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Multi-bump solutions for quasilinear elliptic equations with variable exponents and critical growth in $\mathbb{R}^N$

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In this paper, we are concerned with the existence of multi-bump solutions for the following class of $p(x)$-Laplacian equations:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + (\lambda V(x) + Z(x))|u|^{p(x)-2}u \\ \qquad = \alpha f(x,u) + u^{q(x)-1}, \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \quad u > 0, \end{cases}$$

where $\alpha > 0$ and $\lambda \geq 1$ are two real parameters, the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth, $N > p_+ = \sup_{x\in\mathbb{R}^N} p(x)$, the exponent $q(x)$ can be equal to the critical exponent $p^*(x) = \frac{Np(x)}{N-p(x)}$ at some points of $\mathbb{R}^N$ including at infinity and the potentials V, $Z : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some conditions. We show that if the zero set of V has several isolated connected components $\Omega_1, \ldots, \Omega_k$ such that the interior of Ω_i is not empty and $\partial\Omega_i$ is smooth, then for $\lambda > 0$ large enough there exists, for any non-empty subset $\Gamma \subset \{1, \ldots, k\}$, a bump solution trapped in a neighborhood of $\bigcup_{j\in\Gamma} \Omega_j$. The proofs are based on variational and topological methods.

Keywords: $p(x)$-Laplacian equation; variable exponent Sobolev spaces; critical exponent; multi-bump solutions; variational method.

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1. Introduction and Main Results

In this paper, we study the existence of multi-bump positive solutions for the following class of $p(x)$-Laplacian equations:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + (\lambda V(x) + Z(x))|u|^{p(x)-2}u \\ \qquad = \alpha f(x,u) + u^{q(x)-1}, \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \quad u > 0, \end{cases} \tag{1.1}$$

where $\alpha > 0, \lambda \geq 1$ are two real parameters, $V,\ Z : \mathbb{R}^N \to \mathbb{R}$ are continuous functions with $V \geq 0$, $p, q : \mathbb{R}^N \to (1, \infty)$ are two log-Hölder continuous functions, $N > p_+ = \sup_{x\in\mathbb{R}^N} p(x)$, the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth, the exponent $q(x)$ can be equal to the critical exponent $p^*(x) = \frac{Np(x)}{N-p(x)}$ at some points of $\mathbb{R}^N$ including at infinity.

The study of various mathematical problems with variable exponents has received considerable attention in recent years because it appears in a lot of applications, such as the electrorheological fluids [38], image processing [11], elastic mechanics [43] and the references therein. Besides the importance in applications, the variable exponent problems are also very interesting from the mathematical point of view, because they involve a lot of difficulties, for example, the variable exponent problems possess more complicated nonlinearities than the constant exponent problems. We may refer to the review papers [14, 36, 40] for the advances and the references in this area, to [13, 19, 22, 27, 37] for the variable exponent Lebesgue–Sobolev spaces, and to [13, 17, 23–26, 29, 35, 37, 39, 42] for the $p(x)$-Laplacian equations and the corresponding variational problems. We also refer to the pioneering regularity results and qualitative properties of solutions established by Mingione [30, 31] and to the paper by Pucci and Zhang [34] dedicated to related but general critical equations. The interest for nonlinear problems with critical exponent started after the seminal paper by Brezis and Nirenberg [10].

If $p(x) \equiv 2$, problem (1.1) reduces to the following one:

$$\begin{cases} -\Delta u + (\lambda V(x) + Z(x))u = Q(u), \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0. \end{cases} \tag{1.2}$$

In recent years, many researchers considered the existence and multiplicity of positive solutions for problem (1.2) under various assumptions on the potential and the nonlinearity. For example, in the case when the potential $\lambda V(x) + Z(x)$ is coercive, Miyagaki [32] proved the existence of a positive solution to problem (1.2). For the case when the potential $\lambda V(x)+Z(x)$ is 1-periodic, Alves *et al.* [2] gave the existence of positive solutions. If $\lambda V(x) + Z(x)$ is radial, Alves *et al.* [3] also established the existence of a positive solution. The papers cited above proved only the existence of positive solutions, while for the multiplicity of solutions for problem (1.2) we may refer to [7, 8, 12].

In [15], Ding and Tanaka considered problem (1.2) for the case $Q(u) = u^{p-1}$, where $2 < p < 2^*$, $N \geq 3$. If Ω has k connected components, the authors showed that problem (1.2) has at least $2^k - 1$ solutions, for large λ, established the existence of solutions called multi-bumps. For the case $Q(u) = \alpha u^{p-1} + u^{2^*-1}$, where $\alpha > 0$ and $2 < p < 2^*$, $N \geq 3$, in [4], Alves *et al.* established the similar results. For the case the nonlinearity $Q(u)$ has the exponential critical growth in $\mathbb{R}^2$, in [6], Alves and Souto also gave the existence of multi-bump solutions.

If $p(x) \equiv p$, $2 \leq p < N$ in problem (1.1), when the nonlinear term has subcritical growth, Alves [1] considered the existence of multi-bump solutions. Since the p-Laplacian is not linear, and some properties that occur for Laplacian operator are not standard that they hold for the general case $p \geq 2$, therefore, Alves used different approach in some estimates. Recently, Alves and Ferreira [5] extended the results in [1] to the $p(x)$-Laplace operator. The main difference is related to the fact that for equations involving the $p(x)$-Laplacian operator it is not clear that Moser's iteration method is a good tool to get the estimates for the L^∞-norm. The authors adapted some ideas in [18, 21] to get these estimates.

Motivated by the papers [4, 5], the main goal of this paper is to investigate the existence of multi-bump solutions to problem (1.1). However, since our problem has the variable exponents growth, some estimates for this problem are very delicate and different from those used in the constant exponents problems. Also for this reason, the classical Moser's iteration is not a good tool to obtain the estimates for L^∞-norm. On the other hand, our nonlinearity is critical growth and some arguments developed in [5] cannot be applied. The reader is invited to see that the way how we attach these problems in Sec. 3. As far as we know, there is no result on multi-bump solutions for $p(x)$-Laplace equations with critical growth.

We make the following assumptions on $p(x)$, $q(x)$, $V(x)$, $Z(x)$ and $f(x,u)$ throughout this paper:

- (p) $1 < p_- := \inf_{\mathbb{R}^N} p(x) \leq \sup_{\mathbb{R}^N} p(x) := p_+ < N$.
- (q_1) $1 < q(x) \leq p^*(x) := \frac{Np(x)}{N-p(x)}$, the critical set $\mathcal{A} := \{x \in \mathbb{R}^N : q(x) = p^*(x)\}$ can be non-empty. Moreover, $q(x)$ is critical at infinity in the sense that $q(\infty) = p(\infty)^*$.
- (q_2) $q \ll p^*$, on $\partial\Omega$, that is, $\inf_{x\in\partial\Omega}(p^*(x) - q(x)) > 0$.
- (V_1) The potential well $\Omega = \operatorname{int} V^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and $\overline{\Omega} = V^{-1}(0)$, Ω can be decomposed in k connected components $\Omega_1, \ldots, \Omega_k$ with $\operatorname{dist}(\Omega_i, \Omega_j) > 0$, $i \neq j$.
- (V_2) There exists $M > 0$ such that
$$\lambda V(x) + Z(x) \geq M, \quad \forall\, x \in \mathbb{R}^N, \;\; \lambda \geq 1.$$
- (V_3) There exists $K > 0$ such that
$$|Z(x)| \leq K, \quad \forall\, x \in \mathbb{R}^N.$$

(f_1) $f(x,t) = o(|t|^{p(x)-1})$ as $t \to 0$, uniformly in x.

(f_2) We have

$$\limsup_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{m(x)-1}} < \infty, \quad \text{uniformly in } x \in \mathbb{R}^N,$$

where $m \in C^0(\mathbb{R}^N, \mathbb{R})$ with $p_+ < m_-$, $m_+ \le q_+$ and $m \ll p^* = \frac{Np(x)}{N-p(x)}$. Here, the notation "$m \ll p^*$" means that $\inf_{x\in\mathbb{R}^N}(p^*(x) - m(x)) > 0$.

(f_3) There is a positive constant β with $p_+ < \beta \le \min\{m_-, q_-\}$ such that

$$0 < \beta F(x,t) \le tf(x,t), \quad \forall x \in \mathbb{R}^N, \;\; t > 0,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(f_4) The function $\frac{f(x,t)}{t^{p_+-1}}$ is strictly increasing in $t > 0$, for each $x \in \mathbb{R}^N$.

A typical example of nonlinear term f verifying $(f_1) - (f_4)$ is

$$f(x,t) = |t|^{m(x)-2}t, \quad x \in \mathbb{R}^N \text{ and } \forall t \in \mathbb{R},$$

where $p_+ < \beta \le \min\{m_-, q_-\}$ and $m \ll p^*$.

The main result to be proved in the following theorem.

Theorem 1.1. *Let* (p), (q_1)–(q_2), (V_1)–(V_3), *and* (f_1)–(f_4) *hold. Then, for any non-empty subset* Γ *of* $\{1, 2, \ldots, k\}$, *there exist constants* $\alpha^* > 0$ *and* $\lambda^* = \lambda^*(\alpha^*)$ *such that, for all* $\alpha \ge \alpha_*$ *and* $\lambda \ge \lambda_*$, *problem* (1.1) *has a family* $\{u_\lambda\}$ *of positive solutions which depend on* α *verifying*: *for any sequence* $\lambda_n \to \infty$, *we can extract a subsequence* λ_{n_i} *such that* $u_{\lambda_{n_i}}$ *converges strongly in* $W^{1,p(x)}(\mathbb{R}^N)$ *to a function* u *which satisfies* $u(x) = 0$ *for* $x \notin \Omega_\Gamma$ *and the restriction* $u|_{\Omega_j}$ *is a least energy solution of*

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + Z(x)|u|^{p(x)-2}u = \alpha f(x,u) + u^{q(x)-1}, & x \in \Omega_j, \\ u > 0, \quad x \in \Omega_j, \\ u|_{\partial\Omega_j} = 0, \end{cases}$$

for all $j \in \Gamma$ *and* $\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$.

Corollary 1.2. *Under the assumptions of Theorem* 1.1, *there exist constants* $\alpha^* > 0$ *and* $\lambda^* = \lambda^*(\alpha^*)$ *such that, for all* $\alpha \ge \alpha^*$ *and* $\lambda \ge \lambda^*$, *problem* (1.1) *has at least* $2^k - 1$ *positive solutions.*

We refer to Brezis [9] for some of the main abstract tools used in this paper.

Notation. Throughout this paper, we use the following notations:

- If g is a measurable function, the integral $\int_{\mathbb{R}^N} g(z)dz$ will be denoted by $\int g(z)dz$.
- C denotes any positive constant, whose value is not relevant.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.

2. Preliminaries

Let $p \in L^\infty(\mathbb{R}^N)$ and $p_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^N} p(x) \geq 1$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \,|\, u \text{ is a measurable real-valued function and } \int |u|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} = |u|_{p(x)} = \inf\left\{ \lambda > 0 : \int \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

If $p_- > 1$, the spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are all separable and reflexive Banach spaces. For the basic properties of these spaces, we refer to [13, 19, 27, 37].

For problem (1.1), we shall work in the following subspace of $W^{1,p(x)}(\mathbb{R}^N)$ given by

$$E_\lambda = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) : \int V(x)|u|^{p(x)} dx < \infty \right\}$$

endowed with the norm

$$\|u\|_\lambda = \inf\left\{ \lambda > 0 : \int \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + (\lambda V(x) + Z(x)) \left|\frac{u}{\lambda}\right|^{p(x)} \right) dx \leq 1 \right\}.$$

For $\lambda \geq 1$, we can easily see that E_λ is a Banach space, $E_\lambda \subset W^{1,p(x)}(\mathbb{R}^N)$ and the following inequalities hold:

$$\|u\|_\lambda^{p^-} \leq \varrho_\lambda(u) \leq \|u\|_\lambda^{p^+}, \quad \text{if } \|u\|_\lambda \geq 1,$$

$$\|u\|_\lambda^{p^+} \leq \varrho_\lambda(u) \leq \|u\|_\lambda^{p^-} \quad \text{if } \|u\|_\lambda \leq 1,$$

where $\varrho_\lambda(u) = \int (|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)}) dx$. In particular, for a sequence (u_n) in E_λ,

$$\|u_n\|_\lambda \to 0 \Leftrightarrow \varrho_\lambda(u_n) \to 0, \quad \text{and}$$

$$(u_n) \text{ is bounded in } E_\lambda \Leftrightarrow \varrho_\lambda(u_n) \text{ is bounded in } \mathbb{R}.$$

In view of (V_2), for any open set $\Theta \subset \mathbb{R}^N$ and $u \in E_\lambda$ with $\lambda \geq 1$, we have

$$\begin{aligned}\varrho_{\lambda,\Theta}(u) &= \int_\Theta (|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)})dx \\ &\geq M \int_\Theta |u|^{p(x)}dx = M\varrho_{p(x),\Theta}(u).\end{aligned}$$

The following property is an immediate consequence of the above observation.

Lemma 2.1. *There exist $\delta, \nu > 0$ with $\delta \approx 1$ and $\nu \approx 0$ such that for any open set $\Theta \subset \mathbb{R}^N$*

$$\delta\varrho_{\lambda,\Theta}(u) \leq \varrho_{\lambda,\Theta}(u) - \nu\varrho_{p(x),\Theta}(u), \quad \forall\, u \in E_\lambda, \ \ \lambda \geq 1.$$

Now, we list more facts which will be used later.

Lemma 2.2 (See [19, 27]). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{q(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{q(x)}(\mathbb{R}^N)$,*

$$\int |uv|dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

Lemma 2.3 (See [16]). *Let $\Omega \subset \mathbb{R}^N$ an open domain with the cone property, $p : \overline{\Omega} \to \mathbb{R}$ satisfying (p) and $m \in L^\infty(\Omega)$ and $m_- \geq 1$.*

(i) *If p is Lipschitz continuous and $p \leq m \leq p^*$, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$ is continuous.*
(ii) *If Ω is a bounded, p is continuous and $m \ll p^*$, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$ is compact.*

Lemma 2.4 (See [20]). *Assume that $\Omega \subset \mathbb{R}^N$ is measurable, let (u_n) be a bounded sequence in $L^{p(x)}(\Omega)$ and $u_n \to u \in L^{p(x)}(\Omega)$ a.e. on Ω. If $p(x)$ satisfies (p), then*

$$\lim_{n\to\infty} \int |u_n|^{p(x)} - |u_n - u|^{p(x)} = \int |u|^{p(x)}dx.$$

Lemma 2.5 (See [39]). *Assume that ∞ is critical in the sense that $q(\infty) = p(\infty)^*$. Let $(u_n) \subset D^{1,p(x)}(\mathbb{R}^N)$ be a weakly convergent to $u \in D^{1,p(x)}(\mathbb{R}^N)$. Then there exist two bounded measures μ and ν, an at most enumerable set of indices I, points $x_i \in \mathcal{A}$ (the critical set defined in (q_1)), and positive real numbers μ_i, ν_i, $i \in I$, such that the following convergence hold weakly in the sense of measures,*

$$\begin{aligned}&|\nabla u_n|^{p(x)}dx \rightharpoonup \mu \geq |\nabla u|^{p(x)}dx + \sum \mu_i\delta_{x_i}, \\ &|u_n|^{q(x)}dx \rightharpoonup \nu := |u|^{q(x)}dx + \sum \nu_i\delta_{x_i}, \\ &S_{x_i}\nu_i^{\frac{1}{p(x_i)^*}} \leq \mu_i^{\frac{1}{p(x_i)}}, \quad \textit{for all } i \in I,\end{aligned}$$

where S_{x_i} is the localized Sobolev constant at the point x_i defined as follows:

$$S_{x_i} = \lim_{\epsilon \to 0} S(p(\cdot), q(\cdot), B_{x_i}(\epsilon)) = \sup_{\epsilon > 0} S(p(\cdot), q(\cdot), B_{x_i}(\epsilon)), \tag{2.1}$$

where

$$S(p(\cdot), q(\cdot), B_{x_i}(\epsilon)) = \inf_{u \in W_0^{1,p(\cdot)}(B_{x_i}(\epsilon)), u \neq 0} \frac{|\nabla u|_{L^{p(x)}(B_{x_i}(\epsilon))}}{|u|_{L^{q(x)}(B_{x_i}(\epsilon))}}$$

and $B_{x_i}(\epsilon)$ be a ball centered at x_i with small radius $\epsilon > 0$. Moreover, if we define

$$\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} |u_n|^{q(x)} dx,$$

$$\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} |\nabla u_n|^{p(x)} dx,$$

then

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} dx = \mu(\mathbb{R}^N) + \mu_\infty,$$

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{q(x)} dx = \nu(\mathbb{R}^N) + \nu_\infty,$$

$$S_\infty \nu_\infty^{\frac{1}{q(\infty)}} \leq \mu_\infty^{\frac{1}{p(\infty)}},$$

where S_∞ is the localized Sobolev constant at infinity defined as follows:

$$S_\infty = \lim_{R \to +\infty} S(p(\cdot), q(\cdot), \mathbb{R}^N \backslash B_R) = \sup_{R > 0} S(p(\cdot), q(\cdot), \mathbb{R}^N \backslash B_R),$$

with

$$S(p(\cdot), q(\cdot), \mathbb{R}^N \backslash B_R) = \inf_{u \in W_0^{1,p(\cdot)}(\mathbb{R}^N \backslash B_R), u \neq 0} \frac{|\nabla u|_{L^{p(x)}(\mathbb{R}^N \backslash B_R)}}{|u|_{L^{q(x)}(\mathbb{R}^N \backslash B_R)}}.$$

Lemma 2.6. *The infimum $S = \inf_{x \in \mathcal{A} \cup \{\infty\}} S_x$ is attained at some point of $\mathcal{A} \cup \{\infty\}$.*

Proof. The proof of the lemma can be found in [39], but for reader's convenience we include it. Here, we first prove that the function $x \in \mathcal{A} \to S_x$ is lower semi-continuous. Assume $x_0 \in \mathcal{A}$, $(x_n) \subset \mathcal{A}$ such that $x_n \to x_0$ and fix some $\epsilon > 0$. There exists $N(\epsilon) \in \mathbb{N}$ such that $B_{x_n}(\frac{\epsilon}{3}) \subset B_{x_0}(\epsilon)$ for $n \geq N(\epsilon)$. It follows that

$$S(p(\cdot), q(\cdot), B_\epsilon(x_0)) \leq S(p(\cdot), q(\cdot), B_{\frac{\epsilon}{3}}(x_n)) \leq S_{x_n}$$

for $n \geq N(\epsilon)$. Then $\liminf_{n \to +\infty} S_{x_n} \geq S(p(\cdot), q(\cdot), B_\epsilon(x_0))$ for any $\epsilon > 0$. Letting $\epsilon \to 0$, it yields that $\liminf_{n \to +\infty} S_{x_n} \geq S_{x_0}$.

To prove the lemma, moreover, we need to show that this function is also lower semi-continuous at infinity in the sense that for any sequence $(x_n) \subset \mathcal{A}$ such that $|x_n| \to +\infty$, there holds $\liminf_{n \to +\infty} S_{x_n} \geq S_\infty$. Fix some $R > 0$ and $N_0 \in \mathbb{N}$ such that $|x_n| \geq R+1$. Then, for $n \geq N_0$, $B_{x_n}(\epsilon) \subset \mathbb{R}^N \backslash B_R$ for any $\epsilon < 1$. It follows that for such n and ϵ, $S(p(\cdot), q(\cdot), B_{x_n}(\epsilon)) \geq S(p(\cdot), q(\cdot), \mathbb{R}^N \backslash B_R)$. Taking the limit in ϵ and then in n gives $\liminf_{n\to+\infty} S_{x_n} \geq S(p(\cdot), q(\cdot), \mathbb{R}^N \backslash B_R)$. Taking the limit $R \to +\infty$, we obtain the desired result. □

Remark 2.1. From Lemma 2.2, we know that the infimum $S = \inf_{x\in\mathcal{A}\cup\{\infty\}} S_x$ is attained at some point of $\mathcal{A} \cup \{\infty\}$. So, $S > 0$. Moreover, it is easy to see that $\inf_{x\in\mathcal{A}} S_x \geq S > 0$.

3. A Modified Problem

Since we intend to find positive solutions, throughout this paper we assume that

$$f(x,t) = 0, \quad \forall\, x \in \mathbb{R}^N, \ \ \forall\, t \leq 0.$$

The weak solutions of problem (1.1) are the positive critical points of the functional $J_\lambda : E_\lambda \to \mathbb{R}$ given by

$$\begin{aligned} J_\lambda(u) = &\int \frac{1}{p(x)}(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)})dx \\ &- \alpha \int F(x,u)dx - \int \frac{|u|^{q(x)}}{q(x)}, \end{aligned}$$

where $F(x,t) = \int_0^t f(x,s)ds$.

In order to overcome the difficulties caused by the critical growth of the nonlinearity and the unboundedness of the domain, in this section, we first modify the functional J_λ by adapting the ideas developed in del Pino and Felmer [12] (see also [5]), then we show that, under some energy level, the modified functional satisfies the Palais–Smale (P.S. for shortness) condition.

By (f_1) and (f_2), we have the following elementary observation:

$$f(x,t) \leq \epsilon|t|^{p(x)-1} + C_\epsilon|t|^{m(x)-1}, \quad \forall\, x \in \mathbb{R}^N, \ \ t \in \mathbb{R}, \tag{3.1}$$

and, consequently

$$F(x,t) \leq \epsilon|t|^{p(x)} + C_\epsilon|t|^{m(x)}, \quad \forall\, x \in \mathbb{R}^N, \ \ t \in \mathbb{R}. \tag{3.2}$$

Moreover, since for each $\varsigma > 0$ fixed, we consider the function $a : \mathbb{R}^N \to \mathbb{R}$ given by

$$a(x) = \min\left\{a > 0 : \frac{\alpha f(x,a) + a^{q(x)-1}}{a^{p(x)-1}} = \varsigma\right\}. \tag{3.3}$$

From (f_1) and $q_- > p_+$, it follows that $a_- = \inf_{x\in\mathbb{R}^N} a(x) > 0$.

For technical reasons, we define the function $\tilde{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{f}(x,t) = \begin{cases} 0, & t \leq 0, \\ \alpha f(x,t) + t^{q(x)-1}, & 0 \leq t \leq a(x), \\ \varsigma t^{p(x)-1}, & t \geq a(x), \end{cases}$$

which satisfies the following inequality:

$$\tilde{f}(x,t) \le \varsigma |t|^{p(x)-1}, \quad \forall\, x \in \mathbb{R}^N, \;\; t \in \mathbb{R}. \tag{3.4}$$

Thus

$$\tilde{f}(x,t)t \le \varsigma |t|^{p(x)}, \quad \forall\, x \in \mathbb{R}^N, \;\; t \in \mathbb{R}, \tag{3.5}$$

and

$$\tilde{F}(x,t) \le \frac{\varsigma}{p(x)} |t|^{p(x)}, \quad \forall\, x \in \mathbb{R}^N, \;\; t \in \mathbb{R}, \tag{3.6}$$

where $\widetilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds$.

In virtue of (V_1), for each $j \in \{1, \ldots, k\}$, we can choose a bounded open set Ω'_j with smooth boundary such that

$$\overline{\Omega_j} \subset \Omega'_j, \quad \text{and} \quad \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset, \quad \text{for } i \ne j. \tag{3.7}$$

From now on, we fix a non-empty subset $\Gamma \subset \{1, \ldots, k\}$ and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \quad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j, \quad \chi_\Gamma(x) := \begin{cases} 1, & \text{if } x \in \Omega'_\Gamma, \\ 0, & \text{if } x \notin \Omega'_\Gamma, \end{cases}$$

and the function

$$g(x,t) = \chi_\Gamma(x)(\alpha f(x,t) + t^{q(x)-1}) + (1 - \chi_\Gamma(x))\tilde{f}(x,t) \tag{3.8}$$

and

$$G(x,t) = \int_0^t g(x,s)ds = \chi_\Gamma(x) \left(\alpha F(x,t) + \frac{t^{q(x)}}{q(x)} \right) + (1 - \chi_\Gamma(x))\tilde{F}(x,t), \tag{3.9}$$

and the auxiliary problem

$$(A_\lambda) \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + (\lambda V(x) + Z(x))|u|^{p(x)-2}u = g(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

We remark that $g(x,t) = \alpha f(x,t) + t^{q(x)-1}$ for $0 \le t \le a(x)$ and if u_λ is a solution for (A_λ) satisfying

$$0 < u_\lambda(x) \le a_-, \quad \forall\, x \in \mathbb{R}^N \backslash \Omega'_\Gamma,$$

then it is a solution for the original problem (1.1).

Note that, using (f_1)–(f_4), it is easy to check that

(g_1) $g(x,t) = o(|t|^{p(x)-1})$ as $t \to 0$, uniformly in x.

(g_2) $g(x,t) \le \alpha f(x,t) + t^{q(x)-1}$, for all $t > 0, x \in \mathbb{R}^N$.

(g_3) (i) $0 < \beta G(x,t) \le t g(x,t)$, $\forall\, x \in \Omega'_\Gamma$, $t > 0$;
(ii) $0 \le G(x,t) \le \frac{\varsigma}{p(x)} t^{p(x)}$ and $0 \le t g(x,t) \le \varsigma t^{p(x)}$, $\forall x \in \mathbb{R}^N \backslash \Omega'_\Gamma$, $t > 0$.

(g_4) The function $\frac{g(x,t)}{t^{p(x)-1}}$ is non-decreasing in $t > 0$, for each $x \in \mathbb{R}^N$ and is strictly increasing in $t > 0$, for each $x \in \Omega'_\Gamma$.

Associated with problem (A_λ), we have the energy functional $\Phi_\lambda : E_\lambda \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \int \frac{1}{p(x)}(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)})dx - \int G(x,u)dx,$$

which is $C^1(E_\lambda, \mathbb{R})$ and satisfies the (PS) condition under some energy level, whereas J_λ does not necessarily satisfy this condition. In this way, the mountain pass level is a critical value for Φ_λ.

Lemma 3.1. *The functional Φ_λ satisfies the mountain pass geometry.*

Proof. From (g_2) and (3.2), one has

$$\Phi_\lambda(u) \geq \frac{1}{p_+}\varrho_\lambda(u) - \epsilon\alpha \int |u|^{p(x)}dx - C_\epsilon \alpha \int |u|^{m(x)}dx - \frac{1}{q_-}\int |u|^{q(x)}dx,$$

for any $\epsilon > 0$ and C_ϵ be a constant depending on ϵ. By (V_2), letting $\epsilon < \frac{M}{2\alpha p_+}$, and assuming $\|u\|_\lambda < \min\{1, \frac{1}{C_m}, \frac{1}{C_q}\}$, where $|v|_{m(x)} \leq C_m\|v\|_\lambda$, $|v|_{q(x)} \leq C_q\|v\|_\lambda$, $\forall\, v \in E_\lambda$. Since $p_+ < m_-$ and $p_+ < q_-$, then for $\|u\|_\lambda$ small enough, we have

$$\Phi_\lambda(u) \geq \frac{1}{2p_+}\|u\|_\lambda^{p_+} - C_1\alpha\|u\|_\lambda^{m_-} - C_2\|u\|_\lambda^{q_-} \geq b > 0.$$

Now, choosing $v \in C_0^\infty(\Omega_\Gamma)$ with $v > 0$ in Ω_Γ, one has for $t > 0$

$$\Phi_\lambda(tv) = \int \frac{t^{p(x)}}{p(x)}(|\nabla v|^{p(x)} + Z(x)|v|^{p(x)})dx - \int \alpha F(x, tv)dx - \int \frac{t^{q(x)}}{q(x)}v^{q(x)}dx.$$

If $t > 1$, by (f_3) and $q_- > p_+$, it follows that

$$\begin{aligned}\Phi_\lambda(tv) \leq{}& \frac{t^{p_+}}{p_-}\int (|\nabla v|^{p(x)} + Z(x)|v|^{p(x)})dx - C\alpha t^\beta \int v^\beta dx \\ &- \frac{t^{q_-}}{q_+}\int v^{q(x)}dx \to -\infty\end{aligned}$$

as $t \to +\infty$. The proof is complete. $\square$

Lemma 3.2. *Let $c > 0$ and (u_n) be a* $(\mathrm{PS})_c$ *sequence for Φ_λ, then (u_n) is bounded in E_λ.*

Proof. Assume that (u_n) is a $(\mathrm{PS})_c$ sequence for Φ_λ. Without loss of generality, we set $\|u_n\|_\lambda \geq 1$ for n large, otherwise the proof is complete. On one hand, there is $n_0 \in \mathbb{N}$ such that

$$\Phi_\lambda(u_n) - \frac{1}{\beta}\Phi'_\lambda(u_n)u_n \leq c + 1 + \|u_n\|_\lambda, \quad \text{for } n \geq n_0.$$

On the other hand, from (f_2), (f_3), (3.6) and Lemma 2.1, we have

$$\begin{aligned}
&\Phi_\lambda(u_n) - \frac{1}{\beta}\Phi'_\lambda(u_n)u_n \\
&\quad \geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\varrho_\lambda(u_n) + \int\left(\frac{1}{\beta}g(x,u_n)u_n - G(x,u_n)\right)dx \\
&\quad = \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\varrho_\lambda(u_n) + \int_{\Omega'_\Gamma}\alpha\left(\frac{1}{\beta}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\qquad + \int_{\Omega'_\Gamma}\left(\frac{1}{\beta} - \frac{1}{q(x)}\right)|u_n|^{q(x)}dx + \int_{\mathbb{R}^N\setminus\Omega'_\Gamma}\left(\frac{1}{\beta}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n)\right)dx \\
&\quad \geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\varrho_\lambda(u_n) - \int_{\mathbb{R}^N\setminus\Omega'_\Gamma}\tilde{F}(x,u_n)dx \\
&\quad \geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\delta\varrho_\lambda(u_n).
\end{aligned}$$

Hence

$$c + 1 + \|u_n\|_\lambda \geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\delta\varrho_\lambda(u_n) \geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\delta\|u_n\|_\lambda^{p_-}, \quad \forall\, n \geq n_0,$$

this implies that (u_n) is bounded in E_λ. □

Lemma 3.3. *For $c > 0$, let (u_n) be a $(\mathrm{PS})_c$-sequence for Φ_λ, then for each $\zeta > 0$, there is a number $R = R(\zeta) > 0$ such that*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx \leq \zeta. \tag{3.10}$$

Proof. Let $R > 0$ large such that $\Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$ and $\eta_R \in C^\infty(\mathbb{R}^N)$ satisfying

$$\eta_R(x) := \begin{cases} 0, & x \in B_{\frac{R}{2}}(0), \\ 1, & x \in \mathbb{R}^N\setminus B_R(0), \end{cases}$$

and $0 \leq \eta_R \leq 1$, $|\nabla\eta_R| \leq \frac{C}{R}$, where $C > 0$ is a constant independent on R. From Lemma 3.2, the sequence (u_n) is bounded in E_λ. Moreover, it is easy to verify that the sequence $(u_n\eta_R)$ is also bounded in E_λ. By a simple computation, we have

$$\begin{aligned}
&\int(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})\eta_R dx \\
&\quad = \Phi'_\lambda(u_n)(u_n\eta_R) - \int u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\eta_R dx + \int_{\mathbb{R}^N\setminus\Omega'_\Gamma}\tilde{f}(x,u_n)u_n\eta_R dx.
\end{aligned}$$

Denoting

$$L = \int (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})\eta_R dx.$$

From the definition of η_R, the Hölder inequality and (3.5), it follows that

$$\begin{aligned} L &\le \Phi'_\lambda(u_n)(u_n\eta_R) + \frac{C}{R}\int |u_n||\nabla u_n|^{p(x)-1}dx + \varsigma \int |u_n|^{p(x)}\eta_R dx \\ &\le \Phi'_\lambda(u_n)(u_n\eta_R) + \frac{C}{R}|u_n|_{p(x)}||\nabla u_n|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}} + \frac{\varsigma}{M}L. \end{aligned}$$

Since the sequence (u_n) is bounded in $L^{p(x)}(\mathbb{R}^N)$, and $(|\nabla u_n|^{p(x)-1})$ is bounded in $L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N\setminus B_R(0)} (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx \le o_n(1) + \frac{C}{R}.$$

Fixing $\zeta > 0$ and passing to the limit in the last inequality, it follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx \le \frac{C}{R} < \zeta$$

for some R sufficiently large. We complete the proof. $\square$

Next, for each fixed $j \in \Gamma$, let us denote by $c_j = \inf_{\gamma\in\Lambda_j} \max_{t\in[0,1]} I_j(\gamma(t))$ the minimax level of the mountain pass geometry with the functional $I_j : W_0^{1,p(x)}(\Omega_j) \to \mathbb{R}$ given by

$$I_j(u) = \int_{\Omega_j} \frac{1}{p(x)}(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx - \alpha \int_{\Omega_j} F(x,u)dx - \int_{\Omega_j} \frac{|u|^{q(x)}}{q(x)},$$

where

$$\Lambda_j := \{\gamma \in C([0,1], W_0^{1,p(x)}(\Omega_j)) : \gamma(0) = 0, I_j(\gamma(1)) < 0\}.$$

It is well known that the positive critical points of I_j are weak solutions of the problem

$$(P_\lambda) \begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + Z(x)|u|^{p(x)-2}u = \alpha f(x,u) + u^{q(x)-1}, \quad x \in \Omega_j, \\ u > 0, \quad x \in \Omega_j, \\ u|_{\partial\Omega_j} = 0. \end{cases}$$

In order to prove Theorem 1.1, we shall compare between some energy levels of the functional associated with problem (1.1) with the energy levels associated with other modified problem related to problem (1.1), and study the behavior of some $(\mathrm{PS})_c$ sequence. In this regard, we prove the following results.

Lemma 3.4. *There exists $\alpha^* > 0$ such that, for all $\alpha \ge \alpha^*$, we have*

$$c_j \in \left(0, \frac{1}{k+1}\left(\frac{1}{p_+} - \frac{1}{\beta}\right) \inf_{x\in\mathcal{A}} S_x^N\right), \quad \textit{for all } j \in \{1,\ldots,k\}.$$

Proof. For each $j \in \{1,\dots,k\}$, we choose a nonnegative function $\varphi_j \in W_0^{1,p(x)}(\Omega_j)\backslash\{0\}$. Note that there exits $t_{\alpha,j} \in (0,+\infty)$ depending on α such that

$$c_j \leq I_j(t_{\alpha,j}\varphi_j) = \max_{t\geq 0} I_j(t\varphi_j)$$

and thus, the following equality holds:

$$\begin{aligned}\int_{\Omega_j} & t_{\alpha,j}^{p(x)-1}(|\nabla\varphi_j|^{p(x)} + Z(x)|\varphi_j|^{p(x)})dx \\ &= \alpha\int_{\Omega_j} f(x,t_{\alpha,j}\varphi_j)\varphi_j dx + \int_{\Omega_j} t_{\alpha,j}^{q(x)-1}\varphi_j^{q(x)}dx \\ &\geq \alpha\int_{\Omega_j} f(x,t_{\alpha,j}\varphi_j)\varphi_j dx \\ &\geq \alpha C t_{\alpha,j}^{\beta-1}\int_{\Omega_j}\varphi_j^{\beta}dx. \end{aligned} \tag{3.11}$$

If $t_{\alpha,j} \geq 1$, by (3.11), we have

$$t_{\alpha,j}^{p_+-1}\int_{\Omega_j}(|\nabla\varphi_j|^{p(x)} + Z(x)|\varphi_j|^{p(x)})dx \geq t_{\alpha,j}^{\beta-1}C\alpha\int_{\Omega_j}\varphi_j^{\beta}dx$$

which implies that $(t_{\alpha,j})$ is bounded by $p_+ < \beta$. Thus, up to a subsequence, $t_{\alpha,j} \to t_0 \geq 1$ as $\alpha \to \infty$. On one hand, for large α, there is a constat $C > 0$ such that

$$\int_{\Omega_j} t_{\alpha,j}^{p(x)-1}(|\nabla\varphi_j|^{p(x)} + Z(x)|\varphi_j|^{p(x)})dx \leq C. \tag{3.12}$$

On the other hand, since $t_0 \geq 1$, by the first equality of (3.11), one has

$$\lim_{\alpha\to+\infty}\left(\alpha\int_{\Omega_j} f(x,t_{\alpha,j}\varphi_j)\varphi_j dx + \int_{\Omega_j} t_{\alpha,j}^{q(x)-1}\varphi_j^{q(x)}\right) = \infty,$$

which contradicts with (3.12).

If $t_{\alpha,j} < 1$, up to a subsequence, $t_{\alpha,j} \to t_0 \geq 0$ as $\alpha \to \infty$. If $0 < t_0 < 1$, similar to the above arguments, we may also obtain a contradiction. Thus, we must have $t_0 = 0$ and $t_{\alpha,j} \to 0$ as $\alpha \to +\infty$. Using this limit, one has

$$I_j(t_{\alpha,j}\varphi_j) \to 0, \quad \text{as } \alpha \to +\infty,$$

whence it follows from Remark 2.1 that there exists $\alpha^* > 0$ such that for all $\alpha \geq \alpha^*$,

$$c_j \in \left(0, \frac{1}{k+1}\left(\frac{1}{p_+} - \frac{1}{\beta}\right)\inf_{x\in\mathcal{A}} S_x^N\right), \quad \text{for all } j \in \{1,\dots,k\}. \qquad \square$$

Remark 3.1. In particular, for α large, the above lemma implies that

$$\sum_{j=1}^{k} c_j \in \left(0, \left(\frac{1}{p_+} - \frac{1}{\beta}\right)\inf_{x\in\mathcal{A}} S_x^N\right). \tag{3.13}$$

From Remark 2.1, it is easy to see that $(\frac{1}{p_+} - \frac{1}{\beta})\inf_{x\in\mathcal{A}} S_x^N > 0$.

We shall use the above result to show the following lemma.

Lemma 3.5. *For any $\lambda \geq 1$, the functional Φ_λ satisfies the Palais–Smale condition at any level $c \in (0, (\frac{1}{p_+} - \frac{1}{\beta}) \inf_{x \in \mathcal{A}} S_x^N)$.*

Proof. Let $(u_n) \subset E_\lambda$ be a $(\mathrm{PS})_c$ sequence. Then, from Lemma 3.2, we know that (u_n) is bounded in E_λ. Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } E_\lambda, \\ u_n \to u, & \text{strongly in } L^{h(x)}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for any } 1 \leq h(x) \ll p^*(x), \\ u_n \to u, & \text{for a.e. } x \in \mathbb{R}^N. \end{cases}$$

From $\Phi'_\lambda(u_n)u_n \to 0$, it follows that

$$\int (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx = \int g(x, u_n)u_n dx + o_n(1). \tag{3.14}$$

It is easy to know that the weak limit u is a critical point of Φ_λ, and so

$$\int (|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)})dx = \int g(x, u)u dx. \tag{3.15}$$

Now, we claim that

$$\lim_{n\to\infty} \int g(x, u_n)u_n dx = \int g(x, u)u dx. \tag{3.16}$$

If (3.16) holds, by (3.14) and (3.15), we have that

$$\begin{aligned} &\lim_{n\to\infty} \int (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx \\ &\quad = \int (|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)})dx, \end{aligned}$$

and $u_n \to u$ in E_λ.

Now let us prove (3.16). We first note by Lemma 3.3 that for each $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx < \zeta.$$

This inequality together with (3.5), (3.8) and the Sobolev embedding imply that, for n large enough,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} g(x, u_n)u_n dx &\leq \varsigma \int_{\mathbb{R}^N \setminus B_R(0)} |u_n|^{p(x)} dx \\ &\leq \frac{\varsigma}{M} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)})dx \\ &\leq \frac{\varsigma\zeta}{M}. \end{aligned}$$

On the other hand, we may choose R large enough such that

$$\int_{\mathbb{R}^N\backslash B_R(0)} g(x,u)udx \leq \zeta.$$

By the definition of g, we know that

$$g(x,u_n)u_n \leq \varsigma|u_n|^{p(x)}, \quad \text{for any } x \in \mathbb{R}^N\backslash\Omega'_\Gamma.$$

Since the set $B_R(0)\backslash\Omega'_\Gamma$ is bounded, we can use the above estimate and a variant of the Lebesgue Dominated Convergence theorem (see [33]) to obtain that

$$\lim_{n\to\infty}\int_{B_R(0)\backslash\Omega'_\Gamma} g(x,u_n)u_n dx = \int_{B_R(0)\backslash\Omega'_\Gamma} g(x,u)udx.$$

Finally, we claim that $u_n \to u$ in $L^{q(x)}(\Omega'_\Gamma)$. If it holds, we can use the Sobolev embedding and the Lebesgue Dominated Convergence theorem to conclude that

$$\lim_{n\to\infty}\int_{\Omega'_\Gamma} g(x,u_n)u_n dx = \int_{\Omega'_\Gamma} g(x,u)udx.$$

Using the above information, we conclude that (3.16).

It remains to prove that $u_n \to u$ in $L^{q(x)}(\Omega'_\Gamma)$. By Lemma 2.5, we have an at most enumerable set of indices I, points $x_i \in \mathcal{A}$ (the critical set in (q_1)), and positive real numbers μ_i, ν_i, $i \in I$, such that the following convergences hold weakly in the sense of measures:

$$|\nabla u_n|^{p(x)}dx \rightharpoonup \mu \geq |\nabla u|^{p(x)}dx + \sum \mu_i\delta_{x_i},$$

$$|u_n|^{q(x)}dx \rightharpoonup \nu := |u|^{q(x)}dx + \sum \nu_i\delta_{x_i},$$

$$S_{x_i}\nu_i^{\overline{\frac{1}{p(x_i)^*}}} \leq \mu_i^{\overline{\frac{1}{p(x_i)}}}, \quad \text{for all } i \in I,$$

where S_{x_i} is the localized Sobolev constant at the point x_i defined in Lemma 2.5. It suffices to show that $\{x_i\}_{i\in I}\cap\Omega'_\Gamma = \emptyset$. Suppose, by contradiction, that $x_i \in \Omega'_\Gamma$, for some $i \in I$. For $\epsilon > 0$ small such that $B(x_i, 2\epsilon) \subset \Omega'_\Gamma$, define a function $\phi(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\phi(x) = 1$ in $B(x_i,\epsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N\backslash B(x_i, 2\epsilon)$ and $|\nabla\phi| \leq \frac{2}{\epsilon}$ in $\mathbb{R}^N$. Obviously, $\langle\Phi_\lambda(u_n), u_n\phi\rangle = o_n(1)$, i.e.

$$\begin{aligned}
&-\int u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi dx + o_n(1)\\
&\quad = \int |\nabla u_n|^{p(x)}\phi dx + \int(\lambda V(x)+Z(x))|u_n|^{p(x)}\phi dx\\
&\qquad - \alpha\int f(x,u_n)u_n\phi dx - \int |u_n|^{q(x)}\phi dx. \qquad (3.17)
\end{aligned}$$

Because of the boundedness of (u_n) in E_λ, using the Hölder inequality, we have that

$$\begin{aligned}0 &\le \lim_{\epsilon\to 0}\lim_{n\to\infty}\left|\int u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi dx\right| \\ &\le \lim_{\epsilon\to 0}\lim_{n\to\infty}\int |u_n\nabla\phi||\nabla u_n|^{p(x)-1}dx \\ &\le 2\lim_{\epsilon\to 0}\lim_{n\to\infty}|u_n\nabla\phi|_{p(x)}||\nabla u_n|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}} \\ &\le C\lim_{\epsilon\to 0}\lim_{n\to\infty}|u_n\nabla\phi|_{p(x)}. \end{aligned}\tag{3.18}$$

By a variant of Lebesgue Dominated Convergence theorem (see [33]) and $u_n \to u$ strongly in $L^{p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, we have that

$$\lim_{n\to\infty}\int |u_n\nabla\phi|^{p(x)}dx = \int |u\nabla\phi|^{p(x)}dx.$$

Moreover, by the Hölder inequality and the absolute continuity of the integration, it yields

$$\begin{aligned}\lim_{\epsilon\to 0}\int |u\nabla\phi|^{p(x)}dx &\le 2\lim_{\epsilon\to 0}||u|^{p(x)}|_{L^{\frac{N}{N-p(x)}}(B(x_j,2\epsilon)}||\nabla\phi|^{p(x)}|_{\frac{N}{p(x)}} \\ &\le C\lim_{\epsilon\to 0}||u|^{p(x)}|_{L^{\frac{N}{N-p(x)}}(B(x_j,2\epsilon))} = 0.\end{aligned}\tag{3.19}$$

Combining (3.18) and (3.19), it is easy to see that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi dx = 0.\tag{3.20}$$

Since ϕ has compact support and f has subcritical growth, we can let $n\to\infty$ and $\epsilon\to 0$ to obtain that

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{n\to\infty}\int f(x,u_n)u_n\phi dx &= \lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{B(x_i,2\epsilon)} f(x,u_n)u_n\phi dx \\ &= \lim_{\epsilon\to 0}\int_{B(x_i,2\epsilon)} f(x,u)u\phi dx = 0.\end{aligned}\tag{3.21}$$

Therefore, from (3.17), (3.20) and (3.21), it yields that

$$\begin{aligned}0 &= \lim_{\epsilon\to 0}\lim_{n\to\infty}\left\{\int |\nabla u_n|^{p(x)}\phi dx + \int(\lambda V(x)+Z(x))|u_n|^{p(x)}\phi dx\right. \\ &\qquad \left. -\alpha\int f(x,u_n)u_n\phi dx - \int |u_n|^{q(x)}\phi dx\right\} \\ &\ge \lim_{\epsilon\to 0}\lim_{n\to\infty}\left\{\int |\nabla u_n|^{p(x)}\phi dx - \int |u_n|^{q(x)}\phi dx\right\} \\ &= \mu_i - \nu_i.\end{aligned}$$

Using the above estimate together with Lemma 2.5, we obtain $\nu_i \geq S_{x_i}^{p(x_i)} \nu_i^{\frac{N-p(x_i)}{N}}$. This result implies that

- (1) $\nu_i = 0$ or
- (2)$\nu_i \geq S_{x_i}^N$.

If the second case $\nu_i \geq S_{x_i}^N$ holds, for some $i \in I$, then by (f_3), we have

$$
\begin{aligned}
c &= \lim_{n\to\infty} \left\{ \Phi_\lambda(u_n) - \frac{1}{\beta}\langle \Phi'_\lambda(u_n), u_n\rangle \right\} \\
&= \lim_{n\to\infty} \left\{ \int \left(\frac{1}{p(x)} - \frac{1}{\beta}\right) (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)}) dx \right. \\
&\quad + \alpha \int_{\Omega'_\Gamma} \left(\frac{1}{\beta} f(x,u_n)u_n - F(x,u_n)\right) dx + \int_{\Omega'_\Gamma} \left(\frac{1}{\beta} - \frac{1}{q(x)}\right) |u_n|^{q(x)} dx \\
&\quad \left. + \int_{\mathbb{R}^N\backslash\Omega'_\Gamma} \left(\frac{1}{\beta}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n)\right) dx \right\} \\
&\geq \lim_{n\to\infty} \left\{ \int \left(\frac{1}{p(x)} - \frac{1}{\beta}\right) (|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)}) dx \right. \\
&\quad \left. + \int_{\mathbb{R}^N\backslash\Omega'_\Gamma} \left(\frac{1}{\beta}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n)\right) dx \right\}.
\end{aligned}
$$

If $0 \leq u_n \leq a(x)$, we have from the definition of $\tilde{f}$ and (f_3) that

$$
\frac{1}{\beta}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n) \geq 0, \quad \text{in } \mathbb{R}^N\backslash\Omega'_\Gamma.
$$

If $u_n \geq a(x)$, from the definition of $\tilde{f}$, it follows that

$$
\frac{1}{\beta}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n) = \left(\frac{\varsigma}{\beta} - \frac{\varsigma}{p(x)}\right) |u_n|^{p(x)}, \quad \text{in } \mathbb{R}^N\backslash\Omega'_\Gamma.
$$

Since $\varsigma > 0$ small enough, thus we obtain

$$
\begin{aligned}
&\left(\frac{1}{p_+} - \frac{1}{\beta}\right) \int (\lambda V(x) + Z(x))|u_n|^{p(x)} dx \\
&\qquad + \int_{\mathbb{R}^N\backslash\Omega'_\Gamma} \left(\frac{1}{\beta}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n)\right) dx \geq 0.
\end{aligned}
$$

From the above arguments and Lemma 2.5, one has

$$
\begin{aligned}
c &\geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right) \lim_{n\to\infty} \int |\nabla u_n|^{p(x)} dx \\
&\geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right) \int |\nabla u|^{p(x)} dx + \left(\frac{1}{p_+} - \frac{1}{\beta}\right) \mu_i
\end{aligned}
$$

$$\geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right) S_{x_i}^{p(x_i)} \nu_i^{\frac{N-p(x_i)}{N}}$$
$$\geq \left(\frac{1}{p_+} - \frac{1}{\beta}\right) S_{x_i}^{N},$$

this is impossible. So, $\nu_i = 0$ for all $i \in I$ and

$$u_n \to u \quad \text{in } L^{q(x)}(\Omega'_\Gamma).$$

The proof is complete. $\square$

4. The $(\mathrm{PS})_\infty$ Sequence

Our next step is to study the behavior of a $(\mathrm{PS})_{\infty,c}$ sequence, that is, a sequence $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$ satisfying

$$(u_n) \subset E_{\lambda_n} \quad \text{and} \quad \lambda_n \geq 1, \quad \lambda_n \to \infty,$$
$$\Phi_{\lambda_n}(u_n) \to c, \quad \|\Phi'_{\lambda_n}(u_n)\|^*_{\lambda_n} \to 0.$$

Lemma 4.1. *Assume that the sequence $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$ be a $(\mathrm{PS})_{\infty,c}$ sequence with $c \in (0, (\frac{1}{p_+} - \frac{1}{\beta}) \inf_{x\in\mathcal{A}} S_x^N)$. Then, for some subsequence, still denoted by (u_n), there exists $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$. Moreover,*

(i) *$\varrho_{\lambda_n}(u_n - u) \to 0$ and, so $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$.*
(ii) *$u = 0$ in $\mathbb{R}^N \backslash \Omega_\Gamma$, $u \geq 0$ and $u_{|\Omega_j}$, $j \in \Gamma$, is a nonnegative solution for*

$$(P_j)\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + Z(x)|u|^{p(x)-2}u \\ \qquad = \alpha f(x,u) + |u|^{q(x)-2}u, \quad \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j). \end{cases}$$

(iii) *$\lambda_n \int V(x)|u_n|^{p(x)}dx \to 0$ as $n \to \infty$.*
(iv) *$\varrho_{\lambda_n,\Omega'_j}(u_n) \to \int_{\Omega_j}(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx$, for $j \in \Gamma$.*
(v) *$\varrho_{\lambda_n,\mathbb{R}^N\backslash\Omega_\Gamma}(u_n) \to 0$.*
(vi) *$\Phi_{\lambda_n}(u_n) \to \int_{\Omega_\Gamma} \frac{1}{p(x)}(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx - \alpha\int_{\Omega_\Gamma} F(x,u)dx - \int_{\Omega_\Gamma} \frac{1}{q(x)} \times |u|^{q(x)}dx$.*

Proof. Similar to the proof of Lemma 3.2, for any $(\mathrm{PS})_{\infty,c}$ sequence (u_n), we may prove that $\left(\varrho_{\lambda_n}(u_n)\right)$ is bounded in $\mathbb{R}^+$. Thus, the sequence (u_n) is bounded in $W^{1,p(x)}(\mathbb{R}^N)$, and for some subsequence, still denoted by (u_n), there exists $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } W^{1,p(x)}(\mathbb{R}^N), \\ u_n \to u, & \text{strongly in } L^{h(x)}_{\mathrm{loc}}(\mathbb{R}^N) \text{ for any } 1 \leq h(x) \ll p^*(x), \\ u_n \to u, & \text{for a.e. } x \in \mathbb{R}^N. \end{cases}$$

Now, for each $m \in \mathbb{N}$, we define the set $C_m = \{x \in \mathbb{R}^N : V(x) \geq \frac{1}{m}\}$. For n large, we have

$$\int_{C_m} |u_n|^{p(x)} dx \leq \frac{2m}{\lambda_n} \int_{C_m} (\lambda_n V(x) + Z(x))|u_n|^{p(x)} dx$$
$$\leq \frac{2m}{\lambda_n} \varrho_{\lambda_n}(u_n) \leq \frac{2mC}{\lambda_n}.$$

By the Fatou's lemma and the last inequality, we obtain

$$\int_{C_m} |u|^{p(x)} dx = 0, \quad \forall\, m \in \mathbb{N}.$$

Thus, $u(x) = 0$ in $\bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \backslash \overline{\Omega}$. From this, we can prove that (i)–(vi).

(i) First of all, we know that

$$\begin{aligned}
&\langle \Phi'_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u), u_n - u\rangle \\
&\quad = \int (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u) dx \\
&\qquad + \int (\lambda_n V(x) + Z(x))(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u) dx \\
&\qquad - \int (g(x, u_n) - g(x, u))(u_n - u) dx.
\end{aligned}$$

Here, we note that

$$\langle \Phi'_{\lambda_n}(u_n), u_n - u\rangle \to 0, \quad \text{as } n \to \infty.$$

Moreover, the fact $u = 0$ in $\mathbb{R}^N \backslash \overline{\Omega}$ and $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$ imply that

$$\langle \Phi'_{\lambda_n}(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty.$$

Using the similar arguments explored in Lemma 3.5, we get

$$\begin{aligned}
&\int (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u) dx \\
&\quad + \int (\lambda_n V(x) + Z(x))(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u) dx \to 0.
\end{aligned}$$

Thus, $\varrho_{\lambda_n}(u_n - u) \to 0$, which implies that $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$.

(ii) Since $u \in W^{1,p(x)}(\mathbb{R}^N)$ and $u = 0$ in $\mathbb{R}^N \backslash \overline{\Omega}$, we have $u|_{\Omega_j} \in W_0^{1,p(x)}(\Omega_j)$ for all $j \in \{1, 2, \ldots, k\}$. Moreover, the limits $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$ and $\Phi'_{\lambda_n}(u_n)\varphi \to 0$ for $\varphi \in C_0^\infty(\Omega_j)$ imply that

$$\int_{\Omega_j} (|\nabla u|^{p(x)-2}\nabla u \nabla \varphi + Z(x)|u|^{p(x)-2}u\varphi) dx - \int_{\Omega_j} g(x, u)\varphi dx = 0.$$

This shows that $u|_{\Omega_j}$, $j \in \Gamma$, is a solution for the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + Z(x)|u|^{p(x)-2}u = \alpha f(x,u) + u^{q(x)-1}, & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j). \end{cases}$$

On the other hand, if $j \notin \Gamma$, one has

$$\int_{\Omega_j} (|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx - \int_{\Omega_j} \tilde{f}(x,u)udx = 0.$$

From the above equality, Lemma 2.1 and (3.5), we obtain

$$0 \geq \varrho_{\lambda,\Omega_j}(u) - \varsigma\varrho_{p(x),\Omega_j}(u) \geq \delta\varrho_{\lambda,\Omega_j}(u) \geq 0.$$

Thus, $u = 0$ in $\mathbb{R}^N \backslash \Omega_\Gamma$ and $u \geq 0$ in $\mathbb{R}^N$.

(iii) Since

$$\int \lambda_n V(x)|u_n|^{p(x)}dx = \int \lambda_n V(x)|u_n - u|^{p(x)}dx \leq 2\varrho_{\lambda_n}(u_n - u).$$

From (i), we may get $\lambda_n \int V(x)|u_n|^{p(x)}dx \to 0$ as $n \to \infty$.

(iv) From (i), $\varrho_{\lambda_n}(u_n - u) \to 0$. So, for $\forall\, j \in \Gamma$,

$$\varrho_{p(x),\Omega_j'}(u_n - u) \to 0 \quad \text{and} \quad \varrho_{p(x),\Omega_j'}(\nabla u_n - \nabla u) \to 0.$$

Then by Lemma 2.4, we have

$$\begin{aligned} &\int_{\Omega_j'} (|\nabla u_n|^{p(x)} - |\nabla u|^{p(x)})dx \to 0 \quad \text{and} \\ &\int_{\Omega_j'} Z(x)(|u_n|^{p(x)} - |u|^{p(x)})dx \to 0. \end{aligned} \tag{4.1}$$

From (iii) and $u = 0$ in $\mathbb{R}^N \backslash \Omega_\Gamma$,

$$\int_{\Omega_j'} \lambda_n V(x)(|u_n|^{p(x)} - |u|^{p(x)})dx = \int_{\Omega_j' \backslash \overline{\Omega_j}} \lambda_n V(x)|u_n|^{p(x)}dx \to 0. \tag{4.2}$$

(4.1) and (4.2) imply that

$$\varrho_{\lambda_n,\Omega_j'}(u_n) - \varrho_{\lambda_n,\Omega_j'}(u) \to 0,$$

and

$$\varrho_{\lambda_n,\Omega_j'}(u_n) \to \int_{\Omega_j} (|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx.$$

(v) It is easy to see that $\varrho_{\lambda_n,\mathbb{R}^N \backslash \Omega_\Gamma}(u_n) \to 0$ from (i) and (ii).

(vi) It is clear that

$$\Phi_{\lambda_n}(u_n) = \sum_{j\in\Gamma}\int_{\Omega'_j}\frac{1}{p(x)}(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)})dx$$
$$+ \int_{\mathbb{R}^N\backslash\Omega'_\Gamma}\frac{1}{p(x)}(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)})dx$$
$$- \int G(x, u_n)dx.$$

From (iv), we have for any $j \in \Gamma$

$$\int_{\Omega'_j}\frac{1}{p(x)}(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)})dx$$
$$\to \int_{\Omega_j}\frac{1}{p(x)}(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx.$$

From (v), one has

$$\int_{\mathbb{R}^N\backslash\Omega'_\Gamma}\frac{1}{p(x)}(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)})dx \to 0.$$

Moreover, by (i) and (ii), $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$ and $u = 0$ in $\mathbb{R}^N\backslash\Omega_\Gamma$, it yields that

$$\int G(x, u_n)dx \to \alpha\int_{\Omega_\Gamma} F(x, u)dx + \int_{\Omega_\Gamma}\frac{1}{q(x)}|u|^{q(x)}dx.$$

So, from the above arguments, we have

$$\Phi_{\lambda_n}(u_n) \to \int_{\Omega_\Gamma}\frac{1}{p(x)}(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})dx$$
$$- \alpha\int_{\Omega_\Gamma} F(x, u)dx - \int_{\Omega_\Gamma}\frac{1}{q(x)}|u|^{q(x)}dx. \qquad \square$$

5. The Boundedness of Solutions

In virtue of (q_2) and the continuity of $p(x)$, $q(x)$, we can choose the appropriate smooth bounded domain Ω'_j $(j = 1, \ldots, k)$ in (3.7) such that

$$\overline{\Omega}_j \subset \Omega'_j \quad \text{and} \quad \overline{\Omega}'_j \cap \overline{\Omega}'_i = \emptyset, \quad \text{for } i \neq j,$$

and for any $x \in \Omega'_j\backslash\overline{\Omega}_j$, $q(x) \ll p^*(x)$.

The following lemma plays a fundamental role in the study of the solutions of problem (A_λ).

Lemma 5.1. *Let* (u_λ) *be a family of solutions for* (A_λ) *such that*

$$\sup_{\lambda\geq 1}\Phi_\lambda(u_\lambda) < \left(\frac{1}{p_+}-\frac{1}{\beta}\right)\inf_{x\in\mathcal{A}} S_x^N$$

and $u_\lambda \to 0$ *in* $W^{1,p(x)}(\mathbb{R}^N\backslash\Omega_\Gamma)$ *as* $\lambda\to\infty$. *Then, there exists* $\Lambda_0>0$ *such that* $|u_\lambda|_{L^\infty(\mathbb{R}^N\backslash\Omega'_\Gamma)} \leq a_-$ *for* $\lambda \geq \Lambda_0$. *In particular,* u_λ *is a solution for problem* (1.1) *for* $\lambda\geq\Lambda_0$.

Before to prove the above lemma, we need some technical lemmas.

Lemma 5.2. *There exist* $x_1,\ldots,x_l\in\partial\Omega'_\Gamma$ *and corresponding* $\delta_{x_1},\ldots,\delta_{x_1}>0$ *such that*

$$\partial\Omega'_\Gamma \subset \mathcal{N}(\partial\Omega'_\Gamma) = \bigcup_{i=1}^{l} B_{\frac{\delta_{x_i}}{2}}(x_i).$$

Moreover,

$$q_+^{x_i} \leq (p_-^{x_i})^*, \quad m_+^{x_i}\leq (p_-^{x_i})^*, \tag{5.1}$$

where $q_+^{x_i} = \sup_{B_{\delta_{x_i}}(x_i)} q$, $m_+^{x_i} = \sup_{B_{\delta_{x_i}}(x_i)} m$, $p_-^{x_i} = \inf_{B_{\delta_{x_i}}(x_i)} p$ *and* $(p_-^{x_i})^* = \frac{Np_-^{x_i}}{N-p_-^{x_i}}$.

Proof. Since $q\ll p^*$ on $\partial\Omega$, there exists $\epsilon>0$ such that $\epsilon\leq p^*(x)-q(x)$, for all $x\in\partial\Omega$. By the continuity of p and q, we may choose appropriate Ω'_j, $\forall\, j\in\Gamma$ such that $\epsilon/3 \leq p^*(x)-q(x)$ for all $\partial\Omega'_j$, $\forall\, j\in\Gamma$. So, there exists $\delta>0$ small enough such that

$$\overline{B_\delta(x)} \subset \mathbb{R}^N\backslash\overline{\Omega}'_\Gamma, \quad \forall\, x\in\partial\Omega'_\Gamma.$$

Moreover, for each $x\in\partial\Omega'_\Gamma$, by the continuity of p and q, we can choose $0<\delta_x\leq\delta$ small enough such that

$$q_+^x \leq (p_-^x)^*,$$

where $q_+^x = \sup_{B_{\delta_x}(x)} q$, $p_-^x = \inf_{B_{\delta_x}(x)} p$ and $(p_-^x)^* = \frac{Np_-^x}{N-p_-^x}$. Since $m(x)\ll p^*(x)$ for any $x\in\mathbb{R}^N$. For the former $x\in\partial\Omega'_\Gamma$, we may choose δ_x, if necessary, even smaller such that

$$m_+^x \leq (p_-^x)^*,$$

where $m_+^x = \sup_{B_{\delta_x}(x)} m$.

Since $\partial\Omega'_\Gamma$ is compact, there exist the points $x_1,\ldots,x_l\in\partial\Omega'_\Gamma$ such that

$$\partial\Omega'_\Gamma \subset \bigcup_{i=1}^{l} B_{\frac{\delta_{x_i}}{2}}(x_i). \qquad \square$$

Lemma 5.3. *If u_λ is a solution for (A_λ), then, for each $B_{\delta_{x_i}}(x_i)$, $i = 1, \ldots, l$, given by Lemma 5.2, it satisfies*

$$\int_{A_{k,\bar{\delta},x_i}} |\nabla u_\lambda|^{p_-^{x_i}} dx \\ \leq C\left((k^{q_+} + 2)|A_{k,\tilde{\delta},x_i}| + (\tilde{\delta} - \bar{\delta})^{-(p_-^{x_i})^*} \int_{A_{k,\tilde{\delta},x_i}} (u_\lambda - k)^{(p_-^{x_i})^*} dx\right),$$

where $0 < \bar{\delta} < \tilde{\delta} < \delta_{x_i}$, $k \geq \frac{a_-}{4}$, $C = C(p_-, p_+, m_-, m_+, q_-, q_+, \varsigma, \delta_{x_i}) > 0$ is a constant independent of k, and for any $R > 0$, we denote by A_{k,R,x_i} the set

$$A_{k,R,x_i} = B_R(x_i) \cap \{x \in \mathbb{R}^N : u_\lambda(x) > k\}.$$

Proof. We choose arbitrarily $0 < \bar{\delta} < \tilde{\delta} < \delta_{x_i}$ and $\xi \in C^\infty(\mathbb{R}^N)$ satisfying

$$0 \leq \xi \leq 1, \quad \operatorname{supp} \xi \subset B_{\tilde{\delta}}(x_i), \quad \xi = 1, \quad \text{in } B_{\bar{\delta}}(x_i) \quad \text{and} \quad |\nabla \xi| \leq \frac{2}{\tilde{\delta} - \bar{\delta}}.$$

For $k \geq \frac{a_-}{4}$, we define $\eta = \xi^{p_+}(u_\lambda - k)^+$. By a simple computation, we have

$$\nabla \eta = p_+ \xi^{p_+ - 1}(u_\lambda - k)\nabla \xi + \xi^{p_+} \nabla u_\lambda$$

on the set $\{x \in \mathbb{R}^N : u_\lambda(x) > k\}$. Then, we denote u_λ by u and take η as a test function, and obtain

$$p_+ \int_{A_{k,\tilde{\delta},x_i}} \xi^{p_+ - 1}(u - k)|\nabla u|^{p(x)-2} \nabla u \nabla \xi dx + \int_{A_{k,\tilde{\delta},x_i}} \xi^{p_+} |\nabla u|^{p(x)} dx \\ + \int_{A_{k,\tilde{\delta},x_i}} (\lambda V(x) + Z(x)) u^{p(x)-1} \xi^{p_+}(u - k) dx \\ = \int_{A_{k,\tilde{\delta},x_i}} g(x, u)\xi^{p_+}(u - k) dx.$$

Here, we denote

$$J = \int_{A_{k,\tilde{\delta},x_i}} \xi^{p_+} |\nabla u|^{p(x)} dx.$$

Since $\lambda V(x) + Z(x) \geq M \geq \varsigma$, for $\forall\, x \in \mathbb{R}^N$, we have

$$J \leq p_+ \int_{A_{k,\tilde{\delta},x_i}} \xi^{p_+ - 1}(u - k)|\nabla u|^{p(x)-1} |\nabla \xi| dx \\ - \int_{A_{k,\tilde{\delta},x_i}} \varsigma u^{p(x)-1} \xi^{p_+}(u - k) dx + \int_{A_{k,\tilde{\delta},x_i}} g(x, u)\xi^{p_+}(u - k) dx.$$

From the above inequality, (3.1) and (3.4), it follows that

$$
\begin{aligned}
J \le p_+ &\int_{A_{k,\tilde{\delta},x_i}} \xi^{p_+-1}(u-k)|\nabla u|^{p(x)-1}|\nabla\xi| dx \\
&- \int_{A_{k,\tilde{\delta},x_i}} \varsigma u^{p(x)-1}\xi^{p_+}(u-k)dx \\
&+ \int_{A_{k,\tilde{\delta},x_i}} \varsigma u^{p(x)-1}\xi^{p_+}(u-k)dx \\
&+ \int_{A_{k,\tilde{\delta},x_i}} (C_\varsigma \alpha u^{m(x)-1} + u^{q(x)-1})\xi^{p_+}(u-k)dx,
\end{aligned}
$$

from where it follows

$$
\begin{aligned}
J \le p_+ &\int_{A_{k,\tilde{\delta},x_i}} \xi^{p_+-1}(u-k)|\nabla u|^{p(x)-1}|\nabla\xi| dx \\
&+ \int_{A_{k,\tilde{\delta},x_i}} C_\varsigma u^{m(x)-1}(u-k)dx + \int_{A_{k,\tilde{\delta},x_i}} u^{q(x)-1}(u-k)dx.
\end{aligned}
$$

Using Young's inequality, for $\chi \in (0,1)$, we obtain

$$
\begin{aligned}
J \le{}& \frac{p_+(p_+-1)}{p_-}\chi^{\frac{p_-}{p_+-1}} J + \frac{2^{p_+}p_+}{p_-}\chi^{-p_+} \int_{A_{k,\tilde{\delta},x_i}} \left(\frac{u-k}{\tilde{\delta}-\bar{\delta}}\right)^{p(x)} dx \\
&+ \frac{C_\varsigma(m_+-1)}{m_-}\int_{A_{k,\tilde{\delta},x_i}} u^{m(x)}dx + \frac{C_\varsigma(1+\delta_{x_i}^{m_+})}{m_-}\int_{A_{k,\tilde{\delta},x_i}} \left(\frac{u-k}{\tilde{\delta}-\bar{\delta}}\right)^{m(x)} \\
&+ \frac{(q_+-1)}{q_-}\int_{A_{k,\tilde{\delta},x_i}} u^{q(x)}dx + \frac{(1+\delta_{x_i}^{q_+})}{q_-}\int_{A_{k,\tilde{\delta},x_i}} \left(\frac{u-k}{\tilde{\delta}-\bar{\delta}}\right)^{q(x)}.
\end{aligned}
$$

Writing

$$
Q = \int_{A_{k,\tilde{\delta},x_i}} \left(\frac{u-k}{\tilde{\delta}-\bar{\delta}}\right)^{(p_-^{x_i})^*},
$$

for $\chi \approx 0^+$, by (5.1), we have

$$
\begin{aligned}
J \le{}& \frac{1}{2}\chi^{\frac{p_-}{p_+-1}} J + \frac{2^{p_+}p_+}{p_-}\chi^{-p_+}(|A_{k,\tilde{\delta},x_i}| + Q) \\
&+ \frac{C_\varsigma 2^{m_+}(m_+-1)(1+\delta_{x_i}^{m_+})}{m_-}(|A_{k,\tilde{\delta},x_i}| + Q) \\
&+ \frac{C_\varsigma 2^{m_+}(m_+-1)(1+k^{m_+})}{m_-}|A_{k,\tilde{\delta},x_i}|
\end{aligned}
$$

$$+\frac{C_\varsigma(1+\delta_{x_i}^{m_+})}{m_-}(|A_{k,\tilde{\delta},x_i}|+Q)+\frac{(1+\delta_{x_i}^{q_+})}{q_-}(|A_{k,\tilde{\delta},x_i}|+Q)$$
$$+\frac{2^{q_+}(q_+-1)(1+\delta_{x_i}^{q_+})}{q_-}(|A_{k,\tilde{\delta},x_i}|+Q)$$
$$+\frac{C_\varsigma 2^{q_+}(q_+-1)(1+k^{q_+})}{q_-}|A_{k,\tilde{\delta},x_i}|.$$

Therefore,

$$\int_{A_{k,\tilde{\delta},x_i}}|\nabla u|^{p(x)}\le J\le C[(k^{q_+}+1)|A_{k,\tilde{\delta},x_i}|+Q],$$

for a positive constant $C=C(p_-,p_+,m_-,m_+,q_-,q_+,\varsigma,\delta_{x_i})$ which does not depend on k. Since

$$|\nabla u|^{p_-^{x_i}}-1\le|\nabla u|^{p(x)},\quad\forall\, x\in B_{\delta_{x_i}}(x_i),$$

we obtain

$$\int_{A_{k,\tilde{\delta},x_i}}|\nabla u|^{p_-^{x_i}}\le C[(k^{q_+}+1)|A_{k,\tilde{\delta},x_i}|+Q]+|A_{k,\tilde{\delta},x_i}|$$
$$\le C\left((k^{q_+}+2)|A_{k,\tilde{\delta},x_i}|+(\tilde{\delta}-\bar{\delta})^{-(p_-^{x_i})^*}\int_{A_{k,\tilde{\delta},x_i}}(u_\lambda-k)^{(p_-^{x_i})^*}\,dx\right),$$

for a positive constant $C=C(p_-,p_+,m_-,m_+,q_-,q_+,\varsigma,\delta_{x_i})$ which does not depend on k. □

In order to prove the desired result, the following lemma is needed, see [28].

Lemma 5.4. *Let (J_n) be a sequence of nonnegative numbers satisfying*

$$J_{n+1}\le CB^nJ_n^{1+\eta},\quad n=0,1,2,\ldots,$$

where $C,\eta>0$ and $B>1$. If

$$J_0\le C^{-\frac{1}{\eta}}B^{-\frac{1}{\eta^2}},$$

then $J_n\to 0$, as $n\to\infty$.

Lemma 5.5. *Let (u_λ) be a family of solutions for problem (A_λ) such that*

$$\sup_{\lambda\ge 1}\Phi_\lambda(u_\lambda)<\left(\frac{1}{p_+}-\frac{1}{\beta}\right)\inf_{x\in\mathcal{A}}S_x^N$$

and $u_\lambda\to 0$ in $W^{1,p(x)}(\mathbb{R}^N\backslash\Omega_\Gamma)$ as $\lambda\to\infty$. Then, there exists $\lambda^*>0$ such that $|u_\lambda|_{L^\infty(\mathcal{N}(\partial\Omega_\Gamma'))}\le a_-$ for $\lambda\ge\lambda^*$.

Proof. It is enough to prove the inequality in each ball $B_{\frac{\delta_{x_i}}{2}}(x_i)$, $i = 1, \ldots, l$, given by Lemma 5.2. We set

$$\tilde{\delta}_n = \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2^{n+1}}, \quad \bar{\delta}_n = \frac{\tilde{\delta}_n + \tilde{\delta}_{n+1}}{2},$$

$$k_n = \frac{a_-}{2}\left(1 - \frac{1}{2^{n+1}}\right), \quad \forall\, n = 0, 1, 2, \ldots.$$

Then,

$$\tilde{\delta}_n \downarrow \frac{\delta_{x_i}}{2}, \quad \tilde{\delta}_{n+1} < \bar{\delta}_n < \tilde{\delta}_n, \quad k_n \uparrow \frac{a_-}{2}.$$

Now, we fix

$$J_n = \int_{A_{k_n, \tilde{\delta}_n, x_i}} (u_\lambda(x) - k_n)^{(p_-^{x_i})^*}, \quad n = 0, 1, 2, \ldots$$

and $\xi \in C^1(\mathbb{R})$ such that

$$0 \le \xi \le 1, \quad \xi(t) = 1, \quad \text{for } t \le \frac{1}{2}, \quad \text{and} \quad \xi(t) = 0, \quad \text{for } t \ge \frac{3}{4}.$$

Set

$$\xi_n(x) = \xi\left(\frac{2^{n+1}}{\delta_{x_i}}\left(|x - x_i| - \frac{\delta_{x_i}}{2}\right)\right), \quad x \in \mathbb{R}^N, \ \ n = 0, 1, 2, \ldots,$$

one has $\xi_n = 1$ in $B_{\tilde{\delta}_{n+1}}(x_i)$ and $\xi_n = 0$ in $\mathbb{R}^N \backslash B_{\bar{\delta}_n}(x_i)$. Denote u_λ by u, we have

$$\begin{aligned}
J_{n+1} &\le \int_{A_{k_{n+1}, \bar{\delta}_n, x_i}} ((u(x) - k_{n+1})\xi_n(x))^{(p_-^{x_i})^*}\, dx \\
&= \int_{B_{\delta_{x_i}}(x_i)} ((u(x) - k_{n+1})^+ \xi_n(x))^{(p_-^{x_i})^*}\, dx \\
&\le C(N, p_-^{x_i}) \left(\int_{B_{\delta_{x_i}}(x_i)} |\nabla((u(x) - k_{n+1})^+ \xi_n)(x)|^{p_-^{x_i}}\, dx\right)^{\frac{(p_-^{x_i})^*}{p_-^{x_i}}} \\
&\le C(N, p_-^{x_i}) \left(\int_{A_{k_{n+1}, \bar{\delta}_n, x_i}} |\nabla u|^{p_-^{x_i}}\, dx\right. \\
&\qquad \left. + \int_{A_{k_{n+1}, \bar{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} |\nabla \xi_n|^{p_-^{x_i}}\, dx\right)^{\frac{(p_-^{x_i})^*}{p_-^{x_i}}}.
\end{aligned}$$

Since

$$|\nabla \xi_n| \le C\delta_{x_i} 2^{n+1}, \quad \forall\, x \in \mathbb{R}^N,$$

and writing $J_{n+1}^{\frac{p_-^{x_i}}{(p_-^{x_i})^*}} = \tilde{J}_{n+1}$, we obtain

$$\tilde{J}_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}) \left(\int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} |\nabla u|^{p_-^{x_i}} dx + 2^{np_-^{x_i}} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \right).$$

Using Lemma 5.3,

$$\begin{aligned} \tilde{J}_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}) \Bigg(&(k_{n+1}^{q_+} + 2)|A_{k_{n+1}, \tilde{\delta}_n, x_i}| \\ &+ \left(\frac{2^{n+3}}{\delta_{x_i}} \right)^{(p_-^{x_i})^*} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{(p_-^{x_i})^*} \\ &+ 2^{np_-^{x_i}} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \Bigg) \\ \leq C(N, p_-^{x_i}, \delta_{x_i}) \Bigg(&(k_{n+1}^{q_+} + 2)|A_{k_{n+1}, \tilde{\delta}_n, x_i}| \\ &+ 2^{n(p_-^{x_i})^*} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{(p_-^{x_i})^*} \\ &+ 2^{np_-^{x_i}} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \Bigg). \end{aligned}$$

From Young's inequality

$$\begin{aligned} &\int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} dx \\ &\quad \leq C p_-^{x_i} \left(|A_{k_{n+1}, \tilde{\delta}_n, x_i}| + \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{(p_-^{x_i})^*} \right). \end{aligned}$$

Thus

$$\begin{aligned} \tilde{J}_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}) \Bigg(&\left(\left(\frac{a_-}{2} \right)^{q_+} + 2 + 2^{np_-^{x_i}} \right) |A_{k_{n+1}, \tilde{\delta}_n, x_i}| \\ &+ 2^{n(p_-^{x_i})^*} J_n + 2^{np_-^{x_i}} J_n \Bigg). \end{aligned}$$

Now, since

$$J_n \geq \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_n)^{(p_-^{x_i})^*} dx \geq (k_{n+1} - k_n)^{(p_-^{x_i})^*} |A_{k_{n+1}, \tilde{\delta}_n, x_i}|,$$

it follows that

$$|A_{k_{n+1}, \tilde{\delta}_n, x_i}| \leq \left(\frac{2^{n+3}}{a_-} \right)^{(p_-^{x_i})^*} J_n,$$

and thus,

$$\tilde{J}_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+)(2^{n(p_-^{x_i})^*} J_n + 2^{n(p_-^{x_i}+(p_-^{x_i})^*)} J_n + 2^{n(p_-^{x_i})^*} J_n + 2^{np_-^{x_i}} J_n).$$

Fixing $\alpha = (p_-^{x_i} + (p_-^{x_i})^*)$, it follows that

$$J_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+)\Big(2^{\alpha\frac{(p_-^{x_i})^*}{p_-^{x_i}}}\Big)^n J_n^{\frac{(p_-^{x_i})^*}{p_-^{x_i}}},$$

and thus

$$J_{n+1} \leq CB^n J_n^{1+\eta},$$

where $C = C(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+)$, $B = 2^{\alpha\frac{(p_-^{x_i})^*}{p_-^{x_i}}}$ and $\eta = \frac{(p_-^{x_i})^*}{p_-^{x_i}} - 1$. Now, since $u_\lambda \to 0$ in $W^{1,p(x)}(\mathbb{R}^N \backslash \Omega_\Gamma)$ as $\lambda \to \infty$, there exists $\lambda_i > 0$ such that

$$\int_{A_{\frac{a_-}{4}, \delta_{x_i}, x_i}} \Big(u_\lambda - \frac{a_-}{4}\Big)^{(p_-^{x_i})^*} dx = J_0(\lambda) \leq C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}}, \quad \lambda \geq \lambda_i.$$

From Lemma 5.4, $J_n(\lambda) \to 0$ as $n \to \infty$, for all $\lambda \geq \lambda_i$, and so

$$u_\lambda \leq \frac{a_-}{2} < a_-, \quad \text{in } B_{\frac{\delta_{x_i}}{2}}, \quad \text{for all } \lambda \geq \lambda_i.$$

Now, taking $\lambda^* = \max\{\lambda_1, \ldots, \lambda_l\}$, we obtain that

$$|u_\lambda|_{L^\infty(\mathcal{N}(\partial\Omega'_\Gamma))} \leq a_-, \quad \text{for } \forall\, \lambda \geq \lambda^*. \qquad \square$$

Proof of Lemma 5.1. Fix $\lambda \geq \lambda^*$, where λ^* is given in Lemma 5.5, and define $\tilde{u}_\lambda : \mathbb{R}^N \backslash \Omega'_\Gamma \to \mathbb{R}$ given by

$$\tilde{u}_\lambda(x) = (u_\lambda - a_-)^+(x).$$

From Lemma 5.5, we know that $\tilde{u}_\lambda \in W_0^{1,p(x)}(\mathbb{R}^N \backslash \Omega'_\Gamma)$. Now, we are going to prove that $\tilde{u}_\lambda = 0$ in $\mathbb{R}^N \backslash \Omega'_\Gamma$. It implies

$$|u_\lambda|_{\infty, \mathbb{R}^N \backslash \Omega'_\Gamma} \leq a_-.$$

Here, we may extend $\tilde{u}_\lambda(x) = 0$ in Ω'_Γ and take $\tilde{u}_\lambda$ as a test function, it yields

$$\int_{\mathbb{R}^N \backslash \Omega'_\Gamma} |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda \nabla \tilde{u}_\lambda dx + \int_{\mathbb{R}^N \backslash \Omega'_\Gamma} (\lambda V(x) + Z(x))|u_\lambda|^{p(x)-2} u_\lambda \tilde{u}_\lambda dx = \int_{\mathbb{R}^N \backslash \Omega'_\Gamma} g(x, u_\lambda)\tilde{u}_\lambda dx.$$

Since

$$\int_{\mathbb{R}^N\setminus\Omega'_\Gamma} |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda \nabla \tilde{u}_\lambda dx = \int_{\mathbb{R}^N\setminus\Omega'_\Gamma} |\nabla \tilde{u}_\lambda|^{p(x)} dx,$$

$$\int_{\mathbb{R}^N\setminus\Omega'_\Gamma} (\lambda V(x) + Z(x))|u_\lambda|^{p(x)-2} u_\lambda \tilde{u}_\lambda dx$$

$$= \int_{(\mathbb{R}^N\setminus\Omega'_\Gamma)_+} (\lambda V(x) + Z(x))|u_\lambda|^{p(x)-2}(\tilde{u}_\lambda + a_-)\tilde{u}_\lambda dx,$$

and

$$\int_{\mathbb{R}^N\setminus\Omega'_\Gamma} g(x, u_\lambda)\tilde{u}_\lambda dx = \int_{\mathbb{R}^N\setminus\Omega'_\Gamma} \tilde{f}(x, u_\lambda)\tilde{u}_\lambda dx$$

$$= \int_{(\mathbb{R}^N\setminus\Omega'_\Gamma)_+} \frac{\tilde{f}(x, u_\lambda)}{u_\lambda}(\tilde{u}_\lambda + a_-)\tilde{u}_\lambda,$$

where

$$(\mathbb{R}^N\setminus\Omega'_\Gamma)_+ = \{x \in \mathbb{R}^N\setminus\Omega'_\Gamma : u_\lambda(x) > a_-\}.$$

From the above equalities, we obtain that

$$\int_{\mathbb{R}^N\setminus\Omega'_\Gamma} |\nabla \tilde{u}_\lambda|^{p(x)} dx$$

$$+ \int_{(\mathbb{R}^N\setminus\Omega'_\Gamma)_+} \left((\lambda V(x) + Z(x))|u_\lambda|^{p(x)-2} - \frac{\tilde{f}(x, u_\lambda)}{u_\lambda} \right)(\tilde{u}_\lambda + a_-)\tilde{u}_\lambda = 0.$$

In virtue of (3.4), we have

$$(\lambda V(x) + Z(x))|u_\lambda|^{p(x)-2} - \frac{\tilde{f}(x, u_\lambda)}{u_\lambda} \geq (M - \varsigma)|u_\lambda|^{p(x)-2}$$

$$\geq 0, \quad \text{in } (\mathbb{R}^N\setminus\Omega'_\Gamma)_+.$$

Thus, $\tilde{u}_\lambda = 0$ in $(\mathbb{R}^N\setminus\Omega'_\Gamma)_+$, and $\tilde{u}_\lambda = 0$ in $\mathbb{R}^N\setminus\Omega'_\Gamma$. The proof is complete. □

6. A Special Critical Value for Φ_λ

In this section, for each $\lambda \geq 1$ and $j \in \Gamma$, let us denote by $\Phi_{\lambda,j} : W^{1,p(x)}(\Omega'_j) \to \mathbb{R}$ the functional

$$\Phi_{\lambda,j}(u) = \int_{\Omega'_j} \frac{1}{p(x)}(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)})dx$$

$$- \int_{\Omega'_j} F(x, u)dx - \int_{\Omega'_j} \frac{1}{q(x)}|u|^{q(x)}dx.$$

We know that the critical points of $\Phi_{\lambda,j}$ are related with the weak solutions to the following problems:

$$\begin{cases} -\Delta_{p(x)}u + (\lambda V(x) + Z(x))u = f(x,u) + |u|^{q(x)-2}u, & \text{in } \Omega'_j, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega'_j. \end{cases} \tag{6.1}$$

It is easy to check that the functional $\Phi_{\lambda,j}$ satisfies the mountain pass geometry. In what follows, we denote by $c_{\lambda,j}$ the minimax level related to the above functional defined by

$$c_{\lambda,j} = \inf_{\gamma\in\Lambda_{\lambda,j}} \max_{t\in[0,1]} \Phi_{\lambda,j}(\gamma(t)),$$

where

$$\Lambda_{\lambda,j} = \{\gamma \in C([0,1], W^{1,p(x)}(\Omega'_j)) : \gamma(0) = 0, \Phi_{\lambda,j}(\gamma(1)) < 0\}.$$

If α is large enough, similar to the arguments in Lemmas 3.4 and 3.5, we know that the functional I_j and $\Phi_{\lambda,j}$ satisfy the $(\mathrm{PS})_{c_j}$ and $(\mathrm{PS})_{c_{\lambda,j}}$ conditions, respectively. Therefore, it implies that there exist two nonnegative functions $w_j \in W_0^{1,p(x)}(\Omega_j)$ and $w_{\lambda,j} \in W^{1,p(x)}(\Omega'_j)$ verifying

$$I_j(w_j) = c_j \quad \text{and} \quad I'_j(w_j) = 0,$$

and

$$\Phi_{\lambda,j}(w_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \Phi'_{\lambda,j}(w_{\lambda,j}) = 0.$$

Moreover, we have the following lemma.

Lemma 6.1. (i) $0 < c_{\lambda,j} \le c_j$, *for* $\lambda \ge 1$, $\forall\, j \in \{1,2,\ldots,k\}$.

(ii) c_j ($c_{\lambda,j}$, *respectively*) *is a least energy level for* $I_j(u)$ ($\Phi_{\lambda,j}(u)$, *respectively*), *that is*

$$c_j = \inf\{I_j(u) : u \in W_0^{1,p(x)}(\Omega_j)\backslash\{0\}, I'_j(u)u = 0\},$$

and

$$c_{\lambda,j} = \inf\{\Phi_{\lambda,j}(u) : u \in W^{1,p(x)}(\Omega'_j)\backslash\{0\}, \Phi'_{\lambda,j}(u)u = 0\}.$$

(iii) $c_j = \max_{t>0} I_j(tw_j)$, $c_{\lambda,j} = \max_{t>0} \Phi_{\lambda,j}(tw_{\lambda,j})$.

(iv) $c_{\lambda,j} \to c_j$ *as* $j \to \infty$.

Proof. For any $u \in W_0^{1,p(x)}(\Omega_j)$, we may extend u to $\tilde{u} \in W^{1,p(x)}(\Omega'_j)$ by

$$\tilde{u}(x) := \begin{cases} u(x), & \text{in } \Omega_j, \\ 0, & \text{in } \Omega'_j\backslash\Omega_j, \end{cases}$$

and $W_0^{1,p(x)}(\Omega_j) \subset W^{1,p(x)}(\Omega'_j)$. Thus, we have $\Lambda_j \subset \Lambda_{\lambda,j}$ and

$$\begin{aligned} c_{\lambda,j} &= \inf_{\gamma\in\Lambda_{\lambda,j}} \max_{t\in[0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &\le \inf_{\gamma\in\Lambda_j} \max_{t\in[0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &= \inf_{\gamma\in\Lambda_j} \max_{t\in[0,1]} I_j(\gamma(t)) = c_j. \end{aligned}$$

Thus (i) holds. The proof of (ii) and (iii) is standard by using (f_4).

Now, we prove (iv). Using Lemma 4.1, we may extract a subsequence $\lambda_n \to \infty$ such that

$$w_{\lambda_n,j} \to u_0, \quad \text{strongly in } W^{1,p(x)}(\Omega'_j),$$

where $u_0 \in W_0^{1,p(x)}(\Omega_j)$ is a solution of (P_λ) and

$$\Phi_{\lambda_n,j}(w_{\lambda_n,j}) \to I_j(u_0).$$

By the definition of c_j, we have

$$\limsup_{\lambda\to\infty} c_{\lambda,j} = \limsup_{\lambda\to\infty} \Phi_{\lambda,j}(w_{\lambda,j}) \ge I_j(u_0) \ge c_j.$$

Together with (i), we get (iv). □

In what follows, let us fix $R > 1$ such that

$$\left| I_j\left(\frac{1}{R}w_j\right)\right| < \frac{1}{2}c_j, \quad \forall\, j \in \Gamma$$

and

$$|I_j(Rw_j) - c_j| \ge 1, \quad \forall\, j \in \Gamma.$$

From the definition of c_j, it is easy to check that

$$\max_{s\in[\frac{1}{R^2},1]} I_j(sRw_j) = c_j, \quad \forall\, j \in \Gamma.$$

We consider $\Gamma = \{1, 2, \ldots, l\}(l \le k)$, and the maps

$$\gamma_0(s_1, s_2, \ldots, s_l)(x) = \sum_{j=1}^{l} s_j R w_j(x) \quad \forall (s_1, s_2, \ldots, s_l) \in \left[\frac{1}{R^2}, 1\right]^l, \tag{6.2}$$

$$\Lambda_* = \left\{\gamma \in C\left(\left[\frac{1}{R^2}, 1\right]^l, E_\lambda\backslash\{0\}\right) : \gamma = \gamma_0 \text{ on } \partial\left(\left[\frac{1}{R^2}, 1\right]^l\right)\right\}, \tag{6.3}$$

and

$$b_{\lambda,\Gamma} = \inf_{\gamma\in\Lambda_*} \max_{(s_1,s_2,\ldots,s_l)\in[\frac{1}{R^2},1]^l} \Phi_\lambda(\gamma(s_1, s_2, \ldots, s_l)).$$

We remark that $\gamma_0 \in \Lambda_*$, so $\Lambda_* \neq \emptyset$ and $b_{\lambda,\Gamma}$ is well defined.

Lemma 6.2. *For any* $\gamma \in \Lambda_*$, *there exists* $(t_1, t_2, \ldots, t_l) \in [\frac{1}{R^2}, 1]^l \to \mathbb{R}^l$ *such that*

$$\Phi'_{\lambda,j}(\gamma(t_1, t_2, \ldots, t_l))(\gamma(t_1, t_2, \ldots, t_l)) = 0, \quad \forall\, j \in \{1, 2, \ldots, l\}.$$

Proof. For a given $\gamma \in \Lambda_*$, let us consider the map $\tilde{\gamma} : [\frac{1}{R^2}, 1]^l \to \mathbb{R}^l$ defined by

$$\tilde{\gamma}(s_1, s_2, \ldots, s_l) = (\Phi'_{\lambda,1}(\gamma)(\gamma), \Phi'_{\lambda,2}(\gamma)(\gamma), \ldots, \Phi'_{\lambda,l}(\gamma)(\gamma)),$$

where

$$\Phi'_{\lambda,j}(\gamma)(\gamma) = \Phi'_{\lambda,j}(\gamma(s_1, s_2, \ldots, s_l))(\gamma(s_1, s_2, \ldots, s_l)), \quad \text{for all } j \in \Gamma.$$

For any $(s_1, s_2, \ldots, s_l) \in \partial([\frac{1}{R^2}, 1]^l)$, it follows that

$$\gamma(s_1, s_2, \ldots, s_l) = \gamma_0(s_1, s_2, \ldots, s_l).$$

Then

$$\Phi'_{\lambda,j}(\gamma_0(s_1, s_2, \ldots, s_l))(\gamma_0(s_1, s_2, \ldots, s_l)) = 0.$$

It implies that $s_j \notin \{\frac{1}{R^2}, 1\}$ for all $j \in \Gamma$. Otherwise,

$$\Phi'_{\lambda,j}(\gamma_0(s_1, s_2, \ldots, s_l))(\gamma_0(s_1, s_2, \ldots, s_l)) = 0,$$

for $s_j = \frac{1}{R^2}$ or $s_j = 1$, that is

$$I'_j\left(\frac{1}{R}w_j\right)\left(\frac{1}{R}w_j\right) = 0 \quad \text{or} \quad I'_j(Rw_j)(Rw_j) = 0$$

implying that

$$I_j\left(\frac{1}{R}w_j\right) \geq c_j \quad \text{or} \quad I_j(Rw_j) \geq c_j,$$

which contradicts the choice of R. Thus,

$$(0, 0, \ldots, 0) \notin \tilde{\gamma}\left(\partial\left(\left[\frac{1}{R^2}, 1\right]^l\right)\right).$$

Using this fact, it follows from the topological degree

$$\deg\left(\tilde{\gamma}, \left(\frac{1}{R^2}, 1\right)^l, (0, 0, \ldots, 0)\right) = (-1)^l \neq 0.$$

Hence, there exists $(t_1, t_2, \ldots, t_l) \in (\frac{1}{R^2}, 1)^l$ satisfying

$$\Phi'_{\lambda,j}(\gamma(t_1, t_2, \ldots, t_l))(\gamma(t_1, t_2, \ldots, t_l)) = 0, \quad \text{for all } j \in \{1, 2, \ldots, l\}.$$

The proof is completed. □

In the sequel, the number $c_\Gamma = \sum_{j=1}^{l} c_j \in (0, (\frac{1}{p_+} - \frac{1}{\beta}) \inf_{x \in \mathcal{A}} S_x^N)$ (see Remark 3.1) is very important in the proof of Theorem 1.1. Now, we show the relation among $\sum_{j=1}^{l} c_{\lambda,j}$, $b_{\lambda,\Gamma}$ and c_Γ.

Lemma 6.3. *The following facts hold*:

(i) $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,\Gamma} \leq c_\Gamma$ *for all* $\lambda \geq 1$.
(ii) $\Phi_\lambda(\gamma(s_1, s_2, \ldots, s_l)) < c_\Gamma$ *for all* $\lambda \geq 1$, $\gamma \in \Lambda_*$ *and* $(s_1, s_2, \ldots, s_l) \in \partial([\frac{1}{R^2}, 1]^l)$.

Proof. (i) Since γ_0 defined in (6.2) belongs to Λ_*, we have

$$\begin{aligned} b_{\lambda,\Gamma} &\leq \max_{(s_1,s_2,\ldots,s_l)\in[\frac{1}{R^2},1]^l} \Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) \\ &= \max_{(s_1,s_2,\ldots,s_l)\in[\frac{1}{R^2},1]^l} \sum_{j=1}^{l} I_j(sRw_j) \\ &= \sum_{j=1}^{l} c_j = c_\Gamma. \end{aligned}$$

Fixing $(t_1, t_2, \ldots, t_l) \in [\frac{1}{R^2}, 1]^l$ given in Lemma 6.2 and recalling that $c_{\lambda,j}$ can be characterized by

$$c_{\lambda,j} = \inf\{\Phi_{\lambda,j}(u) : u \in W^{1,p(x)}(\Omega'_j)\backslash\{0\}, \Phi'_{\lambda,j}(u)u = 0\}.$$

It follows that

$$\Phi_{\lambda,j}(\gamma(t_1, t_2, \ldots, t_l)) \geq c_{\lambda,j}, \quad \forall\, j \in \Gamma.$$

On the other hand, from (3.6), $\Phi_{\lambda,\mathbb{R}^N\backslash\Omega'_\Gamma}(u) \geq 0$ for all $u \in W^{1,p(x)}(\mathbb{R}^N\backslash\Omega'_\Gamma)$ which yields

$$\Phi_\lambda(\gamma(s_1, s_2, \ldots, s_l)) \geq \sum_{j=1}^{l} \Phi_{\lambda,j}(\gamma(s_1, s_2, \ldots, s_l)), \quad \forall (s_1, s_2, \ldots, s_l) \in \left[\frac{1}{R^2}, 1\right]^l.$$

Thus

$$\max_{(s_1,s_2,\ldots,s_l)\in[\frac{1}{R^2},1]^l} \Phi_\lambda(\gamma(s_1, s_2, \ldots, s_l)) \geq \Phi_\lambda(\gamma(t_1, t_2, \ldots, t_l)) \geq \sum_{j=1}^{l} c_{\lambda,j}.$$

From the definition of $b_{\lambda,\Gamma}$, we can obtain

$$b_{\lambda,\Gamma} \geq \sum_{j=1}^{l} c_{\lambda,j}.$$

(ii) Since $\gamma(s_1, s_2, \ldots, s_l) = \gamma_0(s_1, s_2, \ldots, s_l)$ on $\partial([\frac{1}{R^2}, 1]^l)$, we have

$$\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) = \sum_{j=1}^{l} I_j(s_j Rw_j).$$

Moreover, $I_j(s_j Rw_j) \leq c_j$ for all $j \in \Gamma$ and for some $j_0 \in \Gamma$, $s_{j_0} \in \{\frac{1}{R^2}, 1\}$ and $I_{j_0}(s_{j_0} Rw_{j_0}) \leq \frac{c_{j_0}}{2}$. Therefore,

$$\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) \leq c_\Gamma - \epsilon,$$

for some $\epsilon > 0$. This completes the proof of Lemma 6.3(ii). ☐

Corollary 6.1. (i) $b_{\lambda,\Gamma} \to c_\Gamma$ *as* $\lambda \to \infty$.

(ii) $b_{\lambda,\Gamma}$ *is a critical value of* Φ_λ *for large* λ.

Proof. (i) For all $\lambda \geq 1$ and for each j, we have $0 < c_{\lambda,j} \leq c_j$. Using the same arguments in the proof of Lemma 4.1, we can prove that $c_{\lambda,j} \to c_j$ as $\lambda \to \infty$ and thus, from Lemma 6.3, $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.

(ii) By Corollaries 6.1 and 3.13, we may choose λ large such that

$$b_{\lambda,\Gamma}, \quad c_\Gamma \in \left(0, \left(\frac{1}{p_+} - \frac{1}{\beta}\right) \inf_{x\in\mathcal{A}} S_x^N\right).$$

Lemma 3.5 implies that any $(\mathrm{PS})_{b_{\lambda,\Gamma}}$ sequence of the functional Φ_λ has a strongly convergent subsequence in E_λ. We can use well-known arguments involving deformation lemma [41] to conclude that $b_{\lambda,\Gamma}$ is a critical level to Φ_λ for large λ. $\square$

7. The Proof of Main Theorem

To prove Theorem 1.1, we need to find positive solution u_λ for a large λ, which approaches a least energy solution in each $\Omega_j (j \in \Gamma)$ and vanishes elsewhere as $\lambda \to \infty$. To this end, we will prove two propositions that, together with the estimates made in the previous section, imply that Theorem 1.1 holds.

Hereafter, we denote by

$$r = R^{p_+} \sum_{j=1}^{l} \left(\frac{1}{p_+} - \frac{1}{\beta}\right)^{-1} c_j,$$

$$\overline{B}_r^\lambda = \{u \in E_\lambda : \varrho_\lambda(u) \leq r\}.$$

For small $\mu > 0$, we define

$$A_\mu^\lambda = \{u \in \overline{B}_r^\lambda : \varrho_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma}(u) \leq \mu, |\Phi_{\lambda,j}(u) - c_j| \leq \mu, \forall j \in \Gamma\}.$$

We also use the notation

$$\Phi_\lambda^{c_\Gamma} = \{u \in E_\lambda : \Phi_\lambda(u) \leq c_\Gamma\}$$

and note that $w = \sum_{j=1}^l w_j \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$ which shows that $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma} \neq \emptyset$. Fixing

$$0 < \mu < \frac{1}{4} \min_{j\in\Gamma} c_j. \tag{7.1}$$

We have the following uniform estimate of $\|\Phi'_\lambda(u)\|$ on the annulus $(A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}$.

Proposition 7.1. *Let* $\mu > 0$ *satisfy* (7.1). *Then there exist* $\sigma_0 > 0$ *and* $\lambda^* \geq 1$ *independent of* λ *such that*

$$\|\Phi'_\lambda(u)\| \geq \sigma_0 \quad \textit{for } \lambda \geq \lambda^* \quad \textit{and all } u \in (A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n \to \infty$ and $u_n \in (A_{2\mu}^{\lambda_n} \setminus A_\mu^{\lambda_n}) \cap \Phi_{\lambda_n}^{c_\Gamma}$ such that $\|\Phi'_{\lambda_n}(u_n)\| \to 0$. Since $u_n \in A_{2\mu}^{\lambda_n}$, this implies that $(\varrho_\lambda(u_n))$ is bounded sequence, so $(\Phi_{\lambda_n}(u_n))$ is also bounded. Thus, up to a subsequence, we may assume that

$$\Phi_{\lambda_n}(u_n) \to c \le c_\Gamma.$$

Applying Lemma 4.1, we can extract a subsequence (u_n) such that $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$ where $u \in W_0^{1,p(x)}(\Omega_\Gamma)$, and $u|_{\Omega_j}$, $j \in \Gamma$ is a nonnegative solution for (P_j) with

$$\varrho_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Gamma}(u_n) \to 0, \quad \Phi_{\lambda_n,j}(u_n) \to I_j(u).$$

Since c_j is the least energy level for I_j, we have two possibilities:

(i) $I_j(u) = c_j$, for all $j \in \Gamma$.
(ii) $I_{j_0}(u) = 0$, that is $u|_{\Omega_{j_0}} \equiv 0$ for some $j_0 \in \Gamma$.

If (i) occurs, then for n large, we have

$$\varrho_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Gamma}(u_n) \le \mu \quad \text{and} \quad |\Phi_{\lambda_n,j}(u_n) - c_j| \le \mu, \quad \forall\, j \in \Gamma.$$

So, $u_n \in A_\mu^{\lambda_n}$ for large n, which is a contradiction to $u_n \in (A_{2\mu}^{\lambda_n}\setminus A_\mu^{\lambda_n})$.

If (ii) occurs, it follows that

$$|\Phi_{\lambda_n,j_0}(u_n) - c_{j_0}| \to c_{j_0} > 4\mu$$

which is a contradiction with the fact that $u_n \in A_{2\mu}^{\lambda_n}$. Thus neither (i) nor (ii) can hold, and the proof is completed. □

Proposition 7.2. *Let $\mu > 0$ satisfy* (7.1) *and $\lambda^* \ge 1$ be a constant given by in Proposition* 7.1. *Then, for $\lambda \ge \lambda^*$, there exists a positive solution u_λ of (A_λ) satisfying $u_\lambda \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$.*

Proof. Assuming by contradiction that there are no critical points in $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, since the Palais–Smale condition holds for Φ_λ in $(0, (\frac{1}{p_+} - \frac{1}{\beta})\inf_{x\in\mathcal{A}} S_x^N)$, there exists a constant $d_\lambda > 0$ such that

$$\|\Phi_\lambda'(u)\| \ge d_\lambda, \quad \text{for all } u \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}.$$

From Proposition 7.1, we also have

$$\|\Phi_\lambda'(u)\| \ge \sigma_0, \quad \text{for all } u \in (A_{2\mu}^\lambda\setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma},$$

where $\sigma_0 > 0$ is independent of λ. In what follows, $\Psi : E_\lambda \to \mathbb{R}$ is a continuous functional verifying

$$\begin{aligned} &\Psi(u) = 1 && \text{for } u \in A_{3\mu/2}^\lambda, \\ &\Psi(u) = 0 && \text{for } u \notin A_{2\mu}^\lambda, \\ &0 \le \Psi(u) \le 1 && \text{for } \forall\, u \in E_\lambda, \end{aligned}$$

and $H : \Phi_\lambda^{c_\Gamma} \to E_\lambda$ verifies

$$H(u) := \begin{cases} -\Psi(u)\dfrac{Y(u)}{\|Y(u)\|}, & u \in A_{2\mu}^\lambda, \\ 0, & u \notin A_{2\mu}^\lambda, \end{cases}$$

where Y is a pseudo-gradient vector field for Φ_λ on $\mathcal{M} = \{u \in E_\lambda : \Phi_\lambda'(u) \neq 0\}$. Thus, using the properties involving Y and Φ_λ, we have the following inequality:

$$\|H(u)\| \leq 1, \quad \forall\, \lambda \geq \lambda^* \quad \text{and} \quad u \in \Phi_\lambda^{c_\Gamma}.$$

Considering the deformation flow $\eta : [0, \infty) \times \Phi_\lambda^{c_\Gamma} \to \Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{d\eta}{dt} = H(\eta) \quad \text{and} \quad \eta(0, u) = u \in \Phi_\lambda^{c_\Gamma},$$

we obtain

$$\frac{d}{dt}\Phi_\lambda(\eta(t,u)) \leq -\frac{1}{2}\Psi(\eta(t,u))\|\Phi_\lambda'(\eta(t,u))\| \leq 0, \tag{7.2}$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda = \|H(\eta)\|_\lambda \leq 1, \tag{7.3}$$

$$\eta(t,u) = u \quad \text{for all } t \geq 0 \quad \text{and} \quad u \in \Phi_\lambda^{c_\Gamma} \backslash A_{2\mu}^\lambda. \tag{7.4}$$

Let $\gamma_0(s_1, s_2, \ldots, s_l) \in \Lambda_*$ be a path defined in (6.3) and we consider $\eta(t, \gamma_0(s_1, s_2, \ldots, s_l))$ for large t. Since for all $(s_1, s_2, \ldots, s_l) \in \partial([\frac{1}{R^2}, 1]^l)$, $\gamma_0(s_1, s_2, \ldots, s_l) \notin A_{2\mu}^\lambda$, thus we have by (7.4) that

$$\eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) = \gamma_0(s_1, s_2, \ldots, s_l), \quad \text{for all } (s_1, s_2, \ldots, s_l) \in \partial\left(\left[\frac{1}{R^2}, 1\right]^l\right)$$

and $\eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) \in \Lambda_*$ for all $t \geq 0$.

Since $\operatorname{supp}(\gamma_0(s_1, s_2, \ldots, s_l)(x)) \subset \overline{\Omega}_\Gamma$ for all $(s_1, s_2, \ldots, s_l) \in \partial([\frac{1}{R^2}, 1]^l)$, then $\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l))$ and $\|\gamma_0(s_1, s_2, \ldots, s_l)\|_{\lambda,j}$ etc. do not depend on $\lambda \geq 0$. On the other hand,

$$\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)(x)) \leq c_\Gamma, \quad \forall\, (s_1, s_2, \ldots, s_l) \in \left[\frac{1}{R^2}, 1\right]^l$$

and $\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) = c_\Gamma$ if and only if $s_j = \frac{1}{R}$, that is $\gamma_0(s_1, s_2, \ldots, s_l)(x)|_{\Omega_j} = w_j$ for $j \in \Gamma$. Thus, we have that

$$m_0 := \max\left\{\Phi_\lambda(u) : u \in \gamma_0\left(\left[\frac{1}{R^2}, 1\right]^l\right) \Big\backslash A_\mu^\lambda\right\} \tag{7.5}$$

is independent of λ and $m_0 < c_\Gamma$.

From (7.4), it is easy to see that for any $t > 0$

$$\|\eta(0, \gamma_0(s_1, s_2, \ldots, s_l)) - \eta(t, \gamma_0(s_1, s_2, \ldots, s_l))\|_\lambda \leq t.$$

Since $\Phi_{\lambda,j}(u) \in C^1(E_\lambda)$ for all $j = 1, 2, \ldots, l$, and from the assumptions (f_1)–(f_4), it is easy to see that for a large number $T > 0$, there exists a positive number $\rho_0 > 0$ which is independent of λ such that for all $j = 1, 2, \ldots, l$ and $t \in [0, T]$,

$$\|\Phi_{\lambda,j}(\eta(t, \gamma_0(s_1, s_2, \ldots, s_l))\|_\lambda \leq \rho_0. \tag{7.6}$$

It is easy to know that there exists $K > 0$ such that

$$|\Phi_{\lambda,j}(u) - \Phi_{\lambda,j}(v)| \leq K\|u - v\|_{\lambda,\Omega'_j}, \quad u, v \in \overline{B}_r^\lambda \text{ and } \forall\, j \in \Gamma.$$

We claim that for large T,

$$\max_{(s_1,s_2,\ldots,s_l)\in[\frac{1}{R^2},1]^l} \Phi_\lambda(\eta(T, \gamma_0(s_1, s_2, \ldots, s_l)(x))) < \max\left\{m_0, c_\Gamma - \frac{1}{2K}\sigma_0\mu\right\}, \tag{7.7}$$

where m_0 is given in (7.5).

In fact, if $\gamma_0(s_1, s_2, \ldots, s_l)(x) \notin A_\mu^\lambda$, then by (7.2), we have $\Phi_\lambda(\eta(T, \gamma_0(s_1, s_2, \ldots, s_l)(x))) \leq m_0$ and thus (7.7) holds. If $\gamma_0(s_1, s_2, \ldots, s_l)(x) \in A_\mu^\lambda$, we need to study the behavior of $\tilde{\eta}(t) = \eta(t, \gamma_0(s_1, s_2, \ldots, s_l))$. We set $\tilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}$ and $T = \frac{\sigma_0\mu}{K\tilde{d}_\lambda}$. Now, we distinguish two cases:

(1) $\tilde{\eta}(t) \in A_{3\mu/2}^\lambda$ for all $t \in [0, T]$.

(2) $\tilde{\eta}(t_0) \in \partial A_{3\mu/2}^\lambda$ for some $t_0 \in [0, T]$.

If case (1) holds, we have $\Psi(\tilde{\eta}(t)) \equiv 1$ and $\|\Phi'_\lambda(\tilde{\eta}(t))\| \geq \tilde{d}_\lambda$ for all $t \in [0, T]$. Thus, by (7.2), we have

$$\begin{aligned}
\Phi_\lambda(\tilde{\eta}(T)) &= \Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) + \int_0^T \frac{d}{ds}\Phi_\lambda(\tilde{\eta}(t)) \\
&\leq \Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) - \frac{1}{2}\int_0^T \Psi(\tilde{\eta}(s))\|\Phi'_\lambda(\tilde{\eta}(s))\|ds \\
&\leq c_\Gamma - \int_0^T \tilde{d}_\lambda ds \\
&= c_\Gamma - \frac{1}{2}\tilde{d}_\lambda T \\
&= c_\Gamma - \frac{1}{2}\sigma_0\mu \leq c_\Gamma - \frac{1}{2}\tau_0\mu.
\end{aligned}$$

If (2) holds, there exists $0 \leq t_1 < t_2 \leq T$ such that

$$\tilde{\eta}(t_1) \in \partial A_u^\lambda, \tag{7.8}$$

$$\tilde{\eta}(t_2) \in \partial A_{3\mu/2}^\lambda, \tag{7.9}$$

$$\tilde{\eta}(t) \in A_{3\mu/2}^\lambda \backslash A_u^\lambda, \quad \text{for } \forall\, t \in [t_1, t_2]. \tag{7.10}$$

We show that

$$\|\tilde{\eta}(t_2) - \tilde{\eta}(t_1)\| \geq \frac{\mu}{2K}.$$

It follows from (7.9), for some $j_0 \in \Gamma$,

$$\|\tilde{\eta}(t_2)\|_{\lambda,\mathbb{R}^2\setminus\Omega'_\Gamma} = \frac{3\mu}{2}$$

or

$$|\Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - c_{j_0}| = \frac{3\mu}{2}.$$

Now, we only consider the latter case, the former case can be obtained in a similar way. From the definition of A_μ^λ,

$$|\Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1)) - c_{j_0}| \leq \mu.$$

Thus, we have

$$\begin{aligned}
\|\tilde{\eta}(t_2) - \tilde{\eta}(t_2)\| &\geq \frac{1}{K}|\Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - \Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1))| \\
&\geq \frac{1}{K}(|\Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - c_{j_0}| - |\Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1)) - c_{j_0}|) \\
&\geq \frac{1}{2K}\mu.
\end{aligned}$$

On the other hand, by $t_2 - t_1 \geq \frac{1}{2K}\mu$ and the mean value theorem, there exists $t_3 \in (t_1, t_2)$ such that

$$|\Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - \Phi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1))| = \left|\Phi'_{\lambda,\Omega'_{j_0}} \cdot \frac{d\tilde{\eta}}{dt}(t_3)\right|(t_2 - t_1),$$

hence

$$\begin{aligned}
\Phi_\lambda(\tilde{\eta}(T)) &= \Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) + \int_0^T \frac{d}{ds}\Phi_\lambda(\tilde{\eta}(t)) \\
&= \Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) - \int_0^T \Psi(\tilde{\eta}(s))\|\Phi'_\lambda(\tilde{\eta}(s))\|_\lambda ds \\
&\leq c_\Gamma - \int_{t_1}^{t_2} \Psi(\tilde{\eta}(s))\|\Phi'_\lambda(\tilde{\eta}(s))\|_\lambda ds \\
&= c_\Gamma - \sigma_0(t_2 - t_1) \\
&\leq c_\Gamma - \frac{1}{2K}\sigma_0\mu,
\end{aligned}$$

and so (7.7) is proved. Now, we recall that $\tilde{\eta}(T) = \eta(T, \gamma_0(s_1, s_2, \ldots, s_l)) \in \Lambda_*$. Thus,

$$b_{\lambda,\Gamma} \leq \Phi_\lambda(\tilde{\eta}(T)) \leq \max\left\{m_0, c_\Gamma - \frac{1}{2K}\sigma_0\mu\right\}. \tag{7.11}$$

But, by Corollary 6.1, we know that $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$, which is a contradiction to (7.11), thus $\Phi_\lambda(u)$ has a critical point $u_\lambda \in A_u^\lambda$ for large λ and we complete the proof of the proposition. $\square$

Proof of Theorem 1.1. From Proposition 7.2, there exists a family (u_λ) of positive solutions to problem (A_λ) verifying the following properties:

(i) For fixed $\mu > 0$, there exists λ^* such that

$$\|u_\lambda\|_{\lambda,\mathbb{R}^2\setminus\Omega'_\Gamma} \le \mu, \quad \forall\, \lambda \ge \lambda^*.$$

Thus, from proof of Lemma 5.5, μ fixed sufficiently small, we conclude that

$$\|u_\lambda\|_{\infty,\mathbb{R}^2\setminus\Omega'_\Gamma} \le a^-, \quad \forall\, \lambda \ge \lambda^*,$$

showing that u_λ is a positive solution to problem (1.1).

(ii) Fixing $\lambda_n \to \infty$ and $\mu_n \to 0$, the sequence (u_{λ_n}) verifies

$$\begin{aligned}
&\Phi'_{\lambda_n}(u_{\lambda_n}) = 0, \quad \forall\, n \in N,\\
&\|u_{\lambda_n}\|_{\lambda_n,\mathbb{R}^2\setminus\Omega'_\Gamma} \to 0,\\
&\Phi'_{\lambda_n,j}(u_{\lambda_n}) \to c_j, \quad \forall\, j \in \Gamma.
\end{aligned}$$

From Lemma 4.1, we have

$$u_{\lambda_n} \to u \quad \text{in } W^{1,p(x)}(\mathbb{R}^N) \quad \text{with } u \in W_0^{1,p(x)}(\Omega_\Gamma)$$

and $u \ge 0$ and $u|_{\Omega_j}$, $j \in \Gamma$, is a least energy solution for

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + Z(x)|u|^{p(x)-2}u = \alpha f(x,u) + |u|^{q(x)-2}u, & \text{in } \Omega_j,\\ u \in W_0^{1,p(x)}(\Omega_j). \end{cases}$$

The proof of Theorem 1.1 is now complete. $\square$

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