Rend. Lincei Mat. Appl. 22 (2011), 189–205
DOI 10.4171/RLM/595

Partial Differential Equations — *Combined effects in quasilinear elliptic problems with lack of compactness*, by Patrizia Pucci and Vicenţiu Rădulescu, communicated on 14 January 2011.

This paper is dedicated to the memory of Professor Giovanni Prodi.

Abstract. — We are concerned with the study of weak solutions to a class of subcritical quasilinear elliptic equations with weight, variable parameter, and lack of compactness. By means of variational methods, we establish that this problem does not have any solution if the positive parameter is small enough, while at least two distinct solutions exist, provided that the parameter is sufficiently large.

Key words: Quasilinear elliptic equations, weighted Sobolev spaces, weak solutions, mountain pass geometry.

2010 Mathematics Subject Classification: 35J60, 35J65, 35J70, 58E05.

1. Introduction and the main result

Nonlinear elliptic equations with convex-concave nonlinearities in bounded domains have been studied starting with the seminal paper by Ambrosetti, Brezis and Cerami [2]. They considered the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a positive parameter, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and $1 < q < 2 < p < 2^*$ $(2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2)$. Ambrosetti, Brezis and Cerami proved that there exists $\lambda_0 > 0$ such that problem (1) admits at least two solutions for all $\lambda \in (0, \lambda_0)$, has one solution for $\lambda = \lambda_0$, and no solution exists provided that $\lambda > \lambda_0$.

In [1], Alama and Tarantello studied the related Dirichlet problem with indefinite weights

$$\begin{cases} -\Delta u - \lambda u = k(x)u^q - h(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded open set with smooth boundary, the functions $h, k \in L^1(\Omega)$ are nonnegative, and $1 < p < q$. For $\lambda \in \mathbb{R}$ in a neighbourhood of λ_1 (the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$), they obtained the solvability of (2) (and corresponding multiplicities) under various assumptions on h and k. More exactly, they proved existence, nonexistence and multiplicity results depending on λ and according to the integrability properties of the ratio k^{p-1}/h^{q-1}.

For more general results of (2) involving variable weights functions in unbounded domains, we refer to Chabrowski [7], Chabrowski and do Ó [8], Liu and Wang [13], and Wu [24].

Related studies can be also found in [5], [8], [9], [21], [22] and [25].

Motivated by these results, we are concerned in this paper with the existence and the multiplicity of solutions in the quasilinear case. More precisely, we consider the problem

$$(3) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = \lambda |u|^{q-2}u - h(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \geq 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $h : \mathbb{R}^N \to \mathbb{R}$ *is a positive continuous function, whose growth is described by the condition*

$$(4) \qquad \int_{\mathbb{R}^N} h(x)^{q/(q-p)}\, dx = H \in \mathbb{R}^+,$$

λ *is a positive parameter and* $2 \leq m < q < p < m^*$, with $m^* = Nm/(N-m)$ if $N > m$ and $m^* = \infty$ if $N \leq m$.

Without altering the proof arguments below, the coefficient 1 of the dominating term $|u|^{m-2}u$ can be replaced by any function $f \in L^\infty(\mathbb{R}^N)$ with $\inf \mathrm{ess} \|f\|_{L^\infty} > 0$. Hence equation (3) is the renormalized form. Problem (3) may be viewed as a prototype of pattern formation in biology and is related to the steady-state problem for a chemotactic aggregation model introduced by Keller and Segel [12]. Problem (3) also plays an important role in the study of activator-inhibitor systems modeling biological pattern formation, as proposed by Gierer and Meihardt [10].

In this paper we use standard notations and terminology. We denote by $W^{1,m}(\mathbb{R}^N)$ the Sobolev space equipped with the norm

$$\|u\|_{W^{1,m}} = \left(\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx \right)^{1/m}.$$

For simplicity we often denote the above norm by $\|u\|$.

By $L_r^p(\mathbb{R}^N)$, $1 \leq p < \infty$, we denote the weighted Lebesgue space

$$L_r^p(\mathbb{R}^N) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} r(x)|u|^p\, dx < \infty \right\},$$

where $r(x)$ is a positive continuous function on $\mathbb{R}^N$, equipped with the norm

$$\|u\|_{r,p} = \left(\int_{\mathbb{R}^N} r(x)|u|^p\,dx\right)^{1/p}.$$

If $r(x) \equiv 1$ on $\mathbb{R}^N$, the norm is denoted by $\|\cdot\|_p$.

In this paper we seek weak solutions of problem (3) in a subspace E of $W^{1,m}(\mathbb{R}^N)$, which is defined by

$$E = \left\{u \in W^{1,m}(\mathbb{R}^N) : \int_{\mathbb{R}^N} h(x)|u|^p\,dx < \infty\right\}.$$

Then E is a Banach space if equipped with the norm

$$\|u\|_E = (\|u\|^m_{W^{1,m}} + \|u\|^m_{r,p})^{1/m}.$$

We define a *weak solution* of problem (3) as a function $u \in E$ with $u(x) \geq 0$ a.e. in $\mathbb{R}^N$, satisfying

$$\int_{\mathbb{R}^N} |\nabla u|^{m-2}\nabla u \nabla v\,dx + \int_{\mathbb{R}^N} |u|^{m-2}uv\,dx$$
$$- \lambda\int_{\mathbb{R}^N} |u|^{q-2}uv\,dx + \int_{\mathbb{R}^N} h(x)|u|^{p-2}uv\,dx = 0,$$

for all $v \in E$.

The main result in the present paper establishes the following properties: the non-existence of nontrivial solutions to problem (3) if λ is small enough; the existence of at least two nontrivial solutions for problem (3) if λ is large enough.

THEOREM 1. *Under the above hypotheses there exists $\lambda^* > 0$ such that*

(i) *if $0 < \lambda < \lambda^*$, then problem* (3) *does not possess any nontrivial weak solution;*
(ii) *if $\lambda > \lambda^*$, then problem* (3) *admits at least two nontrivial weak solutions;*
(iii) *if $\lambda = \lambda^*$, then problem* (3) *has at least one nontrivial weak solution.*

2. Proof of Theorem 1

We point out in what follows the main ideas in the proof:

(a) There exists $\lambda^* > 0$ such that problem (3) does not have any solution for any $\lambda < \lambda^*$. This means that if a solution exists then λ must be sufficiently large. One of the key arguments in this proof is based on the assumption $p > q$. In particular, this proof yields an energy lower bound of solutions in term of λ that will be useful to conclude that problem (3) has a non-trivial solution if $\lambda = \lambda^*$.

(b) There exists $\lambda^{**} > 0$ such that problem (3) admits at least two solutions for all $\lambda > \lambda^{**}$. Next, by the properties of λ^* and λ^{**} we deduce that $\lambda^* = \lambda^{**}$.

2.1. Nonexistence if λ is small

Let $\Phi_\lambda : E \to \mathbb{R}$ be the energy functional given by

$$\Phi_\lambda(u) = \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\|u\|_q^q + \frac{1}{p}\|u\|_{h,p}^p.$$

Then $\Phi_\lambda \in C^1(E, \mathbb{R})$ and for all $u, v \in E$

$$\begin{aligned}\langle \Phi_\lambda'(u), v\rangle = &\int_{\mathbb{R}^N} (|\nabla u|^{m-2}\nabla u \nabla v + |u|^{m-2}uv)\,dx \\ &- \lambda \int_{\mathbb{R}^N} |u|^{q-2}uv\,dx + \int_{\mathbb{R}^N} h(x)|u|^{p-2}uv\,dx.\end{aligned}$$

Weak solutions of problem (3) are found as the critical points of the functional Φ_λ in E.

Let us now assume that $u \in E$ is a weak solution of problem (3). Then

$$\|u\|^m + \|u\|_{h,p}^p = \lambda\|u\|_q^q. \tag{5}$$

To proceed further, we need Young's inequality

$$ab \le \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} \quad \text{for all } a, b > 0,$$

where $\alpha, \beta > 1$ satisfy $1/\alpha + 1/\beta = 1$.

Taking $a = h(x)^{q/p}|u|^q$, $b = \lambda/[h(x)]^{q/p}$, $\alpha = p/q$ and $\beta = p/(p-q)$, we obtain

$$h(x)^{q/p}|u|^q \frac{\lambda}{h(x)^{q/p}} \le \frac{q}{p}(h(x)^{q/p}|u|^q)^{p/q} + \frac{p-q}{p}\left(\frac{\lambda}{h(x)^{q/p}}\right)^{p/(p-q)}.$$

Integrating over $\mathbb{R}^N$ we have

$$\lambda\|u\|_q^q \le \frac{q}{p}\|u\|_{h,p}^p + \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} h(x)^{q/(q-p)}\,dx.$$

The above inequality and relation (5) imply

$$\begin{aligned}\|u\|^m &\le \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} h(x)^{q/(q-p)}\,dx + \frac{q-p}{p}\|u\|_{h,p}^p \\ &\le \frac{p-q}{p}\lambda^{p/(p-q)} H,\end{aligned} \tag{6}$$

being $q < p$.

Since $m < q < m^*$, the Sobolev embedding $W^{1,m}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ is continuous, so that there exists a positive constant C_q such that

$$C_q\|v\|_q^m \le \|v\|^m \quad \text{for all } v \in W^{1,m}(\mathbb{R}^N).$$

On the other hand, being $\|u\|_{h,p} \ge 0$, it follows from (5) that

$$\|u\|^m \le \lambda\|u\|_q^q.$$

Combining the last two inequalities we obtain

$$C_q\|u\|_q^m \le \|u\|^m \le \lambda\|u\|_q^q. \tag{7}$$

Retaining the first and the last terms of (7) we get

$$(C_q\lambda^{-1})^{1/(q-m)} \le \|u\|_q.$$

That inequality combined with (7) leads to

$$C_q(C_q\lambda^{-1})^{m/(q-m)} \le \|u\|^m.$$

By relation (6) and the above inequality we have

$$C_q(C_q\lambda^{-1})^{m/(q-m)} \le \|u\|^m \le \frac{p-q}{p}\lambda^{p/(p-q)}H.$$

Retaining the first and the last term it follows that

$$\lambda > \left(C_q^{q/(m-q)}\frac{p-q}{p}H\right)^{(q-p)(q-m)/q(p-m)},$$

being $H > 0$ by (4). Denoting the term in the right-hand side of the above inequality by μ, we conclude that Theorem 1-(i) holds true, by putting

$$\lambda^* := \sup\{\lambda > 0 : (3) \text{ does not admit any nontrivial weak solution}\}. \tag{8}$$

Clearly $\lambda^* \ge \mu > 0$.

2.2. *Existence if λ is large*

We start with several auxiliary results.

LEMMA 1. *The functional Φ_λ is coercive.*

PROOF. We need the following elementary inequality: for every $k_1 > 0$, $k_2 > 0$ and $0 < s < r$ we have

$$k_1|t|^s - k_2|t|^r \le C_{rs}k_1\left(\frac{k_1}{k_2}\right)^{s/(r-s)} \quad \text{for all } t \in \mathbb{R}, \tag{9}$$

where $C_{rs} > 0$ is a constant depending on r and s.

Taking $k_1 = \lambda/q$, $k_2 = (m-1)h(x)/mp$, $s = q$ and $r = p$ (so that $s < r$ is verified, being $q < p$), in (9) for all $x \in \mathbb{R}^N$ we obtain

$$\begin{aligned}\frac{\lambda}{q}|u(x)|^q - \frac{(m-1)h(x)}{mp}|u(x)|^p &\le C_{pq}\frac{\lambda}{q}\Big(\frac{\lambda/q}{(m-1)h(x)/mp}\Big)^{q/(p-q)} \\ &= \frac{C_{pq}}{q}\Big(\frac{mp}{q(m-1)}\Big)^{q/(p-q)}\lambda^{p/(p-q)}h(x)^{q/(q-p)},\end{aligned}$$

where $C_{pq} > 0$ is a constant depending on p and q. Relabeling the constant $\frac{C_{pq}}{q}\big(\frac{mp}{q(m-1)}\big)^{q/(p-q)}$ by K and integrating the above inequality over $\mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N}\Big(\frac{\lambda}{q}|u|^q - \frac{(m-1)h(x)}{mp}|u|^p\Big)dx \le K\lambda^{p/(p-q)}\int_{\mathbb{R}^N} h(x)^{q/(q-p)}\,dx.$$

By assumption (4) there exists a constant $C_\lambda > 0$ such that

$$\frac{\lambda}{q}\|u\|_q^q - \frac{m-1}{mp}\|u\|_{h,p}^p \le C_\lambda.$$

Therefore

$$\begin{aligned}(10)\quad \Phi_\lambda(u) &= \frac{1}{m}\|u\|^m - \Big[\frac{\lambda}{q}\|u\|_q^q - \frac{m-1}{mp}\|u\|_{h,p}^p\Big] - \frac{m-1}{mp}\|u\|_{h,p}^p + \frac{1}{p}\|u\|_{h,p}^p \\ &\ge \frac{1}{m}\|u\|^m + \frac{1}{mp}\|u\|_{h,p}^p - C_\lambda,\end{aligned}$$

and so Φ_λ is coercive in E. □

LEMMA 2. *If $(u_n)_n$ is a sequence in E such that $(\Phi_\lambda(u_n))_n$ is bounded in $\mathbb{R}$, then there exists a subsequence of $(u_n)_n$, still relabeled $(u_n)_n$, which converges weakly in E to some $u_0 \in E$ and*

$$\Phi_\lambda(u_0) \le \liminf_{n\to\infty}\Phi_\lambda(u_n).$$

PROOF. By (10) and the fact that $(\Phi_\lambda(u_n))_n$ is bounded, it follows that both sequences $(\|u_n\|)_n$ and $(\|u_n\|_{h,p})_n$ are bounded. Therefore, $(\|u_n\|_E)_n$ is bounded and there exists $u_0 \in E$ such that

$$\begin{aligned}&u_n \rightharpoonup u_0 \quad \text{in } W^{1,m}(\mathbb{R}^N)\\ &u_n \to u_0 \quad \text{in } L_h^p(\mathbb{R}^N)\\ &u_n \to u_0 \quad \text{in } L_{\mathrm{loc}}^s(\mathbb{R}^N) \quad \text{for all } s \in [1, m^*).\end{aligned}$$

Let us define

$$F(x,u) = \frac{\lambda}{q}|u|^q - h(x)\frac{|u|^p}{p}$$

and

$$f(x,u) = F_u(x,u) = \lambda|u|^{q-2}u - h(x)|u|^{p-2}u,$$

so that

$$f_u(x,u) = \lambda(q-1)|u|^{q-2} - h(x)(p-1)|u|^{p-2}.$$

Using again inequality (9) for $k_1 = \lambda(q-1)$, $k_2 = h(x)(p-1)$, $s = q-2$, $r = p-2$, we obtain

$$\begin{aligned} f_u(x,u) &= \lambda(q-1)|u|^{q-2} - h(x)(p-1)|u|^{p-2} \\ &\le C \cdot \lambda \cdot (q-1) \cdot \left(\frac{\lambda(q-1)}{h(x)(p-1)}\right)^{(q-2)/(p-q)}, \end{aligned}$$

where C is a positive constant depending only of p and q.

This yields,

$$f_u(x,u) \le C_{pq} \cdot \lambda \cdot \left(\frac{\lambda}{h(x)}\right)^{(q-2)/(p-q)}, \tag{11}$$

where C_{pq} is a positive constant depending only of p and q. According to the definition of Φ_λ and F we obtain the following estimate for $\Phi_\lambda(u_0) - \Phi_\lambda(u_n)$

$$\begin{aligned} \Phi_\lambda(u_0) - \Phi_\lambda(u_n) &= \frac{1}{m}(\|u_0\|^m - \|u_n\|^m) \\ &\quad + \int_{\mathbb{R}^N} [F(x,u_n) - F(x,u_0)]\,dx. \end{aligned} \tag{12}$$

By position

$$\begin{aligned} \int_0^s f_u(x, u_0 + t(u_n - u_0))\,dt &= \frac{1}{u_n - u_0}[f(x, u_0 + s(u_n - u_0)) - f(x,u_0)] \\ &= \frac{1}{u_n - u_0}[F_u(x, u_0 + s(u_n - u_0)) - F_u(x,u_0)]. \end{aligned}$$

Integrating the above relation over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 \left(\int_0^s f_u(x, u_0 + t(u_n - u_0))\,dt\right) ds \\ &\quad = \frac{1}{u_n - u_0}\int_0^1 [F_u(x, u_0 + s(u_n - u_0)) - F_u(x,u_0)]\,ds \\ &\quad = \frac{1}{(u_n - u_0)^2}[F(x,u_n) - F(x,u_0)] - \frac{f(x,u_0)}{u_n - u_0}. \end{aligned}$$

The above equality can be rewritten in the following way

$$F(x,u_n)-F(x,u_0)=(u_n-u_0)^2\int_0^1\Big(\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt\Big)\,ds + (u_n-u_0)f(x,u_0). \tag{13}$$

Introducing relation (13) in (12) we get

$$\begin{aligned}\Phi_\lambda(u_0)-\Phi_\lambda(u_n)&=\frac{1}{m}(\|u_0\|^m-\|u_n\|^m)+\int_{\mathbb{R}^N}(u_n-u_0)f(x,u_0)\,dx\\&\quad+\int_{\mathbb{R}^N}(u_n-u_0)^2\int_0^1\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt\,ds\,dx\\&\le\frac{1}{m}(\|u_0\|^m-\|u_n\|^m)+\int_{\mathbb{R}^N}(u_n-u_0)f(x,u_0)\,dx\\&\quad+C_1\int_{\mathbb{R}^N}(u_n-u_0)^2h(x)^{(q-2)/(q-p)}\,dx,\end{aligned}\tag{14}$$

where the last inequality follows from (11) and $C_1=C_{pq}\lambda^{(p-2)/(p-q)}$. It remains to show that the last two integrals converge to 0 as $n\to\infty$.

We define $J:E\to\mathbb{R}$ by

$$J(v)=\int_{\mathbb{R}^N}f(x,u_0)v\,dx.$$

Obviously, J is linear. We shall show that J is also continuous. Indeed,

$$\begin{aligned}|J(v)|&\le\int_{\mathbb{R}^N}|f(x,u_0)|\cdot|v|\,dx\\&\le\lambda\int_{\mathbb{R}^N}|u_0|^{q-1}|v|\,dx+\int_{\mathbb{R}^N}h(x)|u_0|^{p-1}|v|\,dx.\end{aligned}\tag{15}$$

On the other hand, using Hölder's inequality, it follows that

$$\int_{\mathbb{R}^N}|u_0|^{q-1}|v|\,dx\le\|u_0\|_q^{q-1}\|v\|_q.$$

Since $W^{1,m}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ we deduce that there exists a constant $C>0$ such that

$$\|v\|_q\le C\|v\|_{W^{1,m}(\mathbb{R}^N)}\quad\text{for all } v\in W^{1,m}(\mathbb{R}^N).$$

Combining the last two inequalities with the fact that

$$\|v\|_{W^{1,m}(\mathbb{R}^N)}\le\|v\|_E,$$

we deduce that there exists a positive constant $c_q > 0$ such that

$$\int_{\mathbb{R}^N} |u_0|^{q-1}|v|\,dx \le c_q\|v\|_E. \tag{16}$$

Applying again Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} h(x)|u_0|^{p-1}|v|\,dx &= \int_{\mathbb{R}^N} (h(x)^{(p-1)/p}|u_0|^{p-1})(h(x)^{1/p}|v|)\,dx \\ &\le \|u_0\|_{h,p}^{p-1}\|v\|_{h,p} \le C_0\|v\|_{h,p} \\ &\le C_0\|v\|_E,\end{aligned} \tag{17}$$

where C_0 is a positive constant.

By (15), (16) and (17) we conclude that there exists a positive constant κ such that

$$|J(v)| \le \kappa\|v\|_E \quad \text{for all } v \in E,$$

and so J is continuous.

Since $(u_n)_n$ converges weakly to u_0 in E and J is linear and continuous we deduce that

$$J(u_n) \to J(u_0),$$

in other words

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_0)(u_n - u_0)\,dx = 0. \tag{18}$$

In order to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (u_n - u_0)^2 h(x)^{(q-2)/(q-p)}\,dx = 0, \tag{19}$$

we first note that for all $R > 0$

$$\begin{aligned}&\int_{\mathbb{R}^N} (u_n - u_0)^2 h(x)^{(q-2)/(q-p)}\,dx \\ &\quad= \int_{\{|x|<R\}} (u_n - u_0)^2 h(x)^{(q-2)/(q-p)}\,dx \\ &\qquad+ \int_{\{|x|\ge R\}} (u_n - u_0)^2 h(x)^{(q-2)/(q-p)}\,dx \\ &\quad\le \left(\int_{\{|x|<R\}} h(x)^{q/(q-p)}\,dx\right)^{(q-2)/q} \cdot \left(\int_{\{|x|<R\}} |u_n - u_0|^q\,dx\right)^{2/q} \\ &\qquad+ \left(\int_{\{|x|\ge R\}} h(x)^{q/(q-p)}\right)^{(q-2)/q} \cdot \left(\int_{\{|x|\ge R\}} |u_n - u_0|^q\,dx\right)^{2/q}.\end{aligned} \tag{20}$$

By hypothesis (4) we have

$$\int_{\{|x|<R\}} h(x)^{q/(q-p)}\,dx \le \int_{\mathbb{R}^N} h(x)^{q/(q-p)}\,dx = H < \infty \quad \text{for all } R > 0.$$

On the other hand, for all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\int_{\{|x|\ge R_\varepsilon\}} h(x)^{q/(q-p)}\,dx < \varepsilon.$$

Using the fact that $m < q < m^*$ we deduce that $W^{1,m}(B_{R_\varepsilon}(0))$ is compactly embedded in $L^q(B_{R_\varepsilon}(0))$ and thus

$$\lim_{n\to\infty}\Big(\int_{|x|<R_\varepsilon} |u_n - u_0|^q\,dx\Big)^{2/q} = 0.$$

Since $(u_n - u_0)_n$ is bounded in E, it is also bounded in $L^q(\mathbb{R}^N)$ and so there exists a positive constant $M > 0$ such that

$$\Big(\int_{|x|\ge R_\varepsilon} |u_n - u_0|^q\,dx\Big)^{2/q} \le \|u_n - u_0\|_q^2 \le M.$$

Combining the above information with relation (20), we conclude that for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that for all $n \ge N_\varepsilon$ we have

$$\int_{\mathbb{R}^N} (u_n - u_0)^2 h(x)^{(q-2)/(q-p)}\,dx \le H\varepsilon + M\varepsilon^{(q-2)/q}.$$

Therefore, (19) holds true.

Since $(u_n)_n$ converges weakly to u_0 in $W^{1,m}(\mathbb{R}^N)$ Proposition III.5 in [6] implies

$$\liminf_{n\to\infty} \|u_n\|^m_{W^{1,m}(\mathbb{R}^N)} \ge \|u_0\|^m_{W^{1,m}(\mathbb{R}^N)}.$$

Passing to the limit in (14) and taking into account that (18) and (19) hold true, we obtain

$$\Phi_\lambda(u_0) \le \liminf_{n\to\infty} \Phi_\lambda(u_n).$$

Thus, Φ_λ is weakly lower semi-continuous.

The proof of Lemma 2 is now complete. □

Proof of Theorem 1 continued. Using Lemmas 1, 2 and Theorem 1.2 in [20] we deduce that there exists a global minimizer $u \in E$ of Φ_λ, that is,

$$\Phi_\lambda(u) = \inf_{v\in E} \Phi_\lambda(v).$$

It is obvious that u is a weak solution of problem (3). We prove that $u \not\equiv 0$ in E. To do that we show that $\inf_E \Phi_\lambda < 0$, provided that the parameter λ is sufficiently large.

Let us set

$$\bar{\lambda} = \inf_{u \in E} \left\{ \frac{q}{m} \|u\|^m + \frac{q}{p} \|u\|^p_{h,p} : \|u\|_q = 1 \right\}.$$

We point out that $\bar{\lambda} > 0$. Indeed, for any $u \in E$ with $\|u\|_q = 1$ by Hölder's inequality and by (4) we have

$$1 = \|u\|_q^q \le \left(\int_{\mathbb{R}^N} h(x)^{q/(q-p)} \, dx \right)^{(p-q)/p} \cdot \left(\int_{\mathbb{R}^N} h(x)|u|^p \, dx \right)^{q/p} = H^{(p-q)/p} \|u\|^q_{h,p},$$

so that

$$\bar{\lambda} \ge \frac{q}{p} H^{(q-p)/q} > 0,$$

being $H > 0$ by assumption. Let $\lambda > \bar{\lambda}$. Then there exists a function $u_1 \in E$, with $\|u_1\|_q = 1$, such that

$$\lambda \|u_1\|_q^q = \lambda > \frac{q}{m} \|u_1\|^m + \frac{q}{p} \|u_1\|^p_{h,p}.$$

This can be rewritten as

$$\Phi_\lambda(u_1) = \frac{1}{m} \|u_1\|^m - \frac{\lambda}{q} \|u_1\|_q^q + \frac{1}{p} \|u_1\|^p_{h,p} < 0$$

and consequently $\inf_{u \in E} \Phi_\lambda(u) < 0$. Therefore, there exists $\lambda_0 = \bar{\lambda} > 0$ such that problem (3) has a nontrivial weak solution $u_1 \in E$ for any $\lambda > \lambda_0$, and $\Phi_\lambda(u_1) < 0$. Since $\Phi_\lambda(u_1) = \Phi_\lambda(|u_1|)$ and $|u_1| \in E$, we may assume that $u_1 \ge 0$ a.e. in $\mathbb{R}^N$. □

In the following we are looking for a second nontrivial weak solution for problem (3).

Fix $\lambda \ge \lambda_0$. Set

$$g(x,t) = \begin{cases} 0, & \text{if } t < 0, \\ \lambda t^{q-1} - h(x) t^{p-1}, & \text{if } 0 \le t \le u_1(x), \\ \lambda u_1(x)^{q-1} - h(x) u_1(x)^{p-1}, & \text{if } t > u_1(x), \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s) \, ds.$$

Define the functional $\Psi : E \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{m}\|u\|^m - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

Clearly, $\Psi \in C^1(E,\mathbb{R})$ and

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} (|\nabla u|^{m-2}\nabla u \nabla v + |u|^{m-2}uv)\,dx - \int_{\mathbb{R}^N} g(x,u)v\,dx,$$

for all $u, v \in E$. Moreover, *if u is a critical point of Ψ, then $u \geq 0$ a.e. in $\mathbb{R}^N$.*

Next, we are concerned with the location of critical points of the energy functional Ψ.

Lemma 3. *If u is a critical point of Ψ, then $u \leq u_1$.*

Proof. For a function v we define the positive part $v^+(x) = \max\{v(x), 0\}$. By Theorem 7.6 in [11] we deduce that if $v \in E$ then $v^+ \in E$. We have

$$\begin{aligned}
0 &= \langle \Psi'(u) - \Phi'_\lambda(u_1), (u-u_1)^+\rangle \\
&= \int_{\mathbb{R}^N} (|\nabla u|^{m-2}\nabla u - |\nabla u_1|^{m-2}\nabla u_1)\nabla(u-u_1)^+\,dx \\
&\quad + \int_{\mathbb{R}^N} (|u|^{m-2}u - |u_1|^{m-2}u_1)(u-u_1)^+\,dx \\
&\quad - \int_{\mathbb{R}^N} [g(x,u) - \lambda u_1^{q-1} + h(x)u_1^{p-1}](u-u_1)^+\,dx \\
&= \int_{\{u>u_1\}} (|\nabla u|^{m-2}\nabla u - |\nabla u_1|^{m-2}\nabla u_1)(\nabla u - \nabla u_1)\,dx \\
&\quad + \int_{\{u>u_1\}} (|u|^{m-2}u - |u_1|^{m-2}u_1)(u-u_1)\,dx \\
&\geq \int_{\{u>u_1\}} (|\nabla u|^{m-1} - |\nabla u_1|^{m-1})(|\nabla u| - |\nabla u_1|)\,dx \\
&\quad + \int_{\{u>u_1\}} (|u|^{m-1} - |u_1|^{m-1})(|u| - |u_1|)\,dx \geq 0.
\end{aligned}$$

Thus, we obtain $u \leq u_1$ and the proof of Lemma 3 is complete. □

In the following, via the mountain pass theorem, we determine a critical point $u_2 \in E$ of Ψ such that $\Psi(u_2) > 0$. By the above lemma we shall deduce that $0 \leq u_2 \leq u_1$ in Ω. Therefore,

$$g(x,u_2) = \lambda u_2^{q-1} - h(x)u_2^{p-1} \quad \text{and} \quad G(x,u_2) = \frac{\lambda}{q}u_2^q - \frac{h(x)}{p}u_2^p,$$

so that

$$\Psi(u_2) = \Phi_\lambda(u_2) \quad \text{and} \quad \Psi'(u_2) = \Phi'_\lambda(u_2).$$

More precisely, we find

$$\Phi_\lambda(u_2) > 0 = \Phi_\lambda(0) > \Phi_\lambda(u_1) \quad \text{and} \quad \Phi'_\lambda(u_2) = 0.$$

This shows that u_2 is a weak solution of problem (3) such that $0 \le u_2 \le u_1$, $u_2 \ne 0$ and $u_2 \ne u_1$.

In order to find u_2 described above we prove the following result.

Lemma 4. *There exists $\rho \in (0, \|u_1\|)$ and $a > 0$ such that $\Psi(u) \ge a$ for all $u \in E$, with $\|u\| = \rho$.*

Proof. We have

$$\begin{aligned}
\Psi(u) &= \frac{1}{m}\|u\|^m - \int_{\{u>u_1\}} G(x,u)\,dx - \int_{\{u\le u_1\}} G(x,u)\,dx \\
&= \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{\{u>u_1\}} u_1^q\,dx + \frac{1}{p}\int_{\{u>u_1\}} h(x)u_1^p\,dx \\
&\quad - \frac{\lambda}{q}\int_{\{0\le u\le u_1\}} u^q\,dx + \frac{1}{p}\int_{\{0\le u\le u_1\}} h(x)u^p\,dx \\
&\ge \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\|u\|_q^q.
\end{aligned}$$

On the other hand, the continuous Sobolev embedding of E into $L^q(\mathbb{R}^N)$ implies that there exists a positive constant $L > 0$ such that

$$\|v\|_q \le L\|v\| \quad \text{for all } v \in E.$$

The above inequalities yield

$$\Psi(u) \ge \frac{1}{m}\|u\|^m - L_1\|u\|^q = \|u\|^m\Big(\frac{1}{m} - L_1\|u\|^{q-m}\Big),$$

where $L_1 = \lambda L^q/q$ is a positive constant. Since $q > m$ it is clear that Lemma 4 holds true. $\square$

Lemma 5. *The functional Ψ is coercive.*

Proof. For each $u \in E$ we have

$$
\begin{aligned}
\Psi(u) &= \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{\{u>u_1\}} u_1^q\,dx + \frac{1}{p}\int_{\{u>u_1\}} h(x)u_1^p\,dx \\
&\quad - \frac{\lambda}{q}\int_{\{0\le u\le u_1\}} u^q\,dx + \frac{1}{p}\int_{\{0\le u\le u_1\}} h(x)u^p\,dx \\
&\ge \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{\mathbb{R}^N} u_1^q\,dx \\
&= \frac{1}{m}\|u\|^m - L_2,
\end{aligned}
$$

where L_2 is a positive constant, being $u_1 \neq 0$. The above inequality implies that $\Psi(u) \to \infty$ as $\|u\| \to \infty$, that is, Ψ is coercive, as required. $\square$

Proof of Theorem 1 completed. Using Lemma 4 and the mountain pass theorem (see [4] with the variant given by Theorem 1.15 in [23]) we deduce that there exists a sequence $(v_n)_n \subset E$ such that

$$
\Psi(v_n) \to c > 0 \quad \text{and} \quad \Psi'(v_n) \to 0, \tag{21}
$$

where

$$
c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t))
$$

and

$$
\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = u_1\}.
$$

We also refer to Ambrosetti and Prodi [3], Pucci and Serrin [17] for various extensions of the mountain pass theorem, cf. also the recent survey [16].

By relation (21) and Lemma 5 we obtain that $(v_n)_n$ is bounded and thus passing eventually to a subsequence, still denoted by $(v_n)_n$, we may assume that there exists $u_2 \in E$ such that v_n converges weakly to u_2. Standard arguments based on the Sobolev embeddings will show that

$$
\lim_{n\to\infty} \langle \Psi'(v_n), \varphi\rangle = \langle \Psi'(u_2), \varphi\rangle
$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Taking into account that $E \subset W^{1,m}(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,m}(\mathbb{R}^N)$, the above information implies that u_2 is a weak solution of problem (3).

We conclude that problem (3) admits at least two nontrivial weak solutions for all $\lambda > \lambda_0$.

Put

$$
\lambda^{**} := \inf\{\lambda > 0 : \text{problem (3) admits a nontrivial weak solution}\}.
$$

Then $\lambda^{**} \ge \lambda^* > 0$, where λ^* is the parameter defined in (8).

Let us consider the constrained minimization problem

$$\Lambda := \inf_{v \in E} \left\{ \frac{1}{m} \|v\|^m + \frac{1}{p} \|v\|_{h,p}^p : \|v\|_q^q = q \right\}. \tag{22}$$

Let $(v_n)_n \subset E$ be a minimizing sequence of (22). Then $(v_n)_n$ is bounded in E, hence we can assume, without loss of generality, that it converges weekly to some $v \in E$ with $\|v\|_q^q = q$. Moreover, by lower semi-continuity arguments we have

$$\Lambda = \frac{1}{m} \|v\|^m + \frac{1}{p} \|v\|_{h,p}^p.$$

Thus, $\Phi_\lambda(v) = \Lambda - \lambda$ for all $\lambda > \Lambda$.

To complete the proof of Theorem 1 it is enough to show the following crucial facts:

(a) problem (3) has at least two distinct solutions for any $\lambda > \lambda^{**}$;
(b) $\lambda^{**} = \lambda^*$ and problem (3) admits a nontrivial weak solution if $\lambda = \lambda^*$.

Claim (a) follows by standard monotonicity techniques. Claim (b) uses the same proof as in Filippucci, Pucci and Rădulescu [9, p. 712]. The proof of Theorem 1 is now complete. □

We point out that the proof of Theorem 1-(ii) uses some ideas found in the proofs of Theorems 2.1 and 2.2 in [1]. However, our method in finding the second solution is different, since we use the mountain pass theorem, while in [1] the authors appeal to sub- and super-solutions method. Our idea is frequently used when we deal with quasilinear problems, see e.g., Filippucci, Pucci and Rădulescu [9], Mihăilescu and Rădulescu [14], Perera [15], Pucci and Servadei [18, 19].

On the other hand, we point out that equation (3) can be studied also in the case when p is supercritical using similar arguments, since the $|u|^p$ term in the energy continues to be coercive. In these cases standard regularity results will lead to stronger results in what concerns the smoothness of solutions, since $W^{1,m}$ is embedded into C^1.

Acknowledgments. P. Pucci was supported by the Italian MIUR project titled *"Metodi Variazionali ed Equazioni Differenziali alle Derivate Parziali non Lineari"*. V. Rădulescu acknowledges the support through Grant CNCSIS PCCE-8/2010 *"Sisteme diferenţiale în analiza neliniară şi aplicaţii"*.

References

[1] S. Alama - G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. Anal. 141 (1996), 159–215.

[2] A. Ambrosetti - H. Brezis - G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994), 519–543.

[3] A. Ambrosetti - G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge Studies in Advanced Mathematics, vol. 34, Cambridge University Press, Cambridge, viii+171 pp., 1993.

[4] A. Ambrosetti - P. H. Rabinowitz, *Dual variational methods in critical point theory*, J. Funct. Anal. 14 (1973), 349–381.

[5] T. Bartsch - M. Willem, *On an elliptic equation with concave and convex nonlinearities*, Proc. Amer. Math. Soc. 123 (1995), 3555–3561.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[7] J. Chabrowski, *Elliptic variational problems with indefinite nonlinearities*, Topol. Methods in Nonlinear Anal. 9 (1997), 221–231.

[8] J. Chabrowski - J. M. Bezzera do Ó, *On semilinear elliptic equations involving concave and convex nonlinearities*, Math. Nachr. 233–234 (2002), 55–76.

[9] R. Filippucci - P. Pucci - V. Rădulescu, *Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions*, Comm. in Partial Differential Equations 33 (2008), 706–717.

[10] A. Gierer - H. Meinhardt, *A theory of biological pattern formation*, Kybernetik 12 (1972), 30–39.

[11] D. Gilbarg - N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, xiv+517, 2001.

[12] E. F. Keller - L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. 26 (1970), 399–415.

[13] Z. Liu - Z.-Q. Wang, *Schrödinger equations with concave and convex nonlinearities*, Z. Angew. Math. Phys. 56 (2005), 609–629.

[14] M. Mihăilescu - V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. R. Soc. Lond. Ser. A: Math., Phys. Eng. Sci. 462 (2006), 2625–2641.

[15] K. Perera, *Multiple positive solutions for a class of quasilinear elliptic boundary-value problems*, Electronic J. Differential Equations 7 (2003), 5 pp.

[16] P. Pucci - V. Rădulescu, *The impact of the mountain pass theory in nonlinear analysis: a mathematical survey*, Boll. Un. Mat. Ital. B, Ser. IX, III (2010), 543–582.

[17] P. Pucci - J. Serrin, *Extensions of the mountain pass theorem*, J. Funct. Anal. 59 (1984), 185–210.

[18] P. Pucci - R. Servadei, *On weak solutions for p-Laplacian equations with weights*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 18 (2007), 257–267.

[19] P. Pucci - R. Servadei, *Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights*, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), 505–537.

[20] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, A Series of Modern Surveys in Mathematics, Vol. 34, Springer-Verlag, Berlin, xx+302 pp., 2008.

[21] M. Tang, *Exact multiplicity for semilinear elliptic Dirichlet problems involving concave and convex nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003), 705–717.

[22] S. B. Tshinanga, *On multiple solutions of semilinear elliptic equations on unbounded domains with concave and convex nonlinearities*, Nonlinear Anal. 28 (1997), 809–814.

[23] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser Boston, x+162 pp., 1996.

[24] T.-F. Wu, *Multiple positive solutions for a class of concave-convex elliptic problems in R^N involving sign-changing weight*, J. Funct. Anal. 258 (2010), 99–131.

[25] J. Yang - X. Zhu, *On the existence of nontrivial solutions of a quasilinear elliptic boundary value problem for unbounded domains, I: Positive mass case*, Acta Math. Sci. 7 (1987), 341–359.

Received 30 December 2010,
and in revised form 14 January 2011.

Patrizia Pucci
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
06123 Perugia, Italy
pucci@dmi.unipg.it

Vicenţiu Rădulescu
"Simion Stoilow" Mathematics Institute of the Romanian Academy
P.O. Box 1-764, 014700 Bucharest, Romania
vicentiu.radulescu@imar.ro