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Abstract. This paper deals with the qualitative analysis of solutions to the following (p,q)-fractional equation:

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + V(x)\big(|u|^{p-2}u + |u|^{q-2}u\big) = K(x)\frac{f(u)}{|x|^\beta} \quad \text{in } \mathbb{R}^N,$$

where $1 < q < p$, $0 < s_2 \le s_1 < 1$, $ps_1 = N$, $\beta \in [0,N)$, and $V, K \colon \mathbb{R}^N \to \mathbb{R}$, $f \colon \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying some natural hypotheses. We are concerned both with the case when f has a subcritical growth and with the critical framework with respect to the exponential nonlinearity. By combining a Moser–Trudinger type inequality for fractional Sobolev spaces with Schwarz symmetrization techniques and related variational and topological methods, we prove the existence of nonnegative solutions.

2020 *Mathematics Subject Classification.* Primary: 35J35; Secondary: 35J60, 35J75, 35R11, 58J70.

Key words and phrases. Nonlocal operators; fractional (p,q)-equation; singular exponential nonlinearity; Schwarz symmetrization; Moser–Trudinger inequality.

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1. Introduction

In this paper, we are concerned with the study of a nonlinear nonlocal problem whose features are the following:

(a) the presence of several differential operators with different growth, which generates a *double phase* associated energy;
(b) the reaction combines the multiple effects generated by a Hardy singular potential and a term with subcritical or critical growth with respect to the exponential nonlinearity;
(c) due to the unboundedness of the domain, Cerami sequences do not have the compactness property;
(d) we overcome the lack of compactness by exploiting the special properties of the associated potential;
(e) the proofs combine refined techniques, including a Moser–Trudinger type inequality for fractional Sobolev spaces and Schwarz symmetrization tools.

Summarizing, this paper is concerned with the refined qualitative analysis of solutions for a class of *singular* nonlocal problems driven by differential operators with *unbalanced growth*. The arguments cover both the subcritical and critical cases.

We recall in what follows some basic contributions to the study of unbalanced integral functionals and double phase problems. We first refer to the pioneering contributions of Marcellini [30], [31], [32] who studied lower semicontinuity and regularity properties of minimizers of certain quasiconvex integrals. Problems of this type arise in nonlinear elasticity and are connected with the deformation of an elastic body, cf. Ball [8], [9]. We also refer to Fusco and Sbordone [22] for the study of regularity of minima of anisotropic integrals.

In order to recall the roots of double phase problems, let us assume that Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary. If $u\colon \Omega \to \mathbb{R}^n$ is the displacement and if Du is the $n \times n$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$I(u) = \int_\Omega f(x, Du(x))\, dx, \tag{1.1}$$

where the energy function $f = f(x,\xi)\colon \Omega\times\mathbb{R}^{n\times n} \to \mathbb{R}$ is quasiconvex with respect to ξ. One of the simplest examples considered by Ball is given by functions f of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $n \times n$ matrix ξ, and g, h are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geq c_1|\xi|^p, \qquad \lim_{t\to+\infty} h(t) = +\infty,$$

where c_1 is a positive constant and $1 < p < n$. The condition $p < n$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the integral (1.1) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$, and thus it is a continuous function if $p > n$. In accordance with these problems arising in nonlinear elasticity, Marcellini [30], [31] considered continuous functions $f = f(x, u)$ with *unbalanced growth* that satisfy

$$c_1 |u|^p \le |f(x,u)| \le c_2 (1 + |u|^q) \quad \text{for all } (x,u) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 \le p \le q$. Regularity and existence properties of solutions to nonlinear elliptic equations with p, q-growth conditions were studied in [31].

The study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been continued in a series of remarkable papers by Mingione *et al.* [10], [11], [12]. These contributions are in relationship with the works of Zhikov [48], in order to describe the behavior of phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenisation. In particular, Zhikov considered the following model of functional in relationship to the Lavrentiev phenomenon:

$$\mathcal{P}_{p,q}(u) := \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)\, dx, \quad 0 \le a(x) \le L,\ 1 < p < q.$$

In this functional, the modulating coefficient $a(x)$ dictates the geometry of the composite made by two differential materials, with hardening exponents p and q, respectively.

The functional $\mathcal{P}_{p,q}$ falls in the realm of the so-called functionals with nonstandard growth conditions of (p, q)-type, according to Marcellini's terminology. This is a functional of the type in (1.1), where the energy density satisfies

$$|\xi|^p \le |f(x,\xi)| \le |\xi|^q + 1, \quad 1 \le p \le q.$$

Another significant model example of a functional with (p, q)-growth studied by Mingione *et al.* is given by

$$u \mapsto \int_\Omega |\nabla u|^p \log(1 + |\nabla u|)\, dx, \quad p \ge 1,$$

which is a logarithmic perturbation of the p-Dirichlet energy.

Recent contributions to the study of nonlinear problems with nonstandard growth can be found in [5], [7], [28], [34], [39]–[41] (local case) and [6], [25], [47] (nonlocal case). We also refer to [14], [17] for the study of nonlinear fractional Schrödinger equations involving nonlinearities with critical exponential growth.

2. Statement of the problem and abstract setting

In this paper, we are concerned with the existence of solutions to the following singular (p,q)-fractional equation:

$$(\mathcal{P}) \qquad (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + V(x)\big(|u|^{p-2}u + |u|^{q-2}u\big) = K(x)\frac{f(u)}{|x|^\beta} \quad \text{in } \mathbb{R}^N,$$

where $1 < q < p$, $0 < s_2 \leq s_1 < 1$, $2 \leq N = ps_1$, $\beta \in [0, N)$, and $V, K\colon \mathbb{R}^N \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying some natural assumptions. Let $(-\Delta)_p^s$ denote the fractional p-Laplace operator defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(y)-u(x))}{|x-y|^{N+ps}}\, dy.$$

Problems involving the fractional Laplacian, as in $(\mathcal{P})$, arise from a wide range of real world applications such as optimization, phase transition, anomalous diffusion, image processing, soft thin films, conservation laws and water waves, for a list of more bibliography and other details on this topic we refer to [38]. The main motivation to study problems with leading operators given in $(\mathcal{P})$ comes when $s_1 = s_2 = 1$, which is the local case. Here the leading operator, known as (p,q)-Laplacian, arises while studying the stationary solutions of general reaction-diffusion equation

$$u_t = \operatorname{div}[A(u)\nabla u] + r(x,u), \tag{2.1}$$

where $A(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$.

Problem (2.1) has applications to biophysics, plasma physics and chemical reactions, where u corresponds to the concentration term, the first term on the right-hand side represents diffusion with a diffusion coefficient $A(u)$ and the second term is the reaction, which relates to sources and loss processes. For more details, readers are referred to [29].

In the local case, that is, when $s_1 = s_2 = 1$, problem $(\mathcal{P})$ is motivated by the famous Moser–Trudinger inequality. This differential inequality is important because of the fact that $W^{1,N}(\mathbb{R}^N)$ is embedded into $L^p(\mathbb{R}^N)$ for all $N \leq p < \infty$ but not in $L^\infty(\Omega)$, hence in this case the critical nonlinearity is considered to have exponential type growth condition. Problems of this type were studied by several authors; see e.g. [1], [15], [24]. As far as problems with singular exponential nonlinearity are concerned, Adimurthi and Sandeep [2] proved that the embedding

$$W_0^{1,N}(\Omega) \ni u \mapsto |x|^{-\beta} e^{\alpha|u|^{N/(N-1)}} \in L^1(\Omega)$$

is compact if $\alpha/\alpha_N + \beta/N < 1$ and is continuous if $\alpha/\alpha_N + \beta/N = 1$. Using this result they studied problems having singular exponential-type nonlinearity in a bounded domain.

In the case of the entire Euclidean space, Adimurthi and Yang [3] considered the following singular problem

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\beta} + \varepsilon h(x) \quad \text{in } \mathbb{R}^N,$$

where among other assumptions, f has exponential growth condition and h is in the dual of $W^{1,N}(\mathbb{R}^N)$. Here, the authors established a singular Moser–Trudinger type inequality for whole $\mathbb{R}^N$ and obtained the existence of a mountain pass solution when $\varepsilon > 0$ is small. Subsequently, Yang [45] and Goyal and Sreenadh [26] studied similar singular problems in the whole $\mathbb{R}^N$. In the latter work, the authors proved existence and multiplicity properties of solutions by using the Nehari manifold method.

Regarding the problems involving operators with unbalanced growth conditions, we mention the recent work of Figueiredo and Nunes [19]. Using Nehari manifold analysis the authors proved the existence of a solution for (N,p)-type equations in bounded domains. In [21], Fiscella and Pucci studied the following (N,p)-equation:

$$-\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $1 < q,p < N < \infty$, $N \geq 2$, $h(x) \geq 0$, $\lambda, \gamma > 0$ are parameters and the function f has exponential type growth condition. In this work the authors proved the existence of multiple solutions for small $\lambda > 0$ and large γ. We also refer to Figueiredo and Rădulescu [20] for problems with exponential critical growth driven by the mean curvature operator. Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth have been studied by Ji and Rădulescu [27].

In the nonlocal setting, we mention the work of Giacomoni *et al.* [23]. Here, the authors proved existence of multiple solutions using Nehari manifold analysis for the 1/2-Laplacian problem in bounded domains. Subsequently, Zhang [46] established Moser–Trudinger type inequality in fractional Sobolev–Slobodeckĭ spaces $W^{s,p}(\mathbb{R}^N)$ and proved existence and multiplicity of solutions for the following fractional Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + \varepsilon h(x) \quad \text{in } \mathbb{R}^N,$$

when $\varepsilon > 0$ is sufficiently small. Recently, Mingqi, Rădulescu and Zhang [35], [36] and Xiang, Zhang and Repovš [44] studied fractional Kirchhoff problems with exponential nonlinearity.

Problems of the type ($\mathcal{P}$) involving potential K and exponential-type nonlinearity have been studied by do Ó *et al.* [18] for the case $N = 1$ and $s = 1/2$. In this work, authors considered the following problem:

$$(-\Delta)^{1/2}u + u = K(x)g(u) \quad \text{in } \mathbb{R}.$$

Under certain conditions on K, the authors proved compactness results, which were absent due to unboundedness of the domain, and obtained existence of a nontrivial nonnegative solution in the cases when g possesses subcritical or critical growth condition. Subsequently, this work was generalized by Miyagaki and Pucci [37] for Kirchhoff problems in the one-dimensional case.

3. Main results: subcritical and critical cases

The main purpose in the present paper is to obtain the existence of nontrivial nonnegative solutions to $(\mathcal{P})$ under the following assumptions on $V, K\colon \mathbb{R}^N \to \mathbb{R}$.

(i) The function V is continuous and there exists a constant $V_0 > 0$ such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$.

(ii) The function K satisfies $K \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and is positive in $\mathbb{R}^N$.

(iii) For any sequence $\{A_n\}$ of measurable sets of $\mathbb{R}^N$ with $|A_n| \leq R$, for all $n \in \mathbb{N}$ and some $R > 0$, the following holds

$$\lim_{r\to\infty} \int_{A_n \cap B_r^c(0)} K(x)\,dx = 0 \quad \text{uniformly w.r.t. } n \in \mathbb{N}. \tag{3.1}$$

To define the natural function space that contains all the solutions of problem $(\mathcal{P})$, we first recall some basic fractional Sobolev spaces.

For $1 < p < \infty$ and $0 < s < 1$, the fractional Sobolev space is defined as

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty\}$$

endowed with the norm $\|u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p}$, where

$$[u]_{s,p}^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.$$

Let $\widetilde{W}_V^{s_1,p}(\mathbb{R}^N)$ be the space defined as

$$\widetilde{W}_V^{s_1,p}(\mathbb{R}^N) := \bigg\{u \in W^{s_1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx < \infty\bigg\},$$

which is a reflexive Banach space when endowed with the norm

$$\|u\|_{s_1,p} = \bigg([u]_{s_1,p}^p + \int_{\mathbb{R}^N} V(x)|u(x)|^p\,dx\bigg)^{1/p}$$

and analogously we define $\widetilde{W}_V^{s_2,q}(\mathbb{R}^N)$. From [43], we have the following continuous embedding result

$$\widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \hookrightarrow W^{s_1,p}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N), \quad \text{for all } m \geq p = Ns_1. \tag{3.2}$$

Let $X := \widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \cap \widetilde{W}_V^{s_2,q}(\mathbb{R}^N)$ endowed with the norm

$$\|u\| := \|u\|_{s_1,p} + \|u\|_{s_2,q}.$$

In order to deal with problem $(\mathcal{P})$, we prove the following singular version of the Moser-Trudinger inequality for fractional Sobolev spaces in whole $\mathbb{R}^N$. For this we first obtain a related inequality for bounded domains in the spirit of Adimurthi and Sandeep [2, Theorem 2.1]. Then, using the Schwarz symmetrization technique we prove our theorem. For convenience, we denote

$$\Phi_\alpha(t) = e^{\alpha|t|^{N/(N-s)}} - \sum_{\substack{0\le j<N/s-1\\ j\in\mathbb{N}}} \frac{\alpha^j}{j!}|t|^{jN/(N-s)}, \quad \text{for } t\in\mathbb{R}.$$

We state as follows our first result.

THEOREM 3.1. *Let $N \geq 2$, $s \in (0,1)$ and $p = N/s$. For all $\alpha > 0$, $\beta \in [0,N)$ and $u \in W^{s,p}(\mathbb{R}^N)$, the following inequality holds*

$$\int_{\mathbb{R}^N} \frac{\Phi_\alpha(u)}{|x|^\beta}\,dx < \infty.$$

Furthermore, for all $\alpha < \big(1-\beta/N\big)\alpha_{N,s}$ and $\tau > 0$,

$$\sup\left\{\int_{\mathbb{R}^N} \frac{\Phi_\alpha(u)}{|x|^\beta}\,dx : u\in W^{s,p}(\mathbb{R}^N),\ \|u\|_{s,p,\tau}\le 1\right\} < \infty,$$

where

$$\|u\|_{s,p,\tau} = \left([u]_{s,p}^p + \tau\int_{\mathbb{R}^N}|u|^p\right)^{1/p}$$

and $\alpha_{N,s} > 0$ is defined in Section 4 (see Theorem 4.3).

The function f is said to satisfy a subcritical growth condition with respect the exponential nonlinearity if it satisfies (f2) (see below). We say that the growth is critical if f satisfies (f2)$'$. Furthermore, we assume the following hypotheses:

(f1) The function $f\colon \mathbb{R}\to[0,\infty)$ is continuous with $f(t)=0$ for all $t\le 0$ and

$$\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}} = 0.$$

(f2) (Subcritical growth condition) For all $\alpha > 0$, the following holds

$$\lim_{t\to\infty}\frac{f(t)}{\Phi_\alpha(t)} = 0.$$

(f3) The map $t\mapsto t^{1-p}f(t)$ is nondecreasing in $(0,\infty)$ and

$$\lim_{t\to\infty} t^{-p}F(t) = \infty, \quad \text{where } F(t) = \int_0^t f(\tau)\,d\tau.$$

For the critical growth condition, we assume that f satisfies the following hypotheses in addition to (f1):

(f2)$'$ (Critical growth) There exists $\alpha_0 > 0$ such that

$$\lim_{t\to\infty}\frac{f(t)}{\Phi_\alpha(t)} = 0 \quad \text{for all } \alpha>\alpha_0 \quad\text{and}\quad \limsup_{t\to\infty}\frac{f(t)}{\Phi_\alpha(t)} = \infty \quad \text{for all } \alpha<\alpha_0.$$

(f3)$'$ The map $t\mapsto t^{1-p}f(t)$ is nondecreasing in $(0,\infty)$ and there exists $\delta > p$ such that $F(t)\ge C_\delta t^\delta$ for all $t\in\mathbb{R}^+$, for $C_\delta > 0$ sufficiently large (a lower bound is given in Lemma 4.11).

(AR) (Ambrosetti–Rabinowitz-type condition) There exists $\nu > p$ such that

$$\nu F(t) \leq t f(t) \quad \text{for all } t \in \mathbb{R}^+.$$

Due to the unbounded nature of the domain, Cerami sequences do not have the compactness property. We restore this compactness by exploiting the special property of the potential K, namely (3.1) (see Lemma 4.6). This helps us to prove the strong convergence of Cerami sequences and hence to obtain nontrivial solutions. The existence of such sequences is obtained by using the mountain pass lemma. In the subcritical case, we do not assume the Ambrosetti–Rabinowitz type condition on f, which makes difficult to prove the boundedness of Cerami sequences. The non-homogeneous nature of the leading operator in $(\mathcal{P})$ creates additional difficulty to establish the boundedness of the Cerami sequence and its strong convergence result.

We now state our main theorem, which to the best of our knowledge, is new even for the case $\beta = 0$.

Theorem 3.2. *There exists a nonnegative nontrivial solution of problem* $(\mathcal{P})$ *in the following cases*:

(a) *if* (f1), (f2) *and* (f3) *are satisfied*;

(b) *if* (f1), (f2)$'$, (f3)$'$ *and* (AR) *are satisfied, provided that* C_δ *appearing in* (f3)$'$, *is sufficiently large.*

Remark 3.3. We remark that the results of Theorems 3.1 and 3.2 are valid for equations of the type $(\mathcal{P})$ involving a more general class of operators, for instance, operators of the form

$$\mathcal{L}_{\mathcal{K}_{r,s}} u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x) - u(y)|^{r-2}(u(y) - u(x)) \mathcal{K}_{r,s}(x-y)\, dy,$$

where $(r,s) \in \{(p,s_1),(q,s_2)\}$ with $1 < q < p = N/s_1$ and $0 < s_2 \leq s_1 < 1$. Here, the singular kernel $\mathcal{K}_{r,s} \colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is such that

(a) $m\mathcal{K}_{r,s} \in L^1(\mathbb{R}^N)$, where $m(x) := \min\{1, |x|^r\}$.

(b) There exist $c_p > 0$ and $c_q \geq 0$ such that $\mathcal{K}_{p,s_1}(x) \geq c_p |x|^{-(N+ps_1)}$ and $\mathcal{K}_{q,s_2}(x) \geq c_q |x|^{-(N+qs_2)}$ for $x \in \mathbb{R}^N \setminus \{0\}$.

The corresponding energy space is defined as $X := \widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \cap \widetilde{W}_V^{s_2,q}(\mathbb{R}^N)$, where in the definition of $\widetilde{W}_V^{s,r}(\mathbb{R}^N)$, the term $|x-y|^{-(N+rs)}$ is replaced by $\mathcal{K}_{r,s}(x-y)$. For example, one can take $\mathcal{K}_{r,s}(x) = a_r(x)|x|^{-(N+rs)}$, where $a_r \colon \mathbb{R}^N \to \mathbb{R}$ are non-negative bounded functions, for $r \in \{p,q\}$, where a_p is bounded away from zero.

Notation. We denote $\mathcal{A}_1 \colon W^{s_1,p}(\mathbb{R}^N) \times W^{s_1,p}(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{A}_1(u,v) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps_1}}\, dx\, dy,$$

and analogously $\mathcal{A}_2\colon W^{s_2,p}(\mathbb{R}^N)\times W^{s_2,p}(\mathbb{R}^N)\to\mathbb{R}$.

Definition 3.4. A function $u\in X$ is said to be a solution of problem ($\mathcal{P}$), if for all $v\in X$

$$\mathcal{A}_1(u,v)+\mathcal{A}_2(u,v)+\int_{\mathbb{R}^N}V(x)\big(|u|^{p-2}+|u|^{q-2}\big)uv\,dx-\int_{\mathbb{R}^N}\frac{K(x)f(u)v}{|x|^\beta}\,dx=0.$$

The Euler functional $\mathcal{J}\colon X\to\mathbb{R}$ associated to the problem ($\mathcal{P}$) is defined as

$$\mathcal{J}(u)=\frac{1}{p}\,\|u\|_{s_1,p}^p+\int_{\mathbb{R}^N}V(x)|u|^p\,dx+\frac{1}{q}\,\|u\|_{s_2,q}^q+\int_{\mathbb{R}^N}V(x)|u|^q\,dx-\int_{\mathbb{R}^N}\frac{K(x)F(u(x))}{|x|^\beta}\,dx.$$

4. Some technical results

In this section, we first establish some compact embedding results for the space X. Let us define the following weighted Lebesgue space

$$L_V^p(\mathbb{R}^N):=\bigg\{u:\mathbb{R}^N\to\mathbb{R}:u\text{ measurable and }\|u\|_{p,V}^p=\int_{\mathbb{R}^N}V(x)|u(x)|^p\,dx<\infty\bigg\},$$

which is a Banach space when equipped with the norm $\|\cdot\|_{p,V}$, for $0<V$ in $C(\mathbb{R}^N)$.

Remarks 4.1. (a) Due to the fact $0\le\beta<N$, one can easily get that the embedding $X\hookrightarrow L^m(\mathbb{R}^N;|x|^{-\beta})$ is continuous for all $m\ge p$, that is, for all $m\ge p$ there exists $C_m>0$ such that for all $u\in X$,

$$\int_{\mathbb{R}^N}|u(x)|^m|x|^{-\beta}\,dx\le C_m\|u\|^m.$$

(b) Let $\Omega\subset\mathbb{R}^N$ be a bounded domain. Arguments similar to that of [18, Remark 2.1] imply that X is compactly embedded into $L^m(\Omega)$. Indeed, by [16, Theorem 7.1] we have that $W^{s_1,p}(\mathbb{R}^N)$ is compactly embedded into $L^p(\Omega)$ and then using (3.2) and a standard interpolation argument, we can prove that X is compactly embedded into $L^m(\Omega)$ for all $m\ge p$. The aforementioned compact embedding result with a straightforward verification, yields that X is compactly embedded into $L^m(\Omega;|x|^{-\beta})$ for all $m\ge p$.

Proposition 4.2. *The space X is compactly embedded into $L_K^m(\mathbb{R}^N)$ for all $m\in(p,\infty)$.*

Proof. The proof given here is an adaptation of the proof of [37, Proposition 2.1] for $N=1$. For the convenience of the reader, we provide only a sketch of the proof. For fixed $r>m>p$ and $\varepsilon>0$, there exist $\tau_0=\tau_0(\varepsilon)$ and $\tau_1=\tau_1(\varepsilon)$ with

$0 < \tau_0 < \tau_1$, $C = C(\varepsilon) > 0$ and $C_0 > 0$ depending only on K, such that for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

$$K(x)|t|^m \le \varepsilon C_0\big(V(x)|t|^p + |t|^r\big) + CK(x)\chi_{[\tau_0,\tau_1]}(|t|)|t|^m. \tag{4.1}$$

Let $\{u_n\} \subset X$ be a bounded sequence, then by reflexivity of X, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. From the continuous embedding of X into $L^r(\mathbb{R}^N)$ and boundedness of the sequence $\{\|u_n\|\}$, for some $M > 0$, we have

$$\|u_n\|_{p,V}^p \le M \quad \text{and} \quad \|u_n\|_\gamma^\gamma \le M \quad \text{for all } n \in \mathbb{N} \text{ and } \gamma \in \{m, r\}.$$

Therefore, $Q(u_n) := C_0\big(\|u_n\|_{p,V}^p + \|u_n\|_r^r\big) \le 2C_0M$ for all $n \in \mathbb{N}$. Set

$$A_\varepsilon^n := \big\{x \in \mathbb{R}^N : \tau_0 \le |u_n(x)| \le \tau_1\big\}.$$

Then, by the fact that $\{u_n\}$ is bounded in $L^m(\mathbb{R}^N)$, it is easy to observe that $\{|A_\varepsilon^n|\}$ is bounded with regard to n. Again, by (3.1), for $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\int_{A_\varepsilon^n \cap B_{r_\varepsilon}^c(0)} K(x)\,dx < \frac{\varepsilon}{C\tau_1^m}, \quad \text{for all } n \in \mathbb{N}.$$

Using this together with the observation that $Q(u_n)$ is bounded, (4.1) gives us

$$\int_{B_{r_\varepsilon}^c(0)} K(x)|u_n|^m \le 2C_0M\varepsilon + C\tau_1^m \int_{A_\varepsilon^n \cap B_{r_\varepsilon}^c(0)} K(x)\,dx < (2C_0M + 1)\varepsilon, \tag{4.2}$$

for all $n \in \mathbb{N}$. Moreover, by compact embedding of the space X into $L^\gamma(B_{r_\varepsilon}(0))$ for all $\gamma \ge p$ (see Remark 4.1), we get

$$\lim_{n\to\infty} \int_{B_{r_\varepsilon}(0)} K(x)|u_n|^m = \int_{B_{r_\varepsilon}(0)} K(x)|u|^m. \tag{4.3}$$

Combining relations (4.2) and (4.3), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)|u_n|^m = \int_{\mathbb{R}^N} K(x)|u|^m.$$

From the above equation, it is easy to deduce that $u_n \to u$ in $L_K^m(\mathbb{R}^N)$, as $n \to \infty$. □

We state the following Moser–Trudinger type inequality for fractional Sobolev spaces in case of bounded domain.

Theorem 4.3 ([42], [33]). *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \ge 2)$ with Lipschitz boundary, and $s_1 \in (0,1)$, $ps_1 = N$. Let $\widetilde{W}_0^{s_1,p}(\Omega)$ be the space defined as the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{s_1,p}(\mathbb{R}^N)}$ norm. Then, there exists $\alpha_{N,s_1} > 0$ such that, for $\alpha \in [0, \alpha_{N,s_1})$,*

$$\sup\bigg\{\int_\Omega \exp\big(\alpha|u|^{N/(N-s_1)}\big) : u \in \widetilde{W}_0^{s_1,p}(\Omega),\ \|u\|_{p,s_1} \le 1\bigg\} < \infty.$$

Moreover,

$$\sup\left\{\int_\Omega \exp\left(\alpha|u|^{N/(N-s_1)}\right) : u \in \widetilde{W}_0^{s_1,p}(\Omega), \|u\|_{p,s_1} \le 1\right\} = \infty$$

for $\alpha \in (\alpha^*_{N,s_1}, \infty)$, *where*

$$\alpha^*_{N,s_1} = N\left(\frac{2(N\omega_N)^2\,\Gamma(p+1)}{N!}\sum_{k=0}^{\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\right)^{s_1/(N-s_1)},$$

with ω_N *as the volume of the* N*-dimensional unit ball.*

Similar to the result of [2], we prove the following singular Moser–Trudinger inequality for fractional Sobolev spaces in bounded domains, which will help us to prove our Theorem 3.1.

LEMMA 4.4. *Let* $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ *be a bounded domain with Lipschitz boundary and let* $u \in W_0^{s_1,p}(\Omega)$. *Then, for every* $\alpha > 0$ *and* $\beta \in [0, N)$,

$$\int_\Omega \frac{e^{\alpha|u|^{N/(N-s_1)}}}{|x|^\beta}\,dx < \infty.$$

Moreover, if $\alpha/\alpha_{N,s_1} + \beta/N < 1$, *then*

$$\sup_{\|u\|_{W^{s_1,p}(\Omega)}\le 1}\int_\Omega \frac{e^{\alpha|u|^{N/(N-s_1)}}}{|x|^\beta}\,dx < \infty. \tag{4.4}$$

PROOF. Let $t > 1$ be such that $\beta t < N$, then using Hölder inequality and Theorem 4.3, we deduce that

$$\int_\Omega \frac{e^{\alpha|u|^{N/(N-s_1)}}}{|x|^\beta}\,dx \le \left(\int_\Omega e^{\alpha t'|u|^{N/(N-s_1)}}\,dx\right)^{1/t'}\left(\int_\Omega \frac{1}{|x|^{\beta t}}\,dx\right)^{1/t} < \infty.$$

For the second part of the theorem, we first observe that there exist $\widetilde{\alpha} \in (\alpha, \alpha_{N,s_1})$ and $t > 1$ such that $\alpha/\widetilde{\alpha} + \beta t/N = 1$ (this can be done by first choosing $\widetilde{\alpha} < \alpha_{N,s_1}$ such that $\alpha/\alpha_{N,s_1} + \beta/N < \alpha/\widetilde{\alpha} + \beta/N < 1$). Now, by Hölder inequality, we have

$$\sup_{[u]_{s_1,p}\le 1}\int_\Omega \frac{e^{\alpha|u|^{N/(N-s_1)}}}{|x|^\beta}\,dx \le \sup_{[u]_{s_1,p}\le 1}\left(\int_\Omega e^{\widetilde{\alpha}|u|^{N/(N-s_1)}}\,dx\right)^{\alpha/\widetilde{\alpha}}\left(\int_\Omega \frac{1}{|x|^{N/t}}\right)^{\beta t/N},$$

since $\widetilde{\alpha} < \alpha_{N,s_1}$ and $t > 1$, by Theorem 4.3, we get that the right-hand side quantity is finite. □

Before proving Theorem 3.1, we state the following radial lemma.

Lemma 4.5. *Let $N \geq 2$ and $u \in L^p(\mathbb{R}^N)$, with $1 \leq p < \infty$, be a radially symmetric non-increasing function. Then*

$$|u(x)| \leq |x|^{-N/p}\left(\frac{N}{\omega_{N-1}}\right)^{1/p}\|u\|_p, \quad \text{for } x \neq 0,$$

where ω_{N-1} is the $(N-1)$ dimensional measure of $(N-1)$ sphere.

Proof of of Theorem 3.1. Without loss of generality, we assume $u \geq 0$ and let u^* be the Schwarz symmetrization of u. Then by ([4], [13]), for any continuous and increasing function $G\colon [0,\infty) \to [0,\infty)$, there holds

$$\int_{\mathbb{R}^N} G(u^*(x))\,dx = \int_{\mathbb{R}^N} G(u(x))\,dx.$$

Moreover, for all $u \in W^{s,p}(\mathbb{R}^N)$ and $1 \leq m < \infty$, $u^* \in W^{s,p}(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^*(x)-u^*(y)|^p}{|x-y|^{2N}}\,dx\,dy \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{2N}}\,dx\,dy \tag{4.5}$$

and $\|u^*\|_m = \|u\|_m$. Therefore, by the Hardy–Littlewood inequality for symmetrization and the fact that $\big(1/|x|^\beta\big)^* = 1/|x|^\beta$, we get

$$\int_{\mathbb{R}^N} \frac{\Phi_\alpha(u)}{|x|^\beta}\,dx \leq \int_{\mathbb{R}^N} \frac{\Phi_\alpha(u^*)}{|x|^\beta}\,dx. \tag{4.6}$$

Fix $R > 0$ (to be specified later), we have

$$\int_{|x|>R} \frac{\Phi_\alpha(u^*)}{|x|^\beta}\,dx = \int_{|x|>R} \frac{1}{|x|^\beta}\sum_{j=k_0}^{\infty} \frac{\alpha^j}{j!}|u^*|^{jp'}\,dx, \tag{4.7}$$

where k_0 is the smallest integer such that $k_0 \geq p-1$ and $p' = p/(p-1)$ is the Hölder conjugate of p.

Now we consider the following cases.

Case 1. For all $j \geq k_0 > p-1$. Using Lemma 4.5 and (4.5), we obtain

$$\begin{aligned}\int_{|x|>R} \frac{|u^*|^{jp'}}{|x|^\beta} &\leq \int_{|x|>R} |x|^{-Nj/(p-1)-\beta}\left(\frac{N}{\omega_{N-1}}\right)^{j/(p-1)}\|u^*\|_p^{jp'} \\ &\leq \left(\frac{N}{\omega_{N-1}}\right)^{j/(p-1)}\|u\|_p^{jp'}R^{N-Nj/(p-1)-\beta}.\end{aligned} \tag{4.8}$$

Case 2. For $k_0 = p-1$. Using (4.5), we obtain

$$\int_{|x|>R} \frac{|u^*|^{k_0p'}}{|x|^\beta} \leq \int_{|x|>R} \frac{|u^*(x)|^p}{|x|^\beta} \leq \frac{1}{R^\beta}\int_{\mathbb{R}^N} |u^*(x)|^p\,dx = \frac{\|u\|_p^p}{R^\beta}. \tag{4.9}$$

Then, coupling (4.8) and (4.9) in (4.7), we get

$$\int_{|x|>R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \leq \frac{1}{R^\beta}\left(C_\alpha\|u\|_p^p + \sum_{j=k_0+1}^{\infty} \frac{\alpha^j}{j!}\left(\frac{N}{\omega_{N-1}}\right)^{j/(p-1)}\|u\|_p^{jp'}R^{N-Nj/(p-1)}\right).$$

For fixed $u \in W^{s,p}(\mathbb{R}^N)$, we choose $R > 0$ such that

$$R^{-N/(p-1)} \left(\frac{N}{\omega_{N-1}} \right)^{1/(p-1)} \|u\|_p^{p'} = 1,$$

this implies

$$\int_{|x|>R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \le \frac{1}{R^\beta} C(N, s, \alpha, \|u\|_p) < \infty. \tag{4.10}$$

Next, for $p' = N/(N-s)$, there exists $B = B(N,s) > 0$ such that for all $\varepsilon > 0$,

$$(u+v)^{p'} \le u^{p'} + Bu^{p'-1}v + v^{p'}, \quad \text{and } u^\gamma v^\gamma \le \varepsilon u + \varepsilon^{-\gamma/\gamma'} v \tag{4.11}$$

for all $u, v \ge 0$ and $\gamma, \gamma' > 0$ satisfying $\gamma + \gamma' = 1$. For fixed $x_0 \in \mathbb{R}^N$ with $|x_0| = 1$, define

$$v(x) = \begin{cases} u^*(x) - u^*(Rx_0) & \text{if } x \in B_R(0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_R(0). \end{cases}$$

Then, since u^* is radially decreasing function, we have $v \ge 0$ and by [44, Lemma 2.2], $[v]_{s,p}^p \le [u^*]_{s,p}^p \le [u]_{s,p}^p < \infty$. Therefore, $v \in W^{s,p}(\mathbb{R}^N)$ with $v = 0$ almost everywhere in $\mathbb{R}^N \setminus B_R(0)$. Using (4.11), for $x \in B_R(0)$, we deduce that

$$|u^*(x)|^{p'} = |v + u^*(Rx_0)|^{p'} \le v^{p'} + Bv^{p'-1}u^*(Rx_0) + u^*(Rx_0)^{p'},$$

$$v^{p'-1}u^*(Rx_0) = (v^{p'})^{(p'-1)/p'}(u^*(Rx_0)^{p'})^{1/p'} \le \frac{\varepsilon}{A} v^{p'} + \left(\frac{\varepsilon}{A}\right)^{-1/(p'-1)} u^*(Rx_0)^{p'}.$$

Thus,

$$|u^*(x)|^{p'} \le (1+\varepsilon)\, v^{p'} + C(\varepsilon, s, N)\, u^*(Rx_0)^{p'},$$

where $C(\varepsilon, s, N) = 1 + \big(A/\varepsilon\big)^{1/(p'-1)}$. Therefore, using Lemma 4.4, we obtain

$$\begin{aligned} \int_{|x|\le R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} & \\ \le \int_{|x|\le R} \frac{e^{\alpha|u^*|^{p'}}}{|x|^\beta} &\le e^{\alpha C(\varepsilon,s,N)|u^*(Rx_0)|^{p'}} \int_{|x|\le R} \frac{e^{\alpha(1+\varepsilon)|v|^{p'}}}{|x|^\beta} < \infty. \end{aligned} \tag{4.12}$$

This together with (4.10) and (4.6) proves the first part of the theorem.

For the second part, we consider $u \in W^{s,p}(\mathbb{R}^N)$ such that $\|u\|_{s,p,\tau} \le 1$. From (4.8), we have

$$\begin{aligned} \int_{|x|>R} \frac{|u^*|^{jp'}}{|x|^\beta} &\le \left(\frac{N}{\omega_{N-1}}\right)^{j/(p-1)} \|u\|_p^{jp'} R^{N-Nj/(p-1)-\beta} \\ &\le R^{N-\beta}\left(\frac{N}{\omega_{N-1}}\right)^{j/(p-1)} \tau^{-j/(p-1)} R^{-Nj/(p-1)}, \end{aligned} \tag{4.13}$$

where in the last inequality we used the fact $\|u\|_{s,p,\tau} \le 1$. Choosing $R > 0$ such that

$$R^{-N}\frac{N}{\omega_{N-1}\tau} = 1,$$

then, on account of (4.13), we deduce from (4.7) that

$$\int_{|x|>R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \le R^{N-\beta}\sum_{j=k_0}^{\infty}\frac{\alpha^j}{j!} \le C(N,s,\alpha,\beta,\tau). \tag{4.14}$$

Now, due to the fact that $\|u\|_{s,p,\tau} \le 1$ and $v(x) \le u^*(x)$ in $B_R(0)$, we have

$$\|v\|^p_{s,p,\tau} = [v]^p_{s,p} + \tau\|v\|^p_p \le [u^*]^p_{s,p} + \tau\|u^*\|^p_p \le [u]^p_{s,p} + \tau\|u\|^p_p \le 1,$$

and by using the radial lemma 4.5, we obtain

$$\begin{aligned} u^*(Rx_0)^{p'} &\le |Rx_0|^{-N/(p-1)}\left(\frac{N}{\omega_{N-1}}\right)^{1/(p-1)}\|u^*\|^{p'}_p \\ &\le R^{-N/(p-1)}\left(\frac{N}{\omega_{N-1}\tau}\right)^{1/(p-1)}. \end{aligned}$$

Therefore, from (4.12) and (4.4), we get

$$\begin{aligned} &\int_{|x|\le R}\frac{\Phi_\alpha(u^*)}{|x|^\beta} \\ &\quad\le e^{C(\varepsilon,s,N,\tau,\alpha,\beta)}\int_{|x|\le R}\frac{1}{|x|^\beta}\exp\left\{\alpha(1+\varepsilon)\|v\|^{p'}_{s,p,\tau}\left|\frac{v}{\|v\|_{s,p,\tau}}\right|^{p'}\right\} \\ &\qquad\le C(N,s,\tau,\alpha,\beta), \end{aligned} \tag{4.15}$$

if we choose $\varepsilon > 0$ such that $\alpha(1+\varepsilon) < \big(1-\beta/N\big)\alpha_{N,s}$.

Taking into account (4.14), (4.15) and (4.6), we complete the proof of the second part of the theorem. □

Next, we establish the compactness result under the assumption that (f1) through (f3) hold, that is, the subcritical case.

Lemma 4.6. *Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u$ weakly in X, for some $u \in X$. Then up to a subsequence, the following properties hold*:

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|x|^{-\beta}F(u_n(x))\,dx &= \int_{\mathbb{R}^N} K(x)|x|^{-\beta}F(u(x))dx, \\ \lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|x|^{-\beta}f(u_n(x))\,u_n(x)\,dx &= \int_{\mathbb{R}^N} K(x)|x|^{-\beta}f(u(x))\,u(x)\,dx \\ \lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|x|^{-\beta}f(u_n(x))\,v(x)\,dx &= \int_{\mathbb{R}^N} K(x)|x|^{-\beta}f(u(x))\,v(x)\,dx, \\ &\qquad \textit{for all } v \in X. \end{aligned}$$

PROOF. Set $M = \sup_n \|u_n\|$. For

$$0 < \alpha < \left(1 - \frac{\beta}{N}\right) \frac{\alpha_{N,s_1}}{M^{N/(N-s_1)}},$$

from (f1) and (f2), we get

$$\limsup_{t\to\infty} \frac{f(t)t}{\Phi_\alpha(t)} = \limsup_{t\to\infty} \frac{F(t)}{\Phi_\alpha(t)} = 0, \quad \text{and} \quad \limsup_{t\to 0} \frac{f(t)t}{|t|^p} = \limsup_{t\to 0} \frac{F(t)}{|t|^p} = 0.$$

Therefore, for $\varepsilon > 0$ and $\delta > p$, there exist $\rho_0 = \rho_0(\varepsilon)$, $\rho_1 = \rho_1(\varepsilon)$ with $0 < \rho_0 < \rho_1$, $C = C_\varepsilon > 0$ and $C_0 > 0$ depending only on K, such that for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, the following hold

$$\begin{aligned} |K(x)F(t)| &\le \varepsilon C_0\big(|t|^p + \Phi_\alpha(t)\big) + CK(x)\chi_{[\rho_0,\rho_1]}(|t|)|t|^\delta, \\ |K(x)f(t)t| &\le \varepsilon C_0\big(|t|^p + \Phi_\alpha(t)\big) + CK(x)\chi_{[\rho_0,\rho_1]}(|t|)|t|^\delta. \end{aligned} \tag{4.16}$$

By the embedding results of X into $L^m(\mathbb{R}^N)$ (and hence into $L^m(\mathbb{R}^N; |x|^{-\beta})$, for $0 \le \beta < N$), we have

$$\sup_n \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^\beta} \le \widetilde{M}, \tag{4.17}$$

for some $\widetilde{M} \ge M > 0$. Now, for

$$\alpha < \left(1 - \frac{\beta}{N}\right) \frac{\alpha_{N,s_1}}{M^{N/(N-s_1)}},$$

we have

$$\alpha\|u_n\|^{N/(N-s_1)} \le \alpha M^{N/(N-s_1)} < (1 - \beta/N)\alpha_{N,s_1},$$

therefore, by Theorem 3.1 and the fact that Φ_α is increasing with respect to α, we obtain

$$\sup_n \int_{\mathbb{R}^N} \frac{\Phi_\alpha(u_n)}{|x|^\beta} \le \sup_n \int_{\mathbb{R}^N} \frac{\Phi_{\alpha M^{N/(N-s_1)}}(u_n/\|u_n\|)}{|x|^\beta} \le \widetilde{M}. \tag{4.18}$$

Let $A_\varepsilon^n := \big\{x \in \mathbb{R}^N : \rho_0 \le |u_n(x)| \le \rho_1\big\}$. Then, as $\{|A_\varepsilon^n|\}$ is bounded with respect to n, using (3.1), we deduce that

$$\lim_{r\to\infty} \left| \int_{A_\varepsilon^n \cap B_r^c(0)} \frac{K(x)}{|x|^\beta}\, dx \right| \le \lim_{r\to\infty} \frac{1}{r^\beta} \left| \int_{A_\varepsilon^n \cap B_r^c(0)} K(x)\, dx \right| = 0,$$

uniformly with respect to $n \in \mathbb{N}$. Therefore, for $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{A_\varepsilon^n \cap B_{R_\varepsilon}^c(0)} \frac{K(x)}{|x|^\beta}\, dx < \frac{\varepsilon}{C\rho_1^\delta} \quad \text{for all } n \in \mathbb{N}. \tag{4.19}$$

Taking into account (4.16) through (4.19), we obtain

$$
(4.20)\quad
\begin{aligned}
&\int_{B^c_{R_\varepsilon}(0)} \frac{K(x)F(u_n(x))}{|x|^\beta}\,dx \\
&\qquad \le 2C_0\widetilde{M}\varepsilon + C\rho_1^\delta \int_{A^n_\varepsilon\cap B^c_{R_\varepsilon}(0)} \frac{K(x)}{|x|^\beta}dx < (2C_0\widetilde{M}+1)\varepsilon, \\
&\int_{B^c_{R_\varepsilon}(0)} \frac{K(x)f(u_n(x))u_n(x)}{|x|^\beta}\,dx \\
&\qquad \le 2C_0\widetilde{M}\varepsilon + C\rho_1^\delta \int_{A^n_\varepsilon\cap B^c_{R_\varepsilon}(0)} \frac{K(x)}{|x|^\beta}\,dx < (2C_0\widetilde{M}+1)\varepsilon,
\end{aligned}
$$

for all $n\in\mathbb{N}$. Furthermore, from (f1) and (f2), it is easy to observe that

$$
|f(t)| \le C_1\big(|t|^p + \Phi_\alpha(t)\big), \quad \text{for all } t\in\mathbb{R},
$$

where $C_1>0$ is a constant. Therefore, using the fact that $K\in L^\infty(\mathbb{R}^N)$, we get

$$
\begin{aligned}
&\bigg|\int_{B_{R_\varepsilon}(0)} \frac{K(x)f(u_n)(u_n-u)}{|x|^\beta}\,dx\bigg| \\
&\qquad\le C\bigg(\int_{B_{R_\varepsilon}(0)} \frac{|u_n|^p|u_n-u|}{|x|^\beta} + \int_{B_{R_\varepsilon}(0)} \frac{\Phi_\alpha(u_n)|u_n-u|}{|x|^\beta}\bigg).
\end{aligned}
$$

We choose $\gamma>1$ close to 1 such that

$$
\gamma' > p \quad\text{and}\quad \gamma\alpha < \bigg(1-\frac{\beta}{N}\bigg)\frac{\alpha_{N,s_1}}{M^{N/(N-s_1)}}.
$$

Using Hölder's inequality, Theorem 3.1 and the fact that $\{\|u_n\|\}$ is bounded, we deduce that

$$
\begin{aligned}
&\bigg|\int_{B_{R_\varepsilon}(0)} \frac{K(x)f(u_n)(u_n-u)}{|x|^\beta}\,dx\bigg| \\
&\le C\bigg[\bigg(\int_{\mathbb{R}^N}\frac{|u_n|^{\gamma p}}{|x|^\beta}\bigg)^{1/\gamma} + \bigg(\int_{\mathbb{R}^N}\frac{\Phi_\alpha(u_n)^\gamma}{|x|^\beta}\bigg)^{1/\gamma}\bigg]\bigg(\int_{B_{R_\varepsilon}(0)}\frac{|u_n-u|^{\gamma'}}{|x|^\beta}\bigg)^{1/\gamma'} \\
&\qquad\qquad \le C\bigg(\int_{B_{R_\varepsilon}(0)}\frac{|u_n-u|^{\gamma'}}{|x|^\beta}\bigg)^{1/\gamma'} \to 0
\end{aligned}
$$

as $n\to\infty$, where in the last line we have used the compact embedding result of X as in Remark 4.1. Hence,

$$
\lim_{n\to\infty}\int_{B_{R_\varepsilon}(0)} \frac{K(x)f(u_n)u_n}{|x|^\beta}\,dx = \int_{B_{R_\varepsilon}(0)} \frac{K(x)f(u_n)u_n}{|x|^\beta}\,dx.
$$

Using (f3), one can easily verify that $pF(t)\le f(t)t$ for all $t\in\mathbb{R}^N$. Therefore, by generalized Lebesgue dominated convergence theorem, similar convergence result holds for F also. Thus, using (4.20), we get the required convergence result of the first two integrals of the lemma.

Next, to prove the last convergence result of the lemma, we set

$$E_n := \{x \in \mathbb{R}^N : |u_n(x)| \le 1\} \quad \text{and} \quad E := \{x \in \mathbb{R}^N : |u(x)| \le 1\}.$$

We first claim that the sequence $\{K(x)f(u_n)\chi_{E_n}\}$ is uniformly bounded in $L^{r'}(\mathbb{R}^N; |x|^{-\beta})$, where r' is the Hölder conjugate of r. By (f1), it is easy to see that $|f(t)| \le C|t|^{p-1}$ for all $|t| \le 1$ and some $C > 0$. Therefore,

$$|f(u_n)| \le C|u_n|^{p-1} \quad \text{in } E_n, \text{ for all } n \in \mathbb{N}.$$

Using the fact that $X \hookrightarrow L^p(\mathbb{R}^N; |x|^{-\beta})$ is continuous and $\{\|u_n\|\}$ is bounded, we obtain

$$\int_{E_n} \frac{|K(x)f(u_n)|^{p'}}{|x|^\beta} \le C \int_{E_n} \frac{|u_n|^p}{|x|^\beta} \le C\|u_n\|^p \le C, \quad \text{for all } n \in \mathbb{N}.$$

This together with the pointwise convergence gives us

$$\lim_{n\to\infty} \int_{E_n} \frac{K(x)f(u_n)\phi}{|x|^\beta}\,dx = \int_E \frac{K(x)f(u_n)\phi}{|x|^\beta}\,dx \quad \text{for all } \phi \in L^p(\mathbb{R}^N; |x|^{-\beta}).$$

Now, for any $v \in X$, we have $v \in L^r(\mathbb{R}^N; |x|^{-\beta})$ and hence

$$\lim_{n\to\infty} \int_{E_n} \frac{K(x)f(u_n)v}{|x|^\beta}\,dx = \int_E \frac{K(x)f(u_n)v}{|x|^\beta}\,dx.$$

Similarly, by (f2), for $m \ge 1$, we obtain

$$|f(u_n(x))|^m \le C\Phi_\alpha(u_n(x))^m \le C\Phi_{m\alpha}(u_n(x)) \quad \text{for } x \in E_n^c \text{ and for all } n \in \mathbb{N}.$$

We choose $m > 1$ close to 1 such that

$$m' > p \quad \text{and} \quad m\alpha < \left(1 - \frac{\beta}{N}\right)\frac{\alpha_{N,s_1}}{M^{N/(N-s_1)}}.$$

Then, by Theorem 3.1, we get

$$\int_{E_n^c} \frac{|K(x)f(u_n)|^m}{|x|^\beta} \quad \text{is uniformly bounded.}$$

Therefore, for $v \in X$, we have $v \in L^{m'}(\mathbb{R}^N; |x|^{-\beta})$ and pointwise convergence yields

$$\lim_{n\to\infty} \int_{E_n^c} \frac{K(x)f(u_n)v}{|x|^\beta}dx = \int_{E^c} \frac{K(x)f(u_n)v}{|x|^\beta}dx.$$

This completes proof of the lemma. □

Without loss of generality, we may assume $\alpha_0 = (1 - \beta/N)\alpha_{N,s_1}$, appearing in (f2)′. Then, we have similar compactness result if the conditions (f1), (f2)′ and (f3)′ hold, that is, the critical case.

Corollary 4.7. *Let $\{v_n\} \subset X$ be a sequence such that $v_n \rightharpoonup v$ weakly in X, for some $v \in X$ and $L := \sup_n \|v_n\| \in (0,1)$. Then, the convergence results of the Lemma 4.6 are true in this case, too.*

Proof. Since $L \in (0,1)$, there exists $\alpha_L > \alpha_{N,s_1}(1-\beta/N)$ such that

$$\alpha_L < \left(1-\frac{\beta}{N}\right)\frac{\alpha_{N,s_1}}{L^{N/(N-s_1)}}.$$

Then, by (f1) and (f2)$'$, results similar to (4.16) hold in this case too, with α replaced by α_L. Furthermore, Theorem 3.1 can be applied to obtain boundedness results as in (4.17) and (4.18). Now, rest of the proof follows similar to that of the lemma with α replaced by α_L. □

It is easy to verify that the functional $\mathcal{J}$ is of class $C^1(X)$. Now, we verify the mountain pass geometry for $\mathcal{J}$.

Lemma 4.8. *The functional $\mathcal{J}$ satisfies the following properties*:

(a) *there exists $v_0 \in X \setminus \{0\}$ with $\|v_0\| \geq 2$ such that $\mathcal{J}(v_0) < 0$;*
(b) *there exist $\eta > 0$ and $\rho \in (0,1)$ such that $\mathcal{J}(v) \geq \eta$ for all $v \in X$ with $\|v\| = \rho$.*

Proof. (a) Proof of this part is a standard procedure and follows by the superlinear nature of the nonlinearity F with respect to p.

(b) For fixed $\rho_0 \in (0,1)$, we choose $\alpha > 0$ such that

$$\alpha_{N,s_1}\left(1-\frac{\beta}{N}\right) < \alpha < \left(1-\frac{\beta}{N}\right)\frac{\alpha_{N,s_1}}{\rho_0^{N/(N-s_1)}}.$$

Now, by the fact that $K \in L^\infty(\mathbb{R}^N)$, (f1) and (f2) (or (f2)$'$), for $\delta > p$, we have

$$K(x)F(t) \leq \frac{1}{2^p p C_p^p} t^p + C_2\Phi_\alpha(t)t^\delta \quad \text{for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^N, \tag{4.21}$$

where $C_2 > 0$ is a constant and C_p appears in Remark 4.1 (a). We choose $m > 1$ close to 1 such that

$$m' > p \quad \text{and} \quad m\alpha < \left(1-\frac{\beta}{N}\right)\frac{\alpha_{N,s_1}}{\rho_0^{N/(N-s_1)}},$$

then by Theorem 3.1 and the embedding of X into $L^\gamma(\mathbb{R}^N; |x|^{-\beta})$, for $\gamma \geq p$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\Phi_\alpha(w)|w|^\delta}{|x|^\beta} &\leq \left(\int_{\mathbb{R}^N} \frac{|\Phi_\alpha(w)|^m}{|x|^\beta}\right)^{1/m}\left(\int_{\mathbb{R}^N} \frac{|w|^{\delta m'}}{|x|^\beta}\right)^{1/m'} \\ &\leq C_3\left(\int_{\mathbb{R}^N} \frac{\Phi_{m\alpha}(w)}{|x|^\beta}\right)^{1/r}\|w\|^\delta \leq C_4\|w\|^\delta,\end{aligned} \tag{4.22}$$

for all $w \in X$ with $\|w\| = \rho \leq \rho_0$, where $C_3, C_4 > 0$ are constants independent of w. Therefore, using (4.21), (4.22) and Remark 4.1 (a), we deduce that

$$\begin{aligned}\mathcal{J}(w) &\geq \frac{1}{p}\|w\|_{s_1,p}^p + \frac{1}{q}\|w\|_{s_2,q}^q - \frac{1}{p2^pC_p^p}\int_{\mathbb{R}^N}\frac{|w|^p}{|x|^\beta} - C_2\int_{\mathbb{R}^N}\frac{\Phi_\alpha(w)|w|^\delta}{|x|^\beta} \\ &\geq \frac{2^{1-p}}{p}\|w\|^p - \frac{1}{p2^pC_p^p}C_p^p\|w\|^p - C_4\|w\|^\delta = \frac{2^{-p}}{p}\|w\|^p - C_4\|w\|^\delta,\end{aligned}$$

where we have used the fact that $\|w\|_{s_1,p}, \|w\|_{s_2,q} \leq \|w\| < 1$ and C_i's are positive constants. By the fact that $\delta > p$, there exists $\eta > 0$ and ρ small enough such that $\mathcal{J}(w) \geq \eta$ for all $w \in X$ with $\|w\| = \rho$. □

The mountain pass lemma ensures the existence of a Cerami sequence at the mountain pass level, that is, there exists a sequence $\{u_n\} \subset X$ such that

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\| \to 0, \quad \text{as } n \to \infty,$$

where $c := \inf_{g\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(g(t))$ with

$$\Gamma = \{g \in C([0, 1], X) : g(0) = 0 \text{ and } \mathcal{J}(g(1)) < 0\}.$$

LEMMA 4.9. *Every solution u of* ($\mathcal{P}$) *is nonnegative and if $\{u_n\} \subset X$ is a Cerami sequence, then $\|u_n^-\| \to 0$, as $n \to \infty$.*

PROOF. The proof follows using the inequalities:

$$\begin{aligned}(u(x) - u(y))(u^-(x) - u^-(y)) &\leq -|u^-(x) - u^-(y)|^2,\\ |u(x) - u(y)| &\geq |u^-(x) - u^-(y)|.\end{aligned}$$

Applying these inequalities one can deduce that

$$\mathcal{A}_2(u, u^-) + \int_{\mathbb{R}^N} V(x)|u|^{q-2}uu^- \leq 0.$$

The rest of the proof follows similarly to [37, Lemma 2.9]. □

Following the standard procedure, we can prove the following result.

LEMMA 4.10. *Suppose the function f satisfies* (f1), (f3)′ *and* (AR). *Then any Cerami sequence of $\mathcal{J}$ at level c is bounded.*

LEMMA 4.11. *Let the function f satisfies* (f1), (f2)′, (f3)′ *and* (AR). *Then, for any Cerami sequence $\{u_n\} \subset X$ for $\mathcal{J}$ at the mountain pass level c, the following holds*

$$\sup_{n\in\mathbb{N}} \|u_n\| \in (0, 1),$$

if the constant C_δ, appearing in (f3)′, *is sufficiently large.*

PROOF. Fix $\psi \in C_c^\infty(\mathbb{R}^N)$ with $\|\psi\| > 0$. For $\delta > p$, set $K_0 := \inf_{\text{supp}(\psi)} K > 0$ and $S_\delta = \|\psi\|/\|\psi\|_\delta > 0$. Now, using (f3)′, for $l > 1$, we have

$$\begin{aligned}\mathcal{J}(l\psi) &= \frac{l^p}{p}\|\psi\|_{s_1,p}^p + \frac{l^q}{q}\|\psi\|_{s_2,q}^q - \int_{\mathbb{R}^N} \frac{K(x)F(l\psi(x))}{|x|^\beta}\\ &\leq \frac{l^p}{q}\|\psi\|^p - K_0C_\delta l^\delta S_\delta^{-\delta}\|\psi\|^\delta.\end{aligned}$$

Since $\delta > p$, there exists $l_\delta > 0$ sufficiently large such that $\mathcal{J}(l_\delta\psi) < 0$. Therefore,

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)) \leq \max_{t\in[0,1]} \mathcal{J}(tl_\delta\psi) \leq \sup_{t\in\mathbb{R}^+} \mathcal{J}(t\psi). \tag{4.23}$$

Consider $h\colon [0,\infty) \to \mathbb{R}$ defined by

$$h(t) := \frac{t^p}{p}\|\psi\|^p + \frac{t^q}{q}\|\psi\|^q - C(\delta)t^\delta\|\psi\|^\delta,$$

where $C(\delta) = K_0 C_\delta S_\delta^{-\delta}$. Then an easy computation yields

$$\begin{aligned}\sup_{t\geq 0} h(t) &\leq \sup_{t\geq 0}\bigg(\frac{t^p}{p}\|\psi\|^p - \frac{1}{2}\,C(\delta)t^\delta\|\psi\|^\delta\bigg) + \sup_{t\geq 0}\bigg(\frac{t^q}{q}\|\psi\|^q - \frac{1}{2}C(\delta)t^\delta\|\psi\|^\delta\bigg)\\ &= \bigg(\frac{2}{C(\delta)}\bigg)^{p/(\delta-p)}\bigg(\frac{1}{p}-\frac{1}{\delta}\bigg)\big(\|\psi\|^{p-\delta}\big)^{\delta/(\delta-p)}\\ &\quad + \bigg(\frac{2}{C(\delta)}\bigg)^{q/(\delta-q)}\bigg(\frac{1}{q}-\frac{1}{\delta}\bigg)\big(\|\psi\|^{q-\delta}\big)^{\delta/(\delta-q)}\\ &\leq \bigg(\frac{1}{q}-\frac{1}{\delta}\bigg)\frac{2^{p/(\delta-p)}}{\big(K_0 S_\delta^{-\delta}\big)^\gamma}\,\frac{\|\psi\|^{-\delta}}{C_\delta^{\delta/(\delta-q)}},\end{aligned}$$

where we assumed $C_\delta > 1$ and $\big(K_0 S_\delta^{-\delta}\big)^\gamma = \min\big\{\big(K_0 S_\delta^{-\delta}\big)^p, \big(K_0 S_\delta^{-\delta}\big)^q\big\}$. Therefore, from (4.23), we observe that

$$c \leq \sup_{t\geq 0}\mathcal{J}(t\psi) \leq \sup_{t\geq 0} h(t) \leq \bigg(\frac{1}{q}-\frac{1}{\delta}\bigg)\frac{2^{p/(\delta-p)}}{\big(K_0 S_\delta^{-\delta}\big)^\gamma}\,\frac{\|\psi\|^{-\delta}}{C_\delta^{\delta/(\delta-q)}}. \tag{4.24}$$

By (AR) and the fact that $\{u_n\}$ is a Cerami sequence, we get

$$\begin{aligned} c &= \lim_{n\to\infty}\bigg(\mathcal{J}(u_n) - \frac{1}{\nu}\mathcal{J}'(u_n)u_n\bigg)\\ &\geq \limsup_{n\to\infty}\bigg(\bigg(\frac{1}{p}-\frac{1}{\nu}\bigg)\|u_n\|_{s_1,p}^p + \bigg(\frac{1}{q}-\frac{1}{\nu}\bigg)\|u_n\|_{s_2,q}^q\bigg)\\ &\geq \limsup_{n\to\infty}\bigg(\frac{1}{p}-\frac{1}{\nu}\bigg)\|u_n\|_{s,\gamma}^\gamma,\end{aligned}$$

where $(s,\gamma) \in \{(s_1,p),(s_2,q)\}$. Then, using (4.24), we obtain

$$\limsup_{n\to\infty}\|u_n\|_{s,\gamma}^\gamma \leq \frac{p\nu}{\nu-p}c \leq \frac{p\nu}{\nu-p}\bigg(\frac{1}{q}-\frac{1}{\delta}\bigg)\frac{2^{p/(\delta-p)}}{\big(K_0 S_\delta^{-\delta}\big)^\gamma}\,\frac{\|\psi\|^{-\delta}}{C_\delta^{\delta/(\delta-q)}} < \frac{1}{2^\gamma},$$

provided C_δ is sufficiently large. Then, the proof of the lemma follows by using the definition of $\|u_n\|$. □

In the subcritical case, we prove the boundedness of Cerami sequences. The proof differs from the critical case due to absence of Ambrosetti–Rabinowitz type condition for this case.

Lemma 4.12. *Suppose that* (f1)–(f3) *hold. Then, any Cerami sequence of* $\mathcal{J}$ *at the mountain pass level* c *is bounded.*

Proof. Let $\{v_n\} \subset X$ be a Cerami sequence of $\mathcal{J}$ at level c. Then, as in the proof of lemma 4.11, there exists $t_n \in [0,1]$ such that

$$\mathcal{J}(t_n v_n) = \max_{t\in[0,1]} \mathcal{J}(tv_n). \tag{4.25}$$

We claim that the sequence $\{\mathcal{J}(t_n v_n)\}$ is bounded. The claim is obvious if $t_n = 0$ or 1, therefore we assume $t_n \in (0,1)$. Also, we assume $v_n \geq 0$. Setting

$$H(t) := tf(t) - pF(t) \quad \text{for } t \in \mathbb{R},$$

and since $t^{1-p}f(t)$ is nondecreasing and differentiable (due to (f1) and (f3)), we get that H is nondecreasing in $\mathbb{R}$. Now, from (4.25), we have

$$\frac{d}{dt}\mathcal{J}(tv_n)\bigg|_{t=t_n} = 0,$$

and hence

$$\begin{aligned} p\mathcal{J}(t_n v_n) &= \left(\frac{p}{q} - 1\right) t_n^q \|v_n\|_{s_2,q}^q + \int_{\mathbb{R}^N} \frac{K(x)H(t_n v_n)}{|x|^\beta} \\ &\leq \left(\frac{p}{q} - 1\right) \|v_n\|_{s_2,q}^q + \int_{\mathbb{R}^N} \frac{K(x)H(v_n)}{|x|^\beta} \\ &= p\mathcal{J}(v_n) - \mathcal{J}'(v_n)v_n = pc + o_n(1), \end{aligned}$$

this proves the claim. To prove the lemma, on the contrary, we assume that up to a subsequence $\|v_n\| \to \infty$ as $n \to \infty$ and $\|v_n\| \geq 1$ for all $n \in \mathbb{N}$. Then, there exists $w \in X$ such that $w_n \rightharpoonup w$ weakly in X, where $w_n = v_n/\|v_n\|$. We claim that $w = 0$ almost everywhere in $\mathbb{R}^N$. Indeed, since $\mathcal{J}(v_n) = c + o_n(1)$ and $\|v_n\| \to \infty$, we get

$$\frac{1}{p}\frac{\|v_n\|_{s_1,p}^p}{\|v_n\|^p} + \frac{1}{q}\frac{\|v_n\|_{s_2,q}^q}{\|v_n\|^p} - \int_{\mathbb{R}^N} \frac{K(x)F(v_n)}{\|v_n\|^p|x|^\beta} = o_n(1). \tag{4.26}$$

Since $\lim_{t\to\infty} t^{-p}F(t) = \infty$, for every $\tau > 0$, there exists $\xi > 0$ such that

$$F(t) \geq \tau|t|^p \quad \text{for all } |t| \geq \xi.$$

Therefore, from (4.26) and noting that $q < p$, we obtain

$$o_n(1) + \frac{1}{p} \geq \int_{|v_n|\geq\xi} \frac{K(x)F(v_n)w_n(x)^p}{|v_n(x)|^p|x|^\beta} \geq \tau \int_{\mathbb{R}^N} \frac{K(x)w_n(x)^p}{|x|^\beta}\chi_{\{|v_n|\geq\xi/\|v_n\|\}}.$$

By Fatou's lemma, for all $\tau > 0$, we deduce that

$$\tau\int_{\mathbb{R}^N} \frac{K(x)w(x)^p}{|x|^\beta} \leq \frac{1}{p},$$

which implies $w = 0$ almost everywhere in $\mathbb{R}^N$. Let $T > 0$, then there exists $n_T \in \mathbb{N}$ such that for all $n \geq n_T$, $T\|v_n\|^{-1} \in (0,1)$. Now, from (4.25) and the fact that $\|w_n\| \leq 1$ (follows from $w = 0$), we get

$$\begin{aligned}\mathcal{J}(t_n v_n) \geq \mathcal{J}(T w_n) &= \frac{T^p}{p}\|w_n\|^p_{s_1,p} + \frac{T^q}{q}\|w_n\|^q_{s_2,q} - \int_{\mathbb{R}^N} \frac{K(x)F(Tw_n)}{|x|^\beta} \\ &\geq \frac{2^{1-p}\,T^\gamma}{p}\|w_n\|^p - \int_{\mathbb{R}^N} \frac{K(x)F(Tw_n)}{|x|^\beta},\end{aligned}$$

where $T^\gamma = \min\{T^p, T^q\}$. Then, by the compactness Lemma 4.6, we have

$$\int_{\mathbb{R}^N} \frac{K(x)F(Tw_n)}{|x|^\beta} \to 0, \quad \text{as } n \to \infty.$$

Thus,

$$\liminf_{n\to\infty} \mathcal{J}(t_n v_n) \geq \frac{2^{1-p}T^\gamma}{p},$$

which is a contradiction, if we choose T such that $T = \big(2^p p \sup_n\{\mathcal{J}(t_n v_n)\}\big)^{1/\gamma}$. This proves the lemma. □

5. Proof of the main result

The functional $\mathcal{J}$ satisfies mountain pass geometry in both the cases. Therefore, there exist Cerami sequences $\{u_n\} \subset X$ and $\{v_n\} \subset X$ in the subcritical and critical cases, respectively. Furthermore, $\{u_n\}$ and $\{v_n\}$ are bounded in X. Therefore, up to a subsequence $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in X, for some $u, v \in X$.

5.1. The subcritical case. By the compactness Lemma 4.6, we see that

$$\int_{\mathbb{R}^N} \frac{K(x)f(u_n)}{|x|^\beta}(u_n - u) \to 0 \quad \text{as } n \to \infty.$$

Moreover, since $\langle \mathcal{J}'(u_n), u_n - u\rangle \to 0$ as $n \to \infty$, it follows that

$$\begin{aligned}&\mathcal{A}_1(u_n, u_n - u) + \mathcal{A}_2(u_n, u_n - u) \\ &\qquad + \int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n\big)(u_n - u) = o_n(1).\end{aligned}$$

On the other hand for fixed $u \in X$, it is easy to observe that $\Theta_{u,p} + \Theta_{u,q} \in X'$, where

$$\Theta_{u,p}(v) := \mathcal{A}_1(u, v) + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx$$

for all $v \in X$ and $\Theta_{u,q}$ is analogously defined. Therefore, using the fact that $u_n \rightharpoonup u$ weakly in X, we get

$$\mathcal{A}_1(u, u_n - u) + \mathcal{A}_2(u, u_n - u) + \int_{\mathbb{R}^N} V(x)\big(|u|^{p-2}u + |u|^{q-2}u\big)(u_n - u) = o_n(1).$$

Coupling these, we obtain

$$\begin{aligned}(5.1)\qquad \mathcal{A}_1(u_n, u_n - u) &- \mathcal{A}_1(u, u_n - u) \\ &+ \int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u) \\ &+ \mathcal{A}_2(u_n, u_n - u) - \mathcal{A}_2(u, u_n - u) \\ &+ \int_{\mathbb{R}^N} V(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n - u) = o_n(1).\end{aligned}$$

Now, we consider the cases when $q \geq 2$ and $1 < q < 2$ (note that $p \geq 2$).

Case 1. $q \geq 2$. Using the inequality $|a - b|^l \leq 2^{l-2}(|a|^{l-2}a - |b|^{l-2}b)(a - b)$ for $a, b \in \mathbb{R}^n$ and $l \geq 2$, from (5.1), it follows that

$$[u_n - u]^p_{s_1,p} + \int_{\mathbb{R}^N} V(x)|u_n - u|^p + [u_n - u]^q_{s_2,q} + \int_{\mathbb{R}^N} V(x)|u_n - u|^q \leq o_n(1),$$

that is $\|u_n - u\|^p_{s_1,p} + \|u_n - u\|^q_{s_2,q} = o_n(1)$, hence $u_n \to u$ in X.

Case 2. $1 < q < 2$. As we know that for $a, b \in \mathbb{R}^n$ and $1 < m < 2$, there exists $C_m > 0$ a constant such that

$$|a - b|^m \leq C_m\big(\big(|a|^{m-2}a - |b|^{m-2}b\big)(a - b)\big)^{m/2}(|a|^m + |b|^m)^{(2-m)/2}.$$

Set $a = u_k(x) - u_k(y)$, $b = u(x) - u(y)$ and then using Hölder inequality, we deduce that

$$[u_n - u]^q_{s_2,q} \leq C\big(\mathcal{A}_2(u_n, u_n - u) - \mathcal{A}_2(u, u_n - u)\big)^{q/2}\big([u_n]^q_{s_2,q} + [u]^q_{s_2,q}\big)^{(2-q)/2}$$

and boundedness of $\{u_n\}$ in X, implies

$$[u_n - u]^2_{s_2,q} \leq C\big(\mathcal{A}_2(u_n, u_n - u) - \mathcal{A}_2(u, u_n - u)\big).$$

Therefore, using (5.1) and proceeding similarly as in the previous case, we obtain $u_n \to u$ in $\widetilde{W}^{s_1,p}_V(\mathbb{R}^N)$ as well as in $\widetilde{W}^{s_2,q}_V(\mathbb{R}^N)$, which gives us the required strong convergence of u_n to u in X. Using the fact that $c > 0$ and strong convergence, we get that $u \not\equiv 0$. By Lemma 4.9, u is a nontrivial nonnegative solution of $(\mathcal{P})$.

5.2. The critical case. We observe that if we choose $C_\delta > 0$ such that Lemma 4.11 is satisfied, then the compactness results of Corollary 4.7 hold. Now, we can proceed similarly to prove that $v_n \to v$ in X and $v \not\equiv 0$, hence v is a nontrivial weak solution of problem $(\mathcal{P})$. □

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