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Generalized Choquard Equations Driven by Nonhomogeneous Operators

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Abstract. In this work we prove the existence of solutions for a class of generalized Choquard equations involving the Δ_Φ-Laplacian operator. Our arguments are essentially based on variational methods. One of the main difficulties in this approach is to use the Hardy–Littlewood–Sobolev inequality for nonlinearities involving N-functions. The methods developed in this paper can be extended to wide classes of nonlinear problems driven by nonhomogeneous operators.

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1. Introduction

The stationary Choquard equation

$$-\Delta u+V(x)u=\Big(\int_{\mathbb{R}^N}\frac{|u|^p}{|x-y|^\lambda}\mathrm{d}x\Big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N\geq 3$, $0<\lambda<N$, has appeared in the context of various physical models. In particular, this equation plays particularly an important role in the theory of Bose–Einstein condensation where it accounts for the finite-range many-body interactions.

For $N=3$, $p=2$, and $\lambda=1$, problem (1.1) was investigated by Fröhlich [1] and Pekar [2] in relationship with the quantum theory of a polaron, where free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole). We recall that Choquard [3] used this equation in the Hartree–Fock theory of one-component plasma. This equation was also proposed by Penrose in [4] as a model of self-gravitating matter and is known in that context as the Schrödinger–Newton equation. In fact, the Choquard equation is also known as the Schrödinger–Newton

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equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity.

Recent relevant contributions included in the papers are by Ackermann [5], Alves et al. [6–8], Cingolani, Secchi and Squassina [9], Gao and Yang [10], Lions [11], Ma and Zhao [12], Moroz and van Schaftingen [13–16], van Schaftingen and Xia [17], Wang [18], and their references.

In all the above-mentioned papers, the authors used variational methods to show the existence of solution. This method works well thanks to a Hardy–Littlewood–Sobolev type inequality [19] that has the following statement.

Proposition 1.1 (Hardy–Littlewood–Sobolev inequality). *Let $t, r > 1$ and $0 < \lambda < N$ with $1/t + \lambda/N + 1/r = 2$, $g \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t, N, \lambda, r)$, independent of f, h, such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(x)h(y)}{|x-y|^\lambda}\,dx\,dy\right| \leq C(t,N,\lambda,r)\|g\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}. \tag{1.2}$$

In this paper, we are concerned with the existence of solution for the following class of quasilinear problems:

$$\begin{cases} -\Delta_\Phi u + \phi(|u|)u = \left(\displaystyle\int_{\mathbb{R}^N}\frac{F(u(x))}{|x-y|^\lambda}\,\mathrm{d}x\right) f(u(y)) \quad \text{in } \mathbb{R}^N, \\ \qquad\qquad u \in W^{1,\Phi}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

This problem generalizes (1.1) in a nonhomogeneous setting, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying some natural hypotheses, which will be mentioned in Sect. 3. Let F be the primitive of f, that is,

$$F(t) = \int_0^t f(s)\mathrm{d}s.$$

We denote $\Delta_\Phi = \operatorname{div}(\phi(|\nabla u|)\nabla u)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a N-function of the form

$$\Phi(t) := \int_0^{|t|}\phi(s)s\mathrm{d}s, \tag{1.4}$$

with $\phi : (0, +\infty) \to (0, +\infty)$ being a C^1 function satisfying

(ϕ_1) $$(t\phi(t))' > 0; \quad \forall t > 0$$

and

(ϕ_2) $$\lim_{t\to 0^+} t\phi(t) = 0, \quad \lim_{t\to+\infty} t\phi(t) = +\infty.$$

We also assume that there exist $l, m \in (1, N)$ such that $l \leq m < l^* := \frac{Nl}{N-l}$ and

(ϕ_3) $$l \leq m < l^* := \frac{Nl}{N-l} \quad \text{and} \quad l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m \text{ for all } t > 0.$$

Our purpose is to show that the variational method can be used to establish the existence of solutions for problem (1.3). One of the main difficulties is to show that the energy functional associated with (1.3) given by

$$J(u)=\int_{\mathbb{R}^N}\left(\Phi(|\nabla u|)+\Phi(|u|)\right)\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))F(u(y))}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y$$

is well defined and belongs to $C^1(W^{1,\Phi}(\mathbb{R}^N),\mathbb{R})$. In fact, the main difficulty is to prove that the functional $\Psi: W^{1,\Phi}(\mathbb{R}^N)\to\mathbb{R}$ given by

$$\Psi(u)=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))F(u(y))}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y \tag{1.5}$$

belongs to $C^1(W^{1,\Phi}(\mathbb{R}^N),\mathbb{R})$ with

$$\Psi'(u)v=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x)f(u(y))v(y))}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y,\quad \forall u,v\in W^{1,\Phi}(\mathbb{R}^N).$$

In what follows, we would like to point out that the operator Δ_Φ arises in several physical applications, such as:
Non-Newtonian fluids: $\Phi(t)=\frac{1}{p}|t|^p$ for $p>1$,
Plasma physics: $\Phi(t)=\frac{1}{p}|t|^p+\frac{1}{q}|t|^q$ where $1<p<q<N$ with $q\in(p,p^*)$,
Nonlinear elasticity: $\Phi(t)=(1+t^2)^{\alpha}-1, \alpha\in(1,\frac{N}{N-2})$,
Plasticity: $\Phi(t)=t^p\ln(1+t), 1<\frac{-1+\sqrt{1+4N}}{2}<p<N-1, N\geq 3$,
Generalized Newtonian fluids: $\Phi(t)=\int_0^t s^{1-\alpha}(\sinh^{-1}s)^{\beta}\mathrm{d}s, 0\leq\alpha\leq 1$ and $\beta>0$.

The reader can find more details about the physical applications in [20], [21] and their references. The existence of solution for

$$-\Delta_\Phi u+V(x)\phi(|u|)u=f(u)\quad\text{in}\quad\Omega,\quad\text{and}\quad u=0\quad\text{on}\quad\partial\Omega,$$

with $\Omega\subset\mathbb{R}^N$ being a bounded or unbounded domain has been established in some papers, see for example [22–35] and the references therein.

This paper is organized as follows. In Sect. 2 we recall some facts involving Orlicz–Sobolev spaces. In Sect. 3 we show that Ψ is of class C^1 under certain conditions on f. In Sect. 4 we study the existence of solutions to problem (1.3). Finally, in Sect. 5, we show other problems that can be studied with the approach developed in the present paper.

2. Orlicz–Sobolev Spaces

In this section we recall some results on Orlicz–Sobolev spaces. The results pointed below can be found in [21,28,36,37].

We say that a continuous function $\Phi:\mathbb{R}\to[0,+\infty)$ is a N-function if:

(i) Φ is convex,
(ii) $\Phi(t)=0\Leftrightarrow t=0$,
(iii) $\displaystyle\lim_{t\to 0}\frac{\Phi(t)}{t}=0$ and $\displaystyle\lim_{t\to+\infty}\frac{\Phi(t)}{t}=+\infty$,
(iv) Φ is even.

We say that a N-function Φ verifies the Δ_2-condition, and we denote by $\Phi\in\Delta_2$, if

$$\Phi(2t)\leq K\Phi(t),\quad \forall t\geq t_0, \tag{2.1}$$

for some constants $K, t_0 > 0$. An important fact is that if Φ verifies the Δ_2-condition then for some $s > 0$ there is a constant $C_s > 0$ such that

$$\Phi(st) \leq C_s \Phi(t), \quad \forall t \geq 0.$$

We fix an open set $\Omega \subset \mathbb{R}^N$ and a N-function Φ. We define the Orlicz space associated with Φ as follows

$$L^{\Phi}(\Omega) = \left\{ u \in L^1(\Omega)\colon \int_{\Omega} \Phi\Big(\frac{|u|}{\lambda}\Big) \mathrm{d}x < +\infty \ \text{ for some } \ \lambda > 0 \right\}.$$

The space $L^{\Phi}(\Omega)$ is a Banach space endowed with the Luxemburg norm given by

$$\|u\|_{\Phi} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\Big(\frac{|u|}{\lambda}\Big) \mathrm{d}x \leq 1 \right\}.$$

In the case of $|\Omega| = +\infty$ we will consider that $\Phi \in \Delta_2$ if $t_0 = 0$ in (2.1). The complementary function $\widetilde{\Phi}$ associated with Φ is given by the Legendre transformation, that is,

$$\widetilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\}, \quad \text{for all } s \geq 0.$$

We also have a Young-type inequality given by

$$st \leq \Phi(t) + \widetilde{\Phi}(s), \quad \text{for all } s, t \geq 0.$$

Using the above inequality, it is possible to establish the following Hölder-type inequality:

$$\Big| \int_{\Omega} uv \mathrm{d}x \Big| \leq 2 \|u\|_{\Phi} \|v\|_{\widetilde{\Phi}}, \quad \text{for all } u \in L^{\Phi}(\Omega),\ v \in L^{\widetilde{\Phi}}(\Omega).$$

The following results will be often used and they can be found in [21,28].

Lemma 2.1. *Consider Φ a N-function of the form (1.4) and satisfying $(\phi_1), (\phi_2)$ and (ϕ_3) only with the restriction $1 < l \leq m$. Set*

$$\zeta_0(t) = \min\{t^{\ell}, t^m\} \ \textit{and} \ \zeta_1(t) = \max\{t^{\ell}, t^m\}, \ t \geq 0.$$

Then Φ satisfies

$$\zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \ \rho, t > 0,$$
$$\zeta_0(\|u\|_{\Phi}) \leq \int_{\Omega} \Phi(u)\, dx \leq \zeta_1(\|u\|_{\Phi}), \ u \in L_{\Phi}(\Omega).$$

Lemma 2.2. *If Φ is a N-function of the form (1.4) satisfying (ϕ_1) and (ϕ_2), then*

$$\widetilde{\Phi}(\phi(|t|)t) \leq \Phi(2t) \ \ \forall t \geq 0.$$

For a N-function Φ, the corresponding Orlicz–Sobolev space is defined as the Banach space

$$W^{1,\Phi}(\Omega) = \Big\{ u \in L^{\Phi}(\Omega) \ : \ \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), \quad i = 1, \ldots, N \Big\},$$

endowed with the norm

$$\|u\|_{1,\Phi} = \|\nabla u\|_{\Phi} + \|u\|_{\Phi}.$$

If Φ and $\widetilde{\Phi}$ satisfy the Δ_2-condition, then the spaces $L^{\Phi}(\Omega)$ and $W^{1,\Phi}(\mathbb{R}^N)$ are reflexive and separable. Moreover, the Δ_2-condition also implies that

$$u_n \to u \text{ in } L_\Phi(\Omega) \Longleftrightarrow \int_\Omega \Phi(|u_n - u|) \to 0$$

and

$$u_n \to u \text{ in } W^{1,\Phi}(\Omega) \Longleftrightarrow \int_\Omega \Phi(|u_n - u|) \to 0 \text{ and } \int_\Omega \Phi(|\nabla u_n - \nabla u|) \to 0.$$

An important function related to a N-function Φ is the Sobolev conjugate function Φ_* of Φ defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \mathrm{d}s \quad \text{for} \quad t > 0 \quad \text{when} \quad \int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \mathrm{d}s = +\infty.$$

Let Φ_1 and Φ_2 be N-functions. We say that Φ_1 increases strictly lower than Φ_2, and we denote by $\Phi_1 \prec\prec \Phi_2$, if $\lim\limits_{t\to+\infty} \dfrac{\Phi_1(kt)}{\Phi_2(t)} = 0$ for all $k > 0$.

We say that $\Omega \subset \mathbb{R}^N$ is an admissible domain, if the embedding $W^{1,1}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, \frac{N}{N-1}]$.

The following embedding results can be found in [25,38].

Theorem 2.1. *Let Φ be a N-function, A a N-function and Ω be an admissible domain. Then*

(i) *If $\limsup\limits_{t\to 0} \dfrac{A(t)}{\Phi(t)} < +\infty$ and $\limsup\limits_{t\to+\infty} \dfrac{A(t)}{\Phi_*(t)} < +\infty$, then the embeddings $W^{1,\Phi}(\Omega) \hookrightarrow L^A(\Omega)$ and $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^A(\mathbb{R}^N)$ are continuous,*

(ii) *If $A \prec\prec \Phi_\star$ and $|\Omega| < +\infty$, then the embedding $W^{1,\Phi}(\Omega) \hookrightarrow L^A(\Omega)$ is compact.*

The next result can be found in [25] and it will play an important role in this work.

Theorem 2.2. *Consider $\Phi(t) := \int_0^{|t|} \phi(t)t\, dt$ a N-function with ϕ satisfying $(\phi_1), (\phi_2)$ and (ϕ_3). Let (u_n) be a bounded sequence in $W^{1,\Phi}(\mathbb{R}^N)$ such that there exists $R > 0$ satisfying*

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \Phi(|u_n|) = 0.$$

Then, for any N-function P verifying the Δ_2-condition with

$$\lim_{t\to 0} \frac{P(t)}{\Phi(t)} = 0 \tag{P_1}$$

and

$$\lim_{t\to+\infty} \frac{P(t)}{\Phi_*(t)} = 0, \tag{P_2}$$

we have

$$u_n \to 0 \quad in \quad L^P(\mathbb{R}^N).$$

3. Differentiability of the Functional Ψ

In this section, we will study the differentiability of the functional Ψ given in (1.5). To this end, we must assume some conditions on f.

We will consider $B:\mathbb{R}\to[0,+\infty)$ being a N-function given by $B(t)=\int_0^{|t|} b(s)s\,\mathrm{d}s$, where $b:(0,+\infty)\to(0,+\infty)$ is a function satisfying the following conditions:

$$(H_1)\qquad (tb(t))'>0,\quad \text{for all } t>0,$$

$$(H_2)\qquad \lim_{t\to+\infty} tb(t)=+\infty,$$

and there exist $b_i\in(1,+\infty), i=1,2$ such that

$$(H_3)\qquad b_1\le \frac{b(t)t^2}{B(t)}\le b_2, \forall t>0.$$

The above hypotheses permit to use Lemmas 2.1 and 2.2 by changing Φ by B. More precisely, B satisfies the Δ_2 condition with

$$\zeta_{0,B}(t)B(\rho)\le B(\rho t)\le \zeta_{1,B}(t)B(\rho),\ \ \rho,t>0,$$

and

$$\zeta_{0,B}(\|u\|_B)\le\int_\Omega B(u)\mathrm{d}x\le\zeta_{1,B}(\|u\|_B),\ \ u\in L_B(\Omega),$$

where

$$\zeta_{0,B}(t)=\min\{t^{b_1},t^{b_2}\} \text{ and } \zeta_{1,B}(t)=\max\{t^{b_1},t^{b_2}\},\ \ t\ge 0.$$

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the growth condition

$$|f(t)|\le Cb(|t|)|t|,\quad \forall t\in\mathbb{R}, \tag{f_1}$$

where C is a positive constant.

The primitive of f, that is, $F(t):=\int_0^t f(s)\mathrm{d}s$ is continuous and satisfies

$$|F(t)|\le CB(t),\quad \forall t\in\mathbb{R} \tag{F}$$

for some constant $C>0$.

Let $0<\lambda<N$ and consider $s\in\mathbb{R}$ with

$$\frac{1}{s}+\frac{\lambda}{N}+\frac{1}{s}=2,$$

that is, $s=\frac{2N}{2N-\lambda}$. We will also suppose that the embeddings

$$W^{1,\Phi}(\mathbb{R}^N)\hookrightarrow L^{sb_i}(\mathbb{R}^N),\ \ i=1,2 \tag{E}$$

are continuous. The above condition is not empty, because we can consider the following conditions on B

$$\limsup_{t\to 0^+}\frac{t^{sb_i}}{\Phi(t)}=0$$

and

$$\limsup_{t\to+\infty}\frac{t^{sb_i}}{\Phi_*(t)}=0.$$

If $\Phi(t) = \frac{1}{p}|t|^p$ with $1 < p < N$, we have that $\Delta_\Phi u = \Delta_p u$ and the function f can be of the form

$$f(t) = |t|^{q-2}t + |t|^{\beta-2}t, \quad \text{for all } t \in \mathbb{R}$$

with $sq, s\beta \in (p, p^*)$.

In the proof of the differentiability of Ψ, we will use the following elementary property, whose proof we omit.

Lemma 3.1. *Let E be a normed vector space and $J : E \to \mathbb{R}$ be a functional verifying the following properties:*
(i) *the Fréchet derivative $\frac{\partial J(u)}{\partial v} := \lim\limits_{t\to 0} \frac{J(u+tv)-J(u)}{t}$ exists for all $u, v \in E$;*
(ii) *for each $u \in E$, $\frac{\partial J(u)}{\partial(.)} \in E'$, that is, the application $v \mapsto \frac{\partial J(u)}{\partial v}$ is a continuous linear functional;*
(iii) *we have*

$$u_n \to u \quad in \quad E \Longrightarrow \frac{\partial J(u_n)}{\partial(.)} \to \frac{\partial J(u)}{\partial(.)} \quad in \quad E',$$

that is,

$$u_n \to u \quad in \quad E \Longrightarrow \sup_{\|v\|\le 1} \left| \frac{\partial J(u_n)}{\partial v} - \frac{\partial J(u)}{\partial v} \right| \to 0.$$

Then $J \in C^1(E, \mathbb{R})$ and

$$J'(u)v = \frac{\partial J(u)}{\partial v}, \quad for\ all\ u, v \in E.$$

We are now ready to prove the differentiability of functional Ψ given by (1.5).

Lemma 3.2. *Assume (E), $(H_1) - (H_3)$ and (f_1). Then Ψ given in (1.5) is well defined and belongs to $C^1(W^{1,\Phi}(\mathbb{R}^N), \mathbb{R})$ with*

$$\Psi'(u)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(x))f(u(y))v(y))}{|x-y|^\lambda} \, dx\, dy,$$

for all $u, v \in W^{1,\Phi}(\mathbb{R}^N)$.

Proof. Using the definition of s, the condition (F) and Proposition 1.1, it follows that Ψ is well defined. In the sequel, we will show that Ψ satisfies the assumptions of Lemma 3.1. To this end, we will divide the proof into three steps.

Step 1: Existence of the Fréchet derivative:

Let $u, v \in W^{1,\Phi}(\mathbb{R}^N)$ and $t \in [-1,1]$. Note that

$$\begin{aligned}
&\frac{\Psi(u+tv)-\Psi(u)}{t}\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x)+tv(x))F(u(y)+tv(y))-F(u(x))F(u(y))}{t|x-y|^\lambda}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.1}$$

Denoting by I the integrand in (3.1), we have

$$\begin{aligned}
I=&\frac{F(u(x)+tv(x))(F(u(y)+tv(y))-F(u(y))}{t}\\
&+\tfrac{F(u(y))(F(u(x)+tv(x))-F(u(x)))}{t}.
\end{aligned}$$

By the mean value theorem, there exist $\theta(x,t), \eta(y,t) \in [-1,1]$, such that

$$F(u(y)+tv(y))-F(u(y))=f(u(y)+\eta(y,t)tv(y))v(y)t$$

and

$$F(u(x)+tv(x))-F(u(x))=f(u(x)+\theta(x,t)tv(x))v(x)t.$$

The relation (3.1) allows us to estimate

$$\begin{aligned}
&\left|\frac{\Psi(u+tv)-\Psi(u)}{t}-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x)f(u(y))v(y))}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\right|\\
&\quad\leq |B_1^t|+|B_2^t|,
\end{aligned}$$

where

$$B_1^t:=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x)+tv(x))f(u(y)+\eta(y,t)tv(y))v(y)-F(u(x)f(u(y))v(y))}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y$$

and

$$\begin{aligned}
B_2^t:=&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x)+\theta(x,t)tv(x))v(x)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\\
&-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))f(u(y))v(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

If $x \in \mathbb{R}^N$ and $u(x) \neq 0$, Lemma 2.1 gives

$$\begin{aligned}
|f(u(x))v(x)| &\leq Cb(|u(x)|)|u(x)||v(x)|\\
&\leq C\left(\frac{B(|u(x)|)}{|u(x)|}\right)|v(x)|\\
&\leq C(|u(x)|^{b_1-1}+|u(x)|^{b_2-1})|v(x)|.
\end{aligned}$$

The above estimate also holds if $u(x)=0$. Such estimate, combined with (E) implies that $f(u)v \in L^s(\mathbb{R}^N)$. Since $F(u) \in L^s(\mathbb{R}^N)$, we have by Proposition 1.1,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|F(u(x))f(u(y))v(y)|}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y<+\infty.$$

Thus, by Fubini's theorem,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))f(u(y))v(y)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x))v(x)}{|x-y|^{\lambda}}\mathrm{d}y\mathrm{d}x.$$

Therefore, B_2^t can be rewritten as

$$B_2^t:=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x)+\theta(x,t)tv(x))v(x)-F(u(y)f(u(x))v(x))}{|x-y|}\mathrm{d}x\mathrm{d}y.$$

By Proposition 1.1 we obtain

$$|B_2^t|\leq C\|F(u)\|_{L^{s^+}(\mathbb{R}^N)}\|f(u+\theta(.,t)v)v-f(u)v\|_{L^{s^+}(\mathbb{R}^N)}.$$

Since $\theta(x,t)\in[-1,1]$, the condition (f) combined with Lemma 2.1 implies that

$$\begin{aligned}|f(u(x)+\theta(t,x)tv(x))v(x)-f(u(x))v(x)|^s\leq C((|u(x)|+|v(x)|)^{s(b_1-1)}|v(x)|^s\\ +(|u(x)|+|v(x)|)^{s(b_2-1)}|v(x)|^s),\end{aligned}\tag{3.2}$$

for all $x\in\mathbb{R}^N$. Here C is a constant that does not depend on $t\in[-1,1]$. The embeddings (E) ensure that the right-hand side of the inequality (3.2) is an integrable function. Thus, the Lebesgue's dominated convergence theorem yields

$$\|f(u+\theta(.,t)v)v-f(u)v\|_{L^s(\mathbb{R}^N)}\to 0\ as\ t\to 0.$$

The last limit implies that $B_2^t\to 0$ as $t\to 0$. With respect to B_1^t, we have the estimate below

$$\begin{aligned}|B_1^t|\leq\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|F(u(x))||f(u(y)+\eta(y,t)tv(y))v(y)-f(v(y))v(y)|}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\\ +\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(u(y)+\eta(y,t)tv(y))v(y)||F(u(x)+tv(x))-F(u(x))|}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Arguing as before,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|F(u(x))||f(u(y)+\eta(y,t)tv(y))v(y)-f(v(y))v(y)|}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\to 0$$

as $t\to 0$. On the other hand, the Lebesgue's dominated convergence theorem also yields

$$\|F(u+tv)-F(u)\|_{L^s(\mathbb{R}^N)}\to 0 \text{ as } t\to 0.\tag{3.3}$$

As in (3.2), $\|f(u+\eta(.,t)tv)v\|_{L^s(\mathbb{R}^N)}$ is uniformly bounded by a constant that does not depend on $t\in[-1,1]$. Thus, Proposition 1.1 combined with (3.3) gives

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(u(y)+\eta(y,t)tv(y))v(y)||F(u(x)+tv(x))-F(u(x))|}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\to 0,$$

as $t\to 0$, and so, $B_1^t\to 0$ as $t\to 0$. From the above analysis,

$$\lim_{t\to 0}\frac{\Psi(u+tv)-\Psi(u)}{t}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x)f(u(y))v(y))}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y,$$

showing the existence of the Fréchet derivative $\frac{\partial\Psi(u)}{\partial v}$.

Step 2: $\frac{\partial\Psi(u)}{\partial(.)} \in (W^{1,\Phi}(\mathbb{R}^N))'$ for all $u \in W^{1,\Phi}(\mathbb{R}^N)$.

It is evident that $\frac{\partial\Psi(u)}{\partial v}$ is linear at v for each u fixed. Next, we are going to show that

$$\left|\frac{\partial\Psi(u)}{\partial v}\right| \leq C_u, \quad \forall v \in W^{1,\Phi}(\mathbb{R}^N) \quad \text{with} \quad \|v\|_{W^{1,\Phi}(\mathbb{R}^N)} \leq 1,$$

for some positive constant C_u. From $(f), (F)$ and Proposition 1.1 we have

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))f(u(y))v(y)}{|x-y|^{\lambda(x,y)}}\mathrm{d}x\mathrm{d}y\right| \leq C\|F(u)\|_{L^s(\mathbb{R}^N)}\|f(u)v\|_{L^s(\mathbb{R}^N)}. \tag{3.4}$$

The continuous embeddings of (E) combined with Hölder inequality, (f) and Proposition 2.1 give

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(u(y))v(y)|^s\mathrm{d}y &\leq C\left(\int_{\mathbb{R}^N} |u(y)|^{s(b_1-1)}|v(y)|^s\mathrm{d}y\right.\\
&\quad \left.+\int_{\mathbb{R}^N} |u(y)|^{s(b_2-1)}|v(y)|^s\mathrm{d}y\right)\\
&\quad +C\left(\left(\int_{\mathbb{R}^N} |u(y)|^{sb_1}\mathrm{d}y\right)^{\frac{b_1-1}{b_1}}\left(\int_{\mathbb{R}^N} |v(y)|^{sb_1}\right)^{\frac{1}{b_1}}\right.\\
&\quad \left.+\left(\int_{\mathbb{R}^N} |u(y)|^{sb_2}\mathrm{d}y\right)^{\frac{b_2-1}{b_2}}\left(\int_{\mathbb{R}^N} |v(y)|^{sb_2}\right)^{\frac{1}{b_2}}\right) \leq C_u,
\end{aligned} \tag{3.5}$$

where

$$C_u = K\left(\max\left(\left(\int_{\mathbb{R}^N} |u(y)|^{sb_1}\mathrm{d}y\right)^{\frac{b_1-1}{b_1}}, \left(\int_{\mathbb{R}^N} |u(y)|^{sb_2}\mathrm{d}y\right)^{\frac{b_2-1}{b_2}}\right)\right),$$

with K a constant that does not depend on u and v. Then inequalities (3.4) and (3.5) justify Step 2.

Step 3:

$$u_n \to u \quad \text{in} \quad W^{1,\Phi}(\mathbb{R}^N) \Rightarrow \sup_{\|v\|_{W^{1,\Phi}(\mathbb{R}^N)}\leq 1}\left|\frac{\partial\Psi(u_n)}{\partial v} - \frac{\partial\Psi(u)}{\partial v}\right| \to 0.$$

Consider $v \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|v\|_{W^{1,\Phi}(\mathbb{R}^N)} \leq 1$ and note that

$$\begin{aligned}
\left|\frac{\partial\Psi(u_n)}{\partial v} - \frac{\partial\Psi(u)}{\partial v}\right| &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|F(u_n(x)) - F(u(x))||f(u_n(y))v(y)|}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|F(u(x))||f(u_n(y))v(y) - f(y,u(y))v(y)|}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\\
&:= B_f^n + B_F^n.
\end{aligned}$$

By Proposition 1.1, we obtain

$$B_f^n \leq C\|F(u_n) - F(u))\|_{L^s(\mathbb{R}^N)}\|f(u_n)v\|_{L^s(\mathbb{R}^N)}.$$

Since the sequence $(\|f(u_n)v\|_{L^s(\mathbb{R}^N)})$ is bounded (see (3.5)) and

$$\|F(u_n) - F(u)\|_{L^s(\mathbb{R}^N)} \to 0,$$

it follows that

$$\sup_{\substack{v\in W^{1,\Phi}(\mathbb{R}^N)\\ \|v\|_{W^{1,\Phi}(\mathbb{R}^N)}\leq 1}} \|F(u_n) - F(u))\|_{L^s(\mathbb{R}^N)}\|f(u_n)v\|_{L^s(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.6}$$

Now we will estimate B_F^n. Given $\varepsilon > 0$, fix $R > 0$ large enough such that

$$\int_{B(0,R)^c} |u(x)|^{b_1 s}\mathrm{d}x, \int_{B(0,R)^c} |u(x)|^{b_2 s}\mathrm{d}x < \varepsilon.$$

Since $u_n \to u$ in $L^{b_1 s}(\mathbb{R}^N)$ and $L^{b_2 s}(\mathbb{R}^N)$, there is $n_0 \in \mathbb{N}$ large enough such that

$$\int_{B(0,R)^c} |u_n(x)|^{b_1 s}\mathrm{d}x, \int_{B(0,R)^c} |u_n(x)|^{b_2 s}\mathrm{d}x < \varepsilon, \tag{3.7}$$

for all $n \geq n_0$. Note that by Proposition 1.1

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|F(u(x))||(f(u_n(y)) - f(u(y)))v(y)|}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y &\leq C\|F(u)\|_{L^s(\mathbb{R}^N)} \\ &\quad\times \|(f(u_n) - f(u))v\|_{L^s(\mathbb{R}^N)},\end{aligned} \tag{3.8}$$

where $C_0 > 0$ is a constant that does not depend on $n \in \mathbb{N}$. The condition (f) together with Hölder's inequality yields

$$\begin{aligned}&\int_{B(0,R)^c} |(f(u_n(y)) - f(u(x)))v(y)|^s\mathrm{d}y \\ &\leq C_1\|u_n\|^{s(b_1-1)}_{L^{sb_1}(B(0,R)^c)}\|v\|^s_{L^{sb_1}(B(0,R)^c)} \\ &\quad + C_1\|u_n\|^{s(b_2-1)}_{L^{sb_2}(B(0,R)^c)}\|v\|^s_{L^{sb_2}(B(0,R)^c)},\end{aligned} \tag{3.9}$$

where $C_1 > 0$ is a constant that does not depend on $n \in \mathbb{N}$. Using the embeddings (E), we have $\|v\|^s_{L^{sb_i}(B(0,R)^c)} \leq C_2, i = 1, 2$ where C_2 is a positive constant that does not depend on $v \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|v\|_{W^{1,\Phi}(\mathbb{R}^N)} \leq 1$ and $R > 0$. Thus, from (3.7) and (3.9)

$$\int_{B(0,R)^c} |(f(u_n(y)) - f(u(x)))v(y)|^{q^+}\mathrm{d}y \leq C_3 \max\left\{\varepsilon^{\frac{b_1-1}{b_1}}, \varepsilon^{\frac{b_2-1}{b_2}}\right\}.$$

where C_3 is a positive constant that does not depend on $v \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|v\|_{W^{1,\Phi}(\mathbb{R}^N)} \leq 1$ and $R > 0$. Therefore

$$\sup_{\substack{v\in W^{1,\Phi}(\mathbb{R}^N)\\ \|v\|_{W^{1,\Phi}(\mathbb{R}^N)}\leq 1}} \int_{B(0,R)^c} |(f(u_n(y)) - f(u(y)))v(y)|^s\mathrm{d}y \leq A_\varepsilon, \quad \forall n \geq n_0. \tag{3.10}$$

where

$$A_\varepsilon = C_3 \max\left\{\varepsilon^{\frac{b_1-1}{b_1}}, \varepsilon^{\frac{b_2-1}{b_2}}\right\} \to 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

Now we will estimate the integral

$$\int_{B(0,R)} |f(u_n(y)) - f(u(y))|^s |v(y)|^s \mathrm{d}y.$$

From (f) and Lemma 2.1,

$$|f(u_n(y)) - f(u(y))|^s \le C(1 + |u_n(y)|^{s(b_2-1)} + |u(y)|^{s(b_2-1)}), \forall y \in \mathbb{R}^N.$$

From (E) we have that the embedding $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^{sb_2}(\mathbb{R}^N)$ is continuous which implies that $|f(u_n) - f(u)|^{\frac{sb_2}{b_2-1}}, |v|^{sb_2} \in L^1(B(0,R))$.

From Hölder's inequality we get

$$\begin{aligned}\int_{B(0,R)} |f(u_n(y)) - f(u(y))|^s |v(y)|^s \mathrm{d}y &\le C \||f(u_n) - f(u)|^s\|_{L^{\frac{b_2}{b_2-1}}(B(0,R))} \\ &\quad \times \||v|^s\|_{L^{b_2}(B(0,R))} \\ &\le C_4 \||f(u_n) - f(u)|^s\|_{L^{\frac{b_2}{b_2-1}}(B(0,R))},\end{aligned}$$

where C_4 is a positive constant that does not depend on $n \in \mathbb{N}$ and $v \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|v\|_{W^{1,\Phi}(\mathbb{R}^N)} \le 1$. Recalling that $|f(u_n(y)) - f(u(y))|^s \to 0$ in $L^{\frac{b_2}{b_2-1}}(B(0,R))$, it follows that

$$\sup_{\substack{v \in W^{1,\Phi}(\mathbb{R}^N) \\ \|v\|_{1,\Phi} \le 1}} \int_{B(0,R)} |(f(u_n(y)) - f(u(y)))v(y)|^s \mathrm{d}y \to 0 \text{ as } n \to +\infty. \tag{3.11}$$

From (3.10) and (3.11),

$$\sup_{\substack{v \in W^{1,\Phi}(\mathbb{R}^N) \\ \|v\|_{1,\Phi} \le 1}} \|(f(u_n) - f(u))v\|_{L^s(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty. \tag{3.12}$$

From (3.8) and (3.12),

$$\sup_{\substack{v \in W^{1,\Phi}(\mathbb{R}^N) \\ \|v\|_{1,\Phi} \le 1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(x))||(f(u_n(y)) - f(y,u(y)))v(y)|}{|x-y|^\lambda} \mathrm{d}x\mathrm{d}y \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.13}$$

The step is justified, according to (3.6), and (3.13). Finally, the lemma follows from the previous three steps. □

4. Existence of Nontrivial Solutions

To prove the existence of solution for (1.3) we will need some assumptions. In what follows, we will consider

$$m < sb_i < l^*, \ i = 1, 2,$$

which implies that the embeddings in (E) hold. Moreover, we will consider the condition (f_1) with

$$b_i > m/2, i = 1, 2. \tag{4.1}$$

Finally, we also assume the Ambrosetti–Rabinowitz-type condition: there is $\theta > m$ such that

$$0 < \theta F(t) \leq 2f(t)t, \quad \forall t \neq 0. \tag{f_2}$$

Our main result is the following:

Theorem 4.1. *The problem* (1.3) *has a nontrivial solution under the conditions* (4.1), $(\phi_1)-(\phi_3), (H_1)-(H_3)$ *and* $(f_1)-(f_2)$.

In the proof of Theorem 4.1 we will use variational methods. The energy functional $J : W^{1,\Phi}(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.3) is given by,

$$\begin{aligned} J(u) = &\int_{\mathbb{R}^N} (\Phi(|\nabla u(x)|) + \Phi(|u(x)|))\,\mathrm{d}x \\ &-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{\lambda}}\,\mathrm{d}x\mathrm{d}y, \end{aligned}$$

that is,

$$J(u) = \int_{\mathbb{R}^N} (\Phi(|\nabla u(x)|) + \Phi(|u(x)|))\,\mathrm{d}x - \Psi(u).$$

The analysis developed in the previous section implies that $J \in C^1(W^{1,\Phi}(\mathbb{R}^N), \mathbb{R})$ with

$$\begin{aligned} J'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u(x)|)\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi(|u(x)|)u(x)v(x)\mathrm{d}x \\ - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x)f(u(y))v(y))}{|x-y|^{\lambda(x,y)}}\mathrm{d}x\mathrm{d}y, \end{aligned}$$

for all $u, v \in W^{1,\Phi}(\mathbb{R}^N)$.

Our first lemma establishes the mountain pass geometry.

Lemma 4.1. *The functional J verifies the following properties:*

(i) *There exists $\rho > 0$ small enough such that $J(u) \geq \eta$ for $u \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|u\| = \rho$, for some $\eta > 0$.*

(ii) *There exists $e \in W^{1,\Phi}(\mathbb{R}^N)$ such that $\|e\| > \rho$ and $J(e) < 0$.*

Proof. (i) By (F) and Proposition 1.1 we have

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y\right| \leq C\|F(u)\|^2_{L^s(\mathbb{R}^N)}$$

for all $u \in W^{1,\Phi}(\mathbb{R}^N)$. Note that

$$\begin{aligned} \|F(u)\|_{L^s(\mathbb{R}^N)} &\leq C\left(\int_{\mathbb{R}^N} (|u(x)|^{sb_1} + |u(x)|^{sb_2})\mathrm{d}x\right)^{\frac{1}{s}} \\ &\leq C(\|u\|^{b_1}_{L^{sb_1}(\mathbb{R}^N)} + \|u\|^{b_2}_{L^{sb_2}(\mathbb{R}^N)}), \end{aligned}$$

where C is a constant that does not depend on $u \in W^{1,\Phi}(\mathbb{R}^N)$.

From (E) and Theorem 2.1, it follows that the embeddings $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^{sb_i}(\mathbb{R}^N), i = 1, 2$ are continuous. Therefore

$$\|u\|_{L^{sb_i}(\mathbb{R}^N)} \leq L\|u\|_{1,\Phi}, u \in W^{1,\Phi}(\mathbb{R}^N), i = 1, 2$$

for a positive constant $L > 0$ that does not depend on $u \in W^{1,\Phi}(\mathbb{R}^N)$.

By using the classical inequality

$$(x+y)^\alpha \leq 2^{\alpha-1}(x^\alpha + y^\alpha),\ x,y\geq 0 \quad \text{with} \quad \alpha>1,$$

and Lemma 2.1, we get for $u \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|u\|_{1,\Phi} = \|\nabla u\|_\Phi + \|u\|_\Phi < 1$ that

$$\begin{aligned} J(u) &\geq \Phi(1)\left(\min(\|\nabla u\|_\Phi^l, \|\nabla u\|_\Phi^m) + \min(\|u\|_\Phi^l, \|u\|_\Phi^m)\right) \\ &\quad - C(\|u\|_{1,\Phi}^{2b_1} + \|u\|_{1,\Phi}^{2b_2}) \\ &\geq \overline{C}(\|\nabla u\|_\Phi^m + \|u\|_\Phi^m) - \overline{C}(\|u\|_{1,\Phi}^{2b_1} + \|u\|_{1,\Phi}^{2b_2}) \\ &\geq K\|u\|_{1,\Phi}^m - K(\|u\|_{1,\Phi}^{2b_1} + \|u\|_{1,\Phi}^{2b_2}) \end{aligned}$$

where $C, \overline{C}$, and K are constants that do not depend on u. Since (4.1) holds, the result follows by fixing $\|u\|_{1,\Phi} = \rho$ with $\rho > 0$ small enough.

(ii) The condition (f_2) implies that

$$F(t) \geq C_1 t^{\frac{\theta}{2}} - C_2 \quad \forall t \in \mathbb{R},$$

where $C_1, C_2 \geq 0$ depends only on l and θ. Now, considering a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)\backslash\{0\}$, the last inequality permits to conclude that $J(t\varphi) < 0$ for t large enough. This finishes the proof. □

Using the mountain pass theorem without the Palais–Smale condition (see [39, Theorem 5.4.1]), there is a sequence $(u_n) \subset W^{1,\Phi}(\mathbb{R}^N)$ such that

$$J(u_n) \to d \quad \text{and} \quad J'(u_n) \to 0,$$

where $d > 0$ is the mountain pass level defined by

$$d := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)), \tag{4.2}$$

with

$$\Gamma := \{\gamma \in C(W^{1,\Phi}(\mathbb{R}^N), W^{1,\Phi}(\mathbb{R}^N);\ \gamma(0) = 0, \gamma(1) = e\}.$$

Regarding the above sequence we have the following auxiliary property.

Lemma 4.2. *The sequence (u_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$.*

Proof. Note that

$$J(u_n) - \frac{J'(u_n)u_n}{\theta} \leq d + 1 + \|u_n\|_{1,\Phi},$$

for n large enough and d given in (4.2). On the other hand by (ϕ_3) and (f_2) we have

$$\begin{aligned} J(u_n) - \frac{J'(u_n)u_n}{\theta} &= \int_{\mathbb{R}^N} (\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)\mathrm{d}x \\ &\quad - \frac{1}{\theta}\int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u|^2 + \phi(|u_n(x)|)|u_n(x)|^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u_n(x))}{|x-y|^\lambda}\left(\frac{f(y,u_n(y))u_n(y)}{\theta} - \frac{F(y,u_n(y))}{2}\right)\mathrm{d}x\mathrm{d}y \\ &\geq \left(1 - \frac{m}{\theta}\right)\int_{\mathbb{R}^N} (\Phi(|\nabla u_n|) + \Phi(|u_n|))\ \mathrm{d}x. \end{aligned}$$

The last two inequalities give the boundedness of (u_n) in $W^{1,\Phi}(\mathbb{R}^N)$. □

Since (u_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$, there exists $u \in W^{1,\Phi}(\mathbb{R}^N)$ such that $\nabla u_n \rightharpoonup \nabla u$ in $(L^{\Phi}(\mathbb{R}^N))^N$ up to a subsequence. From (4.1) and Theorem 2.1, we have that the embedding $W^{1,\Phi}(B_R(0)) \hookrightarrow L^{\Phi}(B_R(0))$, where $B_R(0)$ denotes the open ball centered at the origin with radius R is compact.

Since $L^{\Phi}(B_R(0)) \hookrightarrow L^1(B_R(0))$, it follows that $u_n(x) \to u(x)$ a.e in $B_R(0)$ for some subsequence. Since $R > 0$ is arbitrary, we have that $u_n(x) \to u(x)$ a.e in $\mathbb{R}^N$ for some subsequence.

The next two lemmas will be needed to prove that u is a critical point of J.

Lemma 4.3. *The following limits hold for a subsequence:*

(i)

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u_n(x))f(y,u(y))v(y)}{|x-y|^{\lambda}}\,dx\,dy \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))v(y)}{|x-y|^{\lambda}}\,dx\,dy,$$

for all $v \in C_0^{\infty}(\mathbb{R}^N)$,

(ii)

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u_n(x))(f(y,u_n(y))v(y) - f(y,u(y))v(y))}{|x-y|^{\lambda}}\,dx\,dy \to 0,$$

for all $v \in C_0^{\infty}(\mathbb{R}^N)$,

(iii)

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u_n(x))f(y,u_n(y))v(y)}{|x-y|^{\lambda}}\,dx\,dy \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))v(y)}{|x-y|^{\lambda}}\,dx\,dy,$$

for all $v \in C_0^{\infty}(\mathbb{R}^N)$.

Proof. (i) The hypothesis (F), the fact that (u_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$ and the continuous embeddings $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^{sb_i}(\mathbb{R}^N), i = 1, 2$ ensure that $(F(u_n(.)))$ is a bounded sequence in $L^s(\mathbb{R}^N)$. Combining the previous information with the pointwise convergence $F(u_n(x)) \to F(u(x))$ a.e in $\mathbb{R}^N$, we have $F(u_n(.)) \rightharpoonup F(u)$ in $L^s(\mathbb{R}^N)$.

By Proposition 1.1 it follows that the function

$$H(w) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w(x)f(u(y))v(y)}{|x-y|^{\lambda}}, w \in L^s(\mathbb{R}^N),$$

defines a continuous linear functional. Since $F(., u_n) \rightharpoonup F(., u)$ in $L^s(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))v(y)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))f(u(y))v(y)}{|x-y|^{\lambda}}\mathrm{d}x\mathrm{d}y,$$

which proves (i).

(ii) Denote by I the integral described in *(ii)*. Since $(F(u_n(.)))$ is bounded in $L^s(\mathbb{R}^N)$,

$$\begin{aligned}|I| &\leq C\|F(u_n)\|_{L^s(\mathbb{R}^N)}\|f(u_n)v - f(u)v\|_{L^s(\mathbb{R}^N)}\\ &\leq K\|f(u_n)v - f(u)v\|_{L^s(\mathbb{R}^N)},\end{aligned}$$

for some positive constant $K > 0$ that does not depend on $n \in \mathbb{N}$ and $v \in C_0^\infty(\mathbb{R}^N)$. Let $v \in C_0^\infty(\mathbb{R}^N)$ and consider a bounded open set Ω that contains the support of v. Since Ω is bounded the compactness of the embeddings $W^{1,\Phi}(\Omega) \hookrightarrow L^{sb_i}(\Omega), i = 1, 2$ implies that there exist $\overline{c} \in L^{sb_1}(\Omega)$ and $\overline{d} \in L^{sb_2}(\Omega)$ such that

- $u_n(x) \to u(x)$ a.e in $\mathbb{R}^N$,
- $|u_n(x)| \leq \overline{c}(x), \overline{d}(x)$ a.e in Ω.

These information combined with Lebesgue's dominated convergence theorem give

$$\|f(u_n)v - f(u)v\|_{L^s(\mathbb{R}^N)} = \|f(u_n)v - f(u)v\|_{L^s(\Omega)} \to 0.$$

This finishes the proof of *(ii)*. *(iii)* is a direct consequence of *(i)* and *(ii)*. $\square$

The next lemma is crucial to prove that u is a critical point of J.

Lemma 4.4. *There is a subsequence such that*

(i) $\nabla u_n(x) \to \nabla u(x)$ *a.e in* $\mathbb{R}^N$,

(ii) $\phi(|\nabla u_n|)\dfrac{\partial u_n}{\partial x_i} \rightharpoonup \phi(|\nabla u|)\dfrac{\partial u}{\partial x_i}$ *in* $L^{\widetilde{\Phi}}(\mathbb{R}^N)$,

(iii) $\phi(|u_n|)u_n \rightharpoonup \phi(|u|)u$ *in* $L^{\widetilde{\Phi}}(\mathbb{R}^N)$.

Proof. (i) We begin this proof observing that (ϕ_1) yields

$$(\phi(|x|)x - \phi(|y|)y)\,(x - y) > 0, \;\; \forall x, y \in \mathbb{R}^N \text{ with } x \neq y, \tag{4.3}$$

where $(\cdot, \cdot)$ denotes the usual inner product in $\mathbb{R}^N$. Given $R > 0$, let us consider $\xi = \xi_R \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$0 \leq \xi \leq 1, \xi \equiv 1 \text{ in } B_R(0) \text{ and } \operatorname{supp}(\xi) \subset B_{2R}(0).$$

Using (4.3), we get

$$\begin{aligned}0 &\leq \int_{B_R(0)} (\phi(|\nabla u_n(x)|)\nabla u_n - \phi(|\nabla u(x)|)\nabla u(x))\\ &\quad \times (\nabla u_n(x) - \nabla u(x))\,\mathrm{d}x\\ &\leq \int_{B_{2R}(0)} (\phi(|\nabla u_n(x)|)\nabla u_n(x) - \phi(|\nabla u(x)|)\nabla u(x))\\ &\quad \times (\nabla u_n(x) - \nabla u(x))\xi(x)\,\mathrm{d}x\\ &= \int_{B_{2R}(0)} \phi(|\nabla u_n(x)|)\nabla u_n(x)(\nabla u_n(x) - \nabla u(x))\xi(x)\,\mathrm{d}x\\ &\quad - \int_{B_{2R}(0)} \phi(|\nabla u(x)|)\nabla u(x)(\nabla u_n(x) - \nabla u(x))\xi(x)\,\mathrm{d}x.\end{aligned} \tag{4.4}$$

Now, combining the boundedness of $\{(u_n-u)\xi\}$ in $W^{1,\Phi}(\mathbb{R}^N)$ with the limit $\|J'(u_n)\| = o_n(1)$, it follows that

$$\begin{aligned} o_n(1) = J^{'}(u_n)((u_n-u)\xi) = &\int_{B_{2R}(0)} \phi(|\nabla u_n(x)|) \\ &\times \nabla u_n(x)\nabla\left((u_n-u)(x)\xi(x)\right)\ \mathrm{d}x \\ &+ \int_{B_{2R}(0)} \phi(|u_n(x)|)u_n(x)(u_n-u)(x)\xi(x)\ \mathrm{d}x \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))(u_n(y)-u(y))\xi(y)}{|x-y|^\lambda}\ \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4.5}$$

The boundedness of (u_n) in $W^{1,\Phi}(\mathbb{R}^N)$ implies that

$$\begin{aligned} \int_{B_{2R}(0)} \widetilde{\Phi}\left(\phi(|u_n(x)|)u_n(x)\right)\ \mathrm{d}x \le &\int_{B_{2R}(0)} \Phi(2|u_n(x)|) \\ \times \mathrm{d}x \le K &\int_{B_{2R}(0)} \Phi(|u_n(x)|)\ \mathrm{d}x \le \widetilde{C}, \end{aligned} \tag{4.6}$$

where $\widetilde{C}$ is a constant that does not depend on (u_n). Since

$$\begin{aligned} &\|\phi(|\nabla u_n|)u_n\|^{\frac{m}{m-1}}_{L^{\tilde{\Phi}}(B_{2R}(0))} - 1 \le \min(\|\phi(|\nabla u_n|)u_n\|^{\frac{l}{l-1}}_{L^{\tilde{\Phi}}(B_{2R}(0))}, \\ &\|\phi(|\nabla u_n|)u_n\|^{\frac{m}{m-1}}_{L^{\tilde{\Phi}}(B_{2R}(0))}), \end{aligned}$$

it follows from (4.6) that the sequence $(\phi(|u_n|)|u_n|)$ is bounded in $L^{\widetilde{\Phi}}(B_{2R}(0))$. A similar reasoning implies that $(\phi(|\nabla u_n|)|\nabla u_n|)$ is bounded in $L^{\widetilde{\Phi}}(B_{2R}(0))$.

The compact embedding $W^{1,\Phi}(B_{2R}(0)) \hookrightarrow L^{\Phi}(B_{2R}(0))$ implies that $\|u_n - u\|_{L^\Phi(B_{2R}(0))} \to 0$ for a subsequence. It follows that, up to a subsequence, we have

$$\begin{aligned} &\left|\int_{B_{2R}(0)} \phi(|u_n(x)|)u_n(x)(u_n-u)(x)\xi(x)\ \mathrm{d}x\right| \\ &\le \int_{B_{2R}(0)} \phi(|u_n(x)|)|u_n(x)||(u_n-u)(x)|\ \mathrm{d}x \\ &\le C \parallel \phi(|u_n|)|u_n| \parallel_{L^{\tilde{\Phi}}(B_{2R}(0))} \parallel u_n - u \parallel_{L^\Phi(B_{2R}(0))} \\ &\to 0. \end{aligned} \tag{4.7}$$

A similar reasoning implies that

$$\int_{B_{2R}(0)} \phi(|\nabla u_n(x)|)(u_n-u)(x)\nabla u_n(x)\nabla\xi(x)\ \mathrm{d}x \to 0, \tag{4.8}$$

for some subsequence. Standard arguments imply, for a subsequence, that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))(u_n-u)(y)\xi(y)}{|x-y|^\lambda}\ \mathrm{d}x\mathrm{d}y \to 0. \tag{4.9}$$

Note that

$$\begin{aligned}\int_{B_{2R}(0)} \widetilde{\Phi}\left(\xi(x)\phi(|\nabla u(x)|)\left|\frac{\partial u(x)}{\partial x_i}\right|\right)\, \mathrm{d}x &\leq C\int_{B_{2R}(0)} \widetilde{\Phi}\left(\phi(|\nabla u(x)|)\left|\frac{\partial u(x)}{\partial x_i}\right|\right)\, \mathrm{d}x\\ &\leq \tilde{C}\int_{B_{2R}(0)} \Phi(2|\nabla u(x)|)\, \mathrm{d}x\\ &\leq \tilde{K}\int_{B_{2R}(0)} \Phi(|\nabla u(x)|)\, \mathrm{d}x.\end{aligned}$$

Thus from Hölder inequality, it follows that the function

$$L_i(v) := \int_{B_{2R}(0)} \xi(x)\phi(|\nabla u(x)|)\frac{\partial u(x)}{\partial x_i}v(x)\, \mathrm{d}x, i = 1, \ldots, N,$$

with $v \in L^{\Phi}(B_{2R}(0))$ defines a linear continuous functional. Since $\dfrac{\partial u_n}{\partial x_i} \rightharpoonup \dfrac{\partial u}{\partial x_i}$ in $L^{\Phi}(B_{2R}(0))$, we get

$$\int_{B_{2R}(0)} \phi(|\nabla u(x)|)\nabla u(x)(\nabla u_n(x) - \nabla u(x))\xi(x)\, \mathrm{d}x \to 0. \tag{4.10}$$

From (4.4), (4.5), (4.7), (4.8) (4.9), (4.10) we obtain that

$$\int_{B_{2R}(0)} \phi(|\nabla u_n(x)|)\nabla u_n(x)(\nabla u_n(x) - \nabla u(x))\xi(x)\, \mathrm{d}x \to 0.$$

Therefore

$$\int_{B_R(0)} \phi(|\nabla u_n(x)|)\nabla u_n(x)(\nabla u_n(x) - \nabla u(x))\, \mathrm{d}x \to 0.$$

Applying a result found in Dal Maso and Murat [40], it follows that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e in } \quad B_R(0),$$

for each $R > 0$. Since R is arbitrary, there is a subsequence of (u_n), still denoted by itself, such that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e in } \mathbb{R}^N.$$

ii) We have $\phi(|\nabla u_n(x)|)\nabla u_n(x) \to \phi(|\nabla u(x)|)\nabla u(x)$ a.e in $\mathbb{R}^N$. Note that

$$\widetilde{\Phi}(\phi(|\nabla u_n(x)|)|\nabla u_n(x)|) \leq \tilde{C}\Phi(2|\nabla u_n(x)|) \leq \tilde{K}\Phi(|\nabla u_n(x)|).$$

Then the boundedness of (u_n) in $W^{1,\Phi}(\mathbb{R}^N)$ implies that the sequences $\left(\phi(|\nabla u_n|)\dfrac{\partial u_n}{\partial x_i}\right)$, $i = 1, \ldots, N$ are bounded in $L^{\widehat{\Phi}}(\mathbb{R}^N)$. By [41], it follows that

$$\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i} \rightharpoonup \phi(|\nabla u|)\frac{\partial u}{\partial x_i} \quad \text{in} \quad L^{\widetilde{\Phi}}(\mathbb{R}^N).$$

A similar reasoning implies (iii). $\square$

Now, we are ready to prove that u is a critical point of J.

Lemma 4.5. *The function u is a critical point of J, that is, $J^{'}(u) = 0$.*

Proof. First of all, we claim that

$$J^{'}(u_n)v \to J^{'}(u)v, \quad \forall v \in C_0^\infty(\mathbb{R}^N).$$

To verify such limit, note that

$$\begin{aligned} J^{'}(u_n)v = &\int_{\mathbb{R}^N} \phi(|\nabla u_n(x)|)\nabla u_n(x)\nabla v(x) + \phi(|u_n|)u_n(x)v(x) \, \mathrm{d}x \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))v(y)}{|x-y|^{\lambda(x,y)}} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

By Lemmas 4.3 and 4.4,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))v(y)}{|x-y|^{\lambda(x,y)}} \mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))f(u(y))v(y)}{|x-y|^{\lambda(x,y)}} \mathrm{d}x\mathrm{d}y, \tag{4.11}$$

$$\int_{\mathbb{R}^N} \phi(|\nabla u_n(x)|)\nabla u_n(x)\nabla v(x)\mathrm{d}x \to \int_{\mathbb{R}^N} \phi(|\nabla u(x)|)\nabla u(x)\nabla v(x)\mathrm{d}x, \tag{4.12}$$

and

$$\int_{\mathbb{R}^N} \phi(|u_n(x)|)u_n(x)v(x)\mathrm{d}x \to \int_{\mathbb{R}^N} \phi(|u(x)|)u(x)v(x)\mathrm{d}x. \tag{4.13}$$

From the relations (4.11), (4.12) and (4.13) we have the claim. Since $J'(u_n)v \to 0$, the claim ensures that $J^{'}(u)v = 0$, for all $v \in C_0^\infty(\mathbb{R}^N)$. Now, the lemma follows using the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,\Phi}(\mathbb{R}^N)$. □

4.1. Proof of Theorem 4.1

If $u \neq 0$, then u is a nontrivial solution and the theorem is proved. If $u = 0$, we must find another solution $v \in W^{1,\Phi}(\mathbb{R}^N) \setminus \{0\}$ for the equation (1.3). For such purpose, the claim below is crucial in our argument.

Claim 4.1. *There exist $r > 0, \beta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that*

$$\liminf_{n\to+\infty} \int_{B_r(y_n)} \Phi(|u_n(x)|)\,dx \geq \beta > 0.$$

Proof. In fact, if the above claim does not hold, using Theorem 2.2, we derive the limit

$$\int_{\mathbb{R}^N} P(|u_n|)\,\mathrm{d}x \to 0, \tag{4.14}$$

for any N-function P satisfying $(P_1) - (P_2)$. Applying Proposition 1.1,

$$\begin{aligned} &\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))u_n(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y\right| \\ &\leq C\|F(u_n(x))\|_{L^s(\mathbb{R}^N)}\|f(u_n(y))u_n(y)\|_{L^s(\mathbb{R}^N)} \end{aligned}$$

By (f_1), (F), (E) and Theorem 2.2,

$$\int_{\mathbb{R}^N} |F(u_n(x))|^s \mathrm{d}x \to 0$$

and

$$\int_{\mathbb{R}^N} |f(u_n(y))u_n(y)|^s \mathrm{d}y \to 0.$$

Therefore

$$\int_{\mathbb{R}^N} \frac{F(u_n(x))f(u_n(y))u_n(y)}{|x-y|^\lambda} \mathrm{d}x\mathrm{d}y \to 0.$$

The above limit together with the fact that $J'(u_n)u_n = o_n(1)$ give

$$\int_{\mathbb{R}^N} \phi(|\nabla u_n(x)|)|\nabla u_n(x)|^2 + \phi(|u_n(x)|)|\nabla u_n(x)|^2 \,\mathrm{d}x \to 0.$$

From (ϕ_3),

$$\int_{\mathbb{R}^N} \Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|) \,\mathrm{d}x \to 0.$$

This limit leads to $J(u_n) \to 0$, which contradicts the limit $J(u_n) \to d > 0$. $\square$

Using standard arguments, we can assume in Claim 4.1 that $(y_n) \subset \mathbb{Z}^N$. By setting $v_n(x) = u_n(x + y_n)$, it follows that

$$J(v_n) = J(u_n), \|J'(v_n)\| = \|J'(u_n)\| \quad \text{and} \quad \|u_n\|_{1,\Phi} = \|v_n\|_{1,\Phi} \quad \forall n \in \mathbb{N}.$$

From the above information, we have that $J(v_n) \to d$ and $J^{'}(v_n) \to 0$. Since (v_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$, up to a subsequence, $v_n \to v$ in $L^\Phi(B_r(0))$, for some $v \in W^{1,\Phi}$. To verify that $v \neq 0$, note that by Claim 4.1, we have for some subsequence

$$\begin{aligned} 0 < \beta &\leq \lim_{n\to+\infty} \int_{B_r(y_n)} \Phi(|u_n(x)|)\mathrm{d}x \\ &= \lim_{n\to+\infty} \int_{B_r(0)} \Phi(|v_n(x)|)\mathrm{d}x = \int_{B_r(0)} \Phi(|v(x)|)\mathrm{d}x. \end{aligned}$$

Applying the same arguments as in the proofs of Lemmas 4.3, 4.4 and 4.5 for the sequence (v_n) we obtain the desired result.

5. Final Comments

The same arguments used in this paper can be applied to study the existence of solutions for related problems of the following type:

$$\begin{cases} -\Delta_\Phi u + V(x)\phi(|u|)u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F(u(x))}{|x-y|^\lambda}\right) f(u(y)), \text{ in } \mathbb{R}^N, \\ u \in W^{1,\Phi}(\mathbb{R}^N). \end{cases} \tag{5.1}$$

The potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous functions with $\inf_{x\in\mathbb{R}^N} V(x) > 0$ that belongs to one of the following classes:

Class 1: V is periodic: V is a $\mathbb{Z}^N$-periodic function, that is,

$$V(x + y) = V(x), \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad y \in \mathbb{Z}^N.$$

Class 2: V is asymptotic periodic function: There is a $\mathbb{Z}^N$-periodic function $V_p : \mathbb{R}^N \to \mathbb{R}$ such that

$$V(x) < V_p(x), \quad \forall x \in \mathbb{R}^N$$

and

$$|V(x) - V_p(x)| \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Class 3: V is coercive V is a coercive function, that is,

$$V(x) \to +\infty \quad \text{as} \quad |x| \to +\infty.$$

Class 4: V is a Bartsch–Wang-like potential: The potential V verifies the following property

$$\text{med}\left(\{x \in \mathbb{R}^N \,:\, V(x) \le M\}\right) < +\infty, \quad \text{for all } M > 0.$$

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