{V^*} \leqslant \delta_n$. Therefore in this situation, the functional $W_{|S_r}$ has at least n distinct essential values $e_1, e_2, \ldots, e_n$ in $(b_1 - \varepsilon_0, b_n + \varepsilon_0)$.

For completing the proof of Theorem 1 it suffices to show that $e_1, e_2, \ldots, e_n$ are critical values of $W_{|S_r}$. Assuming the contrary, there exists some $j \in \{1, 2, \ldots, n\}$ such that e_j is not a critical value of $W_{|S_r}$. In what follows we are going to prove that this fact implies

(A) There exists $\bar{\varepsilon} > 0$ so that $W_{|S_r}$ has no critical values in $(e_j - \bar{\varepsilon}, e_j + \bar{\varepsilon})$;

(B) For every $a, b \in (e_j - \bar{\varepsilon}, e_j + \bar{\varepsilon})$ with $a < b$, the pair $((W_{|S_r})^b, (W_{|S_r})^a)$ is trivial.

Suppose that (A) is not valid. Then we get the existence of a sequence (d_k) of critical values of $W_{|S_r}$ with $d_k \to e_j$ as $k \to \infty$. Since d_k is a critical value it follows that there exists $u_k \in S_r$ such that

$$W(u_k) = d_k \quad \text{and} \quad \lambda_{W_{|S_r}}(u_k) = 0$$

Using the fact that (PS) holds (see Lemma 2) we can suppose that, up to a subsequence, (u_k) converges to some $u \in S_r$ as $k \to \infty$. Taking into account the continuity of W and the lower semicontinuity of $\lambda_{W_{|S_r}}$ we obtain

$$W(u) = e_j \quad \text{and} \quad \lambda_{W_{|S_r}}(u) = 0$$

which contradicts the assumption that e_j is not a critical value.

To get (B), we notice that on the basis of (A) we can apply the non-critical interval theorem (see Reference [20, Theorem 2.15]) on every interval $[a, b]$ as described in (B). It implies that there exists a continuous map $\chi : S_r \times [0, 1] \to S_r$ such that

$$\chi(u, 0) = u, \quad W(\chi(u, t)) \leqslant W(u) \ \forall (u, t) \in S_r \times [0, 1]$$

$$ \tag{36} $$

$$W(u) \leqslant b \Rightarrow W(\chi(u, 1)) \leqslant a, \quad W(u) \leqslant a \Rightarrow \chi(u, t) = u$$

Define the map $\rho : (W_{|S_r})^b \to (W_{|S_r})^a$ by $p(u) = \chi(u, 1)$. From (36) we have that ρ is well defined and it is a retraction. Set

$$\mathcal{H} : (W_{|S_r})^b \times [0, 1] \to (W_{|S_r})^b, \quad \mathcal{H}(u, t) = \chi(u, t)$$

Again from (36) we see that, for every $u \in (W_{|S_r})^b$,

$$\mathcal{H}(u, 0) = u, \quad \mathcal{H}(u, 1) = \rho(u) \tag{37}$$

and for each $(u, t) \in (W_{|S_r})^a \times [0, 1]$,

$$\mathcal{H}(u, t) = u \tag{38}$$

From (37) and (38) it follows that the pair $((W_{|S_r})^b, (W_{|S_r})^a)$ is trivial.

Combining assertions (A), (B) and Definition 5 it is seen that e_j is not an essential value of $W_{|S_r}$. The achieved contradiction completes the proof. $\qquad \square$

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