DOUBLE PHASE IMPLICIT OBSTACLE PROBLEMS WITH CONVECTION AND MULTIVALUED MIXED BOUNDARY VALUE CONDITIONS*

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Abstract. In this paper we consider a mixed boundary value problem with a nonhomogeneous, nonlinear differential operator (called a double phase operator), a nonlinear convection term (a reaction term depending on the gradient), three multivalued terms, and an implicit obstacle constraint. Under very general assumptions on the data, we prove that the solution set of such an implicit obstacle problem is nonempty (so there is at least one solution) and weakly compact. The proof of our main result uses the Kakutani-Ky Fan fixed point theorem for multivalued operators along with the theory of nonsmooth analysis and variational methods for pseudomonotone operators.

Key words. Clarke's generalized gradient, convection term, convex subdifferential, double phase problem, existence results, implicit obstacle, Kakutani-Ky Fan fixed point theorem, mixed boundary conditions, multivalued mapping

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1. Introduction. Obstacle problems go back to the pioneering work by Stefan [53], who studied the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water. Obstacle problems are also commonly used in physics, biology, and financial mathematics. Some relevant examples include the dam problem, the Hele–Shaw flow, pricing of American options, quadrature domains, random matrices, etc.

This paper is devoted to the study of a nonlinear partial differential system with a nonlinear convection term, three multivalued terms, and an implicit obstacle effect. To this end, let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, such that its boundary $\Gamma := \partial\Omega$ is Lipschitz continuous and it is divided into three mutually disjoint parts Γ_1, Γ_2, and Γ_3 with Γ_1 having positive Lebesgue measure. It should be pointed out that in our setting the parts Γ_2 and Γ_3 can be empty, i.e., Γ_1 could be the whole boundary

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$\Gamma_1 = \Gamma$. Moreover, let $1 < p < q < N$ and let $\mu \colon \overline{\Omega} \to [0, \infty)$ be a given function, $U_1 \colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$, $U_2 \colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ be two multivalued mappings, $\phi \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ be a convex function with respect to the second argument, and $f \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a nonlinear convection function. We study the following problem:

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) & \\
+|u|^{p-2}u + \mu(x)|u|^{q-2}u \in U_1(x, u) + f(x, u, \nabla u) \quad & \text{in } \Omega, \\
u = 0 \quad & \text{on } \Gamma_1, \\
\frac{\partial u}{\partial \nu_a} \in U_2(x, u) \quad & \text{on } \Gamma_2, \\
-\frac{\partial u}{\partial \nu_a} \in \partial_c \phi(x, u) \quad & \text{on } \Gamma_3, \\
L(u) \leq J(u), &
\end{aligned}
$$

(1.1)

where

$$
\frac{\partial u}{\partial \nu_a} := \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \cdot \nu
$$

with ν being the unit normal vector on Γ, $\partial_c \phi(x, u)$ being the convex subdifferential of $s \mapsto \phi(x, s)$, and $L, J \colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ being given functions defined on the Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$; see section 2 for its precise definition.

We point out that problem (1.1) combines several interesting phenomena. First the differential operator involved is the so-called double phase operator, which is given by

(1.2) $\qquad \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \quad \text{for } u \in W^{1,\mathcal{H}}(\Omega).$

The corresponding energy functional related to (1.2) is given by

(1.3) $\qquad \omega \mapsto \int_{\Omega} \left(|\nabla \omega|^p + \mu(x)|\nabla \omega|^q\right) \mathrm{d}x$

and was initially introduced by Zhikov [60] in 1986. Such a functional was used to describe models for strongly anisotropic materials and it also demonstrated its importance in the study of duality theory as well as in the context of the Lavrentiev phenomenon (see Zhikov [61]). The main feature of the variational integral (1.3) is the fact that the energy density changes its ellipticity and growth properties according to the point in the domain. Indeed, the energy density of (1.3) exhibits ellipticity in the gradient of order q on the points x where $\mu(x)$ is positive and of order p on the points x where $\mu(x)$ vanishes. In general, double phase differential operators and corresponding energy functionals interpret various comprehensive natural phenomena and model several problems in mechanics, physics and engineering Sciences. For instance, in elasticity theory, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents p and q (see Zhikov [62]). Further, we mention some famous results in the regularity theory of local minimizers of (1.3) (see, for example, the papers of Baroni, Colombo, and Mingione [5, 6], Colombo and Mingione [15, 16], and Marcellini [35, 34]).

A second interesting phenomenon is the appearance of a nonlinearity that depends on the gradient of the solution. Such terms are said to be convection functions. The difficulty in the study of such terms is their nonvariational character, that

is, the standard variational tools to corresponding energy functionals cannot be applied. In the past years several interesting works have been published with convection terms, we refer to the papers of de Araujo and Faria [18], El Manouni, Marino, and Winkert [19], Faraci, Motreanu, and Puglisi [20], Faraci and Puglisi [21], Figueiredo and Madeira [23], Gasiński and Papageorgiou [24], Liu, Motreanu, and Zeng [32], Marano and Winkert [33], Papageorgiou, Rădulescu, and Repovš [45], and Pucci and Temperini [49].

Further interesting phenomena are the combination of an implicit obstacle effect along with mixed boundary conditions in a very general setting and the appearance of multivalued mappings (which include as special case Clarke's generalized gradients) and convex subdifferentials. We point out that in several critical situations arising in engineering and economic models, such as Nash equilibrium problems with shared constraints, semipermeability problems with free boundary conditions, and transport route optimization with feedback control, the constraint conditions, usually, depend explicitly on the unknown solution. On the other hand, the theory of (variational-) hemivariational inequalities was originally developed by Panagiotopoulos [41, 42] (see also Naniewicz and Panagiotopoulos [40]) in order to study the nonsmooth mechanical problems. The main feature in hemivariational inequalities is to remove the hypotheses on differentiability and convexity of energy functionals with the help of generalized subgradients introduced by Clarke [13]. Nowadays such problems with implicit obstacle effect and Clarke's generalized gradient for homogeneous Dirichlet problems have been considered by many authors, see, for example, the works of Alleche and Rădulescu [1], Aussel, Sultana, and Vetrivel [2], Bonanno, Motreanu, and Winkert [8], Carl and Winkert [11], Migórski, Khan, and Zeng [37, 36], Gwinner [28], Zeng, Migórski, and Khan [58], and the references therein. We also refer to the recent monograph of Carl and Le [10] about multivalued variational inequalities and inclusions.

Let us comment on some relevant special cases of problem (1.1).

(i) Let $j_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 \colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ be such that $x \mapsto j(x,s)$ and $x \mapsto j_2(x,s)$ are measurable in Ω and on Γ_2, respectively, and $s \mapsto j_1(x,s)$ and $s \mapsto j_2(x,s)$ are both locally Lipschitz continuous. Moreover, let $r_1 \colon \mathbb{R} \to \mathbb{R}$ and $r_2 \colon \mathbb{R} \to \mathbb{R}$ be two functions and denote by ∂j_i Clarke's generalized gradient of j_i (with respect to the second variable) for $i = 1, 2$. If U_1 and U_2 are defined by $U_1(x,s) = r_1(s)\partial j_1(x,s)$ for a. a. $x \in \Omega$, $s \in \mathbb{R}$ and $U_2(x,s) = r_2(s)\partial j_2(x,s)$ for a. a. $x \in \Gamma_2$, $s \in \mathbb{R}$, then problem (1.1) becomes
(1.4)

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$$
$$+|u|^{p-2}u + \mu(x)|u|^{q-2}u \in r_1(u)\partial j_1(x,u) + f(x,u,\nabla u) \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma_1,$$
$$\frac{\partial u}{\partial \nu_a} \in r_2(u)\partial j_2(x,u) \qquad \text{on } \Gamma_2,$$
$$-\frac{\partial u}{\partial \nu_a} \in \partial_c \phi(x,u) \qquad \text{on } \Gamma_3,$$
$$L(u) \leq J(u).$$

As far as we know problem (1.4) has not been studied yet. As a special case of our main theorem we state in Theorem 3.11 that the solution set of (1.4) is nonempty and weakly compact.

(ii) If $\Gamma_2 = \emptyset$ and $\Gamma_3 = \emptyset$, i.e., $\Gamma_1 = \Gamma$, then problem (1.1) reduces to the following double phase implicit obstacle system with Dirichlet boundary condition:

(1.5)
$$- \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$$
$$+|u|^{p-2}u + \mu(x)|u|^{q-2}u \in U_1(x,u) + f(x,u,\nabla u) \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma,$$
$$L(u) \le J(u).$$

If $U_1(x,s) = -\partial j(x,s)$ with $j\colon \Omega \times \mathbb{R} \to \mathbb{R}$ being locally Lipschitz continuous with respect to the second argument, problem (1.5) has been recently studied by Zeng et al. [56]. A further special case when f is independent of u or $U_1 \equiv 0$ and $J(u) = \infty$ has been treated by Zeng et al. [54, 57, 55]. Additionally to $U_1 \equiv 0$, if we suppose $J(u) \equiv +\infty$, (1.5) reduces to a single valued equation studied by Gasiński and Winkert [26]. Note that the techniques used in these papers differ from ours.

(iii) Let $\Psi\colon \Omega \to \mathbb{R}$ be a given obstacle. When $J(u) \equiv 0$ and $L(u) := \int_\Omega (u(x) - \Psi(x))^+ \, dx$ for all $u \in W^{1,\mathcal{H}}(\Omega)$, then our problem (1.1) can be rewritten to the following obstacle inclusion problem:

(1.6)
$$- \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$$
$$+|u|^{p-2}u + \mu(x)|u|^{q-2}u \in U_1(x,u) + f(x,u,\nabla u) \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma_1,$$
$$\frac{\partial u}{\partial \nu_a} \in U_2(x,u) \qquad \text{on } \Gamma_2,$$
$$-\frac{\partial u}{\partial \nu_a} \in \partial_c \phi(x,u) \qquad \text{on } \Gamma_3,$$
$$u(x) \le \Psi(x) \qquad \text{in } \Omega.$$

To the best of our knowledge, this special case of (1.1) has not been considered yet. The same can be said when $\Phi\colon \Gamma \to \mathbb{R}$ is a given obstacle on the boundary. Then the last inequality in (1.6) is replaced by $u(x) \le \Phi(x)$ on Γ.

(iv) If $J(u) \equiv +\infty$ for all $u \in W^{1,\mathcal{H}}(\Omega)$, then problem (1.1) turns into the following mixed boundary value problem without obstacle effect:

(1.7)
$$- \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$$
$$+|u|^{p-2}u + \mu(x)|u|^{q-2}u \in U_1(x,u) + f(x,u,\nabla u) \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma_1,$$
$$\frac{\partial u}{\partial \nu_a} \in U_2(x,u) \qquad \text{on } \Gamma_2,$$
$$-\frac{\partial u}{\partial \nu_a} \in \partial_c \phi(x,u) \qquad \text{on } \Gamma_3,$$

which has not been investigated yet. As a corollary we can prove that there exists a solution of (1.7) and the solution set of (1.7) is weakly compact (see Corollary 3.10 below).

The main goal of the paper is to develop a general framework for determining the existence of a (weak) solution to the nonlinear double phase differential inclusion problem (1.1) via employing the Kakutani-Ky Fan fixed point theorem for multivalued operators, the theory of nonsmooth analysis, and variational methods for pseudomonotone operators. In fact, to the best of our knowledge, this is the first work which

combines a double phase phenomena along with an implicit obstacle constraint, a nonlinear convection term (i.e., a reaction term depending on the gradient), and multivalued mixed boundary conditions which include a convex subdifferential operator and an abstract multivalued function.

The features of this paper are the following:

(i) the presence of a nonhomogeneous differential operator with different isotropic growth, which generates a double phase associated energy;

(ii) the analysis developed in this paper is concerned with the combined effects of a nonstandard operator with unbalanced growth, a convection nonlinearity, three multivalued terms, and an implicit obstacle constraint;

(iii) the proofs rely on fixed point methods for multivalued operators in combination with tools from nonsmooth analysis and theory of pseudomonotone operators.

To the best of our knowledge, this is the first paper dealing with multivalued double phase obstacle implicit problems with convection.

In order to give a complete picture of existence results for double phase problems, we also refer to some recent works dealing with different types of problems. In Bahrouni, Rădulescu, and Repovš [3] and Bahrouni, Rădulescu, and Winkert [4] the authors studied a class of double phase energy functionals arising in the theory of transonic flow which are driven by the Baouendi–Grushin operator with variable exponents. Very recently, Biagi, Esposito, and Vecchi [7] considered positive singular solutions of double phase type equations and proved symmetry as well as monotonicity properties of such solutions. A new class of double phase problems with variable growth has been developed in Cencelj, Rădulescu, and Repovš [12] while Colasuonno and Squassina [14] studied existence and properties of related variational eigenvalues. A new type of so-called Finsler double phase operators has been introduced in Farkas and Winkert [22] in order to prove an existence result for corresponding singular problems. Furthermore, a Nehari manifold treatment for finding sign changing solutions can be found in Gasiński and Papageorgiou [25] for a locally Lipschitz right-hand side, in Gasiński and Winkert [27] for nonlinear boundary conditions, and in Liu and Dai [30] for superlinear right-hand sides. Existence and multiplicity results for double phase problems with Robin boundary conditions has been recently studied in Papageorgiou, Rădulescu, and Repovš [44] and Papageorgiou, Vetro, and Vetro [46]. Perera and Squassina [48] used a Morse theoretical approach in order to prove the existence of a solution where they used a cohomological local splitting to get an estimate of the critical groups at zero. Existence results for general quasilinear elliptic equations of double phase type in $\mathbb{R}^N$ have been obtained in Zhang and Rădulescu [59] by using the tools of critical points theory in generalized Orlicz and Sobolev spaces with variable exponent. Finally, we mention the overview articles of Rădulescu [50, 51] about isotropic and anistropic double-phase problems and of Mingione and Rădulescu [39] about recent developments for problems with nonstandard growth and nonuniform ellipticity.

The paper is organized as follows. Section 2 presents a detailed overview about Musielak–Orlicz Sobolev spaces including several useful embedding results. Moreover, we state some fundamental results in nonsmooth analysis as well as the Kakutani-Ky Fan fixed point theorem for multivalued functions. In section 3, we first impose the full assumptions on the data of problem (1.1) and introduce an auxiliary problem defined in (3.5). Then, we apply an existence theorem for a class of mixed variational inequalities involving pseudomonotone operators in which the constraint set is a bounded, closed, and convex set to show that the solution map of the auxiliary problem (3.5) is well-defined and completely continuous (see Theorem 3.4). Moreover, we use the Kakutani-Ky Fan fixed point theorem along with the theory of nonsmooth analysis

to explore the nonemptiness and compactness of the solution set of problem (1.1). Finally, several special cases of our problem (1.1) are discussed and the corresponding existence results are obtained at the end of the paper.

2. Preliminaries. In this section we are going to provide the main definitions and tools which will be needed in what follows. To this end, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ such that Γ is divided into three mutually disjoint parts Γ_1, Γ_2, and Γ_3 with Γ_1 having a positive Lebesgue measure. For any fixed $r \in [1,\infty)$ and for any subset D of $\overline{\Omega}$ we denote the usual Lebesgue spaces by $L^r(D) := L^r(D; \mathbb{R})$ and $L^r(D; \mathbb{R}^N)$ equipped with the norm $\|\cdot\|_{r,D}$ given by

$$\|u\|_{r,D} := \left(\int_D |u|^r \, dx \right)^{\frac{1}{r}} \quad \text{for all } u \in L^r(D).$$

We set $L^r(D)_+ := \{u \in L^r(D) : u(x) \geq 0 \text{ for a.a. } x \in D\}$ and define by $W^{1,r}(\Omega)$ the corresponding Sobolev space endowed with the norm $\|\cdot\|_{1,r,\Omega}$ given by

$$\|u\|_{1,r,\Omega} := \|u\|_{r,\Omega} + \|\nabla u\|_{r,\Omega} \quad \text{for all } u \in W^{1,r}(\Omega).$$

Throughout the paper the symbols " $\xrightarrow{w}$ " and "$\rightarrow$" stand for the weak and the strong convergence, respectively. Furthermore, $r' > 1$ is the conjugate of $r > 1$, that is, $\frac{1}{r} + \frac{1}{r'} = 1$. Moreover, we denote by r^* and r_* the critical exponents to r in the domain and on the boundary, respectively, given by

$$(2.1) \qquad r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N \end{cases} \quad \text{and} \quad r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N, \end{cases}$$

respectively.

Since Γ_1 has positive measure we know from Korn's inequality that the function space

$$W := \left\{ u \in W^{1,p}(\Omega) : u = 0 \text{ for a. a. } x \in \Gamma_1 \right\}$$

equipped with the norm $\|\nabla \cdot\|_{p,\Omega}$, is a reflexive Banach space. In what follows, let $\lambda > 0$ and $\rho > 0$ be the smallest constants such that

$$(2.2) \qquad \|u\|_{p,\Omega}^p \leq \lambda \|\nabla u\|_{p,\Omega}^p \quad \text{and} \quad \|u\|_{p,\Gamma_2}^p \leq \rho \|\nabla u\|_{p,\Omega}^p$$

for all $u \in W$. From Simon [52, formula (2.2)] we have the well-known inequality

$$(2.3) \qquad \left(|x|^{r-2}x - |y|^{r-2}y \right) \cdot (x - y) \geq k(r)|x - y|^r$$

for $r \geq 2$ and for all $x, y \in \mathbb{R}^N$, where $k(r)$ is a positive constant.

In the entire paper, we assume that

$$(2.4) \qquad 1 < p < q < N, \quad q < p^*, \quad \text{and} \quad 0 \leq \mu(\cdot) \in L^\infty(\Omega).$$

Next we introduce the function $\mathcal{H} \colon \Omega \times [0, \infty) \to [0, \infty)$ defined by

$$\mathcal{H}(x, t) = t^p + \mu(x)t^q \quad \text{for all } (x, t) \in \Omega \times [0, \infty)$$

and recall that $M(\Omega)$ stands for the space of all measurable functions $u \colon \Omega \to \mathbb{R}$, where we identify two such functions which differ on a Lebesgue-null set. Then, the Musielak–Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ is given by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\},$$

where the modular function is given by

$$\rho_{\mathcal{H}}(u) := \int_{\Omega} \mathcal{H}(x, |u|)\,\mathrm{d}x = \int_{\Omega} \left(|u|^p + \mu(x)|u|^q\right)\,\mathrm{d}x.$$

We know that $L^{\mathcal{H}}(\Omega)$ is uniformly convex, so it is a reflexive Banach space. Additionally, let us introduce the seminormed function space $L^q_{\mu}(\Omega)$

$$L^q_{\mu}(\Omega) = \left\{u \in M(\Omega) \,:\, \int_{\Omega} \mu(x)|u|^q\,\mathrm{d}x < +\infty\right\}$$

endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_{\Omega} \mu(x)|u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Furthermore, we recall that the Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is given by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. Clearly, $W^{1,\mathcal{H}}(\Omega)$ is a reflexive Banach space.

Let us now introduce a closed subspace V of $W^{1,\mathcal{H}}(\Omega)$ defined by

$$V := \{u \in W^{1,\mathcal{H}}(\Omega) \,:\, u = 0 \text{ on } \Gamma_1\}.$$

It is clear that V is also a reflexive Banach space. In the following we denote the norm of V by $\|u\|_V = \|u\|_{1,\mathcal{H}}$ for all $u \in V$, and by $\|\cdot\|_{V^*}$ we denote the norm of the dual space V^* of V.

The following proposition states the main embeddings for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$; see Gasiński and Winkert [27, Proposition 2.2] or Crespo-Blanco et al. [17, Propositions 2.17 and 2.19].

PROPOSITION 2.1. *Let (2.4) be satisfied and let p^* as well as p_* be the critical exponents to p as given in (2.1) for $r = p$. Then the following embeddings hold:*
  (i) *$L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ are continuous for all $r \in [1,p]$;*
  (ii) *$W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [1,p^*]$ and compact for all $r \in [1,p^*)$;*
  (iii) *$W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is continuous for all $r \in [1,p_*]$ and compact for all $r \in [1,p_*)$;*

(iv) $L^{\mathcal{H}}(\Omega) \hookrightarrow L_{\mu}^{q}(\Omega)$ *is continuous;*
(v) $L^{q}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

Remark 2.2. Note that if we replace the space $W^{1,\mathcal{H}}(\Omega)$ by V in Proposition 2.1, then the embeddings (ii) and (iii) remain valid.

The following proposition is due to Liu and Dai [30, Proposition 2.1].

PROPOSITION 2.3. *Let* (2.4) *be satisfied and let* $y \in L^{\mathcal{H}}(\Omega)$. *Then the following hold:*

 (i) *if* $y \neq 0$, *then* $\|y\|_{\mathcal{H}} = \lambda$ *if and only if* $\rho_{\mathcal{H}}\left(\frac{y}{\lambda}\right) = 1$;
 (ii) $\|y\|_{\mathcal{H}} < 1$ *(resp.* > 1 *and* $= 1$*) if and only if* $\rho_{\mathcal{H}}(y) < 1$ *(resp.* > 1 *and* $= 1$*)*;
 (iii) *if* $\|y\|_{\mathcal{H}} < 1$, *then* $\|y\|_{\mathcal{H}}^{q} \leq \rho_{\mathcal{H}}(y) \leq \|y\|_{\mathcal{H}}^{p}$;
 (iv) *if* $\|y\|_{\mathcal{H}} > 1$, *then* $\|y\|_{\mathcal{H}}^{p} \leq \rho_{\mathcal{H}}(y) \leq \|y\|_{\mathcal{H}}^{q}$;
 (v) $\|y\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(y) \to 0$;
 (vi) $\|y\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(y) \to +\infty$.

Further, we introduce the nonlinear operator $A \colon V \to V^{*}$ given by

$$(2.5) \qquad \langle A(u), v \rangle := \int_{\Omega} \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) \cdot \nabla v \, \mathrm{d}x$$
$$+ \int_{\Omega} \left(|u|^{p-2} u + \mu(x) |u|^{q-2} u \right) v \, \mathrm{d}x$$

for $u, v \in V$ with $\langle \cdot, \cdot \rangle$ being the duality pairing between V and its dual space V^{*}. The following proposition states the main properties of $A \colon V \to V^{*}$. We refer to Liu and Dai [30, Proposition 3.1] or Crespo-Blanco et al. [17, Proposition 3.4] for its proof.

PROPOSITION 2.4. *The operator A defined by* (2.5) *is bounded, continuous, monotone (hence maximal monotone) and of type* (S_{+}), *that is,*

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{and} \quad \limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0$$

imply $u_n \to u$ *in* V.

Subsequently, we recall some notion and results concerning nonsmooth analysis and multivalued analysis. We start by recalling definitions and properties of semicontinuous multivalued operators.

DEFINITION 2.5. *Let* Y *and* Z *be topological spaces, let* $D \subset Y$ *be a nonempty set, and let* $G \colon Y \to 2^{Z}$ *be a multivalued map.*

 (i) *The map* G *is called upper semicontinuous (u.s.c.) at* $y \in Y$ *if for each open set* $O \subset Z$ *such that* $G(y) \subset O$, *there exists a neighborhood* $N(y)$ *of* y *satisfying* $G(N(y)) := \cup_{z \in N(y)} G(z) \subset O$. *If it holds for each* $y \in D$, *then* G *is called to be u.s.c. in* D.
 (ii) *The map* G *is closed at* $y \in Y$ *if for every sequence* $\{(y_n, z_n)\}_{n \in \mathbb{N}} \subset \mathrm{Gr}(G)$ *satisfying* $(y_n, z_n) \to (y, z)$ *in* $Y \times Z$, *it holds* $(y, z) \in \mathrm{Gr}(G)$, *where* $\mathrm{Gr}(G)$ *is the graph of* G *defined by*

$$\mathrm{Gr}(G) := \{(y, z) \in Y \times Z \, : \, z \in G(y)\}.$$

If it holds for each $y \in Y$, *then* G *is called to be closed or* G *has a closed graph.*

The following proposition provides two useful ways to determinate that a multivalued map is u.s.c.

PROPOSITION 2.6. *Let $F: X \to 2^Y$ with X and Y being topological spaces. The following statements are equivalent:*

(i) *F is u.s.c.*

(ii) *For each closed set $C \subset Y$, $F^-(C) := \{x \in X \ : \ F(x) \cap C \neq \emptyset\}$ is closed in X.*

(iii) *For each open set $O \subset Y$, $F^+(O) := \{x \in X \ : \ F(x) \subset O\}$ is open in X.*

Let $(E, \|\cdot\|_E)$ be a real Banach space. Recall that a function $\varphi: E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called to be proper, convex, and lower semicontinuous (l.s.c.) if the following conditions are satisfied:

- $D(\varphi) := \{u \in E \ : \ \varphi(u) < +\infty\} \neq \emptyset$;
- for any $u, v \in E$ and $t \in (0, 1)$, it holds that $\varphi(tu + (1-t)v) \leq t\varphi(u) + (1-t)\varphi(v)$;
- $\liminf_{n \to \infty} \varphi(u_n) \geq \varphi(u)$, where the sequence $\{u_n\}_{n \in \mathbb{N}} \subset E$ is such that $u_n \xrightarrow{w} u$ as $n \to \infty$ for some $u \in E$.

If the map φ is convex, an element $x^* \in E^*$ is called a subgradient of φ at $u \in E$ if

$$(2.6) \qquad \langle x^*, v - u \rangle \leq \varphi(v) - \varphi(u) \quad \text{for all } v \in E.$$

The set of all elements $x^* \in E^*$ which satisfies (2.6) is called the convex subdifferential of φ at u and is denoted by $\partial_c \varphi(u)$.

We call a function $j: E \to \mathbb{R}$ to be locally Lipschitz at $x \in E$ if there is a neighborhood $O(x)$ of x and a constant $L_x > 0$ such that

$$|j(y) - j(z)| \leq L_x \|y - z\|_E \quad \text{for all } y, z \in O(x).$$

Moreover, we denote by

$$j^\circ(x; y) := \limsup_{z \to x, \lambda \downarrow 0} \frac{j(z + \lambda y) - j(z)}{\lambda}$$

the generalized directional derivative of j at the point x in the direction y and $\partial j: E \to 2^{E^*}$ given by

$$\partial j(x) := \{\xi \in E^* \ : \ j^\circ(x; y) \geq \langle \xi, y \rangle_{E^* \times E} \text{ for all } y \in E\} \quad \text{for all } x \in E$$

being the generalized gradient of j at x in the sense of Clarke.

The next proposition summarizes the properties of generalized gradients and generalized directional derivatives of a locally Lipschitz functions (see, for example, Migórski, Ochal, and Sofonea [38, Proposition 3.23]).

PROPOSITION 2.7. *Let $j: E \to \mathbb{R}$ be locally Lipschitz with Lipschitz constant $L_x > 0$ at $x \in E$. Then we have the following:*

(i) *The function $y \mapsto j^\circ(x; y)$ is positively homogeneous and subadditive and satisfies*

$$|j^\circ(x; y)| \leq L_x \|y\|_E \quad \text{for all } y \in E.$$

(ii) *The function $(x, y) \mapsto j^\circ(x; y)$ is u.s.c.*

(iii) *For each $x \in E$, $\partial j(x)$ is a nonempty, convex, and weakly* compact subset of E^* with $\|\xi\|_{E^*} \leq L_x$ for all $\xi \in \partial j(x)$.*

(iv) *$j^\circ(x; y) = \max\{\langle \xi, y \rangle_{E^* \times E} \ : \ \xi \in \partial j(x)\}$ for all $y \in E$.*

(v) *The multivalued function $E \ni x \mapsto \partial j(x) \subset E^*$ is u.s.c. from E into w^*-E^*.*

The following theorem provides a useful criterion to determinate the existence of solutions for a class of mixed variational inequalities involving pseudomonotone operators (see, for example, Liu et al. [31, Proposition 5]).

THEOREM 2.8. *Let V be a reflexive Banach space, $T: V \to 2^{V^*}$ be a pseudomonotone operator in the sense of Brézis, $C \subset V$ be nonempty, bounded, closed, and convex, and $\varphi: V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and l.s.c. function. Then, for a given element $f \in V^*$, there exist $u \in C$ and $u^* \in T(u)$ such that*

$$\langle u^* - f, v - u \rangle + \varphi(v) - \varphi(u) \geq 0 \text{ for all } v \in C.$$

We end this section by recalling the so-called Kakutani-Ky Fan fixed point theorem (see, for example, Papageorgiou, Kyritsi, and Yiallourou [43, Theorem 2.6.7]). We are going to apply this theorem in order to prove the existence of solutions of problem (1.1).

THEOREM 2.9. *Let Y be a reflexive Banach space and $D \subseteq Y$ be a nonempty, bounded, closed, and convex set. Let $\Lambda: D \to 2^D$ be a multivalued map with nonempty, closed, and convex values such that its graph is sequentially closed in $Y_w \times Y_w$ topology. Then Λ has a fixed point.*

3. Main results. This section is devoted to the main results of this paper proving the existence of weak solutions to problem (1.1). First we state the hypotheses on the data of problem (1.1).

H(f): $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies the following conditions:

(i) there exist $a_f, b_f \geq 0$ and a function $\alpha_f \in L^{\frac{q_1}{q_1 - 1}}(\Omega)_+$ satisfying

$$|f(x, s, \xi)| \leq a_f |\xi|^{\frac{p(q_1 - 1)}{q_1}} + b_f |s|^{q_1 - 1} + \alpha_f(x)$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$, where $1 < q_1 < p^*$ and p^* are the critical exponents to p in the domain given in (2.1);

(ii) there exist $c_f, d_f \geq 0$, $\theta_1, \theta_2 \in [1, p]$, and a function $\beta_f \in L^1(\Omega)_+$ such that

$$f(x, s, \xi)s \leq c_f |\xi|^{\theta_1} + d_f |s|^{\theta_2} + \beta_f(x)$$

for a. a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$;

(iii) there exist $e_f, h_f \geq 0$ such that

$$(f(x, s, \xi) - f(x, t, \xi))(s - t) \leq e_f |s - t|^p,$$
$$|f(x, s, \xi_1) - f(x, s, \xi_2)| \leq h_f |\xi_1 - \xi_2|^{p-1}$$

for a. a. $x \in \Omega$, for all $s, t \in \mathbb{R}$, and for all $\xi, \xi_1, \xi_2 \in \mathbb{R}^N$.

H(U_1): $U_1: \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies the following conditions:

(i) $U_1(x, s)$ is a nonempty, bounded, closed, and convex set in $\mathbb{R}$ for a. a. $x \in \Omega$ and all $s \in \mathbb{R}$;

(ii) $x \mapsto U_1(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$;

(iii) $s \mapsto U_1(x, s)$ is u.s.c.;

(iv) there exist $\theta_3 \in [1, p]$, $\alpha_{U_1} \in L^{p'}(\Omega)_+$, and $\beta_{U_1} > 0$ such that

$$|U_1(x, s)| \leq \alpha_{U_1}(x) + \beta_{U_1} |s|^{\theta_3 - 1}$$

for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

H(U_2): $U_2 \colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies the following conditions:
  (i) $U_2(x, s)$ is a nonempty, bounded, closed, and convex set in $\mathbb{R}$ for a. a. $x \in \Gamma_2$ and all $s \in \mathbb{R}$;
  (ii) $x \mapsto U_2(x, s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$;
  (iii) $s \mapsto U_2(x, s)$ is u.s.c.;
  (iv) there exist $\theta_4 \in [1, p]$, $\alpha_{U_2} \in L^{p'}(\Gamma_2)_+$ and $\beta_{U_2} > 0$ such that

$$|U_2(x, s)| \leq \alpha_{U_2}(x) + \beta_{U_2}|s|^{\theta_4 - 1}$$

  for a. a. $x \in \Gamma_2$ and for all $s \in \mathbb{R}$.

H(ϕ): $\phi \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
  (i) $x \mapsto \phi(x, r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$ such that $x \mapsto \phi(x, 0)$ belongs to $L^1(\Gamma_3)$;
  (ii) for a. a. $x \in \Gamma_3$, $r \mapsto \phi(x, r)$ is convex and l.s.c.

H(L): $L \colon V \to \mathbb{R}$ is positively homogeneous and subadditive such that

$$(3.1) \qquad L(u) \leq \limsup_{n \to \infty} L(u_n)$$

  whenever $\{u_n\}_{n \in \mathbb{N}} \subset V$ is such that $u_n \xrightarrow{w} u$ in V for some $u \in V$.

H(J): $J \colon V \to (0, +\infty)$ is weakly continuous, that is, for any sequence $\{u_n\}_{n \in \mathbb{N}} \subset V$ such that $u_n \xrightarrow{w} u$ for some $u \in V$, we have

$$J(u_n) \to J(u).$$

H(0): The inequalities

$$\max\{e_f, h_f \lambda^{\frac{1}{p}}\} < k(p) \quad \text{and} \quad \max\{c_f \delta(\theta_1), d_f \lambda \delta(\theta_2)$$
$$+ \beta_{U_1} \delta(\theta_3) + \beta_{U_2} \delta(\theta_4)\} < 1$$

  hold, where $k(p) > 0$ is given in (2.3), λ, ρ are given in (2.2), and $\delta \colon [1, p] \to \{1, 0\}$ is defined by

$$\delta(\theta) = \begin{cases} 1 & \text{if } \theta = p, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1. The homogeneity and subadditivity of L reveals that L is a convex function. So, it is not difficult to verify that if $L \colon V \to \mathbb{R}$ is l.s.c., then inequality (3.1) holds automatically.

We consider the following multivalued map $K \colon V \to 2^V$ given by

$$(3.2) \qquad K(u) := \{v \in V \colon L(v) \leq J(u)\}$$

for all $u \in V$.

Next, we give the definition of a weak solution to problem (1.1).

DEFINITION 3.2. *A function $u \in V$ is said to be a weak solution of problem (1.1) if there exist functions $\eta \in L^{p'}(\Omega)$, $\xi \in L^{p'}(\Gamma_2)$ with $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$ and $\xi(x) \in U_2(x, u(x))$ for a. a. $x \in \Gamma_2$ and if*

$$\int_\Omega \left(|\nabla u|^{p-2} \nabla u + \mu(x)|\nabla u|^{q-2} \nabla u\right) \cdot \nabla(v - u)\, \mathrm{d}x + \int_{\Gamma_3} \phi(x, v)\, \mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x, u)\, \mathrm{d}\Gamma$$

$$+ \int_\Omega \left(|u|^{p-2}u + \mu(x)|u|^{q-2}u\right)(v - u)\, \mathrm{d}x$$

$$\geq \int_\Omega \eta(x)(v - u)\, \mathrm{d}x + \int_{\Gamma_2} \xi(x)(v - u)\, \mathrm{d}\Gamma + \int_\Omega f(x, u, \nabla u)(v - u)\, \mathrm{d}x$$

is satisfied for all $v \in K(u)$, *where the multivalued function* $K \colon V \to 2^V$ *is defined by* (3.2).

The following lemma states several properties of K, which will be used in what follows.

LEMMA 3.3. *Let* $J \colon V \to (0, +\infty)$ *and* $L \colon V \to \mathbb{R}$ *be two functions such that* $\mathrm{H}(L)$ *and* $\mathrm{H}(J)$ *are satisfied. Then, the following statements hold:*
  (i) *for each* $u \in V$, $K(u)$ *is closed and convex in* V *such that* $0 \in K(u)$;
  (ii) *the graph* $\mathrm{Gr}(K)$ *of* K *is sequentially closed in* $V_w \times V_w$, *that is,* K *is sequentially closed from* V *with the weak topology into the subsets of* V *with the weak topology;*
  (iii) *if* $\{u_n\}_{n \in \mathbb{N}} \subset V$ *is a sequence such that*

$$u_n \xrightarrow{w} u \quad in \ V \quad as \quad n \to \infty$$

  for some $u \in V$, *then for each* $v \in K(u)$ *there exists a sequence* $\{v_n\}_{n \in \mathbb{N}} \subset V$ *such that*

$$v_n \in K(u_n) \quad and \quad v_n \to v \quad in \ V \quad as \quad n \to \infty.$$

Proof. (i) Note that $J(u) > 0$ for each $u \in V$ and L is positively homogeneous. So we have $0 = L(0_V) < J(u)$ for each $u \in V$, and thus $K(u) \neq \emptyset$ for every $u \in V$. The convexity of L (see Remark 3.1) deduces directly that $K(u)$ is convex for each $u \in V$. On the other hand, inequality (3.1) implies that $K(u)$ is closed for each $u \in V$.

(ii) Let $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset V \times V$ be a sequence and $(u, v) \in V \times V$ such that $v_n \in K(u_n)$ for each $n \in \mathbb{N}$ and

$$(u_n, v_n) \xrightarrow{w} (u, v) \quad in \ V \times V \quad as \quad n \to \infty.$$

Then, for each $n \in \mathbb{N}$, it holds that $v_n \in K(u_n)$, namely, $L(v_n) \leq J(u_n)$. Keeping in mind that J is weakly continuous on V (see hypotheses $\mathrm{H}(J)$), we obtain

$$(3.3) \qquad \lim_{n \to \infty} J(u_n) = J(u).$$

However, hypotheses $\mathrm{H}(L)$ turn out

$$(3.4) \qquad L(v) \leq \limsup_{n \to \infty} L(v_n).$$

Combining (3.3) and (3.4) yields

$$L(v) \leq \limsup_{n \to \infty} L(v_n) \leq \limsup_{n \to \infty} J(u_n) = J(u).$$

This means that $v \in K(u)$, that is, $(u, v) \in \mathrm{Gr}(K)$. Therefore, we conclude that $\mathrm{Gr}(K)$ is sequentially closed in $V_w \times V_w$.

(iii) Let $\{u_n\}_{n \in \mathbb{N}} \subset V$ be a sequence such that

$$u_n \xrightarrow{w} u \quad in \ V \quad as \quad n \to \infty$$

for some $u \in V$ and let $v \in K(u)$ be arbitrary. Because of $J(w) > 0$ for every $w \in V$, we construct the sequence $\{v_n\}_{n \in \mathbb{N}} \subset V$ given by

$$v_n = \frac{J(u_n)}{J(u)} v.$$

Due to $v \in K(u)$ (i.e., $L(v) \leq J(u)$), it follows from hypotheses H(L) that

$$L(v_n) = L\left(\frac{J(u_n)}{J(u)}v\right) = \frac{J(u_n)}{J(u)}L(v) \leq \frac{J(u_n)}{J(u)}J(u) = J(u_n).$$

Therefore, $v_n \in K(u_n)$. Moreover, a simply calculation shows

$$\|v_n - v\|_V = \left|\frac{J(u_n)}{J(u)} - 1\right|\|v\|_V = 0.$$

Hence, we get that $v_n \to v$ in V as $n \to \infty$. This completes the proof of the lemma. □

Let $X = L^p(\Omega) \times L^p(\Gamma_2)$ and $(\eta, \xi) \in X^* := L^{p'}(\Omega) \times L^{p'}(\Gamma_2)$, $w \in V$ be arbitrary. We introduce the following nonlinear auxiliary elliptic unilateral obstacle system

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) & \\
+ |u|^{p-2}u + \mu(x)|u|^{q-2}u = \eta(x) + f(x, u, \nabla u) \quad & \text{in } \Omega, \\
u = 0 \quad & \text{on } \partial\Gamma_1, \\
\frac{\partial u(x)}{\partial \nu_a} = \xi(x) \quad & \text{on } \partial\Gamma_2, \\
-\frac{\partial u(x)}{\partial \nu_a} \in \partial_c\phi(x, u) \quad & \text{on } \partial\Gamma_3, \\
L(u) \leq J(w). &
\end{aligned}
$$

(3.5)

By virtue of Definition 3.2, a function $u \in V$ is said to be a weak solution of problem (3.5) if the following holds: $u \in K(w)$ and

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \cdot \nabla(v - u)\,\mathrm{d}x + \int_{\Gamma_3}\phi(x, v)\,\mathrm{d}\Gamma - \int_{\Gamma_3}\phi(x, u)\,\mathrm{d}\Gamma$$

$$+ \int_\Omega\left(|u|^{p-2}u + \mu(x)|u|^{q-2}u\right)(v - u)\,\mathrm{d}x$$

$$\geq \int_\Omega \eta(x)(v - u)\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)(v - u)\,\mathrm{d}\Gamma + \int_\Omega f(x, u, \nabla u)(v - u)\,\mathrm{d}x$$

for all $v \in K(w)$.

Next, we prove the existence and uniqueness of weak solutions to problem (3.5).

THEOREM 3.4. *Let $p \geq 2$. Suppose that (2.4), H(f), H(L), and H(ϕ) are satisfied. If, in addition, the inequality $\max\{e_f, h_f\lambda^{\frac{1}{p}}\} < k(p)$ holds, then, for every $(w, (\eta, \xi)) \in V \times X^*$, there exists a unique weak solution $u \in K(w)$ of problem (3.5).*

Proof. Uniqueness: Let $(w, (\eta, \xi)) \in V \times X^*$ be fixed and assume that the solution set of problem (3.5) associated with $(w, (\eta, \xi)) \in V \times X^*$ is nonempty. Let $u_1, u_2 \in V$ be two weak solutions of problem (3.5) corresponding to $(w, (\eta, \xi)) \in V \times X^*$. This means that $u_i \in K(w)$ and

$$\int_\Omega\left(|\nabla u_i|^{p-2}\nabla u_i + \mu(x)|\nabla u_i|^{q-2}\nabla u_i\right) \cdot \nabla(v - u_i)\,\mathrm{d}x + \int_{\Gamma_3}\phi(x, v)\,\mathrm{d}\Gamma$$

$$- \int_{\Gamma_3}\phi(x, u_i)\,\mathrm{d}\Gamma + \int_\Omega\left(|u_i|^{p-2}u_i + \mu(x)|u_i|^{q-2}u_i\right)(v - u_i)\,\mathrm{d}x$$

$$\geq \int_\Omega\eta(x)(v - u_i)\,\mathrm{d}x + \int_{\Gamma_2}\xi(x)(v - u_i)\,\mathrm{d}\Gamma + \int_\Omega f(x, u, \nabla u_i)(v - u_i)\,\mathrm{d}x$$

for all $v \in K(w)$ and for $i = 1, 2$. Taking $v = u_2$ and $v = u_1$ for the inequalities above with $i = 1$ and $i = 2$, respectively, we sum up the resulting inequalities to get

$$\int_\Omega \left(|\nabla u_1|^{p-2} \nabla u_1 + \mu(x)|\nabla u_1|^{q-2} \nabla u_1 \right) \cdot \nabla (u_1 - u_2) \, dx$$

$$- \int_\Omega \left(|\nabla u_2|^{p-2} \nabla u_2 + \mu(x)|\nabla u_2|^{q-2} \nabla u_2 \right) \cdot \nabla (u_1 - u_2) \, dx$$

$$+ \int_\Omega \left(|u_1|^{p-2} u_1 + \mu(x)|u_1|^{q-2} u_1 \right) (u_1 - u_2) \, dx$$

$$- \int_\Omega \left(|u_2|^{p-2} u_2 + \mu(x)|u_2|^{q-2} u_2 \right) (u_1 - u_2) \, dx$$

$$\leq \int_\Omega (f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2))(u_1 - u_2) \, dx$$

$$= \int_\Omega (f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_1))(u_1 - u_2) \, dx$$

$$+ \int_\Omega (f(x, u_2, \nabla u_1) - f(x, u_2, \nabla u_2))(u_1 - u_2) \, dx.$$

Taking (2.3), (2.4), and H(f)(iii) into account implies

$$k(p) \left(\|\nabla u_1 - \nabla u_2\|_{p,\Omega}^p + \|u_1 - u_2\|_{p,\Omega}^p \right)$$

$$\leq \int_\Omega e_f |u_1 - u_2|^p \, dx + \int_\Omega h_f |\nabla u_1 - \nabla u_2|^{p-1} |u_1 - u_2| \, dx.$$

Applying Hölder's inequality and (2.2) gives

$$k(p) \left(\|\nabla u_1 - \nabla u_2\|_{p,\Omega}^p + \|u_1 - u_2\|_{p,\Omega}^p \right)$$

$$\leq e_f \|u_1 - u_2\|_{p,\Omega}^p + h_f \|\nabla u_1 - \nabla u_2\|_{p,\Omega}^{p-1} \|u_1 - u_2\|_{p,\Omega}$$

$$\leq e_f \|u_1 - u_2\|_{p,\Omega}^p + h_f \lambda^{\frac{1}{p}} \|\nabla u_1 - \nabla u_2\|_{p,\Omega}^p.$$

Hence,

$$\min \left\{ (k(p) - e_f), (k(p) - h_f \lambda^{\frac{1}{p}}) \right\} \left(\|\nabla u_1 - \nabla u_2\|_{p,\Omega}^p + \|u_1 - u_2\|_{p,\Omega}^p \right) \leq 0.$$

By assumption, we know that $\max\{e_f, h_f \lambda^{\frac{1}{p}}\} < k(p)$, thus $u_1 = u_2$. Therefore, for each $(w, (\eta, \xi)) \in V \times X^*$ problem (3.5) has a unique weak solution $u \in V$ provided the solution set of problem (3.5) is nonempty.

Existence: Let $n \in \mathbb{N}$ be fixed such that $K_n := \overline{B(0, n)} \cap K(w) \neq \emptyset$, where $\overline{B(0, n)}$ is the closed ball centered at the origin with radius $n > 0$, that is, $\overline{B(0, n)} := \{u \in V : \|u\|_V \leq n\}$. We first consider the following auxiliary problem: find $u_n \in K_n$ such that

(3.6)
$$\int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x)|\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla (v - u_n) \, dx$$

$$+ \int_\Omega \left(|u_n|^{p-2} u_n + \mu(x)|u_n|^{q-2} u_n \right) (v - u_n) \, dx$$

$$+ \int_{\Gamma_3} \phi(x, v) \, d\Gamma - \int_{\Gamma_3} \phi(x, u_n) \, d\Gamma$$

$$\geq \int_{\Omega} \eta(x)(v - u_n)\, \mathrm{d}x + \int_{\Gamma_2} \xi(x)(v - u_n)\, \mathrm{d}\Gamma$$

$$+ \int_{\Omega} f(x, u_n, \nabla u_n)(v - u_n)\, \mathrm{d}x \quad \text{for all } v \in K_n.$$

Let us consider the function $\varphi \colon V \to \mathbb{R} \cup \{+\infty\}$ defined by

$$(3.7) \qquad \qquad \varphi(u) := \int_{\Gamma_3} \phi(x, u)\, \mathrm{d}\Gamma \quad \text{for all } u \in V.$$

Moreover, let $N_f \colon V \subset L^{q_1}(\Omega) \to L^{q_1}(\Omega)$ be the Nemytskij operator associated to f and let ι be the embedding operator from V to $L^{q_1}(\Omega)$ with its adjoint operator $\iota^* \colon L^{q_1'}(\Omega) \to V^*$. We set $T \colon V \to V^*$

$$Tu = Au - \iota^* N_f(u) \quad \text{for all } u \in V,$$

where $A \colon V \to V^*$ is defined by (2.5). Based on this, it is easy to see that problem (3.6) can be rewritten equivalently to the following one: find $u_n \in K_n$ such that

$$(3.8) \qquad \qquad \langle Tu_n, v - u_n \rangle + \varphi(v) - \varphi(u_n) \geq \langle g, v - u_n \rangle$$

for all $v \in K_n$, where $g \in V^*$ is defined by (because of Riesz's representative theorem)

$$\langle g, w \rangle := \int_{\Omega} \eta(x)w(x)\, \mathrm{d}x + \int_{\Gamma_2} \xi(x)w(x)\, \mathrm{d}\Gamma \quad \text{for all } w \in V.$$

Note that K_n is a bounded, closed, and convex subset of V. We are going to apply Theorem 2.8 to prove that problem (3.8) has at least one solution. By hypothesis $\mathrm{H}(\phi)$ it is obvious to see that φ defined in (3.7) is a proper, convex, and l.s.c. function. Reasoning as in the proof of Theorem 3.2 in Gasiński and Winkert [26], we conclude that T is a bounded and pseudomonotone operator. Therefore, all conditions of Theorem 2.8 are satisfied and so for each $n \in \mathbb{N}$ there exists one solution $u_n \in K_n$ of problem (3.8). Applying the same arguments as in the proof of uniqueness part, we conclude that u_n is the unique solution to problem (3.8).

Furthermore, we claim that there exists a number $N_0 \in \mathbb{N}$ such that the unique solution u_{N_0} of problem (3.8) with $n = N_0$ satisfies the inequality

$$(3.9) \qquad \qquad \|u_{N_0}\|_V < N_0.$$

Let us suppose (3.9) is not true. Then for each $n \in \mathbb{N}$ we have $\|u_n\|_V = n$. So, it holds that $\|u_n\|_V \to +\infty$ as $n \to \infty$.

Recalling that $0 \in K(w)$ (see Lemma 3.3(i)), a simple calculation shows

$$(3.10) \quad \begin{aligned} &\int_{\Omega} |\nabla u_n|^p + \mu(x)|\nabla u_n|^q + |u_n|^p + \mu(x)|u_n|^q\, \mathrm{d}x + \int_{\Gamma_3} \phi(x, u_n)\, \mathrm{d}\Gamma \\ &\leq \int_{\Omega} \eta(x)u_n\, \mathrm{d}x + \int_{\Gamma_2} \xi(x)u_n\, \mathrm{d}\Gamma + \int_{\Gamma_3} \phi(x, 0)\, \mathrm{d}\Gamma + \int_{\Omega} f(x, u_n, \nabla u_n)u_n\, \mathrm{d}x. \end{aligned}$$

Since φ is a proper, convex, and l.s.c. function, there are constants $\alpha_\varphi > 0$ and $\beta_\varphi \geq 0$ satisfying

$$(3.11) \qquad \qquad \varphi(v) \geq -\alpha_\varphi \|v\|_V - \beta_\varphi \text{ for all } v \in V,$$

owing to Brezis [9, Proposition 1.10]. Let $\varepsilon_1, \varepsilon_2 > 0$. Employing hypothesis $H(f)(ii)$, we have

(3.12)
$$\int_\Omega f(x, u_n, \nabla u_n) u_n(x) \, dx$$
$$\leq \int_\Omega c_f |\nabla u_n(x)|^{\theta_1} + d_f |u_n(x)|^{\theta_2} + \beta_f(x) \, dx$$
$$\leq \begin{cases} c_f \|\nabla u\|_{p,\Omega}^p + d_f \|u_n\|_{p,\Omega}^p + \|\beta_f\|_{1,\Omega} & \text{if } \theta_1 = \theta_2 = p, \\ c_f \|\nabla u_n\|_{p,\Omega}^p + \varepsilon_1 \|u_n\|_{p,\Omega}^p + c_1(\varepsilon_1) + \|\beta_f\|_{1,\Omega} & \text{if } \theta_1 = p \text{ and } \theta_2 < p, \\ \varepsilon_2 \|\nabla u_n\|_{p,\Omega}^p + c_2(\varepsilon_2) + d_f \|u_n\|_{p,\Omega}^p + \|\beta_f\|_{1,\Omega} & \text{if } \theta_1 < p \text{ and } \theta_2 = p, \\ \varepsilon_2 \|\nabla u_n\|_{p,\Omega}^p + c_2(\varepsilon_2) + \varepsilon_1 \|u_n\|_{p,\Omega}^p + c_1(\varepsilon_1) + \|\beta_f\|_{1,\Omega} & \text{if } \theta_1 < p \text{ and } \theta_2 < p \end{cases}$$

for some $c_1(\varepsilon_1), c_2(\varepsilon_2) > 0$. Keeping in mind that $(\eta, \xi) \in X^*$, we use Hölder's inequality to get

(3.13) $$\int_\Omega \eta(x) u_n \, dx + \int_{\Gamma_2} \xi(x) u_n \, d\Gamma \leq \|\eta\|_{p',\Omega} \|u_n\|_{p,\Omega} + \|\xi\|_{p',\Gamma_2} \|u_n\|_{p,\Gamma_2}.$$

Combining (3.11), (3.12), and (3.13) with (3.10) and using (2.2), (2.4), as well as Young's inequality we obtain

$$\int_\Omega |\nabla u_n|^p + \mu(x)|\nabla u_n|^q + |u_n|^p + \mu(x)|u_n|^q \, dx \leq c_3$$

for some $c_3 > 0$. Using Proposition 2.3, it is not difficult to see that the inequality above is a contradiction to the fact that $\|u_n\|_V \to +\infty$ as $n \to \infty$. Therefore, there exists a number $N_0 \in \mathbb{N}$ such that (3.9) is satisfied.

For any $z \in K(w)$, we can choose $t \in (0, 1)$ small enough such that $v_t = u_{N_0} + (1-t)z \in K_{N_0}$ thanks to (3.9). Taking $v = v_t$ in (3.8) gives

$$\langle T u_{N_0}, z - u_{N_0} \rangle + \varphi(z) - \varphi(u_{N_0}) \geq \langle g, z - u_{N_0} \rangle.$$

Since $z \in K(w)$ is arbitrary, we see that $u_{N_0} \in K(w)$ is a weak solution of problem (3.5).

In conclusion, we have shown that for every $(w, (\eta, \xi)) \in V \times X^*$ problem (3.5) has a unique solution. □

Particularly, when $J(w) = +\infty$ for all $w \in V$, then problem (3.5) becomes the following nonobstacle nonlinear elliptic system with mixed boundary conditions: find $u \in V$ such that

(3.14)
$$-\text{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)$$
$$+ |u|^{p-2}u + \mu(x)|u|^{q-2}u = \eta(x) + f(x, u, \nabla u) \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma_1,$$
$$\frac{\partial u}{\partial \nu_a} = \xi(x) \qquad \text{on } \Gamma_2,$$
$$-\frac{\partial u}{\partial \nu_a} \in \partial_c \phi(x, u) \qquad \text{on } \Gamma_3.$$

We have the following corollary for problem (3.14).

COROLLARY 3.5. *Let $p \geq 2$. Assume that* (2.4), H(f), *and* H(ϕ) *are fulfilled. If the inequality* $\max\{e_f, h_f \lambda^{\frac{1}{p}}\} < k(p)$ *holds, then for each pair of function* $(\eta, \xi) \in X^*$, *problem* (3.14) *has a unique weak solution.*

We define $\gamma \colon V \to X$ by $\gamma u = (\gamma_1 u, \gamma_2 u)$ for all $u \in V$, where $\gamma_1 \colon V \to L^p(\Omega)$ is the embedding operator of V to $L^p(\Omega)$ and $\gamma_2 \colon V \to L^p(\Gamma_2)$ is the trace operator. From Proposition 2.1 we know that γ is a linear, bounded, and compact operator. Next we introduce the multivalued mapping $U \colon X \to 2^{X^*}$ defined by

(3.15)
$$U(u, v) := \left\{ (\eta, \xi) \in X^* : \eta(x) \in U_1(x, u(x)) \text{ a. a. in } \Omega, \ \xi(x) \in U_2(x, v(x)) \text{ a. a. on } \Gamma_2 \right\}$$

for all $(u, v) \in X$.

LEMMA 3.6. *Assume that* H(U_1) *and* H(U_2) *are satisfied. Then, the following hold:*
  (i) *U is well-defined and for each $(u, v) \in X$ the set $U(u, v)$ is bounded, closed, and convex in X^*;*
  (ii) *U is strongly-weakly u.s.c.*

Proof. (i) Note that U_1 and U_2 satisfy an upper Carathéodory condition, that is, $x \mapsto U_1(x, s)$ and $x \mapsto U_2(x, s)$ are measurable and $s \mapsto U_1(x, s)$ and $s \mapsto U_2(x, s)$ are u.s.c. Therefore, we can apply Theorem 1.3.4 of Kamenskii, Obukhovskii, and Zecca [29] to conclude that for each $(u, v) \in X$ the functions $x \mapsto U_1(x, u(x))$ and $x \mapsto U_2(x, v(x))$ are both measurable in Ω and on Γ_2, respectively. Additionally, from the Yankov–von Neumann–Aumann selection theorem (see, e. g., Papageorgiou and Winkert [47, Theorem 2.7.25]), we are able to find measurable selections $\eta \colon \Omega \to \mathbb{R}$ and $\xi \colon \Gamma_2 \to \mathbb{R}$ such that $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$ and $\xi(x) \in U_2(x, v(x))$ for a. a. $x \in \Gamma_2$. From hypotheses H(U_1)(iv) and H(U_2)(iv) we have that

$$\|\eta\|_{p',\Omega}^{p'} = \int_\Omega |\eta(x)|^{p'} \, \mathrm{d}x \leq \int_\Omega \left(\alpha_{U_1}(x) + \beta_{U_1} |u(x)|^{\theta_3 - 1} \right)^{p'} \, \mathrm{d}x$$

(3.16)
$$\leq \begin{cases} \displaystyle\int_\Omega \left(\alpha_{U_1}(x) + \beta_{U_1} |u(x)|^{p-1} \right)^{p'} \, \mathrm{d}x & \text{if } \theta_3 = p, \\[3mm] \displaystyle\int_\Omega \left(\alpha_{U_1}(x) + c_4 + |u(x)|^{p-1} \right)^{p'} \, \mathrm{d}x & \text{if } \theta_3 < p, \end{cases}$$

$$\leq \begin{cases} \displaystyle\int_\Omega c_5 \left(|\alpha_{U_1}(x)|^{p'} + \beta_{U_1} |u(x)|^p \right) \, \mathrm{d}x & \text{if } \theta_3 = p, \\[3mm] \displaystyle\int_\Omega c_5 \left(|\alpha_{U_1}(x)|^{p'} + 1 + |u(x)|^p \right) \, \mathrm{d}x & \text{if } \theta_3 < p \end{cases}$$

$$< +\infty$$

for some $c_4, c_5 > 0$, and

(3.17)
$$\|\xi\|_{p',\Gamma_2}^{p'} = \int_{\Gamma_2} |\xi(x)|^{p'} \, \mathrm{d}\Gamma \leq \int_{\Gamma_2} \left(\alpha_{U_2}(x) + \beta_{U_2} |v(x)|^{\theta_4 - 1} \right)^{p'} \, \mathrm{d}\Gamma$$

$$\leq \begin{cases} \displaystyle\int_{\Gamma_2} \left(\alpha_{U_2}(x) + \beta_{U_2} |v(x)|^{p-1} \right)^{p'} \, \mathrm{d}\Gamma & \text{if } \theta_4 = p, \\[3mm] \displaystyle\int_{\Gamma_2} \left(\alpha_{U_2}(x) + c_6 + |v(x)|^{p-1} \right)^{p'} \, \mathrm{d}\Gamma & \text{if } \theta_4 < p, \end{cases}$$

$$\leq \begin{cases} \displaystyle\int_{\Gamma_2} c_7 \left(|\alpha_{U_2}(x)|^{p'} + \beta_{U_2}|v(x)|^p \right) \, d\Gamma & \text{if } \theta_4 = p, \\[2ex] \displaystyle\int_{\Gamma_2} c_7 \left(|\alpha_{U_2}(x)|^{p'} + 1 + |v(x)|^p \right) \, d\Gamma & \text{if } \theta_4 < p \end{cases}$$

$$< +\infty$$

for some $c_6, c_7 > 0$, where we have used the Young inequality for the cases $\theta_3 < p$ and $\theta_4 < p$. This means that U is well-defined and for each $(u,v) \in X$ the set $U(u,v)$ is bounded in X^*. Moreover, the closedness and convexity of U_1 and U_2 guarantee that the set $U(u,v)$ is closed and convex in X^* for all $(u,v) \in X$.

(ii) From Proposition 2.6, it is sufficient to prove that for each weakly closed set D of X^*, the set $U^-(D)$ is closed in X. Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset U^-(D)$ be such that $(u_n, v_n) \to (u, v)$ in X as $n \to \infty$ for some $(u, v) \in X$. Without any loss of generality, we may assume that

$$(3.18) \qquad \begin{aligned} u_n(x) &\to u(x) \quad \text{as } n \to \infty \quad \text{for a. a. } x \in \Omega, \\ v_n(x) &\to v(x) \quad \text{as } n \to \infty \quad \text{for a. a. } x \in \Gamma_2. \end{aligned}$$

Let $\{(\eta_n, \xi_n)\}_{n\in\mathbb{N}} \subset X^*$ be a sequence such that $(\eta_n, \xi_n) \in U(u_n, v_n)$ for each $n \in \mathbb{N}$. From (3.16) and (3.17), we can see that $\{(\eta_n, \xi_n)\}_{n\in\mathbb{N}}$ is bounded in X^*. The reflexivity of X permits us to assume that, passing to a subsequence if necessary, $(\eta_n, \xi_n) \xrightarrow{w} (\eta, \xi)$ in X^* as $n \to \infty$ for some $(\eta, \xi) \in D$, owing to the weak closedness of D. Our objective is to prove that $(\eta, \xi) \in U(u, v)$, that is, $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$ and $\xi(x) \in U_2(x, v(x))$ for a. a. $x \in \Gamma_2$. Subsequently, we only show that $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$, and the other proof works similarly.

From Mazur's theorem we know that there exists a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of convex combinations of $\{\eta_n\}_{n\in\mathbb{N}}$ such that

$$(3.19) \qquad \zeta_n \to \eta \quad \text{in } L^{q'}(\Omega) \quad \text{and} \quad \zeta_n(x) \to u(x) \quad \text{for a. a. } x \in \Omega \quad \text{as} \quad n \to \infty.$$

The convexity of U_1 reveals that $\zeta_n(x) \in U_1(x, u_n(x))$ for a. a. $x \in \Omega$. Applying the convergences in (3.18) and (3.19) along with the u.s.c. of U_1 (see hypothesis H(U_1)(iii)), we get that $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$. Likewise, it is true that $\xi(x) \in U_2(x, v(x))$ for a. a. $x \in \Gamma_2$. This means that $(\eta, \xi) \in U(u, v) \cap D$, and thus, $(u, v) \in U^-(D)$. Therefore, we can apply Proposition 2.6 to conclude that U is strongly-weakly u.s.c. $\qquad \square$

By virtue of Theorem 3.4, we are now in a position to consider the solution mapping $S \colon V \times X^* \to V$ of problem (3.5) defined by

$$S(w, (\eta, \xi)) := u(w, (\eta, \xi)) \quad \text{for all } (w, (\eta, \xi)) \in V \times X^*,$$

where $u(w, (\eta, \xi)) \in V$ is the unique solution of problem (3.5) corresponding to $(w, (\eta, \xi)) \in V \times X^*$.

The following lemma indicates that the solution map S is completely continuous, that is, for any sequence $\{(w_n, (\eta_n, \xi_n))\}_{n\in\mathbb{N}} \subset V \times X^*$ and $(u, (\eta, \xi)) \in V \times X^*$ with $(w_n, (\eta_n, \xi_n)) \xrightarrow{w} (w, (\eta, \xi))$ in $V \times X^*$ as $n \to \infty$ it holds that $S(w_n, (\eta_n, \xi_n)) \xrightarrow{w} S(w, (\eta, \xi))$ in V as $n \to \infty$.

LEMMA 3.7. *Let $p \geq 2$. Assume that (2.4), H(f), H(ϕ), H(L), H(J), and H(0) are fulfilled. Then, the solution map $S \colon V \times X^* \to V$ of problem (3.5) is completely continuous.*

Proof. Let $\{(w_n, (\eta_n, \xi_n))\}_{n\in\mathbb{N}} \subset V \times X^*$, $\{u_n\}_{n\in\mathbb{N}} \subset V$ be sequences and $(w, (\eta, \xi)) \in V \times X^*$ such that

$$(w_n, (\eta_n, \xi_n)) \xrightarrow{w} (w, (\eta, \xi)) \text{ in } V \times X^* \text{ as } n \to \infty$$

and $u_n = S(w_n, (\eta_n, \xi_n))$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we have $u_n \in K(w_n)$ and

$$
\begin{aligned}
\int_\Omega & \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right) \cdot \nabla(v - u_n)\,\mathrm{d}x \\
& + \int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n\right)(v - u_n)\,\mathrm{d}x \\
& + \int_{\Gamma_3} \phi(x, v)\,\mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x, u_n)\,\mathrm{d}\Gamma \\
& \geq \int_\Omega \eta_n(x)(v - u_n)\,\mathrm{d}x + \int_{\Gamma_2} \xi_n(x)(v - u_n)\,\mathrm{d}\Gamma \\
& + \int_\Omega f(x, u_n, \nabla u_n)(v - u_n)\,\mathrm{d}x \quad \text{for all } v \in K(w_n).
\end{aligned}
$$

(3.20)

Claim. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V.

Arguing by contradiction, without any loss of generality, we may assume that $\|u_n\|_V \to +\infty$ as $n \to \infty$. Lemma 3.3(i) indicates that $0 \in K(w_n)$ for each $n \in \mathbb{N}$. Taking $v = 0$ in (3.20) yields

$$
\begin{aligned}
& \int_\Omega |\nabla u_n|^p + \mu(x)|\nabla u_n|^q + |u_n|^p + \mu(x)|u_n|^q\,\mathrm{d}x + \int_{\Gamma_3} \phi(x, u_n(x))\,\mathrm{d}\Gamma \\
& \leq \int_\Omega \eta_n(x)u_n(x)\,\mathrm{d}x + \int_{\Gamma_2} \xi_n(x)u_n(x)\,\mathrm{d}\Gamma + \int_\Omega f(x, u_n, \nabla u_n)u_n(x)\,\mathrm{d}x \\
& + \int_{\Gamma_3} \phi(x, 0)\,\mathrm{d}\Gamma \\
& \leq \|\phi(\cdot, 0)\|_{1,\Gamma_3} + \int_\Omega c_f|\nabla u_n(x)|^{\theta_1} + d_f|u_n(x)|^{\theta_2} + \beta_f(x)\,\mathrm{d}x + \|\eta_n\|_{p',\Omega}\|u_n\|_{p,\Omega} \\
& + \|\xi_n\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2},
\end{aligned}
$$

where we have used Hölder's inequality and hypotheses H(f)(ii) and H(ϕ)(i). Using (2.2), (2.4), (3.11), (3.12), and inequality $\max\{c_f\delta(\theta_1), d_f\delta(\theta_2)\} < 1$ we obtain

$$\int_\Omega |\nabla u_n|^p + \mu(x)|\nabla u_n|^q + |u_n|^p + \mu(x)|u_n|^q\,\mathrm{d}x \leq c_8$$

for some $c_8 > 0$ which is independent of n thanks to the boundedness of $\{(\eta_n, \xi_n)\}_{n\in\mathbb{N}}$ in X^*. Combining the inequality above and Proposition 2.3, we get a contradiction. So, the claim is proved.

From the claim we may assume, passing to a subsequence if necessary, that

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{as} \quad n \to \infty$$

for some $u \in V$.

We assert that $u = S(w, (\eta, \xi))$, i.e., u is the unique solution of problem (3.5) corresponding to $(w, (\eta, \xi)) \in V \times X^*$. Lemma 3.3(ii) indicates that $\mathrm{Gr}(K)$ is sequentially

closed in $V_w \times V_w$. Keeping in mind $\{(u_n, w_n)\}_{n \in \mathbb{N}} \subset \mathrm{Gr}(K)$ and $(u_n, w_n) \xrightarrow{w} (u, w)$ in $V \times V$ as $n \to \infty$, we have $u \in K(w)$. On the other hand, it follows from Lemma 3.3(iii) that there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset V$ satisfying $y_n \in K(w_n)$ and

$$y_n \to u \quad \text{in } V \quad \text{as} \quad n \to \infty.$$

Inserting $v = y_n$ in (3.20) and passing to the upper limit as $n \to \infty$ to the resulting inequality, we have

$$\limsup_{n \to \infty} \int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x) |\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla(u_n - y_n) \, \mathrm{d}x$$

$$+ \limsup_{n \to \infty} \int_\Omega \left(|u_n|^{p-2} u_n + \mu(x) |u_n|^{q-2} u_n \right) (u_n - y_n) \, \mathrm{d}x$$

$$\leq \limsup_{n \to \infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n - y_n) \, \mathrm{d}x + \limsup_{n \to \infty} \int_{\Gamma_3} \phi(x, y_n) \, \mathrm{d}\Gamma$$

$$- \liminf_{n \to \infty} \int_{\Gamma_3} \phi(x, u_n) \, \mathrm{d}\Gamma + \limsup_{n \to \infty} \int_{\Gamma_2} \xi_n(x)(y_n - u_n) \, \mathrm{d}\Gamma$$

$$+ \limsup_{n \to \infty} \int_\Omega \eta_n(x)(u_n - y_n) \, \mathrm{d}x.$$

Because the embeddings of V into $L^{q_1}(\Omega)$, V into $L^p(\Omega)$, and V into $L^p(\Gamma_2)$ are all compact (due to $q_1 < p^*$; see Proposition 2.1), the sequences $\{(\eta_n, \xi_n)\}_{n \in \mathbb{N}}$ and $\{f(\cdot, u_n, \nabla u_n)\}_{n \in \mathbb{N}}$ are bounded in X^* and $L^{q_1'}(\Omega)$ (see hypothesis H(f)(i)), respectively. We obtain

$$\limsup_{n \to \infty} \int_\Omega \eta_n(x)(u_n - y_n) \, \mathrm{d}x + \limsup_{n \to \infty} \int_{\Gamma_2} \xi_n(x)(u_n - y_n) \, \mathrm{d}\Gamma$$

$$+ \limsup_{n \to \infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n - y_n) \, \mathrm{d}x = 0.$$

From the convexity and continuity of ϕ (hence, $V \ni u \mapsto \int_{\Gamma_3} \phi(x, u(x)) \, \mathrm{d}\Gamma$ is weakly l.s.c.), it is not difficult to see that

$$\limsup_{n \to \infty} \int_{\Gamma_3} \phi(x, y_n(x)) \, \mathrm{d}\Gamma - \liminf_{n \to \infty} \int_{\Gamma_3} \phi(x, u_n(x)) \, \mathrm{d}\Gamma \leq 0.$$

Taking into account the last three inequalities, we have

$$\limsup_{n \to \infty} \int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x) |\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla(u_n - u) \, \mathrm{d}x$$

$$+ \limsup_{n \to \infty} \int_\Omega \left(|u_n|^{p-2} u_n + \mu(x) |u_n|^{q-2} u_n \right) (u_n - u) \, \mathrm{d}x$$

$$\leq \limsup_{n \to \infty} \int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x) |\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla(u_n - u) \, \mathrm{d}x$$

$$+ \liminf_{n \to \infty} \int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x) |\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla(u - y_n) \, \mathrm{d}x$$

$$+ \limsup_{n \to \infty} \int_\Omega \left(|u_n|^{p-2} u_n + \mu(x) |u_n|^{q-2} u_n \right) (u_n - u) \, \mathrm{d}x$$

$$+ \liminf_{n \to \infty} \int_\Omega \left(|u_n|^{p-2} u_n + \mu(x) |u_n|^{q-2} u_n \right) (u - y_n) \, \mathrm{d}x$$

$$\leq \limsup_{n \to \infty} \int_{\Omega} \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x) |\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla (u_n - y_n) \, dx$$

$$+ \limsup_{n \to \infty} \int_{\Omega} \left(|u_n|^{p-2} u_n + \mu(x) |u_n|^{q-2} u_n \right) (u_n - y_n) \, dx$$

$$\leq 0.$$

The latter together with the convergence $u_n \xrightarrow{w} u$ in V as $n \to \infty$ and Proposition 2.4 (i.e., A is of type (S_+)) deduces that

$$u_n \to u \quad \text{in } V \quad \text{as} \quad n \to \infty.$$

For any fixed $v \in K(w)$, Lemma 3.3(iii) permits us to find a sequence $\{v_n\}_{n \in \mathbb{N}} \subset V$ such that

$$v_n \in K(w_n) \quad \text{and} \quad v_n \to v \quad \text{in } V \quad \text{as} \quad n \to \infty.$$

Taking $v = v_n$ in (3.20) and passing to the upper limit as $n \to \infty$ we obtain

$$\int_{\Omega} \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) \cdot \nabla (v - u) \, dx + \int_{\Gamma_3} \phi(x, v) \, d\Gamma - \int_{\Gamma_3} \phi(x, u) \, d\Gamma$$

$$+ \int_{\Omega} \left(|u|^{p-2} u + \mu(x) |u|^{q-2} u \right) (v - u) \, dx$$

$$\geq \limsup_{n \to \infty} \int_{\Omega} \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x) |\nabla u_n|^{q-2} \nabla u_n \right) \cdot \nabla (v_n - u_n) \, dx$$

$$+ \limsup_{n \to \infty} \int_{\Gamma_3} \phi(x, v_n) \, d\Gamma - \liminf_{n \to \infty} \int_{\Gamma_3} \phi(x, u_n) \, d\Gamma$$

$$+ \limsup_{n \to \infty} \int_{\Omega} \left(|u_n|^{p-2} u_n + \mu(x) |u_n|^{q-2} u_n \right) (v_n - u_n) \, dx$$

$$\geq \limsup_{n \to \infty} \int_{\Omega} \eta_n(x)(v_n - u_n) \, dx + \limsup_{n \to \infty} \int_{\Gamma_2} \xi_n(x)(v_n - u_n) \, d\Gamma$$

$$+ \limsup_{n \to \infty} \int_{\Omega} f(x, u_n, \nabla u_n)(v_n - u_n) \, dx$$

$$\geq \int_{\Omega} \eta(x)(v - u) \, dx + \int_{\Gamma_2} \xi(x)(v - u) \, d\Gamma + \int_{\Omega} f(x, u, \nabla u)(v - u) \, dx.$$

Note that $v \in K(w)$ is arbitrary, so we conclude that u is the unique solution of problem (3.5) corresponding to $(w, (\eta, \xi))$, that is, $u = S(w, (\eta, \xi))$. Since every convergent subsequence of $\{u_n\}_{n \in \mathbb{N}}$ converges to the same limit u, we know that the whole sequence $\{u_n\}_{n \in \mathbb{N}}$ converges strongly to u in V. This means that $S \colon V \times X^* \to V$ is completely continuous. $\square$

The following lemma gives a priori estimates for the weak solutions of problem (1.1).

LEMMA 3.8. *Let $p \geq 2$. Suppose that (2.4), H(f), H(ϕ), H(U_1), H(U_2), H(L), H(J), and H(0) are satisfied. If the solution set of problem (1.1), denoted by Υ, is nonempty, then there exists a constant $M > 0$ such that*

$$\text{(3.21)} \qquad \qquad \|u\|_V \leq M \quad \text{for all } u \in \Upsilon.$$

Proof. Let $u \in V$ be a weak solution of problem (1.1). Then, there exist functions $(\eta, \xi) \in X^*$ with $\eta(x) \in U_1(x, u(x))$ for a. a. $x \in \Omega$ and $\xi(x) \in U_2(x, u(x))$ for a. a. $x \in \Gamma_2$ such that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u \right) \cdot \nabla(v - u) \, dx + \int_{\Gamma_3} \phi(x, v) \, d\Gamma - \int_{\Gamma_3} \phi(x, u) \, d\Gamma$$

$$+ \int_\Omega \left(|u|^{p-2}u + \mu(x)|u|^{q-2}u \right)(v - u) \, dx$$

$$\geq \int_\Omega f(x, u, \nabla u)(v - u) \, dx + \int_\Omega \eta(x)(v - u) \, dx + \int_{\Gamma_2} \xi(x)(v - u) \, d\Gamma$$

for all $v \in K(u)$. Since $0 \in K(u)$, we can take $v = 0$ in the inequality above to get

(3.22)
$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u \right) \cdot \nabla u + |u|^p + \mu(x)|u|^q \, dx$$

$$\leq \int_{\Gamma_3} \phi(x, 0) \, d\Gamma - \int_{\Gamma_3} \phi(x, u) \, d\Gamma + \int_\Omega f(x, u, \nabla u)u \, dx + \int_\Omega \eta(x)u \, dx + \int_{\Gamma_2} \xi(x)u \, d\Gamma.$$

Let $\varepsilon_3, \varepsilon_4 > 0$. From hypotheses H($U_1$)(iv) and H($U_2$)(iv) it follows that

$$\int_\Omega \eta(x)u \, dx \leq \int_\Omega |\eta(x)||u| \, dx \leq \int_\Omega \left(\alpha_{U_1}(x) + \beta_{U_1}|u|^{\theta_3 - 1} \right) |u| \, dx$$

(3.23)
$$\leq \begin{cases} \int_\Omega \alpha_{U_1}(x)|u| + \beta_{U_1}|u|^p \, dx & \text{if } \theta_3 = p, \\ \int_\Omega (\alpha_{U_1}(x) + c_9(\varepsilon_3))|u| + \varepsilon_3|u|^p \, dx & \text{if } \theta_3 < p \end{cases}$$

$$\leq \begin{cases} \|\alpha_{U_1}\|_{p',\Omega}\|u\|_{p,\Omega} + \beta_{U_1}\|u\|_{p,\Omega}^p & \text{if } \theta_3 = p, \\ \|\alpha_{U_1}(\cdot) + c_9(\varepsilon_3)\|_{p',\Omega}\|u\|_{p,\Omega} + \varepsilon_3\|u\|_{p,\Omega}^p & \text{if } \theta_3 < p \end{cases}$$

for some $c_9(\varepsilon_3) > 0$ and

$$\int_{\Gamma_2} \xi(x)u \, d\Gamma \leq \int_{\Gamma_2} |\xi(x)||u| \, d\Gamma \leq \int_{\Gamma_2} (\alpha_{U_2}(x) + \beta_{U_2}|u|^{\theta_4 - 1})|u| \, d\Gamma$$

(3.24)
$$\leq \begin{cases} \int_{\Gamma_2} \alpha_{U_2}(x)|u| + \beta_{U_2}|u|^p \, d\Gamma & \text{if } \theta_4 = p, \\ \int_{\Gamma_2} (\alpha_{U_2}(x) + c_{10}(\varepsilon_4))|u| + \varepsilon_4|u|^p \, d\Gamma & \text{if } \theta_4 < p \end{cases}$$

$$\leq \begin{cases} \|\alpha_{U_2}\|_{p',\Gamma_2}\|u\|_{p,\Gamma_2} + \beta_{U_2}\|u\|_{p,\Gamma_2}^p & \text{if } \theta_4 = p, \\ \|\alpha_{U_2}(\cdot) + c_{10}(\varepsilon_3)\|_{p',\Gamma_2}\|u\|_{p,\Gamma_2} + \varepsilon_4\|u\|_{p,\Gamma_2}^p & \text{if } \theta_4 < p \end{cases}$$

for some $c_{10}(\varepsilon_4) > 0$, where we have used Young's inequality (for the cases $\theta_3 < p$ and $\theta_4 < p$) and Hölder's inequality. Taking (2.4), (3.12), (3.13) and (3.22), (3.23), (3.24) into account, we take ε_3 (for the case $\theta_3 < p$) and ε_4 (for the case $\theta_4 < p$) small enough and use the inequality $\max\{c_f\delta(\theta_1), d_f\delta(\theta_2) + \beta_{U_1}\delta(\theta_3) + \beta_{U_2}\delta(\theta_4)\} < 1$ to infer that there exists a constant $c_{11} > 0$ which is independent of u such that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u \right) \cdot \nabla u + |u|^p + \mu(x)|u|^q \, dx \leq c_{11}.$$

Therefore, it follows that the solution set of problem (1.1) is bounded, namely, there exists a constant $M > 0$ such that (3.21) is satisfied. This completes the proof of the lemma. □

Now we are in the position to state and prove our main result in this paper concerning the nonemptiness and weak compactness of the solution set Υ to problem (1.1).

THEOREM 3.9. *Let $p \geq 2$. Under the assumptions of Lemma* 3.7, *the solution set Υ of problem* (1.1) *is nonempty and weakly compact in V.*

Proof. Nonemptiness: We consider the multivalued mapping $\Xi\colon V \times X^* \to 2^{V \times X^*}$ defined by

$$\Xi(u, (\eta, \xi)) := (S(u, (\eta, \xi)), U(\gamma u)),$$

where U is given in (3.15) and $\gamma u = (\gamma_1 u, \gamma_2 u)$ for all $u \in V$. From Theorem 3.4 and Lemma 3.6, we know that Ξ has nonempty, bounded, closed, and convex values. Observe that if $(u, (\eta, \xi))$ is a fixed point of Ξ, then we have $u = S(u, (\eta, \xi))$ and $(\eta, \xi) \in U(\gamma u)$. By the definitions of S and U it is not difficult to see that u is also a weak solution of problem (1.1). Based on this fact, we are going to apply the Kakutani-Ky Fan fixed point theorem stated as Theorem 2.9 to verify that the fixed point set of Ξ is nonempty.

First, we claim that there exists a constant $M^* > 0$ such that

$$(3.25) \qquad S(\overline{B_V(0, M^*)}, \ U(\gamma \overline{B_V(0, M^*)})) \subset \overline{B_V(0, M^*)},$$

where $\overline{B_V(0, M^*)} := \{u \in V \ : \ \|u\|_V \leq M^*\}$. Arguing by contradiction, suppose there is not a constant M^* such that the inclusion holds. Then for each $n > 0$ there exist $w_n, z_n \in \overline{B_V(0, n)}$ and $(\eta_n, \xi_n) \in X^*$ with $(\eta_n, \xi_n) \in U(\gamma z_n)$ such that

$$u_n = S(w_n, (\eta_n, \xi_n)) \quad \text{and} \quad \|u_n\|_V > n.$$

Hence, for every $n > 0$, we have

$$\int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n \right) \cdot \nabla(v - u_n)\, \mathrm{d}x$$

$$+ \int_\Omega \left(|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n \right)(v - u_n)\, \mathrm{d}x$$

$$+ \int_{\Gamma_3} \phi(x, v)\, \mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x, u_n)\, \mathrm{d}\Gamma$$

$$\geq \int_\Omega \eta_n(x)(v - u_n)\, \mathrm{d}x + \int_{\Gamma_2} \xi_n(x)(v - u_n)\, \mathrm{d}\Gamma + \int_\Omega f(x, u_n, \nabla u_n)(v - u_n)\, \mathrm{d}x$$

for all $v \in K(w_n)$. Choosing $v = 0$ in the inequality above, we obtain

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n \right) \cdot \nabla u_n + |u_n|^p + \mu(x)|u_n|^q\, \mathrm{d}x$$

$$(3.26) \qquad \leq \int_{\Gamma_3} \phi(x, 0)\, \mathrm{d}\Gamma - \int_{\Gamma_3} \phi(x, u_n)\, \mathrm{d}\Gamma + \int_\Omega \eta_n(x)u_n\, \mathrm{d}x + \int_{\Gamma_2} \xi_n(x)u_n\, \mathrm{d}\Gamma$$

$$+ \int_\Omega f(x, u_n, \nabla u_n)u_n\, \mathrm{d}x.$$

Let $\varepsilon_5, \varepsilon_6 > 0$. Applying hypotheses $H(U_1)$(iv) and $H(U_2)$(iv) gives

(3.27)
$$
\int_\Omega \eta_n(x) u_n \, dx \leq \int_\Omega \left(\alpha_{U_1}(x) + \beta_{U_1}|z_n|^{\theta_3-1}\right)|u_n| \, dx
$$
$$
\leq \begin{cases} \int_\Omega \left(\alpha_{U_1}(x) + \beta_{U_1}|z_n|^{p-1}\right)|u_n| \, dx & \text{if } \theta_3 = p, \\[2mm] \int_\Omega \left((\alpha_{U_1}(x) + c_{12}(\varepsilon_5))\,|u_n| + \varepsilon_5|z_n|^{p-1}|u_n|\right) \, dx & \text{if } \theta_3 < p \end{cases}
$$
$$
\leq \begin{cases} \|\alpha_{U_1}\|_{p',\Omega}\|u_n\|_{p,\Omega} + \beta_{U_1}\|z_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} & \text{if } \theta_3 = p, \\[2mm] \|\alpha_{U_1}(\cdot) + c_{12}(\varepsilon_5)\|_{p',\Omega}\|u_n\|_{p,\Omega} + \varepsilon_6\|z_n\|_{p,\Omega}^{p-1}\|u_n\|_{p,\Omega} & \text{if } \theta_3 < p \end{cases}
$$

for some $c_{12}(\varepsilon_5) > 0$, and

(3.28)
$$
\int_{\Gamma_2} \xi_n(x) u_n \, d\Gamma \leq \int_{\Gamma_2} \left(\alpha_{U_2}(x) + \beta_{U_2}|z_n|^{\theta_4-1}\right)|u_n| \, d\Gamma
$$
$$
\leq \begin{cases} \int_{\Gamma_2} \left(\alpha_{U_2}(x) + \beta_{U_2}|z_n|^{p-1}\right)|u_n(x)| \, d\Gamma & \text{if } \theta_4 = p, \\[2mm] \int_{\Gamma_2} \left(\alpha_{U_2}(x) + c_{13}(\varepsilon_6)\right)|u_n| + \varepsilon_6|z_n|^{p-1}|u_n| \, d\Gamma & \text{if } \theta_4 < p \end{cases}
$$
$$
\leq \begin{cases} \|\alpha_{U_2}\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2} + \beta_{U_2}\|z_n\|_{p,\Gamma_2}^{p-1}\|u_n\|_{p,\Gamma_2} & \text{if } \theta_4 = p, \\[2mm] \|\alpha_{U_2}(\cdot) + c_{13}(\varepsilon_6)\|_{p',\Gamma_2}\|u_n\|_{p,\Gamma_2} + \varepsilon_5\|z_n\|_{p,\Gamma_2}^{p-1}\|u_n\|_{p,\Gamma_2} & \text{if } \theta_4 < p \end{cases}
$$

for some $c_{13}(\varepsilon_6) > 0$. Taking (3.11), (3.12) and (3.26), (3.27), (3.28) into account, we are able to find a constant $c_{14} > 0$ which is independent of n such that

$$
\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n\right) \cdot \nabla u_n + |u_n|^p + \mu(x)|u_n|^q \, dx \leq c_{14}
$$

for all $n > 0$, where we have used the fact that $\|z_n\|_V \leq n = \|u_n\|_V$. Letting $n \to \infty$ for the inequality above, it leads to a contradiction. So, there exists a constant $M^* > 0$ such that (3.25) is fulfilled.

From hypotheses $H(U_1)$(iv) and $H(U_2)$(iv), we can see that $U \colon X \to 2^{X^*}$ is a bounded mapping. Let $M_1 > 0$ be such that

$$
\|U(\gamma\overline{B_V(0, M^*)})\|_{X^*} \leq M_1.
$$

Now we introduce a bounded, closed, and convex subset D of $V \times X^*$ given by

$$
D = \{(u, (\eta, \xi)) \in V \times X^* : \|u\|_V \leq M^* \text{ and } \|(\eta, \xi)\|_{X^*} \leq M_1\}.
$$

From this and (3.25) we see that Ξ maps D into itself.

Next, we are going to prove that the graph of Ξ is sequentially closed in $(V \times X^*)_w \times (V \times X^*)_w$. Let $\{(w_n, (\eta_n, \xi_n))\}_{n\in\mathbb{N}} \subset V \times X^*$ and $\{(u_n, (\delta_n, \sigma_n))\}_{n\in\mathbb{N}} \subset V \times X^*$ be sequences such that $(\delta_n, \sigma_n) \in U(\gamma(w_n))$, $u_n = S(w_n, (\eta_n, \xi_n))$, and

$$
(w_n, (\eta_n, \xi_n)) \xrightarrow{w} (w, (\eta, \xi)) \quad \text{in } V \times X^* \quad \text{as } n \to \infty,
$$
$$
(u_n, (\delta_n, \sigma_n)) \xrightarrow{w} (u, (\delta, \sigma)) \quad \text{in } V \times X^* \quad \text{as } n \to \infty
$$

for some $(w, (\eta, \xi)) \in V \times X^*$ and $(u, (\delta, \sigma)) \in V \times X^*$. From Lemma 3.7 we know that

$$u_n = S(w_n, (\eta_n, \xi_n)) \to S(w, (\eta, \xi)) \quad \text{in } V \times X^* \quad \text{as } n \to \infty.$$

This means that $u = S(w, (\eta, \xi))$. Recalling that U is strongly-weakly u.s.c. with bounded, closed, and convex values (see Lemma 3.6), it allows us to apply Theorem 1.1.4 of Kamenskii, Obukhovskii, and Zecca [29] to conclude that the graph of U is strongly-weakly closed. The latter combined with the compactness of γ implies that the graph of $u \mapsto U(\gamma u)$ is weakly-weakly closed. We conclude that the graph of Ξ is closed in $(V \times X^*)_w \times (V \times X^*)_w$.

Therefore, all conditions of Theorem 2.9 are verified. Using this theorem, we conclude that Ξ has at least a fixed point, say, $(u^*, (\eta^*, \xi^*)) \in V \times X$. Hence, $u^* \in V$ is a weak solution of problem (1.1).

Weak compactness. The boundedness of the solution set Υ of problem (1.1) is a direct consequence of Lemma 3.8. Since V is reflexive, we shall verify the weak closedness of the solution set Υ of problem (1.1). Let u be a solution of problem (1.1). Then, by the definitions of weak solutions (see Definition 3.2) and of Ξ, there exist $(\eta, \xi) \in X^*$ such that $u = S(u, (\eta, \xi))$ and $(\eta, \xi) \in U(\gamma u)$, that is, $(u, (\eta, \xi)) \in \Xi(u, (\eta, \xi))$. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of solutions to problem (1.1) such that $u_n \xrightarrow{w} u$ in V as $n \to \infty$. Hence, we are able to find a sequence $\{(\eta_n, \xi_n)\}_{n \in \mathbb{N}} \subset X^*$ such that $(u_n, (\eta_n, \xi_n)) \in \Xi(u_n, (\eta_n, \xi_n))$. Recalling that U is a bounded mapping, it follows that $\{(\eta_n, \xi_n)\}_{n \in \mathbb{N}}$ is bounded in X^*. Passing to a subsequence if necessary, we may suppose that $(\eta_n, \xi_n) \xrightarrow{w} (\eta, \xi)$ in X^* for some $(\eta, \xi) \in X^*$. Recalling that Ξ is weakly sequentially closed, $(u_n, (\eta_n, \xi_n)) \in \Xi(u_n, (\eta_n, \xi_n))$, and $(u_n, (\eta_n, \xi_n)) \xrightarrow{w} (u, (\eta, \xi))$ in $V \times X^*$, it holds that $(u, (\eta, \xi)) \in \Xi(u, (\eta, \xi))$. This means that u is a solution to problem (1.1). Consequently, the solution set of problem (1.1) is weakly compact. □

We end this section by considering special cases of problem (1.1).

If $J(u) \equiv +\infty$ for all $u \in W^{1, \mathcal{H}}(\Omega)$, then problem (1.1) becomes the nonobstacle mixed boundary value problem (1.7). A careful observation gives the following corollary.

COROLLARY 3.10. *Let $p \geq 2$. Assume that (2.4), H(f), H(ϕ), H(U_1), H(U_2), and H(0) are satisfied. Then, the solution set of problem (1.7) is nonempty and weakly compact in V.*

Next, we assume the following conditions.

H(j_1): $j_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $r_1 \colon \mathbb{R} \to \mathbb{R}$ are functions which satisfy the following conditions:

    (i) the function $x \mapsto j_1(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$ with $x \mapsto j_1(x, 0)$ belonging to $L^1(\Omega)$;

    (ii) for a. a. $x \in \Omega$, $s \mapsto j_1(x, s)$ is locally Lipschitz continuous and r_1 is continuous;

    (iii) there exist $\theta_3 \in [1, p]$, $\alpha_{j_1} \in L^{p'}(\Omega)_+$ and $\beta_{j_1} > 0$ such that for all $s \in \mathbb{R}$ and a. a. $x \in \Omega$, we have

$$|r_1(s)\xi| \leq \alpha_{j_1}(x) + \beta_{j_1}|s|^{\theta_3 - 1}$$

    for all $\xi \in \partial j_1(x, s)$.

H(j_2): $j_2 \colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ and $r_2 \colon \mathbb{R} \to \mathbb{R}$ are functions which satisfy the following conditions:

(i) the function $x \mapsto j_2(x, s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$ with $x \mapsto j_2(x, 0)$ belonging to $L^1(\Gamma_2)$;

(ii) for a. a. $x \in \Gamma_2$, $s \mapsto j_2(x, s)$ is locally Lipschitz continuous and r_2 is continuous;

(ii) there exist $\theta_4 \in [1, p]$, $\alpha_{j_2} \in L^{p'}(\Gamma_2)_+$ and $\beta_{j_2} > 0$ such that for all $s \in \mathbb{R}$ and a. a. $x \in \Gamma_2$, we have

$$|r_2(s)\eta| \leq \alpha_{j_2}(x) + \beta_{j_2}|s|^{\theta_4 - 1}$$

for all $\eta \in \partial j_2(x, s)$.

If U_1 and U_2 are given by $U_1(x, s) = r_1(s)\partial j_1(x, s)$ for a. a. $x \in \Omega$, $s \in \mathbb{R}$, and $U_2(x, s) = r_2(s)\partial j_2(x, s)$ for a. a. $x \in \Gamma_2$, $s \in \mathbb{R}$, then problem (1.1) becomes problem (1.4). We have the following result.

THEOREM 3.11. *Let $p \geq 2$. Assume that (2.4), H(f), H(ϕ), H(L), H(J), H(j_1), H(j_2), and H(0) are satisfied. Then, the solution set of problem (1.4) is nonempty and weakly compact in V.*

Proof. It is obvious that the conclusion is a direct consequence of Theorem 3.9. So, we have to verify that the functions U_1 and U_2, defined by $U_1(x, s) = r_1(s)\partial j_1(x, s)$ for a. a. $x \in \Omega$, $s \in \mathbb{R}$, and $U_2(x, s) = r_2(s)\partial j_2(x, s)$ for a. a. $x \in \Gamma_2$, $s \in \mathbb{R}$, fulfill hypotheses H(U_1) and H(U_2), respectively.

It follows from Proposition 2.7 that for a. a. $x \in \Omega$ (resp. for a.a. $x \in \Gamma_2$) and all $s \in \mathbb{R}$ the set $U_1(x, s)$ (resp. $U_2(x, s)$) is nonempty, bounded, closed, and convex in $\mathbb{R}$, namely, conditions H(U_1) and H(U_2) are satisfied. Hypotheses H(j_1)(i) and H(j_2)(ii) indicate that for all $s \in \mathbb{R}$, the functions $x \mapsto U_1(x, s) = r_1(s)\ \partial j_1(x, s)$ and $x \mapsto U_2(x, s) = r_2(s)\partial j_2(x, s)$ are measurable in Ω and on Γ_2, respectively. This means that H(U_1)(i) and H(U_2)(i) hold.

We claim that $s \mapsto r_1(s)\partial j_1(x, s)$ is u.s.c. From Proposition 2.6, it is sufficient to show that $(r_1(\cdot)\partial j_1(x, \cdot))^-(D)$ is closed for each closed set $D \subset \mathbb{R}$. Let $\{s_n\}_{n \in \mathbb{N}} \subset (r_1(\cdot)\partial j_1(x, \cdot))^-(D)$ be such that $s_n \to s$ as $n \to \infty$. Then, there exists a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ satisfying $\eta_n \in r_1(s_n)\partial j_1(x, s_n) \cap D$ for each $n \in \mathbb{N}$. We are able to find a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ such that $\eta_n = r_1(s_n)\xi_n$ and $\xi_n \in \partial j_1(x, s_n)$ for all $n \in \mathbb{N}$ and for a. a. $x \in \Omega$. Recalling that $s_n \to s$, we can apply Proposition 2.7(iii) and (v) to conclude that $\{\xi_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}$. Hence, without any loss of generality, we may assume that $\xi_n \to \xi$ in $\mathbb{R}$ as $n \to \infty$ for some $\xi \in D$, because of the closedness of D. But, the closedness of ∂j_1 (see Proposition 2.7(v)) admits that $\xi \in \partial j_1(x, s)$. This combined with the continuity of r_1 deduces that $\eta_n = r_1(s_n)\xi_n \to r_1(s)\xi \in r_1(s)\partial j_1(x, s)$. This implies that $s \in (r_1(\cdot)\partial j_1(x, \cdot))^-(D)$, that is, $(r_1(\cdot)\partial j_1(x, \cdot))^-(D)$ is closed. Applying Proposition 2.6 we see that $s \mapsto r_1(s)\partial j_1(x, s)$ is u.s.c. Using the same arguments as before, we can also show that $s \mapsto r_2(s)\partial j_2(x, s)$ is u.s.c. Therefore, H(U_1)(iii) and H(U_2)(iii) are verified.

Finally, hypotheses H(U_1)(iv) and H(U_2)(iv) are the consequences of the assumptions H(j_1)(iii) and H(j_2)(iii). Consequently, we apply Theorem 3.9 to obtain the desired conclusion. □

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