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Time-Space Fractional Diffusion Problems: Existence, Decay Estimates and Blow-Up of Solutions

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Abstract. The aim of this paper is to study the following time-space fractional diffusion problem

$$\begin{cases} \partial_t^\beta u + (-\Delta)^\alpha u + (-\Delta)^\alpha \partial_t^\beta u = \lambda f(x,u) + g(x,t) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \backslash \Omega) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $(-\Delta)^\alpha$ is the fractional Laplace operator with $0 < \alpha < 1$, ∂_t^β is the Riemann-Liouville time fractional derivative with $0 < \beta < 1$, λ is a positive parameter, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $g \in L^2(0, \infty; L^2(\Omega))$. Under natural assumptions, the global and local existence of solutions are obtained by applying the Galerkin method. Then, by virtue of a differential inequality technique, we give a decay estimate of solutions. Moreover, the blow-up property of solutions is also investigated.

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1. Introduction and the Main Results

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary. We consider the following time-space fractional problem

$$\begin{cases} \partial_t^\beta u + (-\Delta)^\alpha u + (-\Delta)^\alpha \partial_t^\beta u = \lambda f(x,u) + g(x,t) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \backslash \Omega) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $0<\beta,\alpha<1$, $N>2\alpha$, $\lambda\neq 0$ is a given parameter and $g\in L^2(0,\infty;L^2(\Omega))$ is a given function. We assume that the reaction f is a continuous function on $\Omega\times[0,\infty)$ and $f(x,\xi)=0$ for all $x\in\Omega$ and $\xi\leq 0$. Moreover, for all $\xi>0$, f satisfies

(f_1) $f(x,\xi)=|\xi|^{p-2}\xi$ and $2<p\leq 2^*_\alpha:=\frac{2N}{N-2\alpha}$,
or the following Lipschitz condition:

(f_2) there exists $\mathcal{L}>0$ such that

$$|f(x,\xi_1)-f(x,\xi_2)|\leq\mathcal{L}|\xi_1-\xi_2|\quad\text{for all }\xi_1,\xi_2\in\mathbb{R}\quad\text{and }x\in\Omega.$$

Here, the order β of Riemann-Liouville fractional operator ∂_t^β is defined by

$$\partial_t^\beta u=\partial_t(J^{1-\beta}(u-u(0))),$$

where $J^{1-\beta}$ denotes the $1-\beta$ order Riemann-Liouville fractional integral operator and it is given by

$$J^{1-\beta}(u-u(0))=\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}(u(\tau)-u(0))d\tau.$$

Here Γ is the usual Gamma function. The fractional Laplace operator $(-\Delta)^\alpha$, up to a normalization constant, is defined by

$$(-\Delta)^\alpha\varphi(x)=2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2\alpha}}\,dy,\quad x\in\mathbb{R}^N$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Here, $B_\varepsilon(x)=\{y\in\mathbb{R}^N:|y-x|<\varepsilon\}$. For more properties related to the fractional Laplacian and fractional Sobolev spaces as well as for applications of variational methods to fractional problems, we refer to [3].

The fractional operators and related differential equations have important applications in many areas such as physics [15], mechanics chemistry, population dynamic [4,5], anomalous diffusion [29] and so on. Time fractional differential equations can be used to describe some problems with memory effects. Moreover, both time and space fractional differential equations have been exploited for anomalous diffusion or dispersion where particles spread at a rate inconsistent with Brown motion, see [9]. In the case of time fractional derivatives, particles with "memory effect" propagates slowly, which we call anomalous subdiffusion. Different from the former, spatial fractional diffusion equations are used to describe macroscopic transport and usually result in superdiffusion phenomenon. So far, the works on problems involving the fractional Laplacian and its variants are quite large, here we just list a few, see [8,10,20,21,31–34] and the references cited there.

To the best of our knowledge, until recently there has been still very little works on deal with the existence, decay estimates and blow-up of solutions for time-space fractional problems like (1.1). In [30], Vergara and Zacher considered the following time fractional diffusion problem

$$\begin{cases}\partial_t(k*(u-u_0))-\operatorname{div}(A(x,t)Du)=0,\quad t>0,\ x\in\Omega,\\ u=0\quad t>0.x\in\partial\Omega,\end{cases}$$

where $k*(u-u_0)=\int_0^t k(t-\tau)(u(\tau)-u_0)d\tau$ and $k\in L_{1,loc}(\mathbb{R}_+)$. Some useful fundamental identities were obtained. Based on these identities, the existence and decay estimates of weak solutions were obtained. In particular, the decay estimates of weak solutions were given by using the sub-supersolution method. In [17], Li et al. studied the following time-space fractional Keller-Segel equation

$$ {}_0^C D_t^\beta \rho + (-\Delta)^{\frac{\alpha}{2}} \rho + \nabla \cdot (\rho B(\rho)) = 0, $$

where ${}_0^C D_t^\beta$ denotes the Caputo derivative, $B(\rho) = -s_{n,\gamma} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-\gamma+2}} \rho(y) dy$ is the Riesz potential with a singular kernel. The authors obtained the existence and uniqueness of mild solutions. Moreover, the authors discussed the properties of the mild solutions, such as mass conservation and blow-up behaviors. In [1], Bekkai et al. studied the following Cauchy problem involving the Caputo derivative and the fractional Laplacian

$$ \begin{cases} {}_0^C D_t^\alpha u + (-\Delta)^{\frac{\beta}{2}} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{u(s)} ds, & x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2} $$

$N \geq 1$, $0 < \alpha < 1$, $0 < \beta \leq 2$. First the existence of mild solutions of (1.2) was obatined by the Banach contraction mapping principle. Then the authors proved that the mild solution is also the weak solution. Furthermore, the authors showed the local weak solutions blow up in finite time by choosing suitable test function. See also [6,23,36,39] for similar discussions of the blow-up properties of solutions. Very recently, Fu and Zhang [11] considered the following time-space fractional Kirchhoff problem

$$ \begin{cases} \partial_t^\beta u + M(\|u\|^2_{H_0^\alpha(\Omega)})(-\Delta)^\alpha u = \gamma |u|^\rho u + g(x,t) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \backslash \Omega) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} $$

where $M : [0,\infty) \to [0,\infty)$ is a continuous function. Under suitable assumptions, the authors obtained the global existence of solutions by using the Galerkin method. Furthermore, a decay estimate of solutions was established.

On the other hand, when α, β and s limit to 1, the Eq. (1.1) reduces to the following equation

$$ \partial_t u - \Delta u - \Delta \partial_t u = \lambda |u|^{p-2} u + g(x,t), \quad \partial_t u = \frac{\partial u}{\partial t}, \tag{1.3} $$

which is called pseudo-parabolic equation. Equations like (1.3) can be used to describe many important physical processes, such as unidirectional propagation of nonlinear, long waves [2,27], the aggregation of population [25] and semiconductors [14]. The study of Eq. (1.3) received much more attention in the past years, see [12,18,35].

Recently, Tuan et al. [28] studied the initial boundary value problem and Cauchy problem of Caputo time-fractional pseudo-parabolic equations

$$ \mathbb{D}_t^\alpha (u - m\Delta u) + (-\Delta)^\sigma u = \mathcal{N}(u), \tag{1.4} $$

where $\mathbb{D}_t^\alpha$ denotes the Caputo time fractional derivative. The local well-posedness of Eq. (1.4) was established. Further, the finite time blow-up of solutions was also obtained. In [24], Nguyen et al. considered a class of pseudoparabolic equations with the nonlocal condition and the Caputo derivative and obtained the existence and uniqueness of the mild solution. In [7], Chaoui and Rezgui dealt with a time fractional pseudoparabolic equation with fractional integral condition. By the Rothe time discretization scheme, the existence of weak solution was obtained. Moreover, the uniqueness of weak solution as well as some regularity results were obtained.

Inspired by the above papers, we discuss in this work the existence, uniqueness, decay estimates of weak solutions and solutions that blow up in finite time for problem (1.1) involving the time-space fractional operators. Since our problem is nonlocal, our discussion is more elaborate than the papers in the literature. Comparing with the papers in the literature, the main feature of this paper is that the problem (1.1) contains the Riemann-Liouville time fractional derivative and the fractional Laplacian. Definitely, this paper is the first time to deal with the local existence and global nonexistence of solutions for problems involving the fractional Laplacian and the Riemann-Liouville time fractional derivative.

Definition 1.1. We say that $u \in L^\infty(0,T;H_0^\alpha(\Omega))$ with $\partial_t^\beta u \in L^2(0,T;L^2(\Omega))$ is a weak subsolution (supsolution) of problem (1.1) if $u(x,0) \le (\ge) u_0(x)$ and

$$\begin{aligned}\int_\Omega \varphi \partial_t^\beta u dx &+ \int_{\mathbb{R}^{2N}} \frac{(u(x,t)-u(y,t))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}} dxdy \\ &+ \int_{\mathbb{R}^{2N}} \frac{(\partial_t^\beta u(x,t)-\partial_t^\beta u(y,t))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}} dxdy \\ &\le (\ge) \lambda \int_\Omega f(x,u)\varphi dx + \int_\Omega g\varphi dx\end{aligned}$$

for any $0 \le \varphi \in H_0^\alpha(\Omega)$ and a.e. $t \in (0,T)$. u is a weak solution if and only u is both a subsolution and a supsolution. Here, we call u is a global weak solution of problem (1.1), if the equality in above holds for any $0 < T < \infty$; u is a local weak solution, if there exists $T_0 > 0$ such that the equality in Definition 1.1 holds for $0 < T \le T_0$.

The proof of the following existence results relies on the contract mapping theorem and the Galerkin method.

Theorem 1.2. *Assume that $0 \le u_0 \in H_0^\alpha(\Omega)$, $g \in L^2(0,\infty;L^2(\Omega))$ and $f(x,\xi) = 0$ for all $x \in \Omega$ and $\xi \le 0$. If f satisfies (f_1), then problem (1.1) admits a local nonnegative weak solution. If f satisfies (f_2), then problem (1.1) has a unique global weak solution.*

The following theorem shows the asymptotic behavior of global solutions to problem (1.1).

Theorem 1.3. *Assume that $g \equiv 0$ and $f(x,\xi) = 0$ for all $x \in \Omega$ and $\xi \le 0$. If $0 \le u_0 \in H_0^\alpha(\Omega)$ and $u_0(x) \le \eta_0 \varphi_1(x)$ with $\eta_0 > 0$ for all $x \in \Omega$, and f satisfies (f_2), then the unique solution of problem (1.1) satisfies the following decay estimates*

$$0 \le u(x,t) \le \frac{c_2 \varphi_1(x)}{1+t^\beta} \quad \text{for all } t \ge 0 \quad \text{and } x \in \Omega,$$

where $c_2 > 0$ and $\varphi_1 > 0$ is the eigenfunction corresponding to the first eigenvalue of the fractional Laplacian.

We also discuss the global nonexistence of local solutions for problem (1.1).

Theorem 1.4. *Assume that* $0 \leq u_0 \in H_0^\alpha(\Omega)$, $g = 0$ *and* $f(x, \xi) = 0$ *for all* $x \in \Omega$ *and* $\xi \leq 0$. *Suppose that* f *satisfies* (f_1), *and* $\int_\Omega u_0(x)\varphi_1(x)dx > \left(\frac{\lambda_1}{\lambda}\right)^{1/(p-2)}$, *where* $c_2 > 0$ *and* $\varphi_1 > 0$ *is the eigenfunction corresponding to the first eigenvalue* λ_1. *Then the nonnegative weak solutions of problem* (1.1) *blow up in finite time.*

In what follows, the letters c, c_i, C, C_i, $i = 1, 2, \ldots$, denote positive constants which vary from line to line, but are independent of terms that take part in any limit process. Furthermore, for any $p \geq 1$ we denote $\|u\|_p = \|u\|_{L^p(\Omega)}$.

2. Preliminaries

In this section, we provide some basic results which will be used in the next sections. The fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined as

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2\alpha}} dxdy < \infty \right\}.$$

endowed with the norm

$$\|u\|^2_{H^\alpha(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2\alpha}} dxdy + \|u\|^2_{L^2(\mathbb{R}^N)}.$$

$H_0^\alpha(\Omega)$ is defined as

$$H_0^\alpha(\Omega) = \{u \in H^\alpha(\mathbb{R}^N) : u = 0 \ a.e. \text{ in } \mathbb{R}^N \backslash \Omega\}.$$

in the sequel, we take

$$\|u\|^2_{H_0^\alpha(\Omega)} = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2\alpha}} dxdy.$$

$H_0^\alpha(\Omega)$ is a Hilbert space in which a scalar product is given by

$$\langle u, v \rangle_\alpha = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x)) - v(y))}{|x - y|^{N+2\alpha}} dxdy$$

for any $u, v \in H_0^\alpha(\Omega)$.

Denote by

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots < +\infty$$

the distinct eigenvalues of the fractional Laplace operator and let ω_k be the eigenfunction corresponding to λ_k of the following eigenvalue problem

$$\begin{cases} (-\Delta)^\alpha u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

We obtain for $k \in \mathbb{N}$,

$$\lambda_k = \min_{u \in P_k \backslash \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}} dxdy}{\int_\Omega |u(x)|^2 dx},$$

where $P_1 = H_0^\alpha(\Omega)$ and

$$P_k = \left\{ u \in H_0^\alpha(\Omega) : (u, w_k)_{H_0^\alpha(\Omega)} = 0, \forall j = 1, 2, ..., k - 1 \right\}, \ k \geq 2.$$

Lemma 2.1 ([3]). *Let $2_\alpha^* = \frac{2N}{N-2\alpha}$. For any $q \in [1, 2_\alpha^*]$, the embedding $H_0^\alpha(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. Furthermore, the embedding is compact if $q \in [1, 2_\alpha^*)$.*

The Yosida approximation of the time-fractional derivative operator is an useful tool to deal with problems with Caputo fractional derivative operators. For more details, we refer to [30,37,38]. Let $1 \le p < \infty, 0 < \beta < 1$ and X be a real Banach space. Define fractional derivative operator

$$Bu = \frac{d}{dt}(g_{1-\beta} * u), \qquad D(B) = \{u \in L^p([0,T];X) : g_{1-\beta} * u \in W_0^{1,p}([0,T];X)\},$$

where $g_{1-\beta}$ is given by

$$g_{1-\beta}(t) = \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

Its Yosida approximation B_n defined by $B_n = nB(n+B)^{-1}$ $(n \in \mathbb{N})$ possesses the property that for any $u \in D(B)$, $B_n u \to Bu$ strongly in $L^p([0,T];X)$ as $n \to \infty$. Here, we collect some important properties of $g_{1-\beta}$ and B_n which are listed in the following:

- The kernel $g_{1-\beta,n}$ is nonnegative and nonincreasing for all $n \in N$, and $g_{1-\beta,n} \in W^{1,1}([0,T])$.
- $g_{1-\beta,n} \to g_{1-\beta}$ in $L^1([0,T])$ and $B_n u \to Bu$ in $L^p([0,T];X)$ as $n \to \infty$.

Lemma 2.2 ([30]). *Assume that H is a real Hilbert space and $T > 0$ is a real number. Then for any $k \in W^{1,1}([0,T])$ and $u \in L^2(0,T;H)$, the following identity holds*

$$\begin{aligned}\left(\frac{d}{dt}(k * u)(t), u(t)\right)_H &= \frac{1}{2}\frac{d}{dt}(k * |u(\cdot)|_H^2)(t) + \frac{1}{2}k(t)|u(t)|_H^2 \\ &\quad + \frac{1}{2}\int_0^t [-\dot{k}(s)]|u(t) - u(t-s)|_H^2 ds, \quad \text{a.e. } t \in (0,T).\end{aligned}$$

Remark 1. Obviously, if k is a nonincreasing and nonnegative function in $W^{1,1}([0,T])$, then we obtain for any $u \in L^2(0,T;H)$ that

$$\left(\frac{d}{dt}(k * u)(t), u(t)\right)_H \ge \frac{1}{2}\frac{d}{dt}(k * |u(\cdot)|_H^2)(t), \quad \text{a.e. } t \in (0,T).$$

Lemma 2.3 ([30]). *Let $H \in C^1(\mathbb{R})$ and $k \in W^{1,1}([0,T])$, for a sufficiently smooth function u, then there holds for a.e. $t \in (0,T)$*

$$\begin{aligned}\dot{H}(u(t))\frac{d}{dt}(k * u)(t) &= \frac{d}{dt}(k * H(u))(t) + [-H(u(t)) + \dot{H}(u(t))u(t)]k(t) \\ &\quad + \int_0^t [H(u(t-s)) - H(u(t)) \\ &\quad - \dot{H}(u(t))(u(t) - u(t-s))][-\dot{k}(s)]ds\end{aligned}$$

Definition 2.4 (see [19]). Let $q > 0$ be a real number and $0 < T \le \infty$. We say that a function $\omega : \mathbb{R}^+ \to \mathbb{R}$ $(\mathbb{R}^+ = [0,\infty))$ satisfies a condition (q), if

$$e^{-qt}[\omega(u)]^q \le R(t)\omega(e^{-qt}u^q) \ \text{ for all } u \in \mathbb{R}^+, \ t \in [0,T),$$

where $R(t)$ is a continuous and nonnegative function.

Clearly, if $\omega(u) = u^r, r > 0$, then ω satisfies the condition (q) with any $q > 1$, and $R(t) = e^{(r-1)qt}$.

Lemma 2.5 (see [19, Theorem 1]). *Let a be a nondecreasing, nonnegative C^1-function on $[0,T)$, F be a continuous, nonnegative function on $[0,T)$, $\omega : \mathbb{R}^+ \to \mathbb{R}$ be a continuous, nondecreasing function, $\omega(0) = 0, \omega(u) > 0$ on $[0,T)$, and u be a continuous, nonnegative function on $[0,T)$ with*

$$u(t) \le a(t) + \int_0^t (t-\tau)^{\beta-1} F(\tau)\omega(u(\tau))d\tau, \quad t \in [0,T),$$

where $\beta > 0$. Then the following assertions hold:
(i) *Suppose that $\beta > 1/2$ and ω satisfies the condition (q) with $q = 2$. Then*

$$u(t) \le e^t \{\Omega^{-1}[\Omega(2a(t)^2) + g_1(t)]\}^{1/2}, \quad t \in [0,T_1],$$

where

$$g_1(t) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}} \int_0^t R(\tau)F^2(\tau)d\tau,$$

Γ *is the gamma function,* $\Omega(v) = \int_{v_0}^v \frac{dy}{\omega(y)}$, $v_0 > 0$, Ω^{-1} *is the inverse of* Ω, *and* $T_1 \in \mathbb{R}^+$ *is such that* $\Omega(2a(t)^2) + g_1(t) \in Dom(\Omega^{-1})$ *for all* $t \in [0,T_1]$.
(ii) *Let $\beta \in (0,1/2]$ and ω satisfies the condition (q) with $q = z+2$, where $z = \frac{1-\beta}{\beta}$. Then*

$$u(t) \le e^t \{\Omega^{-1}\left[\Omega(2^{q-1}a(t)^q) + g_2(t)\right]\}^{1/q}, \quad t \in [0,T_1],$$

where

$$g_2(t) = 2^{q-1}K_z^q \int_0^t F(\tau)^q R(\tau)d\tau, \quad K_z = \left[\frac{\Gamma(1-\gamma p)}{p^{1-\gamma p}}\right]^{1/p},$$
$$\gamma = \frac{z}{z+1}, \ p = \frac{z+2}{z+1},$$

$T_1 \in \mathbb{R}^+$ *is such that* $\Omega(2^{q-1}a(t)^q) + g_2(t) \in Dom(\Omega^{-1})$ *for all* $t \in [0,T_1]$.

In particular, if $w(u) = u$, then there holds

Lemma 2.6 (see [19, Theorem 2]). *Let $0 < T \le \infty$, $a(t), F(t)$ be as in Lemma 2.5, and let $u(t)$ be a continuous, nonnegative function on $[0,T)$ with*

$$u(t) \le a(t) + \int_0^t (t-\tau)^{\beta-1} F(\tau)u(\tau)d\tau,$$

where $\beta > 0$. Then the following assertions hold
(i) *If $\beta > 1/2$, then*

$$u(t) \le \sqrt{2}a(t) \exp\left(\frac{2\Gamma(2\beta-1)}{4^\beta} \int_0^t F(\tau)^2 d\tau + t\right), \quad t \in [0,T).$$

(ii) *If $\beta = \frac{1}{z+1}$ for some $z \ge 1$, then*

$$u(t) \le (2^{q-1})^{1/q} a(t) \exp\left(\frac{2^{q-1}}{q} K_z \int_0^t F(\tau)^q d\tau + t\right), \quad t \in [0,T),$$

where K_z is defined by $K_z = [\frac{\Gamma(1-\gamma p)}{p^{1-\gamma p}}]^{1/p}$, $p = \frac{z+2}{z+1}$, $q = z+2$.

Lemma 2.7. *Define an operator*

$$I_{1-\beta}(u)=\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}u(\tau)d\tau$$

for any $u\in L^2(0,T;L^2(\Omega))$. *If* $\beta\geq 1/2$, *then* $I_{1-\beta}:L^2(0,T;L^2(\Omega))\rightarrow L^2(0,T;L^2(\Omega))$ *is a bounded linear operator; If* $0<\beta<1/2$, *then* $I_{1-\beta}:L^r(0,T;L^r(\Omega))\rightarrow L^r(0,T;L^r(\Omega))$ *is a bounded linear operator, where* $r>\frac{1}{\beta}$.

Proof. If $\beta\geq\frac{1}{2}$, then the Hölder inequality implies that

$$\begin{aligned}\left(\int_0^t(t-\tau)^{\beta-1}u(\tau)d\tau\right)^2&\leq\int_0^t(t-\tau)^{2(\beta-1)}d\tau\int_0^t u^2(\tau)d\tau\\&=\frac{1}{2\beta-1}t^{2\beta-1}\int_0^t u^2(\tau)d\tau.\end{aligned}$$

Thus,

$$\int_0^T\int_\Omega(I_{1-\beta}(u))^2dxdt\leq C\int_0^T\int_\Omega u^2dxdt,$$

which yields the desired result.

Now we consider the case $0<\beta<1/2$. By the Hölder inequality, we have

$$\begin{aligned}&\int_0^t(t-\tau)^{\beta-1}u(\tau)d\tau\\&=\int_0^t(t-\tau)^{\beta-1}e^{\tau}e^{-\tau}u(\tau)d\tau\\&\leq\left(\int_0^t(t-\tau)^{q(\beta-1)}e^{q\tau}d\tau\right)^{\frac{1}{q}}\left(\int_0^t u^{q'}(\tau)e^{-q'\tau}d\tau\right)^{\frac{1}{q'}}\\&=\left(\frac{e^{qt}}{q^{1-(1-\beta)q}}\int_0^t\tau^{-(1-\beta)q}e^{-\tau}d\tau\right)^{1/q}\left(\int_0^t u^{q'}(\tau)e^{-q'\tau}d\tau\right)^{\frac{1}{q'}}\\&\leq\left(\frac{e^{qt}}{q^{1-(1-\beta)q}}\Gamma(1-(1-\beta)q)\right)^{1/q}\left(\int_0^t u^{q'}(\tau)d\tau\right)^{\frac{1}{q'}},\end{aligned}$$

where $q>1$ satisfying $1-(1-\beta)q>0$, and $\frac{1}{q}+\frac{1}{q'}=1$. Observe that $1-(1-\beta)q>0$. Thus, we get

$$\int_0^T\int_\Omega(I_{1-\beta}(u))^{q'}dxdt\leq C\int_0^T\int_\Omega|u|^{q'}dxdt,$$

which ends the proof. □

Lemma 2.8 (see [26] Fractional integration by parts). *Let* $\alpha>0$, $p\geq 1$, $q\geq 1$ *and* $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ ($p\neq 1$, $q\neq 1$ *in the case when* $\frac{1}{p}+\frac{1}{q}=1+\alpha$). *If* $\varphi\in L^p(a,b)$ *and* $\psi\in L^q(a,b)$, *then*

$$\int_a^b\varphi(x)(I^\alpha_{a+}\psi)(x)dx=\int_a^b\psi(x)(I^\alpha_{b-}\varphi)(x)dx,$$

where

$$(I_{a+}^{\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\alpha}}dt$$

and

$$(I_{b-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}}dt.$$

Using Lemma 2.8, we can obtain the following result.

Lemma 2.9. *Let $\alpha > 0$ and $1 \leq p$. Assume that $\varphi \in C_0^1(0,T)$ and $\psi \in L^p(0,T)$. Then*

$$\int_0^T \varphi(x)\partial_x^{\alpha}\psi(x)dx = -\frac{1}{\Gamma(\alpha)}\int_0^T\int_x^T \frac{\varphi'(t)}{(t-x)^{1-\alpha}}dt(\psi(x)-\psi(0))dx.$$

In order to show the existence of solutions to problem (1.1), we give some properties of the operator $L : H_0^{\alpha}(\Omega) \to (H_0^{\alpha}(\Omega))'$ defined by

$$\langle L(u), v\rangle = \langle u, v\rangle_{\alpha} = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}dxdy,$$

for all $u, v \in H_0^{\alpha}(\Omega)$.

Lemma 2.10. *The operator $L : H_0^{\alpha}(\Omega) \to (H_0^{\alpha}(\Omega))'$ is a monotone and linear bounded functional. Moreover, $\|L(u)\|_{(H_0^{\alpha}(\Omega))'} \leq [u]_{\alpha}$ for all $u \in H_0^{\alpha}(\Omega)$.*

Proof. Let $u, v \in H_0^{\alpha}(\Omega)$, we have

$$\langle L(u) - L(v), u - v\rangle = [u-v]_{\alpha}^2 \geq 0.$$

Thus, L is monotone. Clearly, L is a linear functional. It remains to show that $\|Lu\|_{(H_0^{\alpha}(\Omega))'} \leq [u]_{\alpha}$. It follows from the Hölder inequality, we have

$$\langle L(u), v\rangle \leq [u]_{\alpha}[v]_{\alpha},$$

which means that $\|L(u)\|_{(H_0^{\alpha}(\Omega))'} \leq [u]_{\alpha}$. The proof is now complete. □

Lemma 2.11. *The operator $L : H_0^{\alpha}(\Omega) \to (H_0^{\alpha}(\Omega))'$ is hemicontinuous.*

Proof. We are going to prove that the map $t \mapsto \langle L(u+tv), w\rangle$ is continuous on $[0,1]$ for all $u, v, w \in H_0^{\alpha}(\Omega)$, i.e,

$$\lim_{t\to 0}\langle L(u+tv), w\rangle = \langle L(u), w\rangle$$

for all $w \in H_0^{\alpha}(\Omega)$. We have,

$$\langle L(u+tv), w\rangle = \langle u+tv, w\rangle_{\alpha}.$$

We define $G_t : [0,1] \to \mathbb{R}$ by

$$G_t(x,y) = \frac{((u+tv)(x) - (u+tv)(y))}{|x-y|^{N+2\alpha}}(w(x)-w(y))$$

and set

$$G(x,y) = \frac{(u(x)-u(y))}{|x-y|^{N+2\alpha}}(w(x)-w(y)).$$

Obviously, $\lim_{t\to 0} G_t(x,y) = G(x,y)$ and there exists $h \in L^1(\mathbb{R}^{2N})$ such that $|G_t(x,y)| \le h(x,y)$. Thus, by the Lebesgue dominated convergence theorem, we obtain the desired result. $\square$

To discuss the compactness of approximate solutions, we need the following Lions-Aubin lemma.

Proposition 2.12 ([16, Theorem 4.1]). *Let $T > 0, \beta \in (0,1)$ and $p \in [1,\infty)$. Let B_0, B, B_1 be Banach spaces. Assume that $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Suppose that $W \subset L^1_{loc}(0,T;B_0)$ satisfies;*

(i) *There exists $C_1 > 0$ such that $\forall u \in W$,*

$$\sup_{t\in(0,T)} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|u\|^p_{B_0}(s)ds \le C_1.$$

(ii) *There exist $r \in (\frac{p}{1+p\gamma},\infty)\bigcap[1,\infty)$ and $C_3 > 0$ such that $\forall u \in W$, there is an assignment of initial value u_0 for u such that the weak Caputo derivative satisfies*

$$\|D^\beta_c u\|_{L^r(0,T;B_1)} \le C_3.$$

Then W is relatively compact in $L^p(0,T,B)$.

Remark 2. The Caputo derivative of an absolutely continuous function u is defined as follows

$$D^\beta_c u = \frac{1}{\Gamma(\beta)} \int_0^t \frac{u'(\tau)}{(t-\tau)^{1-\beta}} d\tau.$$

In view of the definition of the Riemann-Liouville derivative, we know that the Riemann-Liouville derivative and the Caputo derivative have the relationship

$$D^\beta_c u = \partial^\beta_t u.$$

3. Existence and Uniqueness of Weak Solutions

In this section by means of the Galerkin method, we establish the existence of local solutions to the problem (1.1). Assume that $\{\omega_k\}$ is an orthonormal basis in $L^2(\Omega)$ and

$$u_{0m} = \sum_{j=1}^m b_{mj} w_j \to u_0 \quad \text{in } H^\alpha_0(\Omega),$$

then we shall find Galerkin approximation solutions $u_m = u_m(t)$ of the following form

$$u_m(t) = \sum_{j=1}^m a_{mj}(t) w_j, \quad m = 1,2,\ldots,$$

where a_{mj} satisfies that

$$\begin{cases} (\partial^\beta_t u_m, w_j) + \langle u_m, w_j\rangle_\alpha + \langle \partial^\beta_t u_m, w_j\rangle_\alpha = (\lambda f(x,u_m), w_j) + (g, w_j) \\ a_{mj}(0) = b_{mj}, \end{cases} \tag{3.1}$$

for $j = 1, 2, \ldots, m$. Here, $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$. Problem (3.1) is a nonlinear fractional ordinary differential system. Next, we show that problem (3.1) has a unique local solution for every $m \in N$.

First, by Lemmas 2.2 and 2.3, we give a prior estimate for problem (3.1).

Lemma 3.1. *Suppose that $u_m = \sum_{j=1}^{m} a_{mj} w_j$ solves problem (3.1). If $f(x, \xi) = |\xi|^{p-2}\xi$ and $2 < p \le 2_\alpha^*$, then there exist $T^* > 0$ and $\mathcal{C}_1 > 0$ such that*

$$\|u_m\|_{H_0^\alpha(\Omega)} \le \mathcal{C}_1 \quad \text{for all } t \in [0, T^*) \text{ and } m \ge 1. \tag{3.2}$$

Proof. Multiplying (3.1) by $\partial_t^\beta a_{mj}$ and summing j from 1 to m, we have

$$\begin{aligned}
&(\partial_t^\beta u_m, \partial_t^\beta u_m) + ((-\Delta)^\alpha u_m, \partial_t^\beta u_m) + ((-\Delta)^\alpha \partial_t^\beta u_m, \partial_t^\beta u_m) \\
&\quad = \lambda(|u_m|^{p-2} u_m, \partial_t^\beta u_m) + (g, \partial_t^\beta u_m).
\end{aligned} \tag{3.3}$$

By the Yosida approximation of time Riemamm-Liouville fractional derivative, Lemma 2.2 and Hölder's inequality, one can deduce that

$$\begin{aligned}
((-\Delta)^\alpha u_m, \partial_t^\beta u_m) &= (u_m, \partial_t^\beta u_m)_{H_0^\alpha(\Omega)} \\
&= (\frac{d}{dt}(g_{1-\beta,n} * u_m), u_m)_{H_0^\alpha(\Omega)} - g_{1-\beta,n}(u_{0\,m}, u_m)_{H_0^\alpha(\Omega)} - R_{mn}(1) \\
&\ge \frac{1}{2}\frac{d}{dt}(g_{1-\beta,n} * \|u_m\|^2_{H_0^\alpha(\Omega)}) - \frac{1}{2} g_{1-\beta,n} \|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} - R_{mn}(1)
\end{aligned}$$

where

$$R_{mn}(1) = \left(\frac{d}{dt}(g_{1-\beta,n} * (u_m - u_{0m})) - \frac{d}{dt}(g_{1-\beta} * (u_m - u_{0m})), u_m\right)_{H_0^\alpha(\Omega)}.$$

By using the Hölder inequality and Young inequality, we deduce

$$\begin{aligned}
(|u_m|^{p-2} u_m, \partial_t^\beta u_m) &\le \||u_m|^{p-2} u_m\|_{L^{q'}(\Omega)} \|\partial_t^\beta u_m\|_{L^q(\Omega)} \\
&\le \mathcal{S}_* \|u_m\|^{p-1}_{L^{(p-1)q'}(\Omega)} \|\partial_t^\beta u_m\|_{H_0^\alpha(\Omega)} \\
&\le \mathcal{S}_* (C_\varepsilon \|u_m\|^{2(p-1)}_{L^{(p-1)q'}(\Omega)} + \varepsilon \|\partial_t^\beta u_m\|^2_{H_0^\alpha(\Omega)}) \\
&\le \mathcal{S}_* (C_\varepsilon \|u_m\|^{2(p-1)}_{H_0^\alpha(\Omega)} + \varepsilon \|\partial_t^\beta u_m\|^2_{H_0^\alpha(\Omega)})
\end{aligned}$$

where $q = 2_\alpha^*$, $\quad q' = \frac{q}{q-1} = \frac{2N}{N+2\alpha}$ and $\mathcal{S}_*$ is the embedding constant from $H_0^\alpha(\Omega)$ to $L^{2_\alpha^*}(\Omega)$. Here, we have used the fact that $(p-1)q' \le 2_\alpha^*$, thanks to $p \le 2_\alpha^*$. We also get

$$\begin{aligned}
(g, \partial_t^\beta u_m) &\le \|g\|_{L^2(\Omega)} \|\partial_t^\beta u_m\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\|g\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t^\beta u_m\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\partial_t^\beta u_m\|^2_{L^2(\Omega)} + \|\partial_t^\beta u_m\|^2_{H_0^s(\Omega)} + \frac{1}{2}\frac{d}{dt}(g_{1-\beta,n} * \|u_m\|^2_{H_0^\alpha(\Omega)}) \\
&\quad \le \mathcal{S}_* \lambda (C_\varepsilon \|u_m\|^{2(p-1)}_{H_0^\alpha(\Omega)} + \varepsilon \|\partial_t^\beta u_m\|^2_{H_0^\alpha(\Omega)}) \\
&\qquad + \frac{1}{2} g_{1-\beta,n} \|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} + R_{mn}(1) + \frac{1}{2}\|g\|^2_{L^2(\Omega)}
\end{aligned} \tag{3.4}$$

Choose ε small enough such that $S_*\lambda\varepsilon < 1$. Then it follows from (3.4) that

$$\frac{1}{2}\frac{d}{dt}(g_{1-\beta,n} * \|u_m\|^2_{H_0^\alpha(\Omega)}) \\ \leq \mathcal{S}_*\lambda C_\varepsilon \|u_m\|^{2(p-1)}_{H_0^\alpha(\Omega)} + \frac{1}{2}g_{1-\beta,n}\|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} + R_{mn}(1) + \frac{1}{2}\|g\|^2_{L^2(\Omega)}. \tag{3.5}$$

Convolving (3.5) with g_β and letting n go to ∞ and selecting an appropriate subsequence(if necessary), it leads to

$$\|u_m\|^2_{H_0^\alpha(\Omega)} \leq C_1 g_\beta * \|u_m\|^{2(p-1)}_{H_0^\alpha(\Omega)} + \|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} + \frac{t^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\|g\|^2_{L^\infty(0,\infty;L^2(\Omega))}, \tag{3.6}$$

where $C_1 = 2\lambda\mathcal{S}_*C_\varepsilon$,

In Lemma 2.5, let $u(t) = \|u_m\|^2_{H_0^\alpha(\Omega)}$, $w(u) = u^{p-1}$, $F(s) = \frac{C_1}{\Gamma(\beta)}$, $a(t) = \sup_{m\geq 1}\|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} + \frac{t^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\|g\|^2_{L^\infty(0,\infty;L^2(\Omega))}$. When $\beta > \frac{1}{2}$, $R(t) = e^{2(p-2)t}$,

$$g_1(t) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}\int_0^t R(\tau)F^2(\tau)d\tau = C_2(e^{2(p-2)t} - 1),$$

$$\Omega(2a^2) = \int_{v_0}^{2a(t)^2}\frac{1}{y^{p-1}}dy = -\frac{1}{(p-2)(2a(t)^2)^{p-2}} + C_3,$$

$$\Omega^{-1}(v) = \left(v_0^{-\frac{1}{p-2}} - (p-2)v\right)^{-\frac{1}{p-2}},$$

where $C_2 = \frac{4^{\beta-1}C_1^2}{2(p-2)\Gamma(2\beta-1)\Gamma^2(\beta)} > 0$, $C_3 = \frac{1}{(p-2)v_0^{p-2}} > 0$. Then we deduce from (3.6) that

$$u(t) \leq e^t\left(\frac{1}{\frac{1}{(2a(t)^2)^{p-2}} + C_2(p-2) - C_2(p-2)e^{2(p-2)t}}\right)^{\frac{1}{2(p-2)}}.$$

Considering the definition of Ω^{-1} and using $\Omega(2a^2(t)) + g_1(t) \in Dom\Omega^{-1}$, we get

$$-\frac{1}{(p-2)(2a(t)^2)^{p-2}} + C_3 + C_2(e^{2(p-2)t} - 1) < C_3,$$

which implies that

$$\begin{aligned} e^{2(p-2)t} &\leq \frac{1}{C_2(p-2)(2a(t)^2)^{p-2}} + 1 \\ &\leq \frac{1}{C_2(p-2)(2C_0^2)^{p-2}} + 1, \end{aligned}$$

where $C_0 = \sup_{m\geq 1}\|u_{0m}\|^2_{H_0^\alpha(\Omega)}$. Then we have $t < \frac{\ln C_4}{2(p-2)} := T_1$, where

$$C_4 = \frac{1}{C_2(p-2)(2C_0^2)^{p-2}} + 1 > 1.$$

Therefore, for the case $\beta > \frac{1}{2}$, there exists $T_1 > 0$ such that

$$\|u_m\|^2_{H_0^\alpha(\Omega)} \leq C_5,$$

for all $0 < t < T_1$.

When $\beta \in (0, \frac{1}{2})$, $R(t) = e^{(p-2)(z+2)t}$,

$$\Omega(2^{q-1}a^q) = \int_{v_0}^{2^{q-1}a^q} \frac{1}{y^{p-1}} dy = -\frac{1}{(p-2)(2^{q-1}a^q)^{p-2}} + C_3,$$

$$g_2(t) = 2^{q-1}K_z q \int_0^t F^q(\tau)R(\tau)d\tau = C_6(e^{(p-2)(z+2)t} - 1),$$

$$\Omega^{-1}(\Omega(2^{q-1}a^q) + g_2(t)) = (\frac{1}{(2^{q-1}a^q)^{-p+2} + C_6(p-2) - C_6(p-2)e^{(p-2)(z+2)t}})^{\frac{1}{p-2}},$$

then we have

$$u(t) \leq e^t \left(\frac{1}{\frac{1}{(2^{q-1}a^q)^{p-2}} + C_6(p-2) - C_6(p-2)e^{(p-2)(z+2)t}} \right)^{\frac{1}{q(p-2)}},$$

where $z = \frac{1-\beta}{\beta}$, $q = z+2$, $K_z = (\frac{\Gamma(1-\gamma r)}{r^{1-\gamma r}})^{\frac{1}{r}}$, $\gamma = \frac{z}{z+1}$, $r = \frac{z+2}{z+1}$, $C_6 = \frac{C_1^q K_z^q 2^{q-1}}{\Gamma^q(\beta)(p-2)(z+2)}$ > 0.

Consider the definition of Ω^{-1}, we obtain

$$C_6(e^{(p-2)(z+2)t} - 1) - \frac{1}{(p-2)(2^{q-1}a^q)^{p-2}} + C_3 < C_3,$$

which implies that $t < \frac{\ln C_7}{(p-2)(z+2)} := T_2$. Here, $C_7 = \frac{1}{C_6(p-2)(2^{q-1}C_0^q)^{p-2}} + 1 > 0$.

Therefore, for $\beta \in (0, \frac{1}{2})$, there exists $T_2 > 0$ such that $\|u_m\|^2_{H_0^\alpha(\Omega)} \leq C_8$ for all $0 < t < T_2$.

In conclusion, there exist $T^* = \min\{T_1, T_2\}$ and $\mathcal{C} > 0$ such that $\|u_m\|_{H_0^\alpha(\Omega)} \leq \mathcal{C}$ for all $0 < t < T^*$. □

Lemma 3.2. *Suppose that $u_m = \sum_{j=1}^m a_{mj}w_j$ solves problem (3.1). If f satisfies Lipschitz condition (f_1), then for any $T > 0$ there exists $\mathcal{C}_2 > 0$ such that*

$$\|u_m\|_{H_0^\alpha(\Omega)} \leq \mathcal{C}_2 \quad \text{for all } t \in [0, T] \text{ and } m \geq 1.$$

Proof. Since f satisfies the Lipschitz condition, there exists a positive constant C such that

$$|f(x, \xi)| \leq C(1 + |\xi|) \quad \text{for all } \xi \in \mathbb{R}.$$

Then a similar discussion as in Lemma 3.1 gives that

$$\|u_m\|^2_{H_0^\alpha(\Omega)} \leq Cg_\beta * \|u_m\|^2_{H_0^\alpha(\Omega)} + \|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} + \frac{Ct^{1-\beta}}{\Gamma(1-\beta)} + \frac{t^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\|g\|^2_{L^\infty(0,\infty;L^2(\Omega))}.$$

By Lemma 2.6, we get (i) If $\beta > 1/2$, then

$$\|u_m\|^2_{H_0^\alpha(\Omega)} \leq \sqrt{2}a(t) \exp\left(\left(\frac{2C_1^2\Gamma(2\beta-1)}{4^\beta\Gamma(\beta)^2} + 1\right)t\right), \quad t \in [0, T),$$

where $a(t) = \sup_{m\geq 1} \|u_{0\,m}\|^2_{H_0^\alpha(\Omega)} + \frac{C_1 t^{1-\beta}}{\Gamma(1-\beta)} + \frac{t^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\|g\|^2_{L^\infty(0,\infty;L^2(\Omega))}$.

(ii) If $\beta=\frac{1}{z+1}$ for some $z\geq 1$, then

$$\|u_m\|^2_{H_0^\alpha(\Omega)}\leq (2^{q-1})^{1/q}a(t)\exp\left(\left(\frac{2^{q-1}K_zC_1^q}{\Gamma(\beta)^q}+1\right)t\right),\quad t\in[0,T),$$

where $K_z=[\frac{\Gamma(1-\gamma p)}{p^{1-\gamma p}}]^{1/p}$, $\gamma=\frac{z}{z+1}$, $p=\frac{z+2}{z+1}$ and $q=z+2$.

In conclusion, for any $T>0$ there exists $\mathcal{C}_2>0$ such that $\|u_m\|_{H_0^\alpha(\Omega)}\leq\mathcal{C}_2$ for all $t\in[0,T]$ and $m\geq 1$. □

Based on Lemmas 3.1 and 3.2, we obtain the following estimate.

Lemma 3.3. *Suppose that $u_m=\sum_{j=1}^m a_{mj}w_j$ solves (3.1), then there exists $\mathcal{C}_3$ such that*

$$\|\partial_t^\beta u_m\|_{L^2(0,T;L^2(\Omega))}+\|\partial_t^\beta u_m\|_{L^2(0,T:H_0^\alpha(\Omega))}\leq\mathcal{C}_3.$$

Proof. Choosing $\varepsilon=\frac{1}{2\mathcal{S}_*\lambda}$ in (3.4), we get

$$\begin{aligned}&\frac{1}{2}\|\partial_t^\beta u_m\|^2_{L^2(\Omega)}+\frac{1}{2}\|\partial_t^\beta u_m\|^2_{H_0^s(\Omega)}+\frac{1}{2}\frac{d}{dt}(g_{1-\beta,n}*\|u_m\|^2_{H_0^\alpha(\Omega)})\\&\quad\leq\mathcal{S}_*\lambda C_\varepsilon\|u_m\|^{2(p-1)}_{H_0^\alpha(\Omega)}\\&\qquad+\frac{1}{2}g_{1-\beta,n}\|u_{0\,m}\|^2_{H_0^\alpha(\Omega)}+R_{mn}(1)+\frac{1}{2}\|g\|^2_{L^2(\Omega)}.\end{aligned}$$

Integrating above inequality from 0 to T and letting $n\to\infty$, we obtain

$$\begin{aligned}&\|\partial_t^\beta u_m\|^2_{L^2(0,T;(L^2(\Omega)))}+\|\partial_t^\beta u_m\|^2_{L^2(0,T:H_0^\alpha(\Omega))}\\&\quad\leq 2\mathcal{S}_*\lambda C_\varepsilon\int_0^T\|u_m\|^{2p-2}_{H_0^\alpha(\Omega)}dt+\int_0^T g_{1-\beta,n}\|u_{0\,m}\|^2_{H_0^\alpha(\Omega)}dt\\&\qquad+\|g\|^2_{L^2(0,T;L^2(\Omega))}+g_{1-\beta}*\|u_{0,m}\|^2_{H_0^\alpha(\Omega)}.\end{aligned}$$

By Lemma 3.1, it yields

$$\|\partial_t^\beta u_m\|^2_{L^2(0,T;L^2(\Omega))}+\|\partial_t^\beta u_m\|^2_{L^2(0,T:H_0^\alpha(\Omega))}\leq C.$$

For the case f satisfies the Lipschitz condition, by Lemma 3.2 and a similar discussion as above, one can obtain the desired result. □

Now, we prove the local existence of solutions for system (3.1).

Theorem 3.4. *Under the assumptions of Theorem 1.2, system (3.1) has a unique solution for all $t\in[0,T]$, where $0<T<T^*$ if $f=|u|^{p-2}u$ and $0<T<\infty$ if f satisfies (f_1).*

Proof. First, problem (3.1) is equivalent to the problem

$$\begin{cases}\partial_t^\beta\psi(t)+A\psi(t)=BR(\psi(t))+BG(t)\\ \psi(0)=\xi\end{cases}\tag{3.7}$$

where $\psi(t)=(a_{mj}(t))\in\mathbb{R}^m$,$A=diag(\frac{\lambda_j}{1+\lambda_j})_{m\times m}$, $B=diag(\frac{1}{1+\lambda_j})_{m\times m}$, $\xi=(b_{mj})\in\mathbb{R}^m$, $\mu=(\lambda_j)\in\mathbb{R}^m$, $R(\psi(t))_j=\lambda(|u_m|^{p-2}u_m,wj)$, $G_j=(g,w_j)$. By

Laplace transform or convoluting with g_β, we transform (3.7) into the following Volterra type system

$$\psi(t) = \xi + g_\beta * B(R(\psi(t) + G(t))) - g_\beta * A\psi(t). \tag{3.8}$$

Therefore, we only need to prove that system (3.8) admits a unique continuous solution.
Notice

$$\|u_m(t)\|^2_{L^2(\Omega)} = \sum_{j=1}^{m} a^2_{mj}(t), \quad \|\psi(t)\|^2_{\mathbb{R}^m} = \sum_{j=1}^{m} a^2_{mj}(t).$$

Then, $\|u_m(t)\|_{L^2(\Omega)} = \|\psi(t)\|_{R^m}$, $\|u_m(t)\|_{L^2(\Omega)} \le \mathcal{S}\|u_m(t)\|_{H^\alpha_0(\Omega)}$. Let

$$R_0 = \mathcal{S}\mathcal{C}_1.$$

Then we obtain a prior estimate $\|\psi(t)\|_{\mathbb{R}^m} \le R_0$.

Define the operator as

$$\Phi\psi(t) = \xi + g_\beta * B(R(\psi(t) + G(t))) - g_\beta * A\psi(t)$$

for all $\psi \in E_T$, where

$$E_T = \{\psi \in C(0,T;\mathbb{R}^m) : \|\psi\|_{C(0,T;\mathbb{R}^m)} \le 2R_0\}.$$

Set $d(\psi_1,\psi_2) = \max_{t\in[0,T]} \|\psi_1(t) - \psi_2(t)\|_{\mathbb{R}^m}$. Since $C(0,T;\mathbb{R}^m)$ is a Banach space, it follows that (E_T, d) is a complete metric space. Then

$$\|\Phi\psi\|_{\mathbb{R}^m} \le \|\xi\|_{\mathbb{R}^m} + \|Bg_\beta * (R(\psi(t) + G(t)))\|_{\mathbb{R}^m} + \|Ag_\beta * \psi(t)\|_{\mathbb{R}^m}.$$

Observe that

$$\begin{aligned}
&\|g_\beta * B(R(\psi(t) + G(t)))\|^2_{\mathbb{R}^m} \\
&\quad \le \sum_{j=1}^{m}\left(\frac{\lambda}{1+\lambda_j} g_\beta * \int_\Omega |u_m|^{p-2}u_m w_j dx + \frac{1}{1+\lambda_j} g_\beta * \int_\Omega g\omega_j dx\right)^2 \\
&\quad \le \sum_{j=1}^{m}\left(\frac{\lambda\|w_j\|_p}{1+\lambda_j} g_\beta * \|u_m\|^{p-1}_p + \frac{\|w_j\|_2}{1+\lambda_j} g_\beta * \|g\|_2\right)^2 \\
&\quad \le \frac{C^2}{(1-\beta)^2\Gamma^2(1-\beta)}(R_0^{(p-1)} + R_0)^2 t^{2(1-\beta)}
\end{aligned}$$

and

$$\begin{aligned}
\|Ag_\beta * \psi(t)\|^2_{\mathbb{R}^m} \le (g_\beta * \|A\psi(t)\|_{\mathbb{R}^m})^2 &\le \frac{\lambda^2_m t^{2(1-\beta)}}{(1+\lambda_1)^2(1-\beta)^2\Gamma^2(1-\beta)} \sum_{j=1}^{m} a^2_{mj}(t) \\
&\le \frac{\lambda^2_m t^{2(1-\beta)}}{(1+\lambda_1)^2(1-\beta)^2\Gamma^2(1-\beta)} \|\psi(t)\|^2_{\mathbb{R}^m}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\Phi\psi\|_{\mathbb{R}^m} \le \|\xi\|_{\mathbb{R}^m} &+ \frac{C}{(1-\beta)\Gamma(1-\beta)}(R_0^{(p-1)} + R_0)T^{1-\beta} \\
&+ \frac{\lambda_m T^{1-\beta}}{(1+\lambda_1)(1-\beta)\Gamma(1-\beta)} 2R_0.
\end{aligned}$$

Now we assume that T is small enough such that

$$\frac{C}{(1-\beta)\Gamma(1-\beta)}(R_0^{(p-1)}+R_0)T^{1-\beta}+\frac{\lambda_m T^{1-\beta}}{(1+\lambda_1)(1-\beta)\Gamma(1-\beta)}2R_0<R_0.$$

Then $\|\Psi\psi\|_{\mathbb{R}^m}\le 2R_0$, which means that Φ maps from E_T to E_T.

Next we show that Φ is contractive in E_T.

If $2<p\le 2_\alpha^*:=\frac{2N}{N-2\alpha}$, then by the Hölder inequality we have

$$\begin{aligned}
&\|BR(\psi^{(1)})-BR(\psi^{(2)})\|_{\mathbb{R}^m}^2\\
&\quad\le \lambda^2\sum_{j=1}^m\left(\frac{1}{1+\lambda_j}\right)^2\left(\int_\Omega ||u_m^{(1)}|^{p-2}u_m^{(1)}-|u_m^{(2)}|^{p-2}u_m^{(2)}||\omega_j|dx\right)^2\\
&\quad\le \lambda^2C\sum_{j=1}^m\left(\frac{1}{1+\lambda_j}\right)^2\left(\int_\Omega (|u_m^{(1)}|^{p-2}+|u_m^{(2)}|^{p-2})|u_m^{(1)}-u_m^{(2)}||\omega_j|dx\right)^2\\
&\quad\le \lambda^2C\sum_{j=1}^m\left(\frac{1}{1+\lambda_j}\right)^2\||u_m^{(1)}|^{p-2}+|u_m^{(2)}|^{p-2}\|_{L^{\frac{N}{2\alpha}}(\Omega)}^2\|u_m^{(1)}-u_m^{(2)}\|_{L^{\frac{2N}{N-2\alpha}}(\Omega)}^2\|\omega_j\|_2^2\\
&\quad\le \frac{\lambda^2C}{(1+\lambda_1)^2}(\|u_m^{(1)}\|_{H_0^\alpha(\Omega)}^{p-2}+\|u_m^{(2)}\|_{H_0^\alpha(\Omega)}^{p-2})^2\|u_m^{(1)}-u_m^{(2)}\|_{H_0^\alpha(\Omega)}^2\\
&\quad\le \frac{\lambda^2C\lambda_m^{p-1}}{(1+\lambda_1)^2}(\|u_m^{(1)}\|_{L^2(\Omega)}^{p-2}+\|u_m^{(2)}\|_{L^2(\Omega)}^{p-1})^2\|u_m^{(1)}-u_m^{(2)}\|_{L^2(\Omega)}^2\\
&\quad\le \frac{\lambda^2C\lambda_m^{p-1}}{(1+\lambda_1)^2}(\|\psi^{(1)}\|_{\mathbb{R}^m}^{p-2}+\|\psi^{(2)}\|_{\mathbb{R}^m}^{p-2})^2\|\psi^{(1)}-\psi^{(2)}\|_{\mathbb{R}^m}^2\\
&\quad\le \frac{\lambda^2C\lambda_m^{p-1}}{(1+\lambda_1)^2}R_0^{2(p-2)}\|\psi^{(1)}-\psi^{(2)}\|_{\mathbb{R}^m}^2,
\end{aligned}\tag{3.9}$$

thanks to the following basic inequality:

$$||u_m^{(1)}|^{p-2}u_m^{(1)}-|u_m^{(2)}|^{p-2}u_m^{(2)}|\le C(|u_m^{(1)}|^{p-2}+|u_m^{(2)}|^{p-2})|u_m^{(1)}-u_m^{(2)}|.$$

Then

$$\begin{aligned}
&\|g_\beta*BR(\psi^{(1)})-g_\beta*BR(\psi^{(2)})\|_{\mathbb{R}^m}\\
&\quad\le g_\beta*\|BR(\psi^{(1)})-BR(\psi^{(2)})\|_{\mathbb{R}^m}\\
&\quad\le \frac{\lambda C\lambda_m^{\frac{p-1}{2}}}{1+\lambda_1}\frac{R_0^{(p-2)}T^{1-\beta}}{\Gamma(1-\beta)}d(\psi^{(1)},\psi^{(2)}).
\end{aligned}$$

Observe that

$$\begin{aligned}
\|A\psi^{(1)}-A\psi^{(2)}\|_{\mathbb{R}^m}^2&=\sum_{j=1}^m((\psi^{(1)}-\psi^{(2)})\frac{\lambda_j}{1+\lambda_j})^2\\
&\le\left(\frac{\lambda_m}{1+\lambda_m}\right)^2\sum_{j=1}^m(\psi^{(1)}-\psi^{(2)})^2\\
&=\left(\frac{\lambda_m}{1+\lambda_m}\right)^2\|\psi^{(1)}-\psi^{(2)}\|_{\mathbb{R}^m}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}&\|g_\beta * A\psi^{(1)} - g_\beta * A\psi^{(2)}\|_{\mathbb{R}^m}\\ &\quad\le \frac{\lambda_m}{1+\lambda_m}\frac{T^{1-\beta}}{\Gamma(1-\beta)}d(\psi^{(1)},\psi^{(2)}).\end{aligned} \tag{3.10}$$

Gathering (3.9) and (3.10), we arrive at

$$d(\Phi(\psi^{(1)}),\Phi(\psi^{(1)})) \le \mathcal{D}T^{1-\beta}d(\psi^{(1)},\psi^{(2)}), \quad \forall\, \psi^{(1)},\psi^{(2)} \in E_T,$$

where

$$\mathcal{D} := \frac{\lambda C\lambda_m^{\frac{p-1}{2}}}{1+\lambda_1}\frac{R_0^{(p-2)}}{\Gamma(1-\beta)} + \frac{\lambda_m}{1+\lambda_m}\frac{1}{\Gamma(1-\beta)}.$$

Consequently, we prove that Φ is contractive on E_T provided T is small enough such that $\mathcal{D}T^{1-\beta} < 1$. Thus, by the Banach contraction mapping theorem, we know that the map Φ has a unique fixed point on some small interval $[0,T_0]$. Therefore, we prove that system (3.7) has a unique solution on $[0,T_0]$.

On the other hand, if f satisfies the Lipschitz condition, the existence of unique solution of system (3.7) on some small interval $[0,T_0]$ can be proved similarly as above.

Finally, we show that the local solution can be extended to $(0,T]$. Let T_0 and $u_m(T_0)$ be the initial data. Then repeating the same process as above, we can get a unique continuous solution on $[T_0,2T_0]$. Divide $[0,T]$ into $[(k-1)T_0.kT_0]$ with $k=1,2,\dots K$ and $T/K \le T_0$. Then we can obtain a unique continuous solution in $[0,T]$. In conclusion, we show that system (3.1) has a unique solution in $C(0,T;\mathbb{R}^m)$. □

Proof of Theorem 1.2. Gathering Lemma 3.1 and Lemma 3.2, we get

$$\{u_m\} \text{ is bounded in } L^\infty(0,T;H_0^\alpha(\Omega)\cap L^p(\Omega))$$

and

$$\{\partial_t^\beta u_m\} \text{ is bounded in } L^2(0,T;H_0^\alpha(\Omega)).$$

By Proposition 2.12, we deduce that there exist a subsequence (still denoted by $\{u_m\}$) and $u \in L^\infty(0,T;H_0^\alpha(\Omega)\cap L^p(\Omega))$ such that

$$\begin{aligned}&u_m \rightharpoonup u && \text{weakly star in } L^\infty(0,T;H_0^\alpha(\Omega)\cap L^p(\Omega)),\\ &u_m \to u && \text{strongly in } L^2(0,T;L^2(\Omega)),\\ &u_m \to u && \text{a.e. in } (0,T)\times\Omega.\end{aligned} \tag{3.11}$$

By the Vitali convergence theorem, one can show that

$$\lim_{m\to\infty}\int_0^T\int_\Omega f(x,u_m)v\,dx = \int_0^T\int_\Omega f(x,u)v\,dx$$

for all $v \in C^1(0,T;H_0^\alpha(\Omega))$.

Since $\{\partial_t^\beta u_m\}$ is bounded in $L^2(0,T;L^2(\Omega))$, up to a subsequence we may assume that

$$\partial_t^\beta u_m \rightharpoonup \chi \text{ in } L^2(0,T;L^2(\Omega)).$$

Next we show that $\chi = \partial_t^\beta u$. By Lemma 2.8, we obtain

$$\int_0^T \int_\Omega \partial_t^\beta u_m \varphi dxdt = -\int_0^T \int_\Omega \int_t^T \frac{\varphi'(s)}{(s-\tau)^{1-\beta}} ds(u_m - u_m(0))dtdx.$$

It follows from (3.11) that

$$\begin{aligned}\int_0^T \int_\Omega \chi\varphi dxt &= -\int_0^T \int_\Omega \int_t^T \frac{\varphi'(s)}{(s-\tau)^{1-\beta}} ds(u-u_0)dtdx \\ &= \int_0^T \int_\Omega \partial_t^\beta u\varphi dx,\end{aligned}$$

which means that $\chi = \partial_t^\beta u$. Further, by (3.1) we conclude that

$$\begin{aligned}&\int_0^T \int_\Omega \partial_t^\beta uvdxdt + \int_0^T \langle u, v\rangle_\alpha dt + \int_0^T \langle \partial_t^\beta u, v\rangle_\alpha dt \\ &= \lambda \int_0^T \int_\Omega f(x,u)vdxdt + \int_0^T \int_\Omega gvdxdt \end{aligned} \tag{3.12}$$

for any $v \in L^2(0,T; H_0^\alpha(\Omega))$.

Next we show that

$$\begin{aligned}&\int_\Omega \partial_t^\beta u\varphi dx + \langle u,\varphi\rangle_\alpha + \langle \partial_t^\beta u, \varphi\rangle_\alpha \\ &= \lambda \int_\Omega f(x,u)\varphi dx + \int_\Omega g\varphi dx \end{aligned} \tag{3.13}$$

for any $\varphi \in H_0^\alpha(\Omega)$.

For any $t \in (0,T)$, let $\chi_{(0,t)}$ denote the characteristic function in $(0,t)$. Let $\varphi \in H_0^\alpha(\Omega)$. Taking $v = \varphi\chi_{(0,t)}$ in (3.12), we get

$$\begin{aligned}&\int_0^t \int_\Omega \partial_t^\beta u\varphi dxdt + \int_0^t \langle u,\varphi\rangle_\alpha dt + \int_0^t \langle \partial_t^\beta u, \varphi\rangle_\alpha dt \\ &= \lambda \int_0^t \int_\Omega f(x,u)\varphi dxdt + \int_0^t \int_\Omega g\varphi dxdt. \end{aligned} \tag{3.14}$$

Since

$$\int_\Omega \partial_t^\beta u\varphi dx + \langle u,\varphi\rangle_\alpha + \langle \partial_t^\beta u,\varphi\rangle_\alpha - \lambda \int_\Omega f(x,u)\varphi dx - \int_\Omega g\varphi dx \in L^1(0,T),$$

we differentiate (3.14) with respect to t and get that

$$\begin{aligned}&\int_\Omega \partial_t^\beta u\varphi dx + \langle u,\varphi\rangle_\alpha + \langle \partial_t^\beta u, \varphi\rangle_\alpha \\ &= \lambda \int_\Omega f(x,u)\varphi dx + \int_\Omega g\varphi dx\end{aligned}$$

for any $\varphi \in H_0^\alpha(\Omega)$ and a.e. $t \in [0,T]$. Thus, (3.13) holds.

It follows that u is a weak solution of problem (1.1). □

In the next lemma we shall show that under some assumptions the solution of problem (1.1) is nonnegative.

Lemma 3.5. *If $f(x,\xi) \le 0$ for any $\xi \le 0$, $g \ge 0$ and $u_0 \ge 0$ a.e. in Ω. Then the solutions of problem (1.1) are nonnegative.*

Proof. Let u be a weak solution to problem (1.1). Clearly,

$$u^- = \max\{-u, 0\} \in L^\infty(0,T; H_0^\alpha(\Omega)).$$

Taking $v = -u^-$ in Definition 1.1, we obtain

$$\begin{aligned}\int_\Omega \partial_t^\beta u(t)u^-(t)\,dx + \langle u(t), -u^-(t)\rangle_\alpha + \langle \partial_t^\beta u(t), -u^-(t)\rangle_\alpha \\ = -\int_\Omega f(x,u(t))u^-(t)\,dxdt - \int_\Omega gu^-(t)dx.\end{aligned} \tag{3.15}$$

Observe that for a.e. $x, y \in \Omega$,

$$\begin{aligned}&(u(x)-u(y))(-u^-(x)+u^-(y)) \\ &\quad = (u^-(x)-u^-(y))^2 + u^-(x)u^+(y) + u^+(x)u^-(y) \\ &\quad \ge |u^-(x)-u^-(y)|^2, \qquad .\end{aligned}$$

Then

$$\langle u(t), -u^-(t)\rangle_\alpha \ge \|u^-\|^2_{H_0^\alpha}.$$

Moreover, $g(-u^-) \le 0$ and $f(x,u)u^- = 0$ a.e. $x \in \Omega$.

To apply Lemma 2.2, we can use the same regularization discussion as above. For convenience, we omit this process. In view of Lemma 2.2, we have

$$-\left(\frac{d}{dt}(g_{1-\beta} * u(t)), u^-(t)\right) \ge \frac{1}{2}\frac{d}{dt}(g_{1-\beta} * \|u^-(t)\|_2^2)$$

and

$$-\left\langle \frac{d}{dt}(g_{1-\beta} * u(t)), u^-(t)\right\rangle_\alpha \ge \frac{1}{2}\frac{d}{dt}(g_{1-\beta} * \|u^-(t)\|^2_{H_0^\alpha}).$$

Combining these facts with (3.15), it yields

$$\frac{1}{2}\frac{d}{dt}(g_{1-\beta} * \|u^-(t)\|_2^2) + \frac{1}{2}\frac{d}{dt}(g_{1-\beta} * \|u^-(t)\|^2_{H_0^\alpha}) + \|u^-(t)\|^2_\alpha \le 0.$$

This implies that

$$\begin{aligned}&\frac{1}{2}(g_{1-\beta} * \|u^-(t)\|_2^2) + \frac{1}{2}(g_{1-\beta} * \|u^-(t)\|^2_{H_0^\alpha}) + \int_0^t \|u^-(t)\|^2_\alpha dt \\ &\quad \le \frac{1}{2}(g_{1-\beta} * \|u_0^-\|_2^2) + \frac{1}{2}(g_{1-\beta} * \|u_0^-\|^2_{H_0^\alpha}).\end{aligned}$$

Since $u_0 \ge 0$ a.e. in Ω, it leads to

$$g_{1-\beta} * (\|u^-(t)\|_2^2 + \|u^-(t)\|^2_{H_0^\alpha}) \le 0.$$

Then convoluting above inequality with g_β, it leads to

$$\|u^-(t)\|_2^2 \le 0.$$

Thus, we get $u^-(t) = 0$ a.e. in Ω and for any $t > 0$. Hence, $u(x,t) \ge 0$ a.e. in Ω and for any $t > 0$. $\square$

At the end of this section, we study the uniqueness of solutions of problem (1.1).

Theorem 3.6 (Comparison theorem). *Assume that f satisfies (f_2). Let $\underline{u}$ be a subsolution of problem (1.1) and let $\overline{u}$ be a supsolution of problem (1.1). Then $\underline{u} \leq \overline{u}$.*

Proof. We deduce from Definition 1.1 that

$$\begin{aligned}&\int_\Omega \partial_t^\beta(\underline{u}(t)-\overline{u}(t))v\,dx + \langle \underline{u}(t)-\overline{u}(t), v\rangle_\alpha + \langle \partial_t^\beta(\underline{u}(t)-\overline{u}(t)), v\rangle_\alpha \\ &\quad\leq \lambda\int_\Omega (f(x,\underline{u}(t)) - f(x,\overline{u}(t)))v\,dx,\end{aligned}$$

for any $0 \leq v \in H_0^\alpha(\Omega)$. Taking $v = (\underline{u}-\overline{u})^+ = \max\{\underline{u}-\overline{u}, 0\}$ and applying Lemma 2.2 and (f_2), it leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(g_{1-\beta} * \|(\underline{u}(t)-\overline{u}(t))^+\|_2^2) + [(\underline{u}(t)-\overline{u}(t))^+]_\alpha^2 \\ &\quad + \frac{1}{2}\frac{d}{dt}(g_{1-\beta} * (\|(\underline{u}(t)-\overline{u}(t))^+\|_\alpha^2)) \\ &\quad\leq \lambda\mathcal{L}\int_\Omega |(\underline{u}(t)-\overline{u}(t))^+|^2\,dx.\end{aligned}$$

Set $Y(t) = \|(\underline{u}(t)-\overline{u}(t))^+\|_2^2 + [(\underline{u}(t)-\overline{u}(t))^+]_\alpha^2$. Then

$$\frac{d}{dt}(g_{1-\beta} * Y(t)) \leq 2\lambda\mathcal{L}Y(t).$$

Convoluting above inequality with g_β, it follows that

$$Y(t) \leq 2\lambda\mathcal{L}g_\beta * Y(t).$$

By Lemma 2.6 and $Y(0) = 0$, one can get that $Y(t) = 0$. Thus, we get $\underline{u} \leq \overline{u}$ a.e. in $\Omega \times (0,T)$, which ends the proof. □

Lemma 3.7. *Assume that f satisfies (f_2). Then the solution of problem (1.1) is unique.*

Proof. Assume that u_1 and u_2 are two solutions of problem (1.1). Then we deduce from 1.1 that

$$\begin{aligned}&\int_\Omega \partial_t^\beta(u_1(t)-u_2(t))v\,dx + \langle u_1(t)-u_2(t), v\rangle_\alpha + \langle \partial_t^\beta(u_1(t)-u_2(t)), v\rangle_\alpha \\ &\quad= \lambda\int_\Omega (f(x,u_1(t)) - f(x,u_2(t)))v\,dx,\end{aligned}$$

for any $v \in H_0^\alpha(\Omega)$. Taking $v = u_1 - u_2$, we get

$$\begin{aligned}&\int_\Omega (u_1(t)-u_2(t))\partial_t^\beta(u_1(t)-u_2(t))\,dx + [u_1(t)-u_2(t)]_\alpha^2 \\ &\quad + \langle \partial_t^\beta(u_1(t)-u_2(t)), u_1(t)-u_2(t)\rangle_s \\ &\quad= \lambda\int_\Omega (f(x,u_1(t)) - f(x,u_2(t)))u_1(t)-u_2(t))\,dx.\end{aligned}$$

Using a similar discussion as above and applying Lemma 2.2 and (f_2), it leads to

$$\frac{1}{2}\frac{d}{dt}(g_{1-\beta} * \|(u_1(t) - u_2(t)\|_2^2) + [u_1(t) - u_2(t)]_\alpha^2 + \frac{1}{2}\frac{d}{dt}(g_{1-\beta} * ([u_1(t) - u_2(t)]_\alpha^2))$$
$$\leq \lambda\mathcal{L}\int_\Omega |u_1(t) - u_2(t)|^2\, dx.$$

Set $Z(t) = \|(u_1(t) - u_2(t)\|_2^2 + [u_1(t) - u_2(t)]_\alpha^2$. Then

$$\frac{d}{dt}(g_{1-\beta} * Z(t)) \leq 2\lambda\mathcal{L}Z(t).$$

Convoluting above inequality with g_β, it follows that

$$Z(t) \leq \lambda\mathcal{L}g_\beta * Z(t),$$

which together with Lemma 2.2 and $Z(0) = 0$ yields that $Z(t) = 0$. Thus, we get $u_1 = u_2$ a.e. in $\Omega \times (0, T)$, which ends the proof. □

4. Decay Estimates of Solutions

In this section, we give a decay estimate for problem (1.1) in which f satisfies (f_2).

Let φ_1 be the corresponding eigenfunction to the first eigenvalue λ_1 of the fractional Laplacian. Clearly, $\varphi_1 > 0$ and $\varphi_1 \in L^\infty(\Omega)$. Let $0 \leq u_0(x) \leq \eta_0\varphi_1(x)$, where $\eta_0 > 0$. Assume that there exists $h_0 > 0$ such that $f(x, \xi) \leq h_0\xi$ for all $x \in \Omega$ and $\xi \geq 0$.

Set $v(x, t) = \varphi_1(x)\eta(t)$, where η satisfies

$$\begin{cases} \partial_t^\beta \eta(t) + \frac{\lambda_1 - h_0\lambda}{1+\lambda_1}\eta(t) = 0 & \text{for all } t \geq 0, \\ \eta(0) = \eta_0 > 0, \end{cases}$$

where $0 < \lambda < \frac{\lambda_1}{h_0}$.

By [30, Theorem 7.1], there exist $c_1, c_2 > 0$ such that

$$\frac{c_1}{1 + t^\beta} \leq \eta(t) \leq \frac{c_2}{1 + t^\beta} \quad \text{for all } t \geq 0. \tag{4.1}$$

A simple calculation gives that

$$\begin{aligned} &\partial_t^\beta v(x,t) + (-\Delta)^\alpha v(x,t) + (-\Delta)^\alpha \partial_t^\beta v(x,t) \\ &\quad = \varphi_1(x)\partial_t^\beta \eta(t) + \lambda_1\varphi_1(x)\eta(t) + \lambda_1\varphi_1(x)\partial_t^\beta \eta(t) \\ &\quad = \varphi_1(x)(\partial_t^\beta \eta(t) + \lambda_1\eta(t) + \lambda_1\partial_t^\beta \eta(t)) \\ &\quad = \varphi_1(x)\lambda h_0\eta(t) = \lambda h_0 v(x,t) \geq \lambda f(x, v). \end{aligned}$$

Since $v(x, 0) = \varphi_1(x)\eta_0$ and $\varphi_1(x)\eta_0 \geq u_0(x)$, we know that $v(x, t)$ is a supersolution of problem (1.1). Then by Theorem 3.6, we obtain that the unique solution u of problem (1.1) satisfies

$$u(x, t) \leq v(x, t) \quad \text{for a.e. } (x, t) \in \Omega \times (0, \infty).$$

Further, (4.1) implies that

$$0 \leq u(x, t) \leq \frac{c_2\varphi_1(x)}{1 + t^\beta} \quad \text{for all } t \geq 0 \text{ and } x \in \Omega.$$

5. Blow-Up of Solutions

In this section, we consider the blow-up property of solutions of problem (1.1). Some techniques are inspired from [23] and [39]. Let $u_0 \in H_0^\alpha(\Omega)$ satisfy $u_0 \geq 0$. If $2 < p$ and $\int_\Omega u_0\varphi_1 dx > (\frac{\lambda_1}{\lambda})^{1/(p-2)}$, then the solution of problem (1.1) blows up in finite time.

Proof of Theorem 1.4. Taking $\varphi = \varphi_1(x)\varphi_2(t)$ with $\varphi_2 \in C^1(0,T)$ in Definition 1.1 and integrating over $(0,T)$, we get

$$\begin{aligned}&\int_0^T \int_\Omega \partial_t^\beta u(t)\varphi_1\varphi_2\, dxdt + \int_0^T \langle u(t), \varphi_1\varphi_2\rangle_\alpha dt + \int_0^T \langle \partial_t^\beta u(t), \varphi_1\varphi_2\rangle_\alpha dt\\&\quad = \int_0^T \lambda \int_\Omega |u(t)|^{p-1}\varphi_1\varphi_2\, dxdt,\end{aligned}$$

which gives that

$$\begin{aligned}&\int_0^T \int_\Omega \partial_t^\beta u(t)\varphi_1\varphi_2\, dxdt + \lambda_1 \int_0^T \int_\Omega u(t)\varphi_1\varphi_2 dxdt + \lambda_1 \int_0^T \int_\Omega \partial_t^\beta u(t)\varphi_1\varphi_2 dxdt\\&\quad = \int_0^T \lambda \int_\Omega |u(t)|^{p-1}\varphi_1\varphi_2\, dxdt.\end{aligned}$$

Set $H(t) = \int_\Omega u(x,t)\varphi_1(x)dx$. Then

$$\begin{aligned}&(1+\lambda_1)\int_0^T \partial_t^\beta H(t)\varphi_2(t)dt + \lambda_1 \int_0^T H(t)\varphi_2(t)dt\\&\quad = \lambda \int_0^T \int_\Omega |u(t)|^{p-1}\varphi_1\varphi_2\, dxdt.\end{aligned}$$

It follows from Jensen's inequality that

$$\begin{aligned}&(1+\lambda_1)\int_0^T \partial_t^\beta H(t)\varphi_2(t)dt + \lambda_1 \int_0^T H(t)\varphi_2(t)dt\\&\quad \geq \int_0^T \lambda (\int_\Omega u(t)\varphi_1 dx)^{p-1}\varphi_2(t)dt\\&\quad = \lambda \int_0^T H(t)^{p-1}\varphi_2(t)dt,\end{aligned}$$

thanks to $p > 2$. By Lemma 2.9, we have

$$\begin{aligned}&-(1+\lambda_1)\int_0^T (H(t)-H(0)) I_{T-}^\beta \varphi_2'(t)dt + \lambda_1 \int_0^T H(t)\varphi_2(t)dt\\&\quad \geq \int_0^T \lambda (\int_\Omega u(t)\varphi_1 dx)^{p-1}\varphi_2(t)dt\\&\quad = \lambda \int_0^T H(t)^{p-1}\varphi_2(t)dt, \qquad (5.1)\end{aligned}$$

where

$$I_{T-}^{\beta}\varphi_2'(t) = \frac{1}{\Gamma(\beta)}\int_t^T (\tau - t)^{1-\beta}\varphi_2'(\tau)d\tau.$$

On one hand, choose $\varphi_2 = I_{T-}^{\beta}\bar{\varphi}(t)$ with $\bar{\varphi} \in C_0^1(0,T)$ and $\bar{\varphi} \geq 0$. Then we deduce from [13, Lemma 2.21] that

$$\int_0^T (\lambda H^{p-1} - \lambda_1 H) I_{T-}^{\beta}\bar{\varphi}(t)dt \leq (1+\lambda_1)\int_0^T (H - H(0))\bar{\varphi}dt. \tag{5.2}$$

Applying Lemma 2.8 to the left of (5.2), it leads to

$$\int_0^T I_{0+}^{\beta}(\lambda H^{p-1} - \lambda_1 H)\bar{\varphi}(t)dt \leq (1+\lambda_1)\int_0^T (H - H(0))\bar{\varphi}dt.$$

The arbitrary of $\bar{\varphi}$ gives that

$$I_{0+}^{\beta}(\lambda H^{p-1} - \lambda_1 H) + (1+\lambda_1)H(0) \leq (1+\lambda_1)H(t).$$

Since $H(0) > (\frac{\lambda_1}{\lambda})^{1/(p-2)}$, we have $H(t) > (\frac{\lambda_1}{\lambda})^{1/(p-2)}$ as t small enough. Then we have

$$H(t) \geq H(0) > (\frac{\lambda_1}{\lambda})^{1/(p-2)} \text{ for all } t \in [0,T].$$

On the other hand, choose $\varphi_2 = (1 - \frac{t}{T})_+^k$, $t \in [0,T]$, $k > \max\{1, (p-1)\beta/(p-2)\}$, in (5.1). By a direct calculation (see [13]), one can show that

$$-I_{T-}^{\beta}\varphi_2(t) = \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}T^{-\beta}(1 - \frac{t}{T})^{k-\beta}.$$

Then

$$\begin{aligned}-\int_0^T (H(t) - H(0))I_{T-}^{\beta}\varphi_2(t)dt = {}& \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}T^{-\beta}\int_0^T H(t)(1-\frac{t}{T})^{k-\beta}dt \\ & - \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}\frac{H(0)T^{1-\beta}}{(k+1-\beta)}.\end{aligned}$$

By the Hölder inequality, one has

$$\begin{aligned}&\int_0^T H(t)(1-\frac{t}{T})^{k-\beta}dt \\ &\leq \left(\int_0^T H^{p-1}(t)(1-\frac{t}{T})^k dt\right)^{1/(p-1)} \left(\int_0^T (1-\frac{t}{T})^{(k-\beta-\frac{k}{p-1})\frac{p-1}{p-2}}dt\right)^{(p-2)/(p-1)}.\end{aligned}$$

Further, using the Young inequality, for any $\varepsilon > 0$ we get

$$\begin{aligned}&(1+\lambda_1)\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}T^{-\beta}\int_0^T H(t)(1-\frac{t}{T})^{k-\beta}dt \\ &\leq \varepsilon\int_0^T H^{p-1}(t)(1-\frac{t}{T})^k dt + C(\varepsilon)T^{1-(p-1)\beta/(p-2)}.\end{aligned}$$

We know that there exists a constant $C > 0$ such that

$$\int_0^T (\lambda H^{p-1} - \lambda_1 H)\varphi_2 dt + (1+\lambda_1)\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}\frac{H(0)T^{1-\beta}}{k+1-\beta} \leq \varepsilon \int_0^T H^{p-1}\varphi_2 dt + C(\varepsilon)T^{1-(p-1)\beta/(p-2)}.$$

Choosing ε small enough such that $H(0) > (\frac{\lambda_1}{\lambda-\varepsilon})^{-1/(p-2)}$, we get $H(0) \leq CT^{\beta-(p-1)\beta/(p-2)}$ for some $C > 0$. If problem (1.1) has a global weak solution, we obtain $H(0) = 0$ by letting $T \to \infty$, which contradicts $H(0) > (\frac{\lambda_1}{\lambda})^{-1/(p-2)}$. Therefore we show that the global nonexistence of problem (1.1). □

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References

[1] Bekkai, A., Rebiai, B., Kirane, M.: On local existence and blowup of solutions for a time-space fractional diffusion equation with exponential nonlinearity. Math. Methods Appl. Sci. **42**, 1819–1830 (2019)

[2] Benjamin, T.B., Bona, J.L., Mahony, J.J.: Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. Ser. A **272**, 47–78 (1972)

[3] Bisci, G.M., Rădulescu, V.D., Servadei, R.: Variational Methods for Nonlocal Fractional Equations, Encyclopedia of Mathematics and its Applications, vol. 162. Cambridge University Press, Cambridge (2016)

[4] Caffarelli, L.: Some nonlinear problems involving non-local diffusions, ICIAM 07–6th International Congress on Industrial and Applied Mathematics, pp. 43–56. Zürich, Eur. Math. Soc. (2009)

[5] Caffarelli, L.: Non-local diffusions, drifts and games. Nonlinear Partial Differ. Equ. Abel Symposia **7**, 37–52 (2012)

[6] Can, N.H., Kumar, D., Viet, T.V., Nguyen, A.T.: On time fractional pseudo-parabolic equations with nonlocal integral conditions. Math. Methods Appl. Sci. (2021). https://doi.org/10.1002/mma.7196

[7] Chaoui, A., Rezgui, N.: Solution to fractional pseudoparabolic equation with fractional integral condition. Rend. Circ. Mat. Palermo, II. Ser. **67**, 205–213 (2018)

[8] Colombo, F., Gantner, J.: An application of the S-functional calculus to fractional diffusion processes. Milan J. Math. **86**(2), 225–303 (2018)

[9] del Castillo-Negrete, D., Carreras, B.A., Lynch, V.E.: Fractional diffusion in plasma turbulance. Phys. Plasmas **11**, 3854–3864 (2004)

[10] do Ó, J.M., Giacomoni, J., Mishra, P.K.: Nehari manifold for fractional Kirchhoff systems with critical nonlinearity. Milan J. Math. **87**(2), 201–231 (2019)

[11] Fu, Y.Q., Zhang, X.J.: Global existence and asymptotic behavior of weak solutions for time-space fractional Kirchhoff-type diffusion equations. Discrete Contin. Dyn. Syst. B (2021). https://doi.org/10.3934/dcdsb.2021091

[12] Han, Y.Z.: Finite time blowup for a semilinear pseudo-parabolic equation with general nonlinearity. Appl. Math. Lett. **99**, 105986 (2020)

[13] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier Science Ltd, Amsterdam (2006)

[14] Korpusov, M.O., Sveshnikov, A.G.: Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics. Comput. Math. Math. Phys. **43**, 1765–1797 (2003)

[15] Laskin, N.: Fractional quantum mechanics and Lévy path integrals. Phys. Lett. A **268**, 298–305 (2000)

[16] Li, L., Liu, J.G.: Some compactness criteria for weak solutions of time fractional PDEs. SIAM J. Math. Anal. **50**, 3963–3995 (2018)

[17] Li, L., Liu, J.G., Wang, L.Z.: Cauchy problems for Keller-Segel type time-space fractional diffusion equation. J. Differ. Equ. **265**, 1044–1096 (2018)

[18] Long, Q.F., Chen, J.Q.: Blow-up phenomena for a nonlinear pseudo-parabolic equation with nonlocal source. Appl. Math. Lett. **74**, 181–186 (2017)

[19] Medved, M.: A new approach to an analysis of Henry type integral inequalities and their Bihari type versions. J. Math. Anal. Appl. **214**, 349–366 (1997)

[20] Mingqi, X., Rădulescu, V.D., Zhang, B.L.: Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions. Nonlinearity **31**, 3228–3250 (2018)

[21] Mingqi, X., Rădulescu, V.D., Zhang, B.L.: Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity. Calc. Var. Partial Differ. Equ. **57**, 1–27 (2019)

[22] Molica Bisci, G., Rădulescu, V.D.: Ground state solutions of scalar field fractional for Schrödinger equations. Calc. Var. Partial Differ. Equ. **54**, 2985–3008 (2015)

[23] Nabti, A.: Life span of blowing-up solutions to the Cauchy problem for a time-space fractional diffusion equation. Comput. Math. Appl. **78**, 1302–1316 (2019)

[24] Nguyen, A.T., Hammouch, Z., Karapinar, E., Tuan, N.H.: On a nonlocal problem for a Caputo time-fractional pseudoparabolic equation. Math. Methods Appl. Sci. (2021). https://doi.org/10.1002/mma.7743

[25] Padron, V.: Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation. Trans. Am. Math. Soc. **356**, 2739–2756 (2004)

[26] Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Switzerland (1993)

[27] Ting, T.W.: Certain non-steady flows of second-order fluids. Arch. Ration. Mech. Anal. **14**, 1–26 (1963)

[28] Tuan, N.H., Au, V.V., Xu, R.Z.: Semilinear Caputo time-fractional pseudo-parabolic equations. Commun. Pure Appl. Anal. **20**, 583–621 (2021)

[29] Vázquez, J.L.: Nonlinear diffusion with fractional Laplacian operators. Nonlinear Partial Differ. Equ. **7**, 271–298 (2012). (Abel Symp. Springer, Heidelberg)

[30] Vergara, V., Zacher, R.: Optimal decay estimates for time-fractional and other non-local subdiffusion equations via energy methods. SIAM J. Math. Anal. **47**, 210–239 (2015)

[31] Wang, X.C., Xu, R.Z.: Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation. Adv. Nonlinear Anal. **10**, 261–288 (2021)

[32] Wang, F.L., Hu, D., Xiang, M.Q.: Combined effects of Choquard and singular nonlinearities in fractional Kirchhoff problems. Adv. Nonlinear Anal. **10**, 636–658 (2021)

[33] Xiang, M., Rădulescu, V.D., Zhang, B.L.: Nonlocal Kirchhoff problems with singular exponential nonlinearity. Appl. Math. Optim. **84**(1), 915–954 (2021)

[34] Xiang, M., Hu, D., Zhang, B.L., Wang, Y.: Multiplicity of solutions for variable-order fractional Kirchhoff equations with nonstandard growth. J. Math. Anal. Appl. **501**, 124269 (2021)

[35] Xu, R.Z., Su, J.: Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations. J. Funct. Anal. **264**, 2732–2763 (2013)

[36] Xu, Y.Q., Tan, Z.: Blow-up of solutions for a time-space fractional evolution system. Acta Math. Sin. **29**, 1067–1074 (2013)

[37] Zacher, R.: Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients. J. Math. Anal. Appl. **348**, 137–149 (2008)

[38] Zacher, R.: Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces. Funkc. Ekv. **52**, 1–18 (2009)

[39] Zhang, Q.G., Sun, H.R.: The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation, Topological Methods. Nonlinear Anal. **46**, 69–92 (2015)

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