Research Article

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On the best constants in Sobolev inequalities on the solid torus in the limit case $p = 1$

Abstract: In this paper, we analyze the problem of determining the best constants for the Sobolev inequalities in the limiting case where $p = 1$. Firstly, the special case of the solid torus is studied, whenever it is proved that the solid torus is an extremal domain with respect to the second best constant and totally optimal with respect to the best constants in the trace Sobolev inequality. Secondly, in the spirit of Andreu, Mazon and Rossi [3], a Neumann problem involving the 1-Laplace operator in the solid torus is solved. Finally, the existence of both best constants in the case of a manifold with boundary is studied, when they exist. Further examples are provided where they do not exist. The impact of symmetries which appear in the manifold is also discussed.

Keywords: Sobolev inequalities, best constants, limit case, manifolds with boundary, symmetries, solid torus

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1 Introduction

Let (M, g) be a smooth compact n-dimensional Riemannian manifold, $n \geq 3$, with boundary. We define the Sobolev space $H_1^p(M)$ as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_1^p(M)} = \bigg(\int_M |\nabla u|^p \, dv_g\bigg)^{1/p} + \bigg(\int_M |u|^p \, dv_g\bigg)^{1/p},$$

and $\mathring{H}_1^p(M)$ as the closure of $C_0^\infty(M)$ in $H_1^p(M)$. For any $p \in [1, n)$, we denote

$$p^* = \frac{np}{n-p} \quad \text{and} \quad \tilde{p}^* = \frac{(n-1)p}{n-p}.$$

According to Sobolev's theorem (see [7]) the embeddings $H_1^p(M) \hookrightarrow L^q(M)$ and $H_1^p(M) \hookrightarrow L^{\tilde{q}}(\partial M)$ are compact for any $q \in [1, p^*)$ and $\tilde{q} \in [1, \tilde{p}^*)$, respectively, but the embeddings $H_1^p(M) \hookrightarrow L^{p^*}(M)$ and $H_1^p(M) \hookrightarrow L^{\tilde{p}^*}(\partial M)$ are only continuous. So, there exist constants A, B and $\tilde{A}, \tilde{B}$ such that for all $u \in H_1^p(M)$ the following inequalities hold:

$$\bigg(\int_M |u|^{p^*} \, dv_g\bigg)^{1/p^*} \leq A\bigg(\int_M |\nabla u|^p \, dv_g\bigg)^{1/p} + B\bigg(\int_M |u|^p \, dv_g\bigg)^{1/p} \tag{1.1}$$

and

$$\bigg(\int_{\partial M} |u|^{\tilde{p}^*} \, ds_g\bigg)^{1/\tilde{p}^*} \leq \tilde{A}\bigg(\int_M |\nabla u|^p \, dv_g\bigg)^{1/p} + \tilde{B}\bigg(\int_M |u|^p \, dv_g\bigg)^{1/p}. \tag{1.2}$$

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In these two inequalities, we are interested in the values of the best possible constants. Define the sets:

$$\mathcal{A}_p(M) = \inf\{A \in \mathbb{R} : \text{ there exists } B \in \mathbb{R} \text{ such that inequality (1.1) holds for all } u \in H_1^p(M)\},$$
$$\mathcal{B}_p(M) = \inf\{B \in \mathbb{R} : \text{ there exists } A \in \mathbb{R} \text{ such that inequality (1.1) holds for all } u \in H_1^p(M)\}.$$

One can define in the same way the sets $\tilde{\mathcal{A}}_p(M)$ and $\tilde{\mathcal{B}}_p(M)$.

In the case $1 < p < n$, the best constants are known in almost all cases both on Euclidean space (here only appears the first constant) and on Riemannian manifolds (with boundary or without boundary), and a host of literature exists for all these cases, see, e.g., [5, 6, 10, 13, 16–18, 22, 24, 27, 28, 35–38, 41, 50] and the references therein. We point out that Druet and Hebey [24] worked exclusively on the Euclidean space and the manifolds without boundary, and have provided a complete study on the area. In addition, they answered questions like when and under what conditions optimal Sobolev inequalities are true or not. Some more cases towards this direction were studied by Druet [22] and Faget [28] on manifolds without boundary, while Biezuner in [11] answered the same questions on manifolds with boundary. When $p = 1$, some quite remarkable and interesting results are also known. For example, see [3, 41, 43, 46, 48] and the references therein. A complete and thorough study on the best constants in Sobolev inequalities is presented in the books [7, 24, 35, 44].

This paper is organized as follows. In Section 2, a short survey on known results about the best constants is presented; this is done for the sake of completeness so that the paper is self-contained. Section 3 is devoted to the qualitative presentation of the results established in this paper. In Section 4, the case of the solid torus is studied in detail. A Neumann problem involving the Δ_1-Laplace operator in the solid torus is considered in Section 5. Finally, some results are given for smooth compact, Riemannian manifolds with boundary. In particular, the general case of a manifold with boundary is treated in Section 6.1, and in Section 6.2 some results are presented for manifolds in the presence of symmetries.

2 A short survey on the best constants

2.1 The case $1 < p < n$

In this subsection, we recall certain important results concerning the best constants for some Sobolev inequalities, which are well known and they are directly related with the context of the present paper. We now give a short survey of known results about the best constants for the classical as much as for the trace Sobolev inequalities in the case $1 < p < n$. If M is a compact Riemannian manifold with boundary, then $H_1^p(M) \neq \mathring{H}_1^p(M)$, cf. [7]. Therefore, as far as the first best constant of (1.1) is concerned, we have to consider two distinct Sobolev spaces.

(i) When we consider (1.1) on $\mathring{H}_1^p(M)$, the same results for best constants on compact Riemannian manifolds without boundary remain true. Recall that for compact Riemannian manifolds without boundary, the best constant in front of the gradient term in inequality (1.1) is the same as the best constant for the Sobolev embedding for $M = \mathbb{R}^n$ under the Euclidean metric (see [6]), that is,

$$\frac{1}{K(n,p)} = \inf_{\substack{u \in L^{p^*}(\mathbb{R}^n)\setminus\{0\} \\ \nabla u \in L^p(\mathbb{R}^n)}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{(\int_{\mathbb{R}^n} |u|^{p^*} \, dx)^{p/p^*}}.$$

It has been proven that

$$K(n,1) = \frac{1}{n}\Big(\frac{n}{\omega_{n-1}}\Big)^{1/n},$$
$$K(n,p) = \frac{1}{n}\Big(\frac{n(p-1)}{n-p}\Big)^{1-1/p}\Big(\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}\Big)^{1/p},$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$ and Γ is the gamma function. In particular, when $p = 1$ and $M = \mathbb{R}^n$, (1.1) is the usual isoperimetric inequality, see [29, 30, 32]. The exact value of $K(n, 1)$ was computed by Federer and Fleming [30] and by Maz'ya [43]; the extremum functions in this case are the characteristic functions of the balls of $\mathbb{R}^n$. The value of $K(n,p)$ was explicitly computed independently by Aubin [5] and Talenti [50] and is attained by the functions $\varphi(x) = (\lambda + |x|^{p/(p-1)})^{1-n/p}$, where λ is any positive real number. In addition, Aubin [7] proved that the constant $K(n,1)$ is obtained as a limit of $K(n,p)$ as p tends to 1^+.

(ii) Considering inequality (1.1) on $H_1^p(M)$, Cherrier [13] showed that the first best constant is equal to $2^{1/n}K(n,p)$.

Contrary to the first best constant $K(n,p)$ of (1.1), which depends only on the dimension n of the manifold and p, the second best constant $\mathcal{B}_p(M)$ depends on the geometry of the manifold and was computed to be $\mathcal{B}_p(M) = |M|^{-1/n}$, see [35].

Concerning inequality (1.2), Lions [41] proved that the best constant in front of its gradient term in the case of the Euclidean half-space $\mathbb{R}_+^n = \{x = (x', y) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, y \geq 0\}$ is

$$\frac{1}{\tilde{K}(n,p)} = \inf_{\substack{u \in L^{\tilde{p}^*}(\partial R_+^n)\setminus\{0\} \\ \nabla u \in L^p(R_+^n)}} \frac{\int_{R_+^n} |\nabla u|^p \, dx}{\left(\int_{\partial R_+^n} |u|^{\tilde{p}^*} \, dx'\right)^{p/\tilde{p}^*}}.$$

Biezuner [10] used inequality (1.2) and a standard contradiction argument in order to show that there exists a constant $C > 0$ such that

$$\int_M |u|^p \, dv_g \leq C\left(\int_M |\nabla u|^p \, dv_g + B \int_{\partial M} |u|^p \, ds_g\right) \quad \text{for all } u \in H_1^p(M). \tag{2.1}$$

From (1.2) and (2.1) we conclude that there are positive constants A, B, which may depend on M and g, such that

$$\left(\int_{\partial M} |u|^{\tilde{p}^*} \, ds_g\right)^{1/\tilde{p}^*} \leq \tilde{A}\left(\int_M |\nabla u|^p \, dv_g\right)^{1/p} + \tilde{B}\left(\int_{\partial M} |u|^p \, ds_g\right)^{1/p}. \tag{2.2}$$

Inequality (2.1) is a "bridge" to move from (1.2) to (2.2) and to our knowledge has not presented any interest on its best constants.

Regarding the first best constant in inequality (2.2), Biezuner showed that Lions' conclusion still remains valid for any smooth compact n-dimensional Riemannian manifold with boundary and for all $p \in (1, n)$. The explicit value of $\tilde{K}(n,p)$ was computed independently by Escobar [25] and Beckner [8], only in the case where $p = 2$:

$$\tilde{K}(n,2) = \frac{2}{n-2}\omega_{n-1}^{-1/(n-1)}.$$

Unfortunately, their proofs both deeply used the conformal invariance of the associated variational problem, and thus cannot be generalized to other values of p, and the problem was still open. However, Nazaret [47] studying optimal Sobolev trace inequalities on the half-space proved a conjecture made by Escobar [25] about the minimizers in (2.2) and founded that the functions $f(x) = ((\lambda + y)^2 + |x'|^2)^{(p-n)/2(p-1)}$, for all $x = (x', y) \in \mathbb{R}_+^n$ and $\lambda > 0$, are optimal for this inequality. The second best constant of (2.2), for $1 < p < n$, was computed as $\tilde{\mathcal{B}}_p(M) = |\partial M|^{-1/n}$, see [10].

In order to make the paper more self-contained, we introduce at this point some background material from the geometry. (For more details, see [12, 39]). Consider a group G acting on a set X. The *orbit* of a point x in X is the set of elements of X to which x can be moved by the elements of G. (Just as gravity moves a planet around in its orbit, the group action moves an element around in its orbit.) The G-orbit of x is denoted by $O_G(x) = \{\tau(x), \tau \in G\}$. If $Y \subseteq X$, we write $G(Y) = \{\tau(y) : y \in Y \text{ and } \tau \in G\}$. We call the subset Y *invariant* under the action of G if $G(Y) = Y$ and denote it by Y_G. For every $x \in X$, we define the *stabilizer subgroup* of G with respect to x (also called the *isotropy group*) as the set of all elements in G that fix x: $S_G(x) = \{\tau \in G : \tau(x) = x\}$. Moreover, if the set X is equipped with a metric, then the *isometry group* of this metric space is the set of all isometries (i.e., distance-preserving maps) from the metric space onto itself, with the function composition as group operation. Its identity element is the identity function, i.e., the isometry group of a two-dimensional sphere is the orthogonal group $O(3)$. Given a Riemannian manifold (M, g) (complete or not, but connected), we define by $I(M,g)$ its group of isometries. It is well known (see, for instance, [39]) that $I(M,g)$ is a Lie group with respect to the compact open topology, and that $I(M,g)$ acts differentiably on M. Since (this is actually due to Cartan) any closed subgroup of a compact Lie group is a Lie group, we get that any compact subgroup of $I(M,g)$ is a sub-Lie group of $I(M,g)$. It is now classical (see [12, 21]) that $O_G(x)$ is a smooth compact submanifold of M for any $x \in M$. Denote by $|O_G(x)|$ the volume of $O_G(x)$ for the Riemannian metric induced on $O_G(x)$. (In the special case where $O_G(x)$ has finite cardinal, then $|O_G(x)| = \operatorname{card} O_G(x)$.) Let G be a closed subgroup of $I(M,g)$. Assume that for any $x \in M$, $\operatorname{card} O_G(x) = +\infty$, and set $k = \min_{x\in M} \dim O_G(x)$. Then, $k \geq 1$ (see [38]), and it is called *minimum orbit dimension*.

If G denotes some subgroup of $I(M,g)$, we set

$$C_G^\infty(M) = \{u \in C^\infty(M) : u \circ \tau = u \text{ for all } \tau \in G\}$$

and

$$C_{0,G}^\infty(M) = \{u \in C_0^\infty(M) : u \circ \tau = u \text{ for all } \tau \in G\},$$

where $C^\infty(M)$ denotes the space of smooth functions on M and $C_0^\infty(M)$ denotes the space of smooth functions with compact support on M. Similarly, for $p \geq 1$, we set

$$H_{1,G}^p(M) = \{u \in H_1^p(M) : u \circ \tau = u \text{ for all } \tau \in G\}$$

and

$$\mathring{H}_{1,G}^p(M) = \{u \in \mathring{H}_1^p(M) : u \circ \tau = u \text{ for all } \tau \in G\},$$

where $\mathring{H}_{1,G}^p(M)$ is the completion of $C_{0,G}^\infty(M)$ with respect to the norm $\|u\|_{H_1^p(M)}$. If (M,g) is complete and G is compact, by the existence of the a Haar measure on any Lie group, one gets that $H_{1,G}^p(M) = \mathring{H}_{1,G}^p(M)$ for any p.

If k denotes the minimum orbit dimension of G, it is known from Hebey [35] that for a G-invariant Riemannian manifold (M,g) without boundary, the embeddings $H_{1,G}^p(M) \hookrightarrow L^q(M)$ are continuous for any $p \in [1, n-k)$ and $q \in [1, \frac{(n-k)p}{n-k-p}]$ and compact if $q \in [1, \frac{(n-k)p}{n-k-p})$. In this case, the best constant K_G in front of the gradient term in inequality (1.1) was computed by Faget [27]:

$$K_G = \frac{K(n-k,p)}{V^{1/(n-k)}},$$

where V is the minimum volume of orbits of dimension k.

Consider now a compact smooth n-dimensional Riemannian manifold (M,g), $n \geq 3$, with boundary invariant under the action of a subgroup G of the isometry group $I(M,g)$. Let k denote the minimum orbit dimension of G and let V be the minimum of the volume of the k-dimensional orbits. Denote also in this case

$$p^* = \frac{(n-k)p}{n-k-p} \quad \text{and} \quad \tilde{p}^* = \frac{(n-k-1)p}{n-k-p}.$$

Cotsiolis and Labropoulos [18] showed that for any $p \in (1, n-k)$ and for all $u \in H_{1,G}^p(M)$ the first best constants in the inequalities

$$\left(\int_M |u|^{p^*}\,dv_g\right)^{p/p^*} \leq A \int_M |\nabla u|^p\,dv_g + B \int_M |u|^p\,dv_g \tag{2.3}$$

and

$$\left(\int_{\partial M} |u|^{\tilde{p}^*}\,ds_g\right)^{p/\tilde{p}^*} \leq \tilde{A} \int_M |\nabla u|^p\,dv_g + \tilde{B} \int_{\partial M} |u|^p\,ds_g \tag{2.4}$$

are $2^{p/(n-k)}K_G^p$ and $\tilde{K}_G^p$, respectively, where

$$K_G = \frac{K(n-k,p)}{V^{1/(n-k)}} \quad \text{and} \quad \tilde{K}_G = \frac{\tilde{K}(n-k,p)}{V^{(p-1)/(n-k-1)p}}.$$

Since inequalities (2.3) and (2.4) are stronger than (1.1) and (2.2), it follows that the first best constants in inequalities (1.1) and (2.2) in this case are $2^{1/(n-k)}K_G$ and $\tilde{K}_G$. In addition, Cotsiolis and Labropoulos [16, 17] computed the first best constants in the inequalities (1.1) and (2.2) in the special case where the manifold is the 3-dimensional solid torus T, which is invariant under the action of a subgroup $G = O(2) \times I$ of the isometry group $O(3)$.

2.2 The case $p = 1$

The case where $p = 1$ can be said *limiting* because it can be seen as a limit case as p tends to 1. This case is more complicated due to the lack of compactness of the embedding $H_1^1(M) \hookrightarrow L^1(\partial M)$. However, in the direction of interest in this article, there are some important results.

The limit of the Sobolev trace inequality in the Euclidean upper half-space $\mathbb{R}_+^n$ is given by

$$\int_{\partial\mathbb{R}_+^n} |u(x')|\,dx' \leq \tilde{C} \int_{\mathbb{R}_+^n} |\nabla u(x)|\,dx. \tag{2.5}$$

This problem was studied by Motron [46] and Park [48]. The best constant $\tilde{K}(n,1)$, defined by

$$\frac{1}{\tilde{K}(n,1)} = \inf_{\substack{u \in L^1(\partial R_+^n)\setminus\{0\} \\ \nabla u \in L^1(R_+^n)}} \frac{\int_{R_+^n} |\nabla u(x)|\,dx}{\left(\int_{\partial R_+^n} |u(x')|\,dx'\right)},$$

was computed for this inequality to be equal to 1. Moreover, Park proved that if $n = 1$, then the same result is obtained from recognizing the Euler–Lagrange equation for the inequality as the main curvature formula of plane curves. Motron [46] also proved that if Ω is a connected bounded open set of $\mathbb{R}^n$, $n \geq 2$, whose boundary is piecewise C^1, then the second best constant in the inequality

$$\int_{\partial\Omega} |u|\,dS \leq \tilde{A} \int_{\Omega} |\nabla u|\,dV + \tilde{B} \int_{\Omega} |u|\,dV \tag{2.6}$$

is equal to $|\partial\Omega|/|\Omega|$.

Andreu, Mazon and Rossi [3] studied the best constant $\lambda_1(\Omega)$ for the trace map from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, that is, the best constant in the inequality

$$\lambda \int_{\partial\Omega} |u|\,dx' \leq \int_{\Omega} |\nabla u|\,dx + \int_{\Omega} |u|\,dx, \tag{2.7}$$

where Ω is a bounded set in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$. Obviously, if we set $1/\lambda = \tilde{A} = \tilde{B}$, the best constant in (2.7) can be obtained as a special case of (2.6). The authors showed that if $\lambda_1(\Omega) < 1$, then this best constant is attained in $\mathrm{BV}(\Omega)$, the space of functions of bounded variation on Ω, which is defined as the space of functions in $L^1(\Omega)$ whose derivatives in the sense of distributions are bounded measures on Ω. Moreover, the authors proved that this constant can be obtained as a limit when $p \to 1^+$ of the best constant of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ with $p > 1$, that is, of the best constant in the inequality

$$\lambda \int_{\partial\Omega} |u|^p\,dx' \leq \int_{\Omega} |\nabla u|^p\,dx + \int_{\Omega} |u|^p\,dx. \tag{2.8}$$

In order to provide the proofs, they looked at Neumann problems involving the 1-Laplace operator defined by $\Delta_1(u) = \operatorname{div}(Du/|Du|)$ in the context of bounded variation functions (the natural context for this type of problems).

Consider now the case where $p = 1$ and (M, g) is a smooth compact n-dimensional ($n \geq 3$) Riemannian manifold with boundary. Since for $p = 1$, $p^* = \frac{n}{n-1}$ and $\tilde{p}^* = 1$, inequality (2.2) has no interest for $p = 1$. Thus, we are interested in the following two inequalities, arising from (1.1) and (1.2), respectively:

$$\left(\int_M |u|^{n/(n-1)}\,dv_g\right)^{(n-1)/n} \leq A \int_M |\nabla u|\,dv_g + B \int_M |u|\,dv_g \tag{2.9}$$

and

$$\int_{\partial M} |u|\, ds_g \leq \tilde{A} \int_M |\nabla u|\, dv_g + \tilde{B} \int_M |u|\, dv_g, \tag{2.10}$$

where A, B and $\tilde{A}$, $\tilde{B}$ are positive constants.

Traditionally, for the study of best constants it is used the space $W^{1,1}(\Omega)$ of functions in $L^1(\Omega)$ whose gradient in the distributional sense is in $L^1(\Omega)$, i.e.,

$$W^{1,1}(\Omega) = \{u \in L^1(\Omega) : \nabla u \in L^1(\Omega)\}.$$

Although the Sobolev space $W^{1,1}(\Omega)$ is a proper subset of $\mathrm{BV}(\Omega)$, from density results (see [20, 33, 49]), we can derive that if A, B (or $\tilde{A}$, $\tilde{B}$) are such that (2.9) (or (2.10)) is valid for all $u \in W^{1,1}(\Omega)$, then (2.9) (or (2.10)) may be expanded to functions in $\mathrm{BV}(\Omega)$ with the same constants A, B (or $\tilde{A}$, $\tilde{B}$). Concerning the inequality (2.9), it was proved by Andreu, Mazon and Rossi [3] that the best constant is the same in both $W^{1,1}(\Omega)$ and $\mathrm{BV}(\Omega)$. Thus, when we have to solve problems on best constants in the limiting case $p = 1$ on an arbitrary bounded set $\Omega \in \mathbb{R}^n$ we can remain in the space $W^{1,1}(\Omega)$. Furthermore, by Meyers–Serrin's theorem (see [45] or [1, Theorem 3.17]), the equality $H_1^p(\Omega) = W^{1,p}(\Omega)$, $1 \leq p < \infty$, is known to hold on all open subsets Ω of the Euclidean space $\mathbb{R}^n$. It seems to be unknown whether this extends to an arbitrary manifold M. However, by the definition of $H_1^p(M)$, we have $H_1^p(M) \subseteq W^{1,p}(M)$. On the other hand, by Hopf–Rinow's theorem we have that every compact Riemannian manifold (M, g) is geodesically complete. In addition, it is known that if M is geodesically complete, then $C_0^\infty(M)$ is dense in $W^{1,p}(M)$, see [34, Proposition 2.10]. Moreover, $C_0^\infty(M)$ is dense in $H^{1,p}(M)$, see [5, Theorem 1]. In particular, one has $H^{1,p}(M) = W^{1,p}(M)$ but we have been unable to find a direct reference for it.

In the rest of this paper we remain in the space $H_1^p(M)$, and also in the space $\mathrm{BV}(\Omega)$, where it is absolutely necessary (i.e., when we have to solve equations).

3 Qualitative presentation of results

The analysis presented in this paper is divided into three parts, as described in the following.

We first study inequalities (2.9), (2.10), as well as inequality (2.7), and we compute all the best constants in the case where the domain is the solid torus:

$$\overline{T} = \Big\{(x, y, z) \in \mathbb{R}^3 : \big(\sqrt{x^2 + y^2} - l\big)^2 + z^2 \leq r^2,\ l > r > 0\Big\}.$$

One of our main interests is to study the dependence of the best constant in theses inequalities as well as the existence of extremals (functions where the constant is attained) in (2.7) on the geometry. The related problem in the general case was studied by Andreu, Mazon and Rossi [3] and the same problem in a more overall context was studied by Demengel [19]. So, we compute both the best constants in inequalities (2.9) and (2.10), we prove that the solid torus is an extremal domain with respect to the second best constant in inequality (2.10), in the sense that this constant cannot be lowered for all bounded axisymmetric domain Ω in $\mathbb{R}^3$, and we prove that the solid torus is totally optimal with respect to the constants. Moreover, we compute the best constant in the inequality (2.7). The calculation of this best constant allows us to study the corresponding boundary value problem for the 1-Laplace differential operator in the solid torus.

Secondly, the dependence of the existence of a solution to the Neumann problem involving the 1-Laplacian of geometry is also considered. In particular, it is proved that this problem has a solution only in the cases when we have “small” tori. For “big” tori the problem does not have any solution.

Finally, we give some answers to the same problems in the case where the domain is a smooth, compact Riemannian manifold with boundary both in the general case and in the presence of symmetries. More precisely, we are concerned with the following problems.

(a) In the first part, we study the case of a smooth compact Riemannian manifold with boundary. Concerning inequality (2.9), we prove that the best constants are the same as those in the Euclidean case. Regarding inequality (2.10), we prove that the first best constant is equal to 1, remaining the same as that of the Euclidean space. The second best constant is $|\partial M|/|M|$, where $|\partial M|$ denotes the $(n-1)$-dimensional measure of ∂M and $|M|$ the n-dimensional measure of M.

(b) In the second part, we study the impact of the symmetries which appear in the manifold in the general case. Specifically, we compute the best constants in both inequalities (2.9) and (2.10) and we give general theorems concerning the best constants on manifolds in the presence of symmetries for $p = 1$. The values of both best constants in inequality (2.9) are strongly influenced by the geometry. The first best constant depends on the dimension and the volume of the orbit with the minimum volume. The second best constant depends on the volume of the manifold and the dimension of the orbit with minimum volume. Finally, surprising results occur on the best constant in inequality (2.10). For instance, the first best constant remains the same for all smooth, compact Riemannian manifolds and is neither depending on the dimension nor on the geometry. Contrary to the first best constant, the second best constant depends strongly on the geometry.

4 Best constants on the solid torus in the case $p = 1$

Consider the solid torus, represented by

$$\overline{T} = \Big\{(x, y, z) \in \mathbb{R}^3 : \big(\sqrt{x^2 + y^2} - l\big)^2 + z^2 \le r^2,\ l > r > 0\Big\},$$

its boundary

$$\partial T = \Big\{(x, y, z) \in \mathbb{R}^3 : \big(\sqrt{x^2 + y^2} - l\big)^2 + z^2 = r^2,\ l > r > 0\Big\}$$

and the group $G = O(2) \times I$ of $O(3)$. Note that the solid torus $\overline{T} \in \mathbb{R}^3$ is invariant under the action of the group G.

We now recall some background material and results from [16]. Let $\mathcal{A} = \{(\Omega_i, \xi_i) : i = 1, 2\}$ be an atlas on $T = \overline{T} \backslash \partial T$ defined by

$$\Omega_1 = \{(x, y, z) \in T : (x, y, z) \notin H^+_{XZ}\},$$
$$\Omega_2 = \{(x, y, z) \in T : (x, y, z) \notin H^-_{XZ}\},$$

where

$$H^+_{XZ} = \{(x, y, z) \in \mathbb{R}^3 : x > 0,\ y = 0\},$$
$$H^-_{XZ} = \{(x, y, z) \in \mathbb{R}^3 : x < 0,\ y = 0\}$$

and $\xi_i : \Omega_i \to I_i \times D$, $i = 1, 2$, with $I_1 = (0, 2\pi)$, $I_2 = (-\pi, \pi)$. Moreover,

$$D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 < 1\}, \quad \partial D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 = 1\},$$

$\xi_i(x, y, z) = (\omega_i, t, s)$, $i = 1, 2$, with $\cos\omega_i = x/\sqrt{x^2 + y^2}$, $\sin\omega_i = y/\sqrt{x^2 + y^2}$, where

$$\omega_1 = \begin{cases} \arctan\dfrac{y}{x}, & x \ne 0, \\ \dfrac{\pi}{2}, & x = 0, y > 0, \\ 3\dfrac{\pi}{2}, & x = 0, y < 0, \end{cases} \qquad \omega_2 = \begin{cases} \arctan\dfrac{y}{x}, & x \ne 0, \\ \dfrac{\pi}{2}, & x = 0, y > 0, \\ -\dfrac{\pi}{2}, & x = 0, y < 0 \end{cases}$$

and

$$t = \frac{\sqrt{x^2 + y^2} - l}{r}, \quad s = \frac{z}{r}, \quad 0 \le t, s \le 1.$$

Then, the Euclidean metric g on $(\Omega, \xi) \in \mathcal{A}$ and the induced metric on the boundary $\bar{g}$ can be expressed, respectively, as

$$(\sqrt{g} \circ \xi^{-1})(\omega, t, s) = r^2(l + rt) \tag{4.1}$$

and

$$(\sqrt{\bar{g}} \circ \xi^{-1})(\omega, t, s) = r(l + rt). \tag{4.2}$$

If for any G-invariant function u defined on T we define the function

$$\phi(t,s) = (u \circ \xi^{-1})(\omega, t, s), \tag{4.3}$$

then we obtain the following equalities:

$$\int_T |u|^p \, dV = 2\pi r^2 \int_D |\phi(t,s)|^p (l + rt) \, dt \, ds \tag{4.4}$$

$$\int_T |\nabla u|^p \, dV = 2\pi r^{2-p} \int_D |\nabla \phi(t,s)|^p (l + rt) \, dt \, ds, \tag{4.5}$$

$$\int_{\partial T} |u|^p \, dS = 2\pi r \int_{\partial D} |\phi(t,0)|^p (l + rt) \, dt, \tag{4.6}$$

where by ϕ we denote the extension of ϕ on ∂D.

Let $K(2,1) = 1/(2\sqrt{\pi})$ be the best constant of the Sobolev inequality

$$\|\varphi\|_{L^2(\mathbb{R}^2)} \leq K(2,1) \|\nabla \varphi\|_{L^1(\mathbb{R}^2)} \tag{4.7}$$

for the Euclidean space $\mathbb{R}^2$ (see [7]) and let $\tilde{K}(2,1) = 1$ be the best constant in the Sobolev trace embedding

$$\|\varphi\|_{L^1(\partial \mathbb{R}^2_+)} \leq \tilde{K}(2,1) \|\nabla \varphi\|_{L^1(\mathbb{R}^2_+)} \tag{4.8}$$

for the Euclidean half-space $\mathbb{R}^2_+$, see [46, 48].

It is known (see [16, 17]) that the solid torus $\bar{T}$ is invariant under the action of the group $G = O(2) \times I$, all the orbits are of dimension 1, and the "classical" Sobolev inequality in this case states as follows: For any real p such that $1 \leq p < 2$, $p^* = 2p/(2-p)$ and for all $u \in H^1_{1,G}(T)$ there exist two positive constants A and B such that

$$\left(\int_T |u|^{p^*} \, dV \right)^{1/p^*} \leqslant A \left(\int_T |\nabla u|^p \, dV \right)^{1/p} + B \left(\int_T |u|^p \, dV \right)^{1/p}. \tag{4.9}$$

We recall here that the displayed exponent $p^* = 2p/(2-p)$ in the above inequality is the highest possible supercritical exponent (critical of supercritical) because of the symmetry presented by the solid torus.

Aubin [5] proved that in a compact Riemannian manifold M, the best constant in the embedding of the Sobolev space $H^p_1(M)$ in $L^{p^*}(M)$, where $p^* = np/(n-p)$ for $p \in (1,n)$, is equal to $K(n,p)$, the norm of the inclusion $H^p_1 \hookrightarrow L^{p^*}$ on $\mathbb{R}^n$. Thus, for any $\varepsilon > 0$ there exists a constant $B_p(\varepsilon)$ such that every $u \in H^p_1(M)$ satisfies

$$\left(\int_M |u|^{p^*} \, dv_g \right)^{1/p^*} \leq (K(n,p) + \varepsilon) \left(\int_M |\nabla u|^p \, dv_g \right)^{1/p} + B_p(\varepsilon) \left(\int_M |u|^p \, dv_g \right)^{1/p}.$$

A natural question arises: Is the best constant achieved? In other words, does there exist $B_p = B_p(0)$? We can expect a positive answer. Aubin made a conjecture [5] concerning the following inequality of interest, among other very significant inequalities

$$\left(\int_M |u|^{p^*} \, dv_g \right)^{1/p^*} \leq K(n,p) \left(\int_M |\nabla u|^p \, dv_g \right)^{1/p} + B_p \left(\int_M |u|^p \, dv_g \right)^{1/p}, \quad 1 \leq p \leq 2.$$

This conjecture was first proved for $p = 2$ by Hebey and Vaugon [37], then for any p by Druet [22]. On Riemannian manifolds in the presence of symmetries a positive answer is given by Faget [28]. In the case of the solid torus a positive answer is, also, given by Cotsiolis and Labropoulos [17]. However, a new question arises: What happens in the case where $p = 1$? If $p = 1$, then $p^* = 1^* = 2$ and thus, by (4.9), we obtain the following Sobolev inequality:

$$\left(\int_T u^2 \, dV \right)^{1/2} \leqslant A \int_T |\nabla u| \, dV + B \int_T |u| \, dV. \tag{4.10}$$

The question most clearly now is: Is it possible to have an optimal inequality from (4.10) without ε? The answer is positive in the sense that we can not find an arbitrarily small $\varepsilon > 0$ such that inequality (4.10) is valid for $A = K(2,1) + \varepsilon$ for any B and for any $u \in H^1_{1,G}(T)$. In particular, we can state the following theorem.

Theorem 4.1. *Let T be the 3-dimensional solid torus. Then, the following properties are true:*

(i) *There exists $B \in \mathbb{R}$ such that for any $u \in H^1_{1,G}(T)$,*

$$\Big(\int_T u^2\,dV\Big)^{1/2} \leqslant \frac{K(2,1)}{\sqrt{\pi(l-r)}}\int_T |\nabla u|\,dV + B\int_T |u|\,dV. \tag{4.11}$$

(ii) *There exists $A \in \mathbb{R}$ such that for any $u \in H^1_{1,G}(T)$,*

$$\Big(\int_T u^2\,dV\Big)^{1/2} \leqslant A\int_T |\nabla u|\,dV + |T|^{-1/2}\int_T |u|\,dV. \tag{4.12}$$

In addition,

$$\frac{K(2,1)}{\sqrt{\pi(l-r)}} = \frac{1}{2\pi\sqrt{l-r}} \quad \text{and} \quad |T|^{-1/2} = \frac{1}{\pi r\sqrt{2l}}$$

are the best constants for these inequalities.

Proof. We carry out the proof of the theorem in two steps.

Step 1. This first step is devoted to study the first best constant. Cotsiolis and Labropoulos [17] showed that if p is a positive real number such that $1 < p < 2$, then there exists $B = B(p) > 0$ such that for any $u \in H^p_{1,G}(T)$,

$$\Big(\int_T |u|^{2p/(2-p)}\,dV\Big)^{(2-p)/2p} \leq \Big(\frac{K(2,p)}{\sqrt{\pi(l-r)}}\Big)^p \int_T |\nabla u|^p\,dV + B\int_T |u|^p\,dV. \tag{4.13}$$

The constant

$$\frac{K(2,p)}{\sqrt{\pi(l-r)}}$$

is the best constant for which inequality (4.13) remains true for any $u \in H^p_{1,G}(T)$.

Our purpose here is to prove that inequality (4.13) also holds in the case where $p = 1$, namely that for any $u \in H^1_{1,G}(T)$, there exists a positive real number $B > 0$ such that

$$\Big(\int_T u^2\,dV\Big)^{1/2} \leq \frac{K(2,1)}{\sqrt{\pi(l-r)}}\int_T |\nabla u|\,dV + B\int_T |u|\,dV. \tag{4.14}$$

Assume that inequality (4.14) is false. Then, for any $\beta > 0$ there exists $u \in H^1_{1,G}(T)$ such that

$$\Big(\int_T u^2\,dV\Big)^{1/2} > \frac{K(2,1)}{\sqrt{\pi(l-r)}}\int_T |\nabla u|\,dV + \beta\int_T |u|\,dV. \tag{4.15}$$

Thus, by (4.15), we deduce that for any $\beta > 0$ there exists $u \in H^1_{1,G}(T)$ such that

$$I(u) = \frac{(\int_T |\nabla u|\,dV + \beta\int_T |u|\,dV)}{(\int_T u^2\,dV)^{1/2}} < \Big(\frac{K(2,1)}{\sqrt{\pi(l-r)}}\Big)^{-1}. \tag{4.16}$$

By (4.16), we obtain

$$0 < I(u) < \Big(\frac{K(2,1)}{\sqrt{\pi(l-r)}}\Big)^{-1}$$

and so

$$0 \leq \inf_{u\in H^1_{1,G}(T),\,u\not\equiv 0} I(u) < \Big(\frac{K(2,1)}{\sqrt{\pi(l-r)}}\Big)^{-1}. \tag{4.17}$$

The inequality (4.17) allows us, using variation arguments, to prove that for any $\beta > 0$ the infimum

$$\lambda_\beta = \inf_{u \in H^1_{1,G}(T), u \not\equiv 0} I(u)$$

is achieved by a function $u_\beta \geq 0$. Namely, it holds $I(u_\beta) = \lambda_\beta$. Due to (4.17), λ_β is bounded. In addition, we conclude that for sufficiently large β, u_β is not a constant. Otherwise, we would have

$$\lim_{\beta \to +\infty} \frac{\int_T |\nabla u_\beta| \, dV + \beta \int_T |u_\beta| \, dV}{\left(\int_T (u_\beta)^2 \, dV\right)^{1/2}} = \lim_{\beta \to +\infty} \frac{\beta \int_T |u_\beta| \, dV}{\left(\int_T (u_\beta)^2 \, dV\right)^{1/2}} = +\infty,$$

which is false because of (4.17). This implies that for sufficiently large β, we have $\int_T |\nabla u_\beta| \, dV \neq 0$ and then $u_\beta \not\equiv 0$.

For any $p \in [1, 2)$, we define

$$I_p^\beta(u) \equiv \frac{\int_T |\nabla u|^p \, dV + \beta \int_T |u|^p \, dV}{\left(\int_T |u|^{2p/(2-p)} \, dV\right)^{(2-p)/2p}}.$$

Since the function $f(p) = I_p^\beta(u)$ with $p \in [1, 2)$ is continuous on p for any $u \in H^p_{1,G}(T)$, we conclude that $\lim_{p \to 1} I_p^\beta(u) = I(u)$. On the other hand (cf. [7]), we have $\lim_{p \to 1} K(n, p) = K(n, 1)$. Thus, by (4.13), we obtain that for any $p \in [1, 2)$,

$$\inf_{u \in H^p_{1,G}(T), u \not\equiv 0} \lim_{p \to 1} I_p^\beta(u) \geq \lim_{p \to 1} \left(\frac{K(2, p)}{\sqrt{\pi(l - r)}} \right)^{-1}$$

or

$$\inf_{u \in H^1_{1,G}(T), u \not\equiv 0} I(u) \geq \left(\frac{K(2, 1)}{\sqrt{\pi(l - r)}} \right)^{-1}. \tag{4.18}$$

As it is impossible to apply (4.17) and (4.18) simultaneously, we have reached a contradiction.

It remains to prove that the constant $K(2, 1)/\sqrt{\pi(l - r)}$ is the best possible for the inequality (4.14). Since we proved above that for functions the first best constant of the Sobolev inequality (4.9) is greater or equal to $K(2, 1)/\sqrt{\pi(l - r)}$, to complete the proof we must exclude the first case.

We define the *smallish* torus $T_\delta = \{P \in \mathbb{R}^3 : d(P, \mathcal{O}) < \delta\}$, where $\mathcal{O}$ is the orbit of minimum length $2\pi(l - r)$ and $d(\cdot \mathcal{O})$ is the distance to the orbit. Since, for functions belonging to $H^1_{1,G}(T \cap T_\delta)$ the first best constant of the Sobolev inequality (4.9) in T has the same value and in $T \cap T_\delta$ (see [17, Theorem 3.1]), in the sequel of this proof first we will stay in $H^1_{1,G}(T \cap T_\delta)$.

Assume now by contradiction that for some arbitrarily small but fixed ε and for all $\beta > 0$ the inequality

$$\left(\int_{T \cap T_\delta} u^2 \, dV \right)^{1/2} \leqslant \left(\frac{K(2, 1)}{\sqrt{\pi(l - r)}} + \varepsilon \right) \int_{T \cap T_\delta} |\nabla u| \, dV + \beta \int_{T \cap T_\delta} |u| \, dV$$

holds for all $u \in H^1_{1,G}(T \cap T_\delta)$ or, equivalently,

$$I^\beta(u) = \frac{\left(\int_{T \cap T_\delta} |\nabla u| \, dV + \beta \int_{T \cap T_\delta} |u| \, dV\right)}{\left(\int_{T \cap T_\delta} u^2 \, dV\right)^{1/2}} < \left(\frac{K(2, 1)}{\sqrt{\pi(l - r)}} + \varepsilon \right)^{-1}. \tag{4.19}$$

If we define

$$\mathcal{J}^\beta = \inf_{u \in H^1_{1,G}(T \cap T_\delta), u \not\equiv 0} I^\beta(u), \tag{4.20}$$

it follows that for all $\beta > 0$, there exists $\theta(\varepsilon) > 0$ such that

$$\mathcal{J}^\beta < \left(\frac{K(2, 1)}{\sqrt{\pi(l - r)}} + \varepsilon \right)^{-1} = \frac{\sqrt{\pi(l - r)}}{K(2, 1)} - \theta(\varepsilon). \tag{4.21}$$

Let now a minimizing sequence $(u_j) \in H^1_{1,G}(T \cap T_\delta)$ of $I^\beta(u)$. For all u_j, we define on D (the unit disk of $\mathbb{R}^2$) the functions

$$\phi_j(t,s) \equiv (u_j \circ \xi^{-1})(\omega, t, s), \tag{4.22}$$

and for any $\phi_j \in H^p_1(D)$ and any $\lambda \geq 0$, we set

$$\phi_{j_\lambda}(t,s) \equiv \phi_j(\lambda t, \lambda s). \tag{4.23}$$

Consider now the λ-parametric sequence ϕ_{j_λ} defined by (4.23) and the λ-parametric sequence u_{j_λ} defined by (4.22) to be $u_{j_\lambda} = \phi_{j_\lambda} \circ \xi$. By (4.4) and (4.5) we obtain, respectively,

$$\int_{T \cap T_\delta} |u_{j_\lambda}|^2 \, dV = 2\pi \Big(\frac{\delta}{\lambda}\Big)^2 \int_D |\phi_j|^2 \Big(l - r + \frac{\delta}{\lambda} t\Big) \, dt \, ds, \tag{4.24}$$

$$\int_{T \cap T_\delta} |\nabla u_{j_\lambda}| \, dV = 2\pi \frac{\delta}{\lambda} \int_D |\nabla \phi_j| \Big(l - r + \frac{\delta}{\lambda} t\Big) \, dt \, ds. \tag{4.25}$$

Set

$$\mathrm{I}^\beta_{j_\lambda} \equiv \frac{\big(\int_{T\cap T_\delta} |\nabla u_{j_\lambda}| \, dV + \beta \int_{T\cap T_\delta} |u_{j_\lambda}| \, dV\big)}{\big(\int_{T\cap T_\delta} u^2_{j_\lambda} \, dV\big)^{1/2}}. \tag{4.26}$$

By (4.24)–(4.26) and a direct computation, we obtain successively

$$\mathrm{I}^\beta_{j_\lambda} = \frac{2\pi \int_D |\nabla \phi_{j_\lambda}| (l - r + \delta \frac{t}{\lambda}) \, dt \, ds}{\big(2\pi \int_D |\phi_{j_\lambda}|^2 (l - r + \delta \frac{t}{\lambda}) \, dt \, ds\big)^{1/2}} + \frac{\delta}{\lambda} \frac{\beta 2\pi \int_D |\phi_{j_\lambda}| (l - r + \delta \frac{t}{\lambda}) \, dt \, ds}{\big(2\pi \int_D |\phi_{j_\lambda}|^2 (l - r + \delta \frac{t}{\lambda}) \, dt \, ds\big)^{1/2}}. \tag{4.27}$$

Letting $\lambda \to \infty$ in equality (4.27) yields

$$I^\beta_j \equiv \lim_{\lambda \to \infty} I^\beta_{j\lambda} = \sqrt{2\pi(l - r)} \frac{\int_D |\nabla \phi_j| \, dt \, ds}{\big(\int_D |\phi_j|^2 \, dt \, ds\big)^{1/2}}. \tag{4.28}$$

Letting $j \to \infty$ in (4.28), and because of (4.20), we obtain the equality

$$\mathcal{I}^\beta = \sqrt{2\pi(l - r)} \frac{\int_D |\nabla \phi| \, dt \, ds}{\big(\int_D |\phi|^2 \, dt \, ds\big)^{1/2}}, \tag{4.29}$$

where the function ϕ is defined, in the same way as the ϕ_j in (4.22), to be $\phi(t,s) \equiv (u \circ \xi^{-1})(\omega, t, s)$ and u is the limit of $(u_j)_{j=1,2,\dots}$, defined above.

Finally, it is known that the best constant of the Sobolev inequality for functions defined in $H^1_1(D)$ is equal to $\sqrt{2}K(2,1)$ (see [7, Lemma 2.31], and [13] for a complete proof). Thus, by (4.21) and (4.29) we obtain

$$\sqrt{2\pi(l - r)} \frac{1}{\sqrt{2}K(2,1)} < \frac{\sqrt{\pi(l - r)}}{K(2,1)} - \theta(\varepsilon),$$

which is a contradiction.

Step 2. In this second step, we compute the second best constant in inequality (4.11). By taking $u = 1$ in (4.10), we obtain that $B \geq |T|^{-1/2}$. In particular,

$$\mathcal{B}_1(T) \geq |T|^{-1/2}. \tag{4.30}$$

Let $u \in H^1_{1,G}(T)$ and $\bar{u} = \frac{1}{|T|} \int_T u \, dV$. Since $\bar{u}$ is a constant function and because $(u - \bar{u}) \circ \tau = u \circ \tau - \bar{u} \circ \tau = u - \bar{u}$ for any $\tau \in G$, we conclude that $(u - \bar{u}) \in H^1_{1,G}(T)$. Setting $(u - \bar{u}) \circ \xi^{-1} = \phi^*$ in (4.4), we obtain

$$\begin{aligned} \Big(\int_T (u - \bar{u})^2 \, dV\Big)^{1/2} &= \Big(2\pi r^2 \int_D (\phi^*)^2 (l + rt) \, dt \, ds\Big)^{1/2} \\ &\leq (2\pi r^2 (l + r))^{1/2} \Big(\int_D (\phi^*)^2 \, dt \, ds\Big)^{1/2}. \end{aligned} \tag{4.31}$$

Moreover, by the Sobolev–Poincaré inequality there exists a positive real number C such that for any $\phi^* \in H_1^1(D)$, the following inequality holds:

$$\left(\int_D (\phi^* - \bar{\phi}^*)^2 \, dt \, ds\right)^{1/2} \leq C \int_D |\nabla \phi^*| \, dt \, ds, \tag{4.32}$$

where $\bar{\phi}^* = \frac{1}{|T|} \int_D \phi^* \, dt \, ds$.

We may assume that $\bar{\phi}^* = 0$. Actually, if $\bar{\phi}^* = \eta \neq 0$ instead of ϕ^* we could take the function $\phi^* - \eta$ and then

$$\frac{1}{|T|} \int_D (\phi^* - \eta) \, dt \, ds = \frac{1}{|T|} \int_D \phi^* \, dt \, ds - \frac{1}{|T|} \int_D \eta \, dt \, ds = \eta - \frac{\eta}{|T|} |T| = 0.$$

Thus, if $\bar{\phi}^* = 0$, inequality (4.32) yields

$$\left(\int_D (\phi^*)^2 \, dt \, ds\right)^{1/2} \leq C \int_D |\nabla \phi^*| \, dt \, ds. \tag{4.33}$$

Relation (4.5) yields

$$\int_D |\nabla \phi^*| \, dt \, ds \leq \frac{1}{2\pi r(l - r)} \int_T |\nabla u| \, dV. \tag{4.34}$$

Combining inequalities (4.33) and (4.34), we obtain

$$\left(\int_D (\phi^*)^2 \, dt \, ds\right)^{1/2} \leq C_1 \int_T |\nabla u| \, dV, \tag{4.35}$$

where $C_1 = C/(2\pi r(l - r))$. Thus, by (4.35) and (4.31), we have

$$\left(\int_T (u - \bar{u})^2 \, dV\right)^{1/2} \leq C_2 \int_T |\nabla u| \, dV, \tag{4.36}$$

where $C_2 = \sqrt{2\pi r^2 (l + r)} C_1$.

By (4.36) and since $\|u\|_p \leq \|u - \bar{u}\|_p + \|\bar{u}\|_p$, we obtain

$$\left(\int_T u^2 \, dV\right)^{1/2} \leq \left(\int_T (u - \bar{u})^2 \, dV\right)^{1/2} + \left(\int_T \bar{u}^2 \, dV\right)^{1/2}$$

or, equivalently,

$$\left(\int_T u^2 \, dV\right)^{1/2} \leq \left(\int_T (u - \bar{u})^2 \, dV\right)^{1/2} + |\bar{u}| \left(\int_T dV\right)^{1/2}.$$

From the last inequality, since $\bar{u} = \frac{1}{|T|} \int_T u \, dV$, we deduce that

$$\left(\int_T u^2 \, dV\right)^{1/2} \leq \left(\int_T (u - \bar{u})^2 \, dV\right)^{1/2} + |T|^{-1/2} \left|\int_T u \, dV\right|. \tag{4.37}$$

By (4.37) and using (4.36), we find

$$\left(\int_T u^2 \, dV\right)^{1/2} \leq C \int_T |\nabla u| \, dV + |T|^{-1/2} \left|\int_T u \, dV\right|,$$

from which arises

$$\left(\int_T u^2 \, dV\right)^{1/2} \leq C \int_T |\nabla u| \, dV + |T|^{-1/2} \int_T |u| \, dV. \tag{4.38}$$

Combining inequality (4.38) with (4.30) we conclude that $\mathcal{B}_1(T) = |T|^{-1/2}$. □

The following property is the natural extension of Theorem 4.1 in the general case where $1 < p < 2$.

Corollary 4.2. *Let T be the solid torus and let p be a real number such that $1 < p < 2$. Then, the following hold:*

(i) *There exists $B \in \mathbb{R}$ such that for all $u \in H^p_{1,G}(T)$,*

$$\Big(\int_T |u|^{2p/(2-p)}\,dV\Big)^{(2-p)/2p} \le \frac{K(2,p)}{\sqrt{\pi(l-r)}}\Big(\int_T |\nabla u|^p\,dV\Big)^{1/p} + B\Big(\int_T |u|^p\,dV\Big)^{1/p}. \tag{4.39}$$

(ii) *There exists $a \in \mathbb{R}$ such that for all $u \in H^p_{1,G}(T)$,*

$$\Big(\int_T |u|^{2p/(2-p)}\,dV\Big)^{(2-p)/2p} \le A\Big(\int_T |\nabla u|^p\,dV\Big)^{1/p} + |T|^{-1/2}\Big(\int_T |u|^p\,dV\Big)^{1/p}. \tag{4.40}$$

In addition,

$$\frac{K(2,p)}{\sqrt{\pi(l-r)}} \quad \text{and} \quad |T|^{-1/2}$$

are the best constants for these inequalities.

Proof. For the second best constant in inequality (4.40), the calculation follows the same steps as in the proof of Theorem 4.1 and is completed with the help of the inequality

$$\Big|\int_T u\,dV\Big| \le \Big|\int_T |u|\,dV\Big| \le |T|^{1-(1/p)}\Big(\int_T |u|^p\,dV\Big)^{1/p},$$

which arises directly from the Hölder inequality.

Concerning the calculation of the first best constant in inequality (4.39), see [17, Theorem 3.1]. □

The last result concerning the case of the torus is as follows.

Theorem 4.3. *Let T be the solid torus. For all $u \in H^1_{1,G}(T)$, the following inequality holds:*

$$\int_{\partial T} |u|\,dS \le \int_T |\nabla u|\,dV + \frac{2}{r}\int_T |u|\,dV. \tag{4.41}$$

In particular, $1 = \tilde{K}(2,1)$ and $2/r$ are the best constants for this inequality and the function $u_0 = (1/|\partial T|)\chi_T$ is the only extremal function for this inequality.

Furthermore, for all $\phi \in H^1_1(D_r)$, the following inequality holds:

$$\int_{\partial D_r} |\phi|\,ds \le \int_{D_r} |\nabla\phi|\,dv + \frac{2}{r}\int_{D_r} |\phi|\,dv, \tag{4.42}$$

where

$$D_r = \{(t,s) \in \mathbb{R}^2 : t^2 + s^2 < r^2\}$$

is the disk of radius r which rotating around the z-axis and remaining coplanar with it produces the solid torus T.

The proof of this theorem makes use of the following auxiliary property.

Lemma 4.4. *Let T be the solid torus. Suppose that there exist real numbers $\tilde{A}$ and $\tilde{B}$ such that for all $u \in H^1_{1,G}(T)$ the following inequality holds:*

$$\int_{\partial T} |u|\,dS \le \tilde{A}\int_T |\nabla u|\,dV + \tilde{B}\int_T |u|\,dV. \tag{4.43}$$

Then, $\tilde{A} \ge 1 = \tilde{K}(2,1)$.

The proof of Lemma 4.4 is provided in the Appendix.

Proof of Theorem 4.3. In order to prove (4.41), it is equivalent to show firstly that for all $u \in H^1_{1,G}(T)$, there exists a constant $\tilde{A} \in \mathbb{R}$ such that

$$\int_{\partial T} |u|\, dS \leq \tilde{A} \int_T |\nabla u|\, dV + \frac{2}{r} \int_T |u|\, dV, \tag{4.44}$$

and secondly, that for all $u \in H^1_{1,G}(T)$, there exists a constant $\tilde{B} \in \mathbb{R}$ such that

$$\int_{\partial T} |u|\, dS \leq \int_T |\nabla u|\, dV + \tilde{B} \int_T |u|\, dV. \tag{4.45}$$

Proof of inequality (4.44). By [46, Proposition 3.10], the best second constant for the inequality

$$\int_{\partial T} |u|\, dS \leq \tilde{A} \int_T |\nabla u|\, dV + \tilde{B} \int_T |u|\, dV \tag{4.46}$$

is

$$\frac{|\partial T|}{|T|} = \frac{4\pi^2 rl}{2\pi^2 r^2 l} = \frac{2}{r}. \tag{4.47}$$

This fact means that for all $u \in H^1_{1,G}(T)$, there exists $\tilde{A} > 0$ such that the inequality (4.44) holds.

Proof of inequality (4.45). We first establish an auxiliary inequality, that is, for any $\varepsilon > 0$ and all $u \in H^1_{1,G}(T)$, there exists a constant $\tilde{B} \in \mathbb{R}$ such that

$$\int_{\partial T} |u|\, dS \leqslant (1+\varepsilon) \int_T |\nabla u|\, dV + C' \int_T |u|\, dV. \tag{4.48}$$

The proof is rather classic but we give a brief outline of the arguments. Let $P_j \in T$, O_{P_j} be its orbit under the action of subgroup $G = O(2) \times I$ and $l_j = \sqrt{x_j^2 + y_j^2}$ be the distance of O_{P_j} from the z-axis. For any $\varepsilon_0 > 0$, consider $\delta_j = \varepsilon_0 l_j < 1$, and the set $T_j = \{Q \in \mathbb{R}^3 : d(Q, O_{P_j}) < \delta_j\}$. Choose a finite covering of $\overline{T}$ by sets of the type T_j such that the following hold:

(a) If $P_j \in T$, then the entire T_j lies in T and the entire coordinate of $\xi_j(\Omega_j)$ on D lies in D.

(b) If $P_j \in \partial T$, then the coordinate of $\xi_j(\Omega_j \cap \partial T)$ on $\bar{D}$ lies in ∂D.

Now, for T we build a partition of unity that covers T in the following way. For each j we consider $h_j \in C^\infty_{0,G}(D)$, $h_j \geq 0$. Then, h_j can be seen as a function defined on $I \times D$ and depending only on D variables.

Let

$$\eta_j = \frac{h_j \circ \xi_j}{\sum_{j=1}^N (h_j \circ \xi_j)}.$$

The η_j form a new partition of unity relative to T_j, is G-invariant, and $\eta_j \circ \xi_j^{-1}$ depends only on D variables. Moreover, since the covering of T is finite, there exists a positive constant H depending on the chosen covering, namely such that $|\nabla \eta_j| \leqslant H$ for all $j \in \mathbb{N}$. Thus, for any $u \in H^1_{1,G}(T)$, if we set $\phi_j = (\eta_j u) \circ \xi_j^{-1}$, because of (4.31), we obtain

$$\int_{\partial T} |u|\, dS \leq \sum_{j=1}^N \int_{\partial T} |\eta_j u|\, dS \leq \sum_{j=1}^N (1+\varepsilon_0) 2\pi \delta_j l_j \int_{\partial D} |\phi_j(t,0)|\, dt. \tag{4.49}$$

By relation (4.49) and the Sobolev embedding theorem in $\partial \mathbb{R}^2_+$, we deduce that

$$\int_{\partial T} |u|\, dS \leq \frac{1+\varepsilon_0}{1-\varepsilon_0} \sum_{j=1}^N \int_T (\eta_j |\nabla u| + H|u|)\, dV \leqslant \frac{1+\varepsilon_0}{1-\varepsilon_0} \int_T (|\nabla u| + HN|u|)\, dV,$$

and if we set $\varepsilon = O(\varepsilon_0)$ and $C' = HN(1+\varepsilon_0)/(1-\varepsilon_0)$, then we obtain inequality (4.48).

Combining inequality (4.48) with Lemma 4.4 and taking into account that we can choose ε_0 arbitrarily small, we deduce that $1 = \tilde{A}_1(T)$. Furthermore, it is easy to verify that the constant function $u_0 = (1/|\partial T|)\chi_T$ is an extremal function for the inequality (4.45), and the first part of the theorem is proved.

For the first best constant in inequality (4.42) we need to repeat the same steps as in the case of inequality (4.41). Regarding the second best constant, by using the same argument as for inequality (4.41), we find that is equal to

$$\frac{|\partial D_r|}{|D_r|} = \frac{2\pi r}{\pi r^2} = \frac{2}{r}. \tag{4.50}$$

Finally, with a simple substitution we can prove that the constant function $\phi_0 = (1/|\partial D|)\chi_D$ is an extremal function for this inequality. □

Remark 4.5. Observing equalities (4.47) and (4.50), we see that the second best constants in inequalities (4.41) and (4.42) are the same. This geometrical aspect means that the solid torus behaves exactly like the disk from which is produced by rotation in the yz-plane about the z-axis far from z-axis. This result confirms in some sense the fact that each axisymmetric object E of the three-dimensional Euclidean space is identified by its description in $\mathbb{R}^2$ and so, for simplicity, we may view E as a subset of $\mathbb{R}^2$.

Remark 4.6. The solid torus is an extremal domain with respect to the second best constant $2/r$ in the inequality of Theorem 4.3, in the sense that this constant cannot be lowered for all bounded axisymmetric domains Ω in $\mathbb{R}^3$, since

$$\frac{2}{r} = \frac{|\partial T|}{|T|} = \frac{|\partial D_r|}{|D_r|}, \tag{4.51}$$

and because of the isoperimetric equality (see [29]),

$$\frac{|\partial D_r|}{|D_r|} = \inf_{\Omega\in\mathbb{R}^3}\left\{\frac{|\partial\Omega|}{|\Omega|}\right\}. \tag{4.52}$$

Remark 4.7. Since the first best constant of the second inequality in Theorem 4.3 is equal to 1 for all manifolds, we conclude that the solid torus is totally optimal with respect to the constants.

5 A Neumann problem involving the 1-Laplace operator in the solid torus

5.1 Mathematical background. Auxiliary results

At this point we need some background material concerning functions in the space $\mathrm{BV}(\Omega)$ (see [3, 9]), where Ω is a bounded set in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$.

A function $u \in L^1(\Omega)$ whose gradient, in the sense of distributions, is a (vector valued) Radon measure with finite total variation in Ω is called a *function of bounded variation*. Thus, $u \in \mathrm{BV}(\Omega)$ if and only if there are Radon measures $\mu_1, \mu_2, \dots, \mu_n$ defined in Ω with finite total mass in Ω and

$$\int_\Omega uD_i\varphi\,dx = -\int_\Omega \varphi\,d\mu_i \quad \text{for all } \varphi \in C_0^\infty(\Omega),\ i = 1, 2, \dots, n.$$

The gradient of u is the vector measure $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ denoted by Du with finite total variation

$$\sup\left\{\int_\Omega u\,\mathrm{div}\,\psi\,dx : \psi \in C_0^\infty(\Omega, \mathbb{R}^n),\ |\psi(x)| \le 1 \text{ for } x \in \Omega\right\}$$

and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$.

The function space $\mathrm{BV}(\Omega)$ is a Banach space when endowed with the norm

$$\|\nabla u\|_{\mathrm{BV}} = \int_\Omega |u| + \int_\Omega |Du|.$$

A measurable set $E \in \mathbb{R}^n$ is said to be of *finite perimeter* in Ω if $\chi_E \in \mathrm{BV}(\Omega)$, and in this case the perimeter of E in Ω is defined as $P(E, \Omega) = |D\chi_E|$. We shall use the notion $P(E) = P(E, \mathbb{R}^n)$. If E has a smooth boundary, then $P(E)$ and the classical measure $|\partial E|$ of the boundary correspond. It is well known (see, e.g., [2, 26, 51]) that for a given function $u \in \mathrm{BV}(\Omega)$ there exists a sequence $u_j \in W^{1,1}(\Omega)$ such that u_j strict converges to u, that is,

$$u_j \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \int_\Omega |\nabla u_j|\,dx \to \int_\Omega |Du|.$$

Moreover, there exists a trace operator τ which sends $\mathrm{BV}(\Omega)$ into $L^1(\Omega)$, namely for all $u \in \mathrm{BV}(\Omega)$,

$$\|\tau(u)\|_{L^1(\partial\Omega)} \leq C\|u\|_{\mathrm{BV}(\Omega)}$$

for some constant C depending only on Ω. The trace operator τ is continuous between $\mathrm{BV}(\Omega)$, endowed with the topology induced by the strict convergence, and $L^1(\partial\Omega)$. In the sequel we write $\tau(u) = u$.

For further information concerning functions of bounded variation we refer to [26, 51], and for a short generalization on Riemannian manifolds a good reference is [23].

We now recall several results from Andreu, Mazon and Rossi [3] and Anzellotti [4]. Let

$$X(\Omega) = \{z \in L^\infty(\Omega, \mathbb{R}^n) : \mathrm{div}(z) \in L^1(\Omega)\}.$$

If $z \in X(\Omega)$ and $w \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ is defined by

$$\langle (z, Dw), \varphi\rangle = -\int_\Omega \mathrm{div}(z) w\varphi - \int_\Omega zw \cdot \nabla\varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Then, (z, w) is a Radon measure in Ω with

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \quad \text{for all } w \in H_1^1(\Omega) \cap L^\infty(\Omega)$$

and

$$\left|\int_B (z, Dw)\right| \leqslant \int_B |(z, Dw)| \leqslant \|z\|_\infty \int_B |Dw| \quad \text{for any Borel set } B \subseteq \Omega.$$

In addition, a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. More precisely, it is proved that there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty \quad \text{and} \quad \gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{for } x \in \partial\Omega \text{ if } z \in C^1(\bar{\Omega}, \mathbb{R}^n),$$

where ν denotes the outward unit normal along $\partial\Omega$.

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$. Moreover, we have the following Green formula relating the function $[z, \nu]$ and the measure (z, Dw): For $z \in X_1(\Omega)$ and $w \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega \mathrm{div}(z) w\,dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w\,d\mathcal{H}^{n-1}, \tag{5.1}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. For a proof of this result we refer to [4].

5.2 Resolution of the problem

Consider the solid torus defined by

$$T = \left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - l\right)^2 + z^2 < r^2,\ l > r > 0\right\}.$$

We are interested in the following variation problem:

$$\lambda_1(T) = \inf\left\{\int_T |\nabla u|\,dV + \int_T |u|\,dV : u \in H_1^1(T),\ \int_{\partial T} |u|\,dS = 1\right\}. \tag{5.2}$$

In view of the results presented previously, this problem is equivalent to

$$\lambda_1(T) = \inf\left\{ \int_T |Du| + \int_T |u|\,dV : u \in \mathrm{BV}(T),\ \int_{\partial T} |u|\,dS = 1 \right\}. \tag{5.3}$$

Andreu, Mazon and Rossi [3] studied the dependence of the best constant $\lambda_1(\Omega)$ and its extremals on the domain. Here, we are interested to study the dependence of the existence of extremals on the best constant $\lambda_1(T)$, and therefore the geometrical characteristics of the torus. We note that since the variation method fails due the lack of compactness of the embedding $H_1^1(T) \hookrightarrow L^1(\partial T)$, the study of the problem is not trivial.

For $1 < p < 2$, let us consider the variation problem

$$\lambda_p(T) = \inf\left\{ \int_T |\nabla u|^p\,dV + \int_T |u|^p\,dV : u \in H_1^p(T),\ \int_{\partial T} |u|^p\,dS = 1 \right\}. \tag{5.4}$$

Due to the compactness of the embedding $H_1^p(T) \hookrightarrow L^p(\partial T)$, it is known (see [31]) that problem (5.8) has a minimizer in $H_1^p(T)$ and the extremals are weak solutions of the following problem:

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } T, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda_p(T)|u|^{p-2}u & \text{on } \partial T, \end{cases} \tag{5.5}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\partial u/\partial\nu$ is the outer unit normal derivative (see [42]).

Therefore, it seems natural to search for an extremal for $\lambda_1(T)$ as the limit of extremals for $\lambda_p(T)$ when $p \to 1^+$. Unfortunately, there is no hope to prove an existence result in $H_1^1(T)$, since this space is not reflexive. The convenient space in which one must look for solutions is the space $\mathrm{BV}(T)$. So, the extremals in this limit case are solutions of the following Neumann problem involving the 1-Laplacian operator $\Delta_1 u = \operatorname{div}(Du/|Du|)$:

$$\begin{cases} \Delta_1 u = \dfrac{u}{|u|} & \text{in } T, \\ \dfrac{Du}{|Du|}\cdot\nu = \lambda_1(T)\dfrac{u}{|u|} & \text{on } \partial T, \end{cases} \tag{5.6}$$

in the context of bounded variation functions, where Du denotes the gradient of u in this space.

We now recall the definition of the solution of problem (5.6) (see [3]).

Definition 5.1. A function $u \in \mathrm{BV}(T)$ is said to be a solution of problem (5.6) if there exist $z \in X_1(T)$ with $\|Z\|_\infty \le 1$, $\tau \in L^\infty(T)$ with $\|\tau\|_\infty \le 1$ and $\theta \in L^\infty(\partial T)$ with $\|\theta\|_\infty \le 1$ such that

$$\operatorname{div}(z) = \tau \quad \text{in } \mathcal{D}'(T), \tag{5.7}$$

$$\tau u = |u| \quad \text{a.e. in } T \quad \text{and} \quad (z, Du) = |Du| \text{ as measures}, \tag{5.8}$$

$$[z,\nu] = \lambda_1(T)\theta \quad \text{and} \quad \theta u = |u|\mathcal{H}^{n-1} \quad \text{a.e. on } \partial T. \tag{5.9}$$

Proposition 5.2. *The problems* (5.3) *and* (5.6) *are equivalent in the sense that if υ is a solution of* (5.6) *and* $\int_{\partial T}|\upsilon|\,dS \neq 0$, *then*

$$w = \frac{\upsilon}{\int_{\partial T}|\upsilon|\,dS}$$

is a minimizer of (5.3).

Proof (cf. [3]*).* Multiplying (5.7) by υ and integrating by parts due to (5.1), (5.8) and (5.9), we obtain

$$\int_T |\upsilon|\,dV = \int_T \tau\upsilon\,dV = \int_T \operatorname{div}(z)\upsilon\,dV = -\int_T (z, D\upsilon) + \int_{\partial T} [z,\nu]\upsilon\,dS = -\int_T |D\upsilon| + \lambda_1(T)\int_{\partial T} |\upsilon|\,dS.$$

Therefore,

$$\lambda_1(T) = \int_T |Dw| + \int_T |w|\,dV. \qquad \square$$

The result of Proposition 5.2 certifies the equivalence of the above problems in the sense noticed above, namely if problem (5.3) has a solution u, then problem (5.6) has the solution $w = \upsilon / \int_{\partial T} |\upsilon| \, dS$. Also, in [3], necessary and sufficient conditions of the existence of a solution to the above problems are given. In this paper, as already mentioned above, we are not interested in solving these problems in the torus. We use the above results in order to study the dependence of the existence of the solutions to the above problems in the geometry of the torus. In the following theorem, it is confirmed that the above problems have a solution only in the cases when we have "small" tori. For "big" tori the problems have no solutions. In addition, it is proved that in the case when $r = 2$ (where r is the range of the circle which is rotated in the yz-plane about the z-axis far from z-axis), the torus behaves as the disk D_2 of radius 2 in $\mathbb{R}^2$, since in this case $\lambda_1(T) = \lambda_1(D_2)$.

Theorem 5.3. *If $r \leq 2$, there exists a nonnegative function of bounded variation which is a solution of problem* (5.3)*. In particular, for $r = 2$, this problem is equivalent to the same problem considered in the disk D_2 in $\mathbb{R}^2$, in the sense that if υ is a solution to the first one, then the function $\phi = \upsilon \circ \xi^{-1}$, defined by* (4.3)*, is a solution of the second problem. In addition, problem* (5.7) *has a solution if $r \leq 2$ and has no solution if $r > 2$.*

Proof. We study three cases concerning the range of r.

Let $r < 2$. Since $|T|/|\partial T| = r/2$, we obtain (see (4.50)) that $|T|/|\partial T| < 1$ for all $r < 2$ and then by [3, Theorem 1], we conclude that $\lambda_1(T) < 1$. Thus, by [3, Theorem 2], we deduce that there exists a nonnegative function of bounded variation which is a minimizer of the variational problem (5.3) and a solution of problem (5.6), see Proposition 5.2.

For $r = 2$, because of Theorem 4.3, $\lambda_1(T) = \lambda_1(D_2)$, and since $|D_2|/|\partial D_2| = 1$, the variational problem (5.7) is equivalent to the same problem considered in the disk D_2, for which the function $\phi_0 = (1/|\partial D_2|)\chi_{D_2}$ is a minimizer (see [46, Example 1]). Thus, the function $u_0 = (1/|\partial T|)\chi_T$ ($u_0 = \phi_0 \circ \xi$ by Definition (4.3)) is a minimizer of problem (5.7), being the only minimizer in the case $|T|/|\partial T| = 1$, see (4.51).

If $r > 2$, then $|T|/|\partial T| > 1$ and the variational problem (5.3) does not have any minimizer, see [46]. □

6 Best constants on Riemannian manifolds with boundary

6.1 The general case

This part of the paper is devoted to the study of the classical Sobolev inequality on manifolds with boundary, to the Sobolev trace inequality and also to the existence and calculation of best constants, when they exist. The proofs of the related theorems are not difficult and probably are classical in the sense that in general these have been used in the case of manifolds without boundary and the presence of the boundary does not affect them. So, we do not give in detail the proofs of these theorems but we outline the arguments as briefly as possible by making the necessary adjustments for the case of manifolds with boundary. In addition, we note that the study of these inequalities is necessary because they have never been studied in the past and we do not know the values of the best constants. Furthermore, we will give some counter-examples demonstrating that in some cases there are no best constants for the above Sobolev inequalities.

Concerning the first best constant for the classical Sobolev inequality on manifolds with boundary in the case $p = 1$, the following theorem holds.

Theorem 6.1. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary, $n \geq 3$. For any $\varepsilon > 0$, there exists $B \in \mathbb{R}$ such that for any $u \in H_1^1(M)$,*

$$\left(\int_M |u|^{n/(n-1)} \, d\upsilon_g \right)^{(n-1)/n} \leq (2^{1/n} K(n, 1) + \varepsilon) \int_M |\nabla u| \, d\upsilon_g + B \int_M |u| \, d\upsilon_g. \tag{6.1}$$

Moreover, $2^{1/n} K(n, 1)$ is the best constant for this inequality.

Proof. Our first purpose is to establish the first best constant in inequality (6.1). The proof of this theorem uses some ideas from the proof of [35, Theorem 4.5], which are adapted to our case on manifolds with boundary.

Let us sketch the proof. Fix $\varepsilon > 0$. For any P in M and any $\varepsilon_0 > 0$, there exists some chart (Ω, ξ) on P such that

$$1 - \varepsilon_0 \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon_0.$$

Choosing ε_0 small enough, by [35, Theorem 4.5] we can assume that for any smooth function u with compact support in Ω,

$$\Big(\int_M |u|^{n/(n-1)}\,dv_g\Big)^{(n-1)/n} \leq (K(n,1)+\varepsilon_0)\int_M |\nabla u|\,dv_g. \tag{6.2}$$

Since M is compact, it can be covered by a finite number of charts (Ω_k, ξ_k), $k = 1, \dots, N$. Denote by (α_k) a smooth partition of unity subordinated to the covering (Ω_k), and set

$$\eta_k = \frac{\alpha_k^2}{\sum_{m=1}^M \alpha_m^2} \quad \text{for } k = 1, \dots, N.$$

Then, $\eta_k \in C^1(M)$, η_i has compact support in Ω_i for any i and there exists $H \in \mathbb{R}$ such that for any k, $|\nabla \eta_k| \leq H$. Furthermore, for any $u \in C_0^\infty(M)$, after some standard computations we can write

$$\Big(\int_M |u|^{n/(n-1)}\,dv_g\Big)^{(n-1)/n} \leq \sum_{i=1}^N \Big(\int_M |\eta_k u|^{n/(n-1)}\,dv_g\Big)^{(n-1)/n}. \tag{6.3}$$

Thus, by (6.3), because of (6.2) and following standard steps, we obtain that for any $u \in C_0^\infty(M)$,

$$\Big(\int_M |u|^{n/(n-1)}\,dv_g\Big)^{(n-1)/n} \leqslant (K(n,1)+\varepsilon)\int_M (|\nabla u| + NH|u|)\,dv_g, \tag{6.4}$$

where $\varepsilon = O(\varepsilon_0)$.

On the other hand, using [35, Proposition 4.2], we know that if there are real numbers A, B such that inequality (2.5) holds for all $u \in H_1^1(M)$, then $A \geq K(n,1)$, which is the best constant in the classical Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} |u|^{n/(n-1)}\,dx\Big)^{(n-1)/n} \leqslant K(n,1)\int_{\mathbb{R}^n} |\nabla u|\,dx,$$

that holds for all $u \in C_0^\infty(\mathbb{R}^n)$.

Since inequality (6.4) holds for all $u \in C_0^\infty(M)$, we have $\mathcal{A}_1(M) = K(n,1)$.

In order to complete the proof, we need to prove that inequality (6.4) holds for all $u \in H_1^1(M)$. Let (Ω_i, ξ_i), $i = 1, \dots, N$ be a finite atlas of M, each Ω_i being homeomorphic either to a ball of $\mathbb{R}^n$ or to a half ball of $\mathbb{R}^n_+$. We choose the atlas so that in each chart the metric tensor is bounded. Consider a C^∞ partition of unity $\{\alpha_i\}$ subordinated to the covering Ω_i. Then, for all $u \in H_1^1(M)$, $\alpha_i u$ has support in Ω_i. When Ω_i is homeomorphic to a ball, the proof is that of the first part. When Ω_i, is homeomorphic to a half ball, the proof is similar, but in this case the best constant is $2^{1/n}K(n,1)$ (see [7, Lemma 2.31] and [13] for a complete proof). □

As regards the existence of the second best constant the situation is confusing in the sense that it exists for certain manifolds while for others it does not seem to be possible to formulate a relevant theorem that clarifies the situation. For instance, we proved in Theorem 4.3 that in inequality (4.12) the second best constant exists and is equal to $|T|^{-1/2}$. Also, it is well known (see [35, Theorem 4.1]) that for any smooth compact Riemannian n-manifold without boundary, $n \geq 2$, we have that for any $u \in H_1^1(M)$, there exists $A \in \mathbb{R}$ such that

$$\Big(\int_M |u|^{n/(n-1)}\,dv_g\Big)^{(n-1)/n} \leq A\int_M |\nabla u|\,dv_g + |M|^{-1/n}\int_M |u|\,dv_g, \tag{6.5}$$

which means that $\mathcal{B}_1(M) = |M|^{-1/n}$.

We can therefore conclude that on the second constant in the case of the torus (which is a manifold with boundary), the same theorem as in the case of manifolds without boundary is valid. However, as demonstrated in the following example, we can not formulate a theorem that relates to all the manifolds with boundary and calculate the value of the second constant.

Example 6.2. Let $M = M_1 \cup M_2 \cup M_3$ be a smooth manifold with boundary ∂M, where M_1 and M_2 are two smooth disjoint bounded domains in $\mathbb{R}^n$ connected smoothly by a small thin "tube" M_3. Consider now a smooth function u which is equal to 1 on M_1 and 0 on M_2. Then, we can adjust the sizes of M_1 and M_2 in such a way that inequality (6.5) becomes false.

Proof. By the definition of u, there exists $H > 0$ such that $|\nabla u| \leq H$ in $M = M_1 \cup M_2 \cup M_3$. Furthermore, we can choose the small thin "tube" such that $|M_3| = \varepsilon$ for any arbitrarily small $\varepsilon > 0$ (that is, in the three-dimensional case, M_3 is a cylinder of radius a and of length b and then $|M_3| = \pi a^2 b$. Thus, for any $\varepsilon > 0$ and for any arbitrary b, we can choose $a = \sqrt{\varepsilon/\pi b}$ and then $|M_3| = \varepsilon$).

Suppose now that in our case the inequality (6.5) holds. Therefore,

$$
\begin{aligned}
& \Big(\int_{M_1\cup M_3} dv_g\Big)^{(n-1)/n} \leq A\int_{M_3}|\nabla u|\,dv_g + |M_1\cup M_2\cup M_3|^{-1/n}\int_{M_1\cup M_3} dv_g \\
\Rightarrow\quad & |M_1\cup M_3|^{(n-1)/n} \leq AH|M_3| + |M_1\cup M_2\cup M_3|^{-1/n}|M_1\cup M_3| \\
\Rightarrow\quad & (|M_1|+\varepsilon)^{(n-1)/n} \leq \varepsilon AH + (|M_1|+|M_2|+\varepsilon)^{-1/n}(|M_1|+\varepsilon) \\
\Rightarrow\quad & \Big(\frac{|M_1|+|M_2|+\varepsilon}{|M_1|+\varepsilon}\Big)^{1/n} \leq \varepsilon AH\frac{(|M_1|+|M_2|+\varepsilon)^{1/n}}{|M_1|+\varepsilon} + 1 \\
\Rightarrow\quad & 1 + \frac{|M_2|}{|M_1|+\varepsilon} \leq (1+O(\varepsilon))^n,
\end{aligned}
$$

where $O(\varepsilon) = \varepsilon AH(|M_1|+|M_2|+\varepsilon)^{1/n}/(|M_1|+\varepsilon)$.

Obviously, since we can choose M_2 as large as we want, the last inequality does not always hold, and our assertion is proved. □

As demonstrated by the counterexample 6.2, the results related to the value of the second best constant in inequality (6.5) in some cases fail, however in all cases the existence and the value of this depends on the "shape" of the manifold M. Although, another case when this constant exists, is presented in the following proposition.

Proposition 6.3. *Suppose that M is not a connected manifold and $M = \bigcup_{j=1}^J M_j$, where J is a positive integer, M_j is connected, and such that the second best constant in inequality* (2.5) *exists for all M_j, $j \in \{1, 2, \ldots, J\}$. Then, the second best constant is given by*

$$\mathcal{B}_1(M) = \sup_{1\leqslant j\leqslant J}|M_j|^{-1/n}.$$

Proof. By the definition of $\mathcal{B}_1(M)$ we have $\mathcal{B}_1(M) \leq \sup_{1\leqslant j\leqslant J}|M_j|^{-1/n}$.

For the reverse inequality, let M_{j_0} be the component of M of minimum n-dimensional measure $|M_{j_0}|$. Then, $|M_{j_0}|^{-1/n} = \sup_{1\leqslant j\leqslant J}|M_j|^{-1/n}$. Fix $u \in H_1^1(M)$ equal to 1 in M_{j_0} and equal to 0 outside M_{j_0}. Then, the inequality

$$\Big(\int_M |u|^{n/(n-1)}\,dv_g\Big)^{(n-1)/n} \leq A\int_M|\nabla u|\,dv_g + B\int_M|u|\,dv_g \tag{6.6}$$

implies that $B \geq |M_{j_0}|^{-1/n}$, and since it is true for all $B > 0$ such that (6.6) holds, we therefore deduce that $\mathcal{B}_1(M) \geq |M_{j_0}|^{-1/n}$. Thus, we have $\mathcal{B}_1(M) = \sup_{1\leqslant j\leqslant J}|M_j|^{-1/n}$. □

Our second result in this part is the following theorem, which concerns the first best constant in Sobolev trace inequality with $p = 1$.

Theorem 6.4. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary, $n \geq 3$. For any $\varepsilon > 0$, there exists $\tilde{B} \in \mathbb{R}$ such that for all $u \in H_1^1(M)$,*

$$\int_{\partial M}|u|\,ds_g \leq (1+\varepsilon)\int_M|\nabla u|\,dv_g + \tilde{B}\int_M|u|\,dv_g. \tag{6.7}$$

In particular, $1 = \tilde{K}(n,1)$ is always the best constant for this inequality.

The proof of this theorem makes use of the following auxiliary property.

Lemma 6.5. *Let (M, g) be a smooth compact Riemannian n-dimensional manifold with boundary, $n \geq 3$. Suppose that there exist real numbers $\tilde{A}$, $\tilde{B}$ such that the inequality*

$$\int_{\partial M} |u|\, ds_g \leq \tilde{A} \int_{M} |\nabla u|\, dv_g + \tilde{B} \int_{M} |u|\, dv_g \tag{6.8}$$

holds for any $u \in H_1^1(M)$. Then, $\tilde{A} \geq 1 = \tilde{K}(n, 1)$, where $\tilde{K}(n, 1)$ is the best constant in the classical Sobolev trace inequality

$$\int_{\partial \mathbb{R}_+^n} |u|\, dx' \leq \tilde{K}(n, 1) \int_{\mathbb{R}_+^n} |\nabla u|\, dx, \tag{6.9}$$

which holds for all $u \in H_1^1(\mathbb{R}_+^n)$.

The proof of Lemma 6.5 is provided in the Appendix.

Proof of Theorem 6.4. In attempting to compute the first best constant in inequality (6.7) we choose a finite covering of M consisted by geodesic balls $B_k = B_k(P_k)$, $k = 1, \dots, N$ in the following way:
(i) If the center P_k of the ball lies in the interior of the manifold, then the entire ball lies in its interior, and then B_k is a normal geodesic neighborhood with normal geodesic coordinates $x_1, \dots, x_n$.
(ii) If the center P_k of the ball lies in the boundary of the manifold, then B_k is a Fermi neighborhood with Fermi coordinates $x_1, \dots, x_{n-1}, y$.
In all these neighborhoods, the following holds:

$$1 - \varepsilon_0 \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon_0,$$

where ε_0 can be as small as we want, depending on the chosen covering.

Let $(\eta_k)_{k=1,2,\dots,N}$ be a partition of unity associated to the covering B_k. Then, for any $u \in H_1^1(M)$ we obtain

$$\int_{\partial M} |u|\, ds_g = \int_{\partial M} \Big| \sum_{k=1}^{N} (\eta_k u) \Big|\, ds_g \leq \sum_{k=1}^{N} \Big(\int_{\partial M} |\eta_k u|\, ds_g \Big). \tag{6.10}$$

Furthermore, by (6.10) and because of the Sobolev embedding theorem on $\partial \mathbb{R}_+^n$, passing the integration in the Euclidean space and returning to the manifold we deduce that

$$\int_{\partial M} |u|\, ds_g \leq \frac{1 + \varepsilon_0}{1 - \varepsilon_0} \int_{M} \Big(|\nabla u| + \sum_{k=1}^{N} |\nabla \eta_k||u| \Big)\, dv_g. \tag{6.11}$$

Let C be a positive constant depending on the chosen finite covering of the compact manifold M such that $|\nabla \eta_k| \leq C$ for all k. Then, by (6.11), we have

$$\int_{\partial M} |u|\, ds_g \leqslant \frac{1 + \varepsilon_0}{1 - \varepsilon_0} \int_{M} (|\nabla u| + CN|u|)\, dv_g$$

or

$$\int_{\partial M} |u|\, ds_g \leqslant (1 + \varepsilon) \int_{M} |\nabla u|\, dv_g + C' \int_{M} |u|\, dv_g, \tag{6.12}$$

where $\varepsilon = O(\varepsilon_0)$ and $C' = CN(1 + \varepsilon_0)/(1 - \varepsilon_0)$.

Since inequality (6.12) holds for all $u \in H_1^1(M)$ we deduce by Lemma 6.5 that $\tilde{\mathcal{A}}_1(M) = 1 = \tilde{K}(n, 1)$. □

Suppose now that we are interested in studying the existence of the second best constant in the Sobolev trace inequality with $p = 1$ and to calculate its value if it exists. This problem is answered in the case of a connected bounded open set of $\mathbb{R}^n$, see [46, Proposition 3.10]. However, in the following counterexample it is proved that this result is not always true even if the manifold is connected.

Example 6.6. On a huge sphere $\mathbb{S}^n$ consider a cap M_1 around a point P, the complement M_2 of a bigger cap around the same point P and a thin "tube" M_3 connecting smoothly M_1 and M_2. Let us now consider the smooth function u on $M = M_1 \cup M_2 \cup M_3$ which is equal to 1 on M_1 and 0 on M_2. Then, we can adjust the sizes of M_1 and M_2 in such a way that inequality (6.9) becomes false.

The proof of Example 6.6 is omitted since it is similar to that of Example 6.2.

Remark 6.7. If the manifold M is not connected, then the result of Theorem 4.3 concerning the second best constant fails. However, if M is not a connected manifold and $M = \bigcup_{i=1}^{p} M_i$, where p is a positive integer and M_i is connected for all $i \in \{1, 2, \dots, p\}$, then the second best constant for inequality (2.10) is given by

$$\tilde{\mathcal{B}}_1(M) = \sup_{1 \leqslant i \leqslant p} \frac{|\partial M_i|}{|M_i|},$$

see [46, Proposition 3.12 and the example on page 81].

6.2 Best constants on Riemannian manifolds in the presence of symmetries

This part of the paper is devoted to manifolds which present symmetries. The following two examples may be regarded as representatives of these manifolds.

Example 6.8. Consider the three-dimensional solid torus

$$T = \Big\{(x, y, z) \in \mathbb{R}^3 : \big(\sqrt{x^2 + y^2} - l\big)^2 + z^2 < r^2,\ l > r > 0\Big\}$$

with the metric induced by the $\mathbb{R}^3$ metric. Let $G = O(2) \times I$ be the group of rotations around axis z. Then, all the G-orbits of the T are circles, thus of dimension 1, the orbit of minimum volume is the circle of radius $l - r$, and the volume of it is equal to $2\pi(l - r)$. Therefore, T is a compact 3-dimensional manifold with boundary, invariant under the action of the subgroup G of the isometry group $O(3)$.

Example 6.9 ([14, 15]). Let $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$, $k \geq 2$, $m \geq 1$ and $\overline{\Omega} \subset (\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^m$. Denote by $G_{k,m} = O(k) \times \mathrm{Id}_m$ the subgroup of the isometry group $O(n)$ of the type

$$\tau : (x_1, x_2) \to (\sigma(x_1), x_2), \quad \sigma \in O(k), x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^m,$$

and suppose that $\overline{\Omega}$ is invariant under the action of $G_{k,m}$ ($\tau(\overline{\Omega}) = \overline{\Omega}$ for all $\tau \in G_{k,m}$). Then, $\overline{\Omega}$ is a compact n-dimensional manifold with boundary, invariant under the action of the subgroup $G_{k,m}$ of the isometry group $O(n)$.

Considering that we studied the case of the solid torus, we need some background material and results concerning the "decomposition" of a manifold with boundary which presents symmetries.

In the following, we assume the notations and background material from Hebey and Vaugon [38] and Cotsiolis and Labropoulos [18]. We remind that, given $(\widetilde{M}, g)$ a Riemannian manifold (complete or not, but connected), we denote by $I(\widetilde{M}, g)$ its group of isometries. Let (M, g) be a compact n-dimensional, $n \geq 3$, Riemannian manifold with boundary G-invariant under the action of a subgroup G of the isometry group $I(M, g)$. We assume that (M, g) is a smooth bounded open subset of a slightly larger Riemannian manifold $(\widetilde{M}, g)$ (see [40]), invariant under the action of a subgroup G of the isometry group of $(\widetilde{M}, g)$.

The first results we need are the following two properties.

Lemma 6.10 ([38]). *Let $(\widetilde{M}, g)$ be a Riemannian n-manifold (complete or not), and let G be a compact subgroup of $I(\widetilde{M}, g)$. Let $P \in \widetilde{M}$ and set $k = \dim O_P$. Assume $k \geq 1$. There exists a coordinate chart (Ω, ξ) of $\widetilde{M}$ at P such that the following properties hold:*
(i) *$\xi(\Omega) = U \times W$, where U is some open subset of $\mathbb{R}^k$ and W is some open subset of $\mathbb{R}^{n-k}$.*
(ii) *For any $Q \in \Omega$, we have $U \times \Pi_2(\xi(Q)) \subset \xi(O_Q \cap \Omega)$, where $\Pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is the second projection.*

Lemma 6.11 ([38]). *Let M be a compact subset of $\widetilde{M}$ covered by a finite number of charts (Ω_m, ξ_m), where $m = 1, \dots, M$, and $k = \min_{P \in \widetilde{M}} \dim O_P \geq 1$. The following properties are valid:*

(i) *$\xi_m(\Omega_m) = U_m \times W_m$, where U_m is some open subset of $\mathbb{R}^{k_m}$ and W_m is some open subset of $\mathbb{R}^{n-k_m}$, and $k_m \in \mathbb{N}$ satisfies $k \leq k_m < n$.*

(ii) *U_m and W_m are bounded, and W_m has smooth boundary.*

(iii) *For any $Q \in \Omega_m$, we have $U_m \times \Pi_2(\xi_m(Q)) \subset \xi_m(O_Q \cap \Omega_m)$, where $\Pi_2 \colon \mathbb{R}^{k_m} \times \mathbb{R}^{n-k_m} \to \mathbb{R}^{n-k}$ is the second projection.*

(iv) *There exists $\varepsilon_m > 0$ with $(1 - \varepsilon_m)\delta_{ij} \leq g^m_{ij} \leq (1 + \varepsilon_m)\delta_{ij}$ as bilinear forms, where the g^m_{ij} are the components of g in (Ω_m, ξ_m).*

Let $P \in M$ and $O_P = \{\tau(P), \tau \in G\}$ be its orbit of dimension k, $0 \leqslant k < n$. According to [35, §9] and [27], the map $\Phi \colon G \to O_P$, defined by $\Phi(\tau) = \tau(P)$, is of rank k and there exists a submanifold H of G of dimension k with $\mathrm{Id} \in H$, such that Φ restricted to H is a diffeomorphism from H onto its image denoted by $\mathcal{V}_P$.

Let N be a submanifold of M of dimension $(n - k)$, such that $T_P\Phi(H) \oplus T_PN = T_PM$. Using the exponential map at P, we build a $(n - k)$-dimensional submanifold $\mathcal{W}_P$ of N, orthogonal to O_P at P and such that for any $Q \in \mathcal{W}_P$, the minimizing geodesics of (M, g) joining P and Q are all contained in $\mathcal{W}_P$.

Let $\Psi \colon H \times \mathcal{W}_P \to M$ be the map defined by $\Psi(\tau, Q) = \tau(Q)$. Using the local inverse theorem, there exist a neighborhood $\mathcal{V}_{(\mathrm{Id},P)} \subset H \times \mathcal{W}_P$ of (Id, P) and a neighborhood $\mathcal{M}_P \subset M$ such that $\Psi^{-1} = (\Psi_1 \times \Psi_2)$ from $\mathcal{M}_P$ onto $\mathcal{V}_{(\mathrm{Id},P)}$ is a diffeomorphism.

Up to restricting $\mathcal{V}_P$, we choose a normal chart $(\mathcal{V}_P, \varphi_1)$ around P for the metric $\tilde{g}$ induced on O_P with $\varphi_1(\mathcal{V}_P) = U \subset \mathbb{R}^k$. In the same way, we choose a geodesic normal chart $(\mathcal{W}_P, \varphi_2)$ around P for the metric $\tilde{\tilde{g}}$ induced on $\mathcal{W}_P$ with $\varphi_2(\mathcal{W}_P) = W \subset \mathbb{R}^{n-k}$.

We denote $\xi_1 = \varphi_1 \circ \Phi \circ \Psi_1$, $\xi_2 = \varphi_2 \circ \Psi_2$, $\xi = (\xi_1, \xi_2)$ and $\Omega = \mathcal{M}_P$.

From the above and due to Lemma 6.11, the following properties hold, see [27].

Lemma 6.12. *Let (M, g) be a compact Riemannian n-manifold with boundary, G be a compact subgroup of $I(M, g)$ and $P \in M$ with orbit of dimension k, $0 \leqslant k < n$. Then, there exists a chart (Ω, ξ) around P such that the following properties are satisfied:*

(i) *$\xi(\Omega) = U \times W$, where $U \subset \mathbb{R}^k$ and $W \subset \mathbb{R}^{n-k}$.*

(ii) *U, W are bounded, and W has smooth boundary.*

(iii) *(Ω, ξ) is a normal chart of M around of P, $(\mathcal{V}_P, \varphi_1)$ is a normal chart around P of the submanifold O_P and $(\mathcal{W}_P, \varphi_2)$ is a normal geodesic chart around P of the submanifold $\mathcal{W}_P$.*

(iv) *For any $\varepsilon > 0$, (Ω, ξ) can be chosen such that*

$$1 - \varepsilon \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon \quad \text{on } \Omega \text{ for } 1 \leq i, j \leq n,$$
$$1 - \varepsilon \leq \sqrt{\det(\tilde{g}_{ij})} \leq 1 + \varepsilon \quad \text{on } \mathcal{V}_P \text{ for } 1 \leq i, j \leq k.$$

(v) *For any $u \in C^\infty_G(M)$, $u \circ \xi^{-1}$ depends only on W variables.*

We say that we choose a neighborhood of O_P when we choose $\delta > 0$ and we consider

$$O_{P,\delta} = \{Q \in \widetilde{M} : d(Q, O_P) < \delta\}.$$

Such a neighborhood of O_P is called *tubular neighborhood*.

Let $P \in M$ and O_P be its orbit of dimension k. Since the manifold M is included in $\widetilde{M}$, we can choose a normal chart (Ω_P, ξ_P) around P such that Lemma 6.12 holds for some $\varepsilon_0 > 0$. For any $Q = \tau(P) \in O_P$, where $\tau \in G$, we build a chart around Q, denoted by $(\tau(\Omega_P), \xi_P \circ \tau^{-1})$ and "isometric" to (Ω_P, ξ_P). The orbit O_P is then covered by such charts. We denote by $(\Omega_{P,m})_{m=1,\dots,M}$ a finite extract covering. Then, we can choose $\delta > 0$ small enough, depending on P and ε_0 such that the tubular neighborhood $O_{P,\delta} = \{Q \in \widetilde{M} : d(Q, O_P) < \delta\}$ (where $d(\cdot\,, O_P)$ is the distance to the orbit) has the following properties:

(i) $\overline{O_{P,\delta}}$ is a submanifold of $\widetilde{M}$ with boundary,

(ii) $d^2(\cdot\,, O_P)$ is a C^∞ function on $O_{P,\delta}$,

(iii) $O_{P,\delta}$ is covered by $(\Omega_m)_{m=1,\dots,M}$.

Clearly, M is covered by $\bigcup_{P\in M} O_{P,\delta}$. We denote by $(O_{j,\delta})_{j=1,\dots,J}$ a finite extract covering of M, where all $O_{j,\delta}$ are covered by $(\Omega_{jm})_{m=1,\dots,M_j}$. Therefore,

$$M \subset \bigcup_{j=1}^{J} \bigcup_{m=1}^{M_j} \Omega_{jm} = \bigcup_{i=1}^{\sum_{j=1}^{J} M_j} \Omega_i .$$

So we obtain a finite covering of M consisting of Ω_i, $i = 1, \dots, \sum_{j=1}^{J} M_j$. We choose such a covering in the following way:

(i) If P lies in the interior of M, then there exist j, $1 \le j \le J$, and m, $1 \le m \le M_j$, such that the tubular neighborhood $O_{j,\delta}$ and Ω_{jm} with $P \in \Omega_{jm}$ lie entirely in the interior of M, that is, if $P \in M\backslash\partial M$, then $O_{j,\delta} \subset M\backslash\partial M$ and $\Omega_{jm} \subset M\backslash\partial M$.

(ii) If P lies on the boundary ∂M of M, then there exist j, $1 \le j \le J$, such that the tubular neighborhood $O_{j,\delta}$ intersects the boundary ∂M, and m, $1 \le m \le M_j$, such that Ω_{jm} with $P \in \Omega_{jm}$ intersects a part of the boundary ∂M. Then, the Ω_{jm} cover a patch of the boundary of M and the whole of the boundary is covered by charts around $P \in \partial M$.

Let N denote the projection of the image of M through the charts (Ω_{jm}, ξ_{jm}), $j = 1, \dots, J$, $m = 1, \dots, M_j$, on $\mathbb{R}^{n-k}$. Then, $(N, \bar{g})$ is a $(n-k)$-dimensional compact submanifold with boundary of $\mathbb{R}^{n-k}$ and N is covered by (W_i), $i = 1, \dots, \sum_{j=1}^{J} M_j$, where W_i is the component of $\xi_i(\Omega_i)$ on $\mathbb{R}^{n-k}$ for all $i = 1, \dots, \sum_{j=1}^{J} M_j$. Let p be the projection of $\xi_i(P)$, $P \in M$ on $\mathbb{R}^{n-k}$. Thus, one of the following properties holds:

(i) If $p \in N\backslash\partial N$, then $W_i \subset N\backslash\partial N$ and W_i is a normal geodesic neighborhood with normal geodesic coordinates $(y_1, \dots, y_{n-k})$.

(ii) If $p \in \partial N$, then W_i is a Fermi neighborhood with Fermi coordinates $(y_1, \dots, y_{n-k-1}, t)$.

In these neighborhoods of N we have

$$1 - \varepsilon_0 \leqslant \sqrt{\det(\bar{g}_{ij})} \leqslant 1 + \varepsilon_0 \quad \text{for } 1 \le i, j \le n-k,$$

where ε_0 can be as small as we want, depending on the chosen covering.

Set

$$O_j = O_{j,\delta} = \{Q \in \widetilde{M} : d(Q, O_{P_j}) < \delta\} \quad \text{and} \quad (\Omega_{jm}, \xi_{jm}) = (\Omega_m, \xi_m).$$

Lemma 6.13 ([28]). *Let (M, g) be a compact Riemannian n-dimensional manifold and G be a compact subgroup of the isometry group of M. Then, there exists an orbit of minimum dimension k and of minimum volume.*

Lemma 6.14. *Let $O_j = \{Q \in \widetilde{M} : d(Q, O_{P_j}) < \delta\}$ be an arbitrary tubular neighborhood of M, $V_j = \mathrm{Vol}(O_j)$, $\phi = \upsilon \circ \xi^{-1}$ and c be a positive constant. Then, for any $\upsilon \in H^1_{1,G}(O_j \cap \partial M)$, $\upsilon \geqslant 0$ the following inequalities are valid:*

$$(1 - c\varepsilon_0)V_j \int_{\partial N} \phi \, ds_{\bar{g}} \leqslant \int_{\partial M} \upsilon \, dS_g \leqslant (1 + c\varepsilon_0)V_j \int_{\partial N} \phi \, ds_{\bar{g}}, \tag{6.13}$$

$$(1 - c\varepsilon_0)V_j \int_{N} \phi \, d\upsilon_{\bar{g}} \leqslant \int_{M} \upsilon \, dV_g \leqslant (1 + c\varepsilon_0)V_j \int_{N} \phi \, d\upsilon_{\bar{g}}, \tag{6.14}$$

$$(1 - c\varepsilon_0)V_j \int_{N} |\nabla_{\bar{g}}\phi| \, d\upsilon_{\bar{g}} \leqslant \int_{M} |\nabla_g \upsilon| \, dV_g \leqslant (1 + c\varepsilon_0)V_j \int_{N} |\nabla_{\bar{g}}\phi| \, d\upsilon_{\bar{g}}. \tag{6.15}$$

The proof of Lemma 6.14 is provided in the Appendix.

The following theorem concerns the exact value of the first best constant of the classical Sobolev inequality for $p = 1$, in the case where the manifold is invariant under the action of a compact group G of the isometries without finite subgroup.

Theorem 6.15. *Let (M, g) be a smooth compact Riemannian n-dimensional manifold with boundary, $n \ge 3$, invariant under the action of a subgroup G of the isometry group $I(M, g)$. Let k denote the minimum orbit dimension of G and let V denote the minimum of the volume of the k-dimensional orbits. Then, for any $\varepsilon > 0$, there* *exists a real constant B such that for all $u \in H^1_{1,G}(M)$, the following inequality holds:*

$$\left(\int_M |u|^{p^*}\,dv_g\right)^{1/p^*} \leqslant (2^{1/(n-k)}K_G+\varepsilon)\int_M |\nabla u|\,dv_g + B\int_M |u|\,dv_g, \tag{6.16}$$

where

$$p^* = \frac{n-k}{n-k-1} \quad \text{and} \quad K_G = \frac{K(n-k,1)}{V^{1/(n-k)}}.$$

Moreover, $2^{1/(n-k)}K_G$ is the best constant for this inequality.

Regarding the existence of a second best constant of the classical Sobolev inequality with $p = 1$, for reasons similar to those of the general case, it cannot be formulated a global theorem devoted to the calculation of it. However, in some cases this constant can be computed. For example, as in the case of the solid torus (see Theorem 4.1).

We present now our last two theorems in which the exact values of the best constants for trace Sobolev inequalities are calculated for $p = 1$, in the case that the manifold is invariant under the action of a compact group G of the isometries without finite subgroup, when they exist.

Theorem 6.16. *Let (M, g) be a smooth compact Riemannian n-manifold with boundary, $n \geq 3$, invariant under the action of a subgroup G of the isometry group $I(M, g)$. Let k denote the minimum orbit dimension of G and let N be the compact manifold with boundary which is the projection of M on $\mathbb{R}^{n-k}$. Then, for any $\varepsilon > 0$, there exists a real constant $\tilde{B}$ such that for all $u \in H^1_{1,G}(M)$ the following inequality holds:*

$$\int_{\partial M} |u|\,ds_g \leqslant (1+\varepsilon)\int_M |\nabla u|\,dv_g + \tilde{B}\int_M |u|\,dv_g. \tag{6.17}$$

In addition, $1 = \tilde{K}(n-k, 1)$ is the best first constant for this inequality.

Theorem 6.17. *Let (M, g) be a smooth compact Riemannian n-manifold with boundary, $n \geq 3$, invariant under the action of a subgroup G of the isometry group $I(M, g)$. Let k denote the minimum orbit dimension of G and let N be the compact manifold with boundary which is the projection of M on $\mathbb{R}^{n-k}$. If N is connected, then there exists a real constant $\tilde{A}$ such that for all $u \in H^1_{1,G}(M)$ the following inequality holds:*

$$\int_{\partial M} |u|\,ds_g \leqslant \tilde{A}\int_M |\nabla u|\,dv_g + \frac{|\partial N|}{|N|}\int_M |u|\,dv_g. \tag{6.18}$$

In addition, $|\partial N|/|N|$ is the best second constant for this inequality.

Remark 6.18. If the manifold N is not connected the result of the Theorem 6.17 concerning the second best constant fails (see [46]).

Proof of Theorem 6.17. Let $(O_{j,\delta})_{j=1,\dots,J}$ be a finite covering of M and $(\eta_j)_{j=1,\dots,J}$ be a partition of unity associated to this covering. Then, by [46, Proposition 3.10] and Lemma 6.14, for any $u \in H^1_{1,G}(M)$, if we set $\eta_j|u| \circ \xi = |\phi_j|$, we obtain

$$\begin{aligned}
\int_{\partial M} |u|\,ds_g &= \int_{\partial M}\left(\sum_{j=1}^J \eta_j\right)|u|\,ds_g = \sum_{j=1}^J \int_{\partial M}(\eta_j|u|)\,ds_g \\
&\leqslant \sum_{j=1}^J (1+\varepsilon)V_j \int_{\partial N}|\phi_j|\,ds_{\tilde{g}} \leqslant (1+\varepsilon)\sum_{j=1}^J V_j\left(\tilde{A}\int_N |\nabla\phi_j|\,dv_{\tilde{g}} + \frac{|\partial N|}{|N|}\int_N |\phi_j|\,dv_{\tilde{g}}\right) \\
&\leqslant (1+\varepsilon)\sum_{j=1}^J V_j\left(\tilde{A}\frac{1}{(1-\varepsilon)V_j}\int_M |\nabla(\eta_j|u|)|\,dv_g\right) + (1+\varepsilon)\sum_{j=1}^J V_j\left(\frac{1}{(1-\varepsilon)V_j}\frac{|\partial N|}{|N|}\int_M (\eta_j|u|)\,dv_g\right) \\
&\leqslant \frac{1+\varepsilon}{1-\varepsilon}\left(\tilde{A}\int_M |\nabla u|\,dv_g + \frac{|\partial N|}{|N|}\int_M |u|\,dv_g\right).
\end{aligned} \tag{6.19}$$

Relation (6.19) implies that $\tilde{B}_{1,G}(M) \geq |\partial N|/|N|$. In particular, $\tilde{\mathcal{B}}_{1,G}(M) \geq |\partial N|/|N|$.

Suppose by contradiction that for any $\alpha \in \mathbb{N}$ there exists $u_\alpha \in H^1_{1,G}(M)$ such that

$$\int_{\partial M} |u_\alpha|\, ds_g \geqslant \alpha \int_M |\nabla u_\alpha|\, dv_g + \frac{|\partial N|}{|N|} \int_M |u_\alpha|\, dv_g. \tag{6.20}$$

Without loss of generality we can assume that all the functions u_α are defined in the orbit O_j. Thus, by (6.20), if we set $u_\alpha \circ \xi_j = (\phi_j)_\alpha$, we deduce that

$$\begin{aligned}
(1+\varepsilon)V_j \int_{\partial N} |(\phi_j)_\alpha|\, ds_{\tilde g} &\geqslant (1-\varepsilon)V_j\Big(\alpha \int_N |\nabla(\phi_j)_\alpha|\, dv_{\tilde g} + \frac{|\partial N|}{|N|}\int_N |(\phi_j)_\alpha|\, dv_{\tilde g}\Big)\\
\Rightarrow \int_{\partial N} |(\phi_j)_\alpha|\, ds_{\tilde g} &\geqslant \frac{1-\varepsilon}{1+\varepsilon}\Big(\alpha \int_N |\nabla(\phi_j)_\alpha|\, dv_{\tilde g} + \frac{|\partial N|}{|N|}\int_N |(\phi_j)_\alpha|\, dv_{\tilde g}\Big)\\
\Rightarrow \int_{\partial N} |(\phi_\alpha)_j|\, ds_{\tilde g} &\geqslant (\alpha-1) \int_N |\nabla(\phi_j)_\alpha|\, dv_{\tilde g} + \frac{1-\varepsilon}{1+\varepsilon}\frac{|\partial N|}{|N|}\int_N |(\phi_j)_\alpha|\, dv_{\tilde g}.
\end{aligned} \tag{6.21}$$

Inequality (6.21) is false since ε can be chosen arbitrarily small and since the constant $|\partial N|/|N|$ is optimal (see [46]) for the inequality

$$\int_{\partial N} |\phi|\, ds_{\tilde g} \leq \tilde A \int_N |\nabla\phi|\, dv_{\tilde g} + \frac{|\partial N|}{|N|}\int_N |\phi|\, dv_{\tilde g}. \qquad \square$$

We omit the proofs of Theorems 6.15 and 6.16 since they rely on similar arguments as in the case of the torus, in combination with Lemmas 6.11–6.14.

Remark 6.19. We cannot formulate a global theorem that concerns the trace Sobolev inequality on manifolds with boundary in the presence of symmetries, namely to establish an inequality where to the positions of $\tilde A$ and $\tilde B$ to put the best constant $1 = \tilde K(n-k, 1)$ and$|\partial N|/|N|$, respectively. In some cases, such as on the solid torus or on the disk of $\mathbb{R}^2$, there are extremals for this inequality (see Theorem 4.1).

Remark 6.20. The parameter ε that appears in Theorems 4.1, 4.3, 6.15 and 6.16 controls in some sense the thinness of the cover that we use in each case through the related partition of unity. Thus, its existence is absolutely necessary because we do not know if the inequalities are valid without this parameter. Although in some cases, Sobolev inequalities exist without ε (see, e.g., [17, 22, 28, 37]), but in general we cannot make it disappear.

A Appendix

Proof of Lemma 4.4. Suppose by contradiction, that there exist $\tilde A' < \tilde K(2,1)$ and $\tilde B'$ such that for all u in $H^1_{1,G}(T)$ the following inequality holds:

$$\int_{\partial T} |u|\, dS \leq \tilde A' \int_T |\nabla u|\, dV + \tilde B' \int_T |u|\, dV. \tag{A.1}$$

Consider a transformation of the disk $F\colon D \to \mathbb{R}^2_+$. Such a transformation is, for example,

$$F(t,s) = \Big(\frac{4t}{t^2+(1+s)^2}, \frac{2(1-t^2-s^2)}{t^2+(1+s)^2}\Big),$$

see [25]. Choose a finite covering of $\bar D$ consisting of disks D_k, centered on p_k, such that the following hold:
(a) If $p_k \in D$, then the entire D_k lies in D.
(b) If $p_k \in \partial D$, then D_k is a Fermi neighborhood.
In these neighborhoods we have

$$1 - \varepsilon_0 \leqslant \sqrt{\det(\tilde g_{\alpha\beta})} \leqslant 1 + \varepsilon_0. \tag{A.2}$$

Fix now a point $P_0 \in \partial T$ that belongs to the orbit of minimum range $l - r$. For any $\varepsilon_0 > 0$, we can choose $\delta = \varepsilon_0(l - r) < 1$ and

$$T_\delta = \{Q \in \mathbb{R}^3 : d(Q, O_{P_0}) < \delta\}$$

such that if $I \times U \subset I \times D$ is the image of a neighborhood of $P_0 \in \partial T$ through the chart ξ of T and $V \subset \mathbb{R}^2_+$ is the image of U through F, then (A.2) holds. It follows by (A.1) that for any $u \in C_0^\infty(T_\delta)$, we have

$$\int_{\partial T_\delta} |u|\,dS \leqslant \tilde{A}' \int_{T_\delta} |\nabla u|\,dV + \tilde{B}' \int_{T_\delta} |u|\,dV.$$

Relations (4.4)–(4.6) yield successively

$$\begin{aligned}
&\int_{\partial D} |\phi|(l - r + \delta t)\,dt \leqslant \tilde{A}' \int_D |\nabla\phi|(l - r + \delta t)\,dt\,ds + \tilde{B}' \int_D |\phi|(l - r + \delta t)\,dt\,ds \\
\Rightarrow\quad &(1 - \varepsilon_0) \int_{F(\partial D)} (|\phi|\sqrt{\tilde{g}}) \circ F^{-1}\,dx' \le (1 + \varepsilon_0)\tilde{A}' \int_{F(D)} (|\nabla\phi|\sqrt{\tilde{g}}) \circ F^{-1}\,dx + (1 + \varepsilon_0)\delta\tilde{B}' \int_{F(D)} (|\phi|\sqrt{\tilde{g}}) \circ F^{-1}\,dx, \\
\Rightarrow\quad &(1 - \varepsilon_0)^2 \int_{\partial\mathbb{R}^2_+} |\Phi|\,dx' \leqslant (1 + \varepsilon_0)^2\Big(\tilde{A}' \int_{\mathbb{R}^2_+} |\nabla\Phi|\,dx + \delta\tilde{B}' \int_{\mathbb{R}^2_+} |\phi|\,dx\Big) \\
\Rightarrow\quad &\int_{\partial\mathbb{R}^2_+} |\Phi|\,dx' \leqslant \Big(\frac{1 + \varepsilon_0}{1 - \varepsilon_0}\Big)^2\Big(\tilde{A}' \int_{\mathbb{R}^2_+} |\nabla\Phi|\,dx + \delta\tilde{B}' \int_{\mathbb{R}^2_+} |\phi|\,dx\Big).
\end{aligned} \tag{A.3}$$

By (A.3) we deduce that for ε_0 small enough, the following inequality holds:

$$\tilde{A}'' = \Big(\frac{1 + \varepsilon_0}{1 - \varepsilon_0}\Big)^2 \tilde{A}' < \tilde{K}(n, 1) = 1.$$

So, for ε_0 small enough and for all $\Phi \in C_0^\infty(D)$, we have

$$\int_{\partial\mathbb{R}^2_+} |\Phi|\,dx' \leqslant \tilde{A}'' \int_{\mathbb{R}^2_+} |\nabla\Phi|\,dx + \tilde{B}'' \int_{\mathbb{R}^2_+} |\Phi|\,dx. \tag{A.4}$$

Let $\Psi \in C_0^\infty(\mathbb{R}^2_+)$ and $\Psi_\lambda(x) = \lambda\Psi(\lambda x)$, $\lambda > 0$. Then,

$$\int_{\partial\mathbb{R}^2_+} |\Psi_\lambda|\,dx' = \int_{\partial\mathbb{R}^2_+} |\Psi|\,dx', \quad \int_{\mathbb{R}^2_+} |\nabla\Psi_\lambda|\,dx = \int_{\mathbb{R}^2_+} |\nabla\Psi|\,dx \quad \text{and} \quad \int_{\mathbb{R}^2_+} |\Psi_\lambda|\,dx = \frac{1}{\lambda} \int_{\mathbb{R}^2_+} |\Psi|\,dx.$$

Thus, since $\Psi_\lambda \in C_0^\infty(D)$ for $\lambda > 0$ sufficiently large, relation (A.4) yields the following inequality:

$$\int_{\partial\mathbb{R}^2_+} |\Psi|\,dx' \leqslant A'' \int_{\mathbb{R}^2_+} |\nabla\Psi|\,dx + B''\frac{1}{\lambda} \int_{\mathbb{R}^2_+} |\Psi|\,dx.$$

Taking $\lambda \to \infty$, we obtain that the inequality

$$\int_{\partial\mathbb{R}^2_+} |\Psi|\,dx' \leqslant A'' \int_{\mathbb{R}^2_+} |\nabla\Psi|\,dx$$

holds for all $\Psi \in C_0^\infty(\mathbb{R}^2_+)$ with $\tilde{A}'' < 1$. This contradicts inequality (4.33), which asserts that $\tilde{K}(n, 1) = 1$ is the best constant for the Sobolev trace inequality in $\mathbb{R}^2_+$. ☐

Proof of Lemma 6.5. Suppose by contradiction that there exist a Riemannian n-manifold (M, g) and real numbers $A < 1 = \tilde{K}(n, 1)$ and $\tilde{B}$ such that inequality (4.7) is true for all $u \in H_1^1(M)$. Let $P_0 \in \partial M$. Given $\varepsilon > 0$, let $B_\delta(0) \subset \mathbb{R}^n_+$ be the imagine of a convex neighborhood centered at P_0 through a chart (Ω, ξ) of M, which can be chosen such that

$$1 - \varepsilon \le \sqrt{\det(g_{ij})} \le 1 + \varepsilon.$$

Thus, by (4.11), if we choose ε small enough, it follows that there are real numbers $A' < 1$ and B' such that for all $u \in C_0^\infty(B_0(\delta))$,

$$\int_{\partial\mathbb{R}^n_+} |u|\,dx' \le A' \int_{\mathbb{R}^n_+} |\nabla u|\,dx + B' \int_{\mathbb{R}^n_+} |u|\,dx. \tag{A.5}$$

Fix $u \in C_0^\infty(\mathbb{R}^n_+)$ and set $u_\lambda(x) = \lambda^{n-1}u(\lambda x)$, where λ is a positive real number. If we choose λ sufficiently large, then $u_\lambda \in C_0^\infty(B_0(\delta))$, and thus by (A.5) we obtain

$$\int_{\partial\mathbb{R}^n_+} |u_\lambda|\,dx' \le A' \int_{\mathbb{R}^n_+} |\nabla u_\lambda|\,dx + B' \int_{\mathbb{R}^n_+} |u_\lambda|\,dx. \tag{A.6}$$

By rescaling, we obtain

$$\int_{\partial\mathbb{R}^n_+} |u_\lambda|\,dx' = \int_{\partial\mathbb{R}^n_+} |u|\,dx', \quad \int_{\mathbb{R}^n_+} |\nabla u_\lambda|\,dx = \int_{\mathbb{R}^n_+} |\nabla u|\,dx \quad \text{and} \quad \int_{\mathbb{R}^n_+} |u_\lambda|\,dx = \frac{1}{\lambda}\int_{\mathbb{R}^n_+} |u|\,dx.$$

Thus, by (A.6), we deduce that

$$\int_{\partial\mathbb{R}^n_+} |u|\,dx' \le A' \int_{\mathbb{R}^n_+} |\nabla u|\,dx + B'\frac{1}{\lambda} \int_{\mathbb{R}^n_+} |u|\,dx. \tag{A.7}$$

Taking $\lambda \to \infty$ in (A.7), we obtain that for all $u \in C_0^\infty(\mathbb{R}^n_+)$ the following inequality holds:

$$\int_{\partial\mathbb{R}^n_+} |u|\,dx' \le A' \int_{\mathbb{R}^n_+} |\nabla u|\,dx \tag{A.8}$$

with $A' < 1$. This contradicts inequality (4.11), which establishes that $1 = \tilde{K}(n,1)$ is the best constant for the Sobolev trace inequality in $\mathbb{R}^n_+$ (see [46, 48]), and the lemma is proved. □

Proof of Lemma 6.14. Let $P \in \partial M$, O_P be its orbit and $\upsilon \in H^1_{1,G}(O_j \cap M)$, $\upsilon \geqslant 0$. Then, by [18, Lemma 3.3 (4)], it follows that

$$\begin{aligned}
\int_{\partial M} \upsilon\,ds_g &= \int_{\partial M}\Bigl(\sum_{m=1}^{M_j}\beta_m\Bigr)\upsilon\,dS_g = \sum_{m=1}^{M_j}\int_{\partial M}\beta_m\upsilon\,dS_g = \sum_{m=1}^{M_j}\int_{\partial M\cap\Omega_m}\beta_m\upsilon\,dS_g\\
&= \sum_{m=1}^{M_j}\int_{\xi(\partial M\cap\Omega_m)}\sqrt{\det(g^m_{kl})}\beta_m\upsilon\circ\xi_m^{-1}\,dx\,ds_{\bar g}\\
&= \sum_{m=1}^{M_j}\int_{U_m\times(\partial N\cap W_m)}\sqrt{\det(g^m_{kl})}\beta_m\upsilon\circ\xi_m^{-1}\,dx\,ds_{\bar g}\\
&\leqslant (1+\varepsilon_0)\sum_{m=1}^{M_j}\int_{U_m\times(\partial N\cap W_m)}\beta_m\upsilon\circ\xi_m^{-1}\,dx\,ds_{\bar g}.
\end{aligned} \tag{A.9}$$

Since $\beta_m \circ \xi_m^{-1}$ is independent of the W_m's variables for each m, we denote by β_{1m} the function $\beta_m \circ \xi_m^{-1}$ and regard this function as defined on U_m. In the same way, we denote by υ_{2m} the function $\upsilon \circ \xi_m^{-1}$ which is considered as defined on W_m, since according to [18, Lemma 3.3] it depends only on the W_m's variables. Thus, by relation (A.9), we obtain

$$\int_{\partial M} \upsilon\,dS_g \leqslant (1+\varepsilon_0)\sum_{m=1}^{M_j}\int_{U_m}\beta_{1m}\,dx\int_{\partial N\cap W_m}\upsilon_{2m}\,ds_{\bar g}. \tag{A.10}$$

As the charts (Ω_m, ξ_m) are isometric to each other and since υ is G-invariant, $\int_{\partial N\cap W_m}\upsilon_{2m}\,ds_{\bar g}$ does not depend on m. Thus, relation (A.10) leads to

$$\int_{\partial M} \upsilon\,dS_g \leqslant (1+\varepsilon_0)\int_{\partial N\cap W}\upsilon_2\,ds_{\bar g}\sum_{m=1}^{M_j}\int_{U_m}\beta_{1m}\,dx. \tag{A.11}$$

Moreover, according to [18, Lemma 3.3 (3)] we have

$$(1-\varepsilon_0)\int_{U_m}\beta_{1m}\,dx \leqslant \int_{U_m}\beta_{1m}\sqrt{\det(\tilde{g}^m_{kl})}\circ\varphi^{-1}_{1m}\,dx = \int_{\mathcal{V}_j}\beta_{1m}\circ\varphi^{-1}_{1m}\,dv_{\tilde{g}}. \tag{A.12}$$

Finally, by (A.11) and (A.12), we deduce that

$$\begin{aligned}\int_{\partial M} v\,dS_g &\leqslant \frac{1+\varepsilon_0}{1-\varepsilon_0}\int_{\partial N\cap W} v_2\,ds_{\bar{g}}\sum_{m=1}^{M_j}\int_{\mathcal{V}_{jm}}\beta_{1m}\circ\varphi^{-1}_{1m}\,dv_{\bar{g}}\\ &= \frac{1+\varepsilon_0}{1-\varepsilon_0}\int_{\partial N\cap W} v_2\,ds_{\bar{g}}\int_{O_j}\sum_{m=1}^{M_j}\beta_{1m}\circ\varphi^{-1}_{1m}\,dv_{\bar{g}}\\ &= \frac{1+\varepsilon_0}{1-\varepsilon_0}V_j\int_{\partial N\cap W_j} v_2\,ds_{\bar{g}}\\ &= \frac{1+\varepsilon_0}{1-\varepsilon_0}V_j\int_{\partial N} v_2\,ds_{\bar{g}}\\ &\leqslant (1+c_1\varepsilon_0)V_j\int_{\partial N} v_2\,ds_{\bar{g}},\end{aligned} \tag{A.13}$$

where $c_1 \geqslant 2/(1-\varepsilon_0)$.

Following the same arguments as in the proof of (A.13) we can show that

$$\int_{\partial M} v\,dS_g \geqslant (1-c_2\varepsilon_0)V_j\int_{\partial N} v_2\,ds_{\bar{g}}, \tag{A.14}$$

where $c_2 \geqslant 2/(1+\varepsilon_0)$.

Set now $c \geq \max(c_1, c_2)$ and inequality (6.13) is proved. The proof of (6.14) is analogous.

For the proof of (6.15), under the same considerations as before and always in the same spirit we get successively

$$\begin{aligned}\int_M |\nabla_g v|\,dV_g &= \int_M\Big(\sum_{m=1}^{M_j}\beta_m\Big)|\nabla_g v|\,dV_g = \sum_{m=1}^{M_j}\int_M \beta_m|\nabla_g v|\,dV_g\\ &= \sum_{m=1}^{M_j}\int_{M\cap\Omega_m}\beta_m|\nabla_g v|^p\,dV_g\\ &= \sum_{m=1}^{M_j}\int_{\xi(M\cap\Omega_m)}\sqrt{\det(g^m_{kl})}\beta_m|\nabla_g v|\circ\xi^{-1}_m\,dx\,dv_{\bar{g}}\\ &= \sum_{m=1}^{M_j}\int_{U_m\times(N\cap W_m)}\sqrt{\det(g^m_{kl})}\beta_m|g^{kl}_m\partial_k\partial_l v|\circ\xi^{-1}_m\,dx\,dv_{\bar{g}}\\ &\geqslant \frac{1-\varepsilon_0}{1+\varepsilon_0}\sum_{m=1}^{M_j}\int_{U_m\times(N\cap W_m)}(\beta_m\circ\xi^{-1}_m)|\nabla_e(v\circ\xi^{-1}_m)|\,dx\,dv_{\bar{g}}\\ &\geqslant \frac{(1-\varepsilon_0)^2}{1+\varepsilon_0}\sum_{m=1}^{M_j}\int_{U_m\times(N\cap W_m)}(\beta_m\circ\xi^{-1}_m)|\bar{g}^{kl}_m\partial_k\partial_l(v\circ\xi^{-1}_m)|\,dx\,dv_{\bar{g}}\\ &\geqslant \frac{(1-\varepsilon)^2}{1+\varepsilon}\sum_{m=1}^{M_j}\int_{U_m\times(N\cap W_m)}(\beta_m\circ\xi^{-1}_m)|\nabla_{\bar{g}}(v\circ\xi^{-1}_m)|\,dx\,dv_{\bar{g}}.\end{aligned} \tag{A.15}$$

Since $\upsilon \circ \xi_m^{-1}$ depends only on the W_m's variables, we have $|\nabla_{\bar{g}}(\upsilon \circ \xi_m^{-1})| = |\nabla_{\bar{g}} \upsilon_2|$ and by (A.15), we obtain

$$\begin{aligned}\int_M |\nabla_g \upsilon| \, dV_g &\geqslant \frac{(1-\varepsilon_0)^2}{1+\varepsilon_0} \sum_{m=1}^{M_j} \Big(\int_{U_m} \beta_{1m} \, dx \Big) \Big(\int_{N \cap W_m} |\nabla_{\bar{g}} \upsilon_{2m}| \, d\upsilon_{\bar{g}} \Big) \\ &= \frac{(1-\varepsilon_0)^2}{1+\varepsilon_0} \sum_{m=1}^{M_j} \Big(\int_{U_m} \beta_{1m} \, dx \Big) \Big(\int_N |\nabla_{\bar{g}} \upsilon_2| \, d\upsilon_{\bar{g}} \Big) \\ &= \frac{(1-\varepsilon_0)^2}{1+\varepsilon_0} \int_N |\nabla_{\bar{g}} \upsilon_2| \, d\upsilon_{\bar{g}} \\ &\geqslant (1 - c_3 \varepsilon_0) \int_N |\nabla_{\bar{g}} \upsilon_2| \, d\upsilon_{\bar{g}}, \end{aligned} \tag{A.16}$$

where

$$c_3 \geqslant \frac{1}{\varepsilon_0} \Big(1 - \frac{(1-\varepsilon_0)^2}{1+\varepsilon_0} \Big).$$

Inequality (A.10) is the first part of inequality (A.2). Following the same arguments, we can establish the second part of inequality (6.15). □

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