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ABSTRACT

In this paper, we study the existence of infinitely many solutions for a fractional Kirchhoff–Schrödinger–Poisson system. Based on variational methods, especially the fountain theorem for the subcritical case and the symmetric mountain pass theorem established by Kajikiya for the critical case, we obtain infinitely many solutions for the system under certain assumptions. The novelties of this article lie in the appearance of the possibly degenerate Kirchhoff function and weak assumptions on the nonlinear term which are quite mild.

I. INTRODUCTION AND MAIN RESULTS

In this paper, we are concerned with the multiplicity of solutions for the nonlinear fractional Schrödinger–Poisson system of Kirchhoff type,

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi(x)u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi(x) = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where

$$[u]_s^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy,$$

$s, t \in (0,1)$ with $2t+4s > 3$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function satisfying certain assumptions, the potential function $V : \mathbb{R}^3 \to \mathbb{R}^+$ is continuous, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition, λ is a positive parameter, and $(-\Delta)^s$ is the fractional Laplace operator which, up to a normalization constant, is defined as

$$(-\Delta)^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{3+2s}} dy, \quad x \in \mathbb{R}^3,$$

along functions $\varphi \in C_0^\infty(\mathbb{R}^3)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ and with radius $\varepsilon > 0$. It is worth pointing out that the fractional Laplace operator $(-\Delta)^\tau$ becomes the classic Laplace operator $-\Delta$ as $\tau \to 1^-$; see Ref. 17, Proposition 4.4. From a probabilistic point of view, the fractional Laplace operator could be viewed as the infinitesimal generator of a Lévy process; cf. Ref. 9. This operator arises in the description of various phenomena in the applied sciences, such as plasma physics,[23] flame propagation,[12] finance,[16] free boundary obstacle problems,[13] Signorini problems,[37] Hamilton-Jacobi equation with critical fractional diffusion,[38] or phase transitions in the Gamma convergence framework.[1] For more details on the nonlocal fractional Laplace operator, we refer the readers to Refs. 17 and 29 and the references therein.

For our problem, we assume that the Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function satisfying the following conditions:

(M_1) M is nondecreasing, and there exists $m_0 > 0$ such that $\inf_{t\in\mathbb{R}_0^+} M(t) \geq m_0 > 0$.
($\widetilde{M_1}$) For any $\tau > 0$, there exists $\kappa = \kappa(\tau) > 0$ such that $M(t) \geq \kappa$ for all $t \geq \tau$.
(M_2) There exists $\theta \in [2, \frac{3}{3-2s})$ such that $\theta\mathcal{M}(t) := \theta \int_0^t M(\tau)d\tau \geq M(t)t$ for all $t \in \mathbb{R}_0^+$.
(M_3) $\widetilde{\mathcal{M}}(\omega t) \leq \widetilde{\mathcal{M}}(t)$ for all $\omega \in [0, 1]$, where $\widetilde{\mathcal{M}}(t) = \theta\mathcal{M}(t) - M(t)t$ for all $t \in \mathbb{R}_0^+$.

A typical example for M is given by $M(t) = b_0 + b_1 t^{\theta-1}$ for $t \geq 0$, where $\theta \in [2, \frac{3}{3-2s})$, $b_0 \geq 0$, $b_1 \geq 0$, and $b_0 + b_1 > 0$. Note that (M_2) implies that $s > \frac{3}{4}$, and hence this leads to $2t + 4s > 3$. The Kirchhoff problem is called non-degenerate if $M(0) > 0$, while it is named degenerate if $M(0) = 0$; see Ref. 33 for some physical explanations about degenerate Kirchhoff problems. For the physical background of the fractional Kirchhoff model, we refer to Ref. 20, Appendix A.

Obviously, assumptions (M_1)–(M_3) are automatic in the non-degenerate case. Meanwhile, ($\widetilde{M_1}$), (M_2), and (M_3) cover the degenerate case. It is worth stressing that the degenerate case is rather interesting and is treated in well-known papers in the Kirchhoff theory; see, for example, Ref. 18. In the vast literature on degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of u via $M([u]_s^2)$.

In recent years, Kirchhoff–type problems, which arise in various models of physical and biological systems, have received more and more attention. More precisely, Kirchhoff established a model given by

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, p_0, h, E, L are constants which represent some physical meanings, respectively. Here (1.2) extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Note that the presence of the nonlocal Kirchhoff function M makes (1.2) no longer a pointwise identity. Recently, *Fiscella and Valdinoci* in Ref. 20 first proposed a stationary Kirchhoff model involving the fractional Laplacian by taking into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string; see Ref. 20, Appendix A for more details. In this case, M measures the change in the tension on the string caused by the change in its length during the vibration. From this point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a seemingly feasible model.

On the one hand, the study of a system like (1.1) has been motivated by the following Schrödinger–Poisson type system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

which was introduced by *Benci* and *Fortunato* in Ref. 7 as a physical model describing solitary waves for nonlinear Schrödinger type equations interacting with an unknown electrostatic field. The first equation of (1.3) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term ϕu is nonlocal and concerns the interaction with the electric field. For more details on the physical background of system (1.3), we refer the readers to Refs. 8 and 35 and the references cited there.

In the last decades, many researchers have devoted to the existence and multiplicity of solutions for the system like (1.1) via critical point theory under various assumptions on the potential V and the nonlinearity; for example, see Ref. 25. In particular, *Li et al.* considered the following Schrödinger–Maxwell system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Using the variant fountain theorem introduced by *Zou* in Ref. 50, under certain assumptions on V and f, the authors got infinitely many large solutions for the above system. We refer to Refs. 15 and 39 for the applications of the same method. *Zhao et al.* in

Ref. 47 studied the following Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} \left[a+b\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)|u|^2)dx\right](-\Delta u+V(x)u)+\lambda l(x)\phi u=f(x,u), & x\in\mathbb{R}^3,\\ -\Delta\phi=\lambda l(x)u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.4}$$

where $a > 0$, $b \geq 0$, and $\lambda \geq 0$. Indeed, they obtained infinitely many solutions of (1.4) by using the symmetric mountain pass theorem established by *Kajikiya* in Ref. 22.

On the other hand, *Zhang et al.* in Ref. 48 considered the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u+\lambda\phi u=g(u) & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=\lambda u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ and g satisfies subcritical or critical growth conditions. By using a perturbation approach, the authors in Ref. 48 obtained the existence of positive solutions for small λ and studied the asymptotic of solutions for $\lambda \to 0^+$. In Ref. 40, *Teng* studied the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u+V(x)u+\phi u=\mu|u|^{q-1}u+|u|^{2_s^*-2}u & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $\mu > 0$ is a parameter, $1 < q < 2_s^*-1$, and $2s + 4t > 3$. In that paper, when μ is large enough, the existence of a nontrivial ground state solution was obtained by using the method of Pohozaev–Nehari manifold and the arguments of Brézis–Nirenberg, the monotonic trick and global compactness Lemma. However, only the existence of solutions in the above papers was investigated. In Ref. 43, *Wei* studied the following fractional Schrödinger–Maxwell equations:

$$\begin{cases} (-\Delta)^s u+V(x)u+\phi u=f(x,u) & \text{in } \mathbb{R}^3,\\ (-\Delta)^s\phi=K_s u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where K_s is a positive constant only depending on s. Consequently, with the help of the fountain theorem, the existence of infinitely many (but possibly sign changing) solutions under suitable assumptions on the nonlinearity term was obtained. We refer the interested reader to Refs. 14 and 36 for more recent results about the fractional Schrödinger–Poisson system. In Ref. 44, *Xiang and Wang* first considered the following fractional Schrödinger–Poisson–Kirchhoff system:

$$\begin{cases} M\left([u]_s^2+\int_{\mathbb{R}^3}V(x)|u|^2dx\right)[(-\Delta)^s u+V(x)u]+\phi\rho(x)|u|^{2_{s,t}^*-2}u\\ =\lambda h(x)|u|^{p-2}u+|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=\rho(x)|u|^{2_{s,t}^*} \quad \text{in } \mathbb{R}^3, \end{cases}$$

where $[u]_s$ is the Gagliardo norm of u and M satisfies $(\widetilde{M}_1)$, (M_2), and the following hypothesis: there exists $m_0 > 0$ such that $M(\eta) \geq m_0\eta^{\theta-1}$ for all $\eta \in [0, 1]$. Concerning the multiplicity of solutions, the authors in Ref. 44 just considered the existence of two non-negative solutions for the above system by the mountain pass theorem and the Ekeland variational principle. By utilizing the minimax argument, *Ambrosio* in Ref. 3 obtained the existence of solutions for the fractional Kirchhoff–Schrödinger–Poisson system with Berestycki-Lions type nonlinearities.

In the scalar case, we just collect some recent advances related to our problems and methods in the literature. By employing the symmetric mountain pass theorem, *Molica Bisci*[28] obtained the existence of infinitely many solutions for the fractional Laplacian problem with zero boundary condition. In Ref. 45, *Xiang et al.* used the fountain theorem to study the existence of infinitely many solutions for subcritical Kirchhoff type equations involving the fractional Laplacian with homogeneous Dirichlet boundary conditions. See also Ref. 11 for some related results obtained by the fountain theorem. The existence of infinitely many solutions is still proved in Refs. 10, 24, 27 and 33 by using Krasnoselskii's genus theory under degenerate frameworks. Moreover, to get infinitely many solutions, Krasnoselskii's genus theory is used in Ref. 19 for a critical Kirchhoff type fractional problem but just on the non-degenerate case. In Ref. 42, applying Kajikiya's new version of the symmetric mountain pass lemma, the existence of infinitely many solutions for a critical Kirchhoff type fractional equation was proved under a non-degenerate situation. Finally, the symmetric mountain pass theorem was applied to study a fractional Schrödinger–Kirchhoff equation in Ref. 31, a degenerate Kirchhoff-type Schrödinger–Choquard equation in Ref. 26, and a subcritical degenerate Kirchhoff system on a bounded domain Ω in Ref. 46 (see also Refs. 30 and 49).

Motivated by the above studies, we are interested in multiplicity of solutions for (1.1) in the Kirchhoff context. In the non-degenerate case, we will use the fountain theorem to study the existence of infinitely many solutions for problem (1.1) in the subcritical case. In the possibly degenerate case, we will apply the symmetric mountain pass theorem established by Kajikiya to investigate problem (1.1) for the critical case. To the best of our knowledge, there are few results in the literature about the fractional Kirchhoff–Schrödinger–Poisson systems like (1.1). Here we need to overcome the lack of compactness due to the presence of critical exponents as well as the possibly degenerate nature of the Kirchhoff function.

Throughout the paper, we suppose on the potential function V that

(V_1) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) > V_0 > 0$, where V_0 is a constant.
(V_2) $\mathrm{meas}\{x \in \mathbb{R}^3 : -\infty < V(x) \le h\} < +\infty$ for all $h \in \mathbb{R}$.

Note that if V is coercive, a.e. $\lim_{|x|\to\infty} V(x) = +\infty$, then assumption ($V_2$) is satisfied.

Moreover, we impose the following assumptions on the nonlinearity f. Let us denote $F(x, t) = \int_0^t f(x, \mu)d\mu$ and let the real number θ be given in (M_2).

(F_1) $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(x, \cdot)$ is continuous for almost all $x \in \mathbb{R}^3$ and $f(\cdot, t)$ is measurable for all $t \in \mathbb{R}$.
(F_2) There exist non-negative functions $\rho(x) \in L^2 \cap L^\infty(\mathbb{R}^3)$ and $\sigma(x) \in L^\infty(\mathbb{R}^3)$ such that, for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$|f(x, t)| \le \rho(x) + \sigma(x)|t|^{q-1},\ q \in (2\theta, 2_s^*).$$

(F_3) $\lim_{|t|\to\infty} \frac{F(x;t)}{|t|^{2\theta}} = \infty$ uniformly for a.e. $x \in \mathbb{R}^3$.
(F_4) There exist $\nu \ge 1$ and $\mathcal{C} > 0$ such that

$$\nu\mathcal{F}(x, t) \ge \mathcal{F}(x, \eta t) - \mathcal{C} \quad \text{for all} \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \eta \in [0, 1],$$

where $\mathcal{F}(x, t) = f(x, t)t - 2\theta F(x, t)$.
(F_5) There exist $\mu > 2\theta$ and $\varsigma > 0$ such that, for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$\mu F(x, t) \le tf(x, t) + \varsigma t^2.$$

Before stating our main results, we introduce some notations. The fractional Sobolev space $H^s(\mathbb{R}^3)$ can be described by means of the Fourier transform as follows:

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)|\hat{u}|^2 d\xi < \infty\right\},$$

which is endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)\hat{u}\bar{\hat{v}}d\xi, \quad \|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)|\hat{u}|^2 d\xi.$$

In view of Plancherel's theorem (see, for example, Ref. 17, Sec. 3), we have

$$(u, v) = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v + uv)dx, \quad \|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2)dx.$$

The homogeneous fractional Sobolev space $D^s(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}|^2 d\xi < \infty\}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $[u]_s^2$. According to Theorem 6.5 in Ref. 17, $H^s(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $2 \le p \le 2_s^*$, and for any $s \in (0, 1)$, there exists a best constant $S_s > 0$ such that

$$S_s = \inf_{u\in D^s(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{2/2_s^*}}. \tag{1.7}$$

The natural solution space for problem (1.1) is E, which is defined as

$$E = \left\{u \in H^s(\mathbb{R}^3) : \|u\|_E = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)|u|^2)dx\right)^{\frac{1}{2}} < \infty\right\}.$$

In the subcritical case, we will apply the fountain theorem to study the existence of infinitely many solutions for problem (1.1). As a consequence, we obtain the following results.

Theorem 1.1. *Let s, t ∈ (0, 1) with s > 3/4. Assume that* (V_1) *and* (V_2), (M_1)–(M_3), *and* (F_1)–(F_4) *hold. If* $f(x, -t) = -f(x, t)$ *holds for all* $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, *then for any* $\lambda > 0$, *problem* (1.1) *has a sequence of nontrivial weak solutions* $\{u_n\}_n \subset E$ *such that* $I_\lambda(u_n) \to \infty$ *as* $n \to \infty$.

Theorem 1.2. *Let s, t ∈ (0, 1) with s > 3/4. Assume that* (V_1) *and* (V_2), (M_1) *and* (M_2), *and* (F_1)–(F_3), (F_5) *hold. If* $f(x, -t) = -f(x, t)$ *holds for all* $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, *then for any* $\lambda > 0$, *problem* (1.1) *has a sequence of nontrivial weak solutions* $\{u_n\}_n \subset E$ *such that* $I_\lambda(u_n) \to \infty$ *as* $n \to \infty$.

Remark 1.1. The hypothesis (F_4) without the constant $\mathcal{C}$, which was proposed by Jeanjean in Ref. 21, is slightly stronger than (F4). Moreover, the condition (F5) is much weaker than (F5) with $\varsigma = 0$, while the latter assumption was proposed by *Ambrosetti* and *Rabinowiz* in Ref. 2.

Next, we consider the critical case in the possibly degenerate Kirchhoff setting. More precisely, we consider a special case $\lambda f(x, u) = \lambda h(x)|u|^{p-2}u + |u|^{2_s^*-2}u$, that is,

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi(x)u = \lambda h(x)|u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi(x) = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

where $2_s^* = 6/(3-2s)$ and $1 < p < 2$. Now we are in a position to state the corresponding result as follows.

Theorem 1.3. *Let s, t ∈ (0, 1) with s > 3/4. Supposed that* (V_1) *and* (V_2) *and* $(\widetilde{M_1}), (M_2)$ *hold. Then there exists* $\lambda_0 > 0$ *such that if* $0 < \lambda < \lambda_0$, *problem* (1.8) *has a sequence of solutions* $\{u_n\}_n \subset E$ *with* $I_\lambda(u_n) < 0$, $I_\lambda(u_n) \to 0$, *and* $\lim_{n\to\infty} u_n \to 0$.

Remark 1.2. As far as we know, Theorems 1.1–1.3 are new even in the Laplacian case. Although the methods adopted in this article are used before, we need to study carefully some properties of the term $\phi(x)u$ and the effect of the (degenerate) Kirchhoff term.

It is natural to ask the following question: what about the existence of infinitely many solutions for problem (1.8) if $2 \le p < 2_s^*$? It is still open to be solved in the future.

The paper is organized as follows. In Sec. II, we introduce some notations and preliminaries and give the variational formulation for problem (1.1). In Sec. III, we prove Theorems 1.1 and 1.2 in the subcritical case by using the fountain theorem under the Cerami condition. In Sec. IV, we will apply the symmetric mountain pass lemma established by Kajikiya to prove Theorem 1.3 in the critical case.

II. PRELIMINARIES AND VARIATIONAL SETTING

In the following, we outline the variational framework for problem (1.1) and investigate some properties of the nonlocal term ϕ_u appearing in problem (1.1).

It is well known that problem (1.1) can be reduced to a single equation with a nonlocal term. Since $s, t \in (0, 1)$ satisfy $2t + 4s > 3$, there holds $\frac{12}{3+2t} < \frac{6}{3-2s}$ and thus $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. For all $u \in H^t(\mathbb{R}^3)$, let us define the linear functional L_u by

$$L_u(v) = \int_{\mathbb{R}^3} u^2 v\, dx, \quad \forall v \in D^t(\mathbb{R}^3).$$

Then, from the Hölder inequality and (1.7), there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} |L_u(v)| &\le \left(\int_{\mathbb{R}^3} |u(x)^2|^{\frac{6}{3+2t}} dx\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v(x)|^{\frac{6}{3-2t}} dx\right)^{\frac{3-2t}{6}} \\ &\le C_1 S_t^{-\frac{1}{2}} \|u\|^2_{H^t(\mathbb{R}^3)} [v]_t = C_2 \|u\|^2_{H^t(\mathbb{R}^3)} [v]_t. \end{aligned} \tag{2.1}$$

Hence, from the Lax–Milgram theorem, for every $u \in H^t(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in D^t(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} u^2 v\, dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t \cdot (-\Delta)^{\frac{t}{2}} v\, dx \tag{2.2}$$

for any $v \in D^t(\mathbb{R}^3)$, that is, ϕ_u^t is a weak solution of

$$(-\Delta)^t \phi_u^t = u^2 \quad \text{in } \mathbb{R}^3,$$

and the representation formula

$$\phi_u^t = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy = c_t \frac{1}{|x|^{3-2t}} * u^2, \quad \forall x \in \mathbb{R}^3 \tag{2.3}$$

holds, which is called the t-Riesz potential, where

$$c_t = \frac{\Gamma(3-2t)}{\pi^{3/2} 2^{2t} \Gamma(t)}.$$

Then, $\phi_u^t > 0$ for all $u \neq 0$. Moreover, by (2.1) and (2.2) and the Sobolev inequality, there exist $C_3, C_4, C_5 > 0$ such that

$$[\phi_u^t]_t \leq C_3 \|u\|^2_{H^t(\mathbb{R}^3)}, \quad \|\phi_u^t\|_{L^{2_t^*}(\mathbb{R}^3)} \leq C_4 [\phi_u^t]_t, \tag{2.4}$$

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|^{3-2t}} u^2(x) u^2(y) dx dy = \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq C_5 \|u\|^4_{H^t(\mathbb{R}^3)}. \tag{2.5}$$

Substituting ϕ_u^t in problem (1.1), we get the following fractional Schrödinger equation:

$$M([u]_s^2)(-\Delta)^s u + V(x) u + \phi_u^t(x) u = \lambda f(x,u) \quad \text{in } \mathbb{R}^3. \tag{2.6}$$

Obviously, solutions of problem (2.6) can be obtained by looking for critical points of the functional $I : H^s(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\mathcal{M}([u]_s^2) + \int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t(x) u^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u) dx.$$

In addition, it is standard to show that (2.4) and (2.5) imply that I_λ is a well-defined C^1 functional, and for all $v \in H^s(\mathbb{R}^3)$, we get

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^3} (M([u]_s^2)(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x) uv) dx + \int_{\mathbb{R}^3} \phi_u^t uv dx - \lambda \int_{\mathbb{R}^3} f(x,u) v dx.$$

Hence, if $u \in H^s(\mathbb{R}^3)$ is a critical point of I, then the pair (u, ϕ_u^t), with ϕ_u^t as in (2.3), is a (weak) solution of problem (1.1).

Let us define the operator $\Phi : H^t(\mathbb{R}^3) \to D^t(\mathbb{R}^3)$ as follows: $\Phi[u] = \phi_u^t$. In the next lemma, we summarize some properties of Φ, which is useful for the study of our problem. The proof follows the same lines of Refs. 40, 41, and 48.

Lemma 2.1. For any $u \in H^t(\mathbb{R}^3)$, we have that

(1) *Φ is continuous;*
(2) *Φ maps bounded sets into bounded sets;*
(3) *if $u_n \rightharpoonup u$ in $H^t(\mathbb{R}^3)$, then $\Phi[u_n] \rightharpoonup \Phi[u]$ in $D^t(\mathbb{R}^3)$;*
(4) *$\Phi[\theta u] = \theta^2 \Phi[u]$ for all $\theta \in \mathbb{R}$.*
(5) *if $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 2_t^*$, then*

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n v dx \to \int_{\mathbb{R}^3} \phi_u^t(x) uv dx \quad \text{for all } v \in E$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^t(x) u^2 dx.$$

Since $V(x)$ satisfies the conditions (V_1) *and* (V_2), we can recall the following continuous or compact embedding theorem in Ref. 32, Lemma 1.

Lemma 2.2. Let $0 < s < 1$ with $s < N/2$. Suppose that (V_1) and (V_2) hold. If $r \in [2, 2_s^*]$, then the embeddings

$$E \hookrightarrow D^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$$

are continuous with $[u]_s \le C\|u\|_E$ for all $u \in E$, where C is a generic constant. In particular, for $r \in [2, 2_s^*]$, there exists a constant $C_r > 0$ such that $\|u\|_{L^r(\mathbb{R}^3)} \le C_r\|u\|_E$ for all $u \in E$. If $r \in [2, 2_s^*)$, then the embedding

$$E \hookrightarrow\hookrightarrow L^r(\mathbb{R}^3)$$

is compact.

III. THE SUBCRITICAL CASE

Now, we prove that the functional I_λ satisfies the Cerami condition [$(C)_c$-condition for short], i.e., for $c \in \mathbb{R}$, any sequence $\{u_n\}_n \subset E$ such that $I_\lambda(u_n) \to c$ and $\|I'_\lambda(u_n)\|_{E'}(1 + \|u_n\|_E) \to 0$ as $n \to \infty$ has a convergent subsequence. Here, E' is a dual space of E. This plays a key role in obtaining the existence of nontrivial weak solutions for the given problem.

Lemma 3.1. Let $s, t \in (0, 1)$ satisfy $2t + 4s > 3$. Assume that (M_1)–(M_3), (V_1) and (V_2), and (F_1)–(F_4) hold. Then the functional I_λ satisfies the $(C)_c$-condition for any $\lambda > 0$.

Proof. For $c \in \mathbb{R}$, let $\{u_n\}_n$ be a $(C)_c$-sequence in E. This implies that

$$c = I_\lambda(u_n) + o_n(1) \ \text{ and } \ \langle I_\lambda(u_n), u_n\rangle = o_n(1), \tag{3.1}$$

where $o_n(1) \to 0$ as $n \to \infty$. If $\{u_n\}_n$ is bounded in E, it follows from the proceeding as in the proof of Lemma 6 in Ref. 32 that $\{u_n\}_n$ converges strongly to u in E. Hence, it suffices to verify that the sequence $\{u_n\}_n$ is bounded in E. We argue by contradiction. Suppose that the sequence $\{u_n\}_n$ is unbounded in E. We may assume that

$$\|u_n\|_E > 1 \ \text{ and } \ \|u_n\|_E \to \infty, \ \text{ as } \ n \to \infty. \tag{3.2}$$

Define a sequence $\{v_n\}_n$ by $v_n = \frac{u_n}{\|u_n\|_E}$. Then it is obvious that $\{v_n\}_n \subset E$ and $\|v_n\|_E = 1$. Hence, up to a subsequence, still denoted by $\{v_n\}_n$, by Lemma 2.2, we have that as $n \to \infty$,

$$\begin{array}{ll} v_n \rightharpoonup v & \text{in E}, \\ v_n \to v & \text{a.e. in } \mathbb{R}^3, \\ v_n \to v & \text{in } L^r(\mathbb{R}^3) \text{ for } 2 \le r < 2_s^*. \end{array} \tag{3.3}$$

According to (3.1), it is easy to see that

$$\begin{aligned} c &= I_\lambda(u_n) + o_n(1) \\ &= \frac{1}{2}\left[\mathcal{M}([u_n]_s^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx\right] + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \lambda\int_{\mathbb{R}^3} F(x, u_n)dx + o_n(1). \end{aligned}$$

Since $\|u_n\|_E \to \infty$ as $n \to \infty$, by (M_1) and (M_2), we assert that

$$\begin{aligned} \int_{\mathbb{R}^3} F(x, u_n)dx &= \tfrac{1}{2\lambda}\left[\mathcal{M}([u_n]_s^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx\right] + \tfrac{1}{4\lambda}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \tfrac{c}{\lambda} + o_n(1) \\ &\ge \tfrac{M([u_n]_s^2)[u_n]_s^2}{\theta} + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx - \tfrac{c}{\lambda} + o_n(1) \\ &\ge \tfrac{\min\{m_0,1\}}{\theta}\|u\|_E^2 - \tfrac{c}{\lambda} + o_n(1) \\ &\to \infty \end{aligned} \tag{3.4}$$

as $n \to \infty$. Hence by (3.4), we get

$$0 < \frac{1}{2\lambda} \le \int_{\mathbb{R}^3} \limsup_{n\to\infty} \frac{|F(x, u_n)|}{\mathcal{M}([u_n]_s^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx} dx. \tag{3.5}$$

The assumption (F_3) implies that there exists $t_0 > 1$ such that $F(x, t) > |t|^{2\theta}$ for all $x \in \mathbb{R}^3$ and $|t| > t_0$. From assumptions (F_1) and (F_2), we have that there exists a positive constant C such that $|F(x, t)| \ge C$ for all $(x, t) \in \mathbb{R}^3 \times [-t_0, t_0]$. Therefore we can choose a real number C_0 such that $F(x, t) \ge C_0$ for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, and hence

$$\frac{F(x, u_n) - C_0}{\mathcal{M}([u_n]_s^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx} \ge 0 \tag{3.6}$$

for all $x \in \mathbb{R}^3$. Set $\Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. By (3.3), we know that $|u_n(x)| = |v_n(x)| \cdot \|u_n\|_E \to \infty$ as $n \to \infty$ for all $x \in \Omega$. So it follows from assumptions (M_1), (M_2), (F_3), (2.5), (3.2), and (3.4) that for all $x \in \Omega$, we have that

$$\begin{aligned} &\lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}([u_n]_s^2)+\int_{\mathbb{R}^3} V(x)|u_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx} \\ &\geq \lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}(1)\|u_n\|_E^{2\theta}+\int_{\mathbb{R}^3} V(x)|u_n|^2dx+\frac{1}{2}\|u_n\|_E^4} \\ &\geq \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|_E^{2\theta}+\frac{1}{2}\|u_n\|_E^{2\theta}} \\ &= \lim_{n\to\infty} \frac{1}{\mathcal{M}(1)+3/2}\cdot\frac{F(x,u_n)}{|u_n|^{2\theta}}\cdot|v_n|^{2\theta} \\ &= \infty, \end{aligned} \tag{3.7}$$

where we use the fact that $\mathcal{M}([u_n]_s^2) \leq \mathcal{M}(\|u_n\|_E^2) \leq \mathcal{M}(1)\|u_n\|_E^{2\theta}$, which is easily deduced from assumptions (M_1) and (M_2). Hence we get that meas$(\Omega) = 0$. Indeed, if meas$(\Omega) \neq 0$, according to (2.5), (3.4)–(3.7), and Fatou's lemma, we deduce that

$$\begin{aligned} \frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^3} F(x,u_n)dx}{\lambda\int_{\mathbb{R}^3} F(x,u_n)dx+c-o_n(1)} \\ &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^3} F(x,u_n)dx}{\frac{1}{2}(\mathcal{M}([u_n]_s^2)+\int_{\mathbb{R}^3} V(x)|u_n|^2dx)+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx} \\ &\geq \liminf_{n\to\infty}\int_\Omega \frac{F(x,u_n)-C_0}{\frac{1}{2}(\mathcal{M}([u_n]_s^2)+\int_{\mathbb{R}^3} V(x)|u_n|^2dx)+\frac{1}{4}\|u_n\|_E^4}dx \\ &\geq \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)-C_0}{\frac{1}{2}(\mathcal{M}([u_n]_s^2)+\int_{\mathbb{R}^3} V(x)|u_n|^2dx)+\frac{1}{4}\|u_n\|_E^4}dx = \infty, \end{aligned} \tag{3.8}$$

which is a contradiction. Thus, $v(x) = 0$ for almost all $x \in \mathbb{R}^3$. Furthermore, by (3.3), we get for $2 \leq r < 2_s^*$,

$$v_n \to 0 \text{ in } L^r(\mathbb{R}^3) \text{ and } v_n(x) \to 0 \ \ a.e. \text{ in } \mathbb{R}^3 \text{ as } n \to \infty. \tag{3.9}$$

As in Ref. 21, we choose a sequence $\{t_n\}_n \subset [0,1]$ such that $I_\lambda(t_n u_n) = \max_{t\in[0,1]} I_\lambda(tu_n)$. For any positive integer m, we can choose $\tau = \sqrt{2m}$ such that $\tau\|u_n\|_E^{-1} \in (0,1)$ as n is large enough. Since $v_n \to 0$ in $L^r(\mathbb{R}^3)$ and (F_1) by the continuity of the Nemiskii operator, we know that $F(\cdot,\tau v_n) \to 0$ in $L^1(\mathbb{R}^3)$, which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} F(x,\tau v_n)dx = 0. \tag{3.10}$$

Hence, for large enough n, it follows from (3.10) and (M_1) that

$$I_\lambda(t_n u_n) \geq I_\lambda(\tau\|u_n\|_E^{-1}u_n) = I_\lambda(\tau v_n) \geq \min\{m_0,1\}m - \int_{\mathbb{R}^3} F(x,\tau v_n)dx,$$

from which we deduce that $I_\lambda(t_n u_n) \to +\infty$ as $n \to \infty$. But $I_\lambda(0) = 0$, $I_\lambda(u_n) \to c$, so $t_n \in (0,1)$ and

$$\langle I_\lambda(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\Big|_{t=t_n} I_\lambda(tu_n) = 0.$$

Now using (F_4) and (M_3), we get

$$\begin{aligned} \frac{1}{\nu}I_\lambda(t_n u_n) &= \frac{1}{\nu}\left[I_\lambda(t_n u_n) - \frac{1}{2\theta}\langle I'_\lambda(t_n u_n), t_n u_n\rangle\right] + o_n(1) \\ &\leq \frac{1}{2\nu}\left[\frac{1}{2\theta}\widetilde{\mathcal{M}}([t_n u_n]_s^2) + \left(\frac{1}{2}-\frac{1}{2\theta}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx\right. \\ &\quad \left.+\left(\frac{1}{4}-\frac{1}{2\theta}\right)\int_{\mathbb{R}^3}\phi_{u_n^2}(u_n)^2dx + \int_{\mathbb{R}^3}\frac{1}{2\theta}\mathcal{F}(x,t_n u_n)dx\right] + o_n(1) \\ &\leq \frac{1}{\nu}\left[\frac{1}{2\theta}\widetilde{\mathcal{M}}([u_n]_s^2) + \left(\frac{1}{2}-\frac{1}{2\theta}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2dx\right. \\ &\quad \left.+\left(\frac{1}{4}-\frac{1}{2\theta}\right)\int_{\mathbb{R}^3}\phi_{u_n^2}u_n^2dx + \int_{\mathbb{R}^3}\frac{1}{2\theta}(\nu\mathcal{F}(x,u_n)+\mathcal{C})dx\right] + o_n(1) \\ &\leq I_\lambda(u_n) - \frac{1}{2\theta}\langle I'_\lambda(u_n),u_n\rangle + \frac{\mathcal{C}}{2\nu\theta} + o_n(1) = c + \frac{\mathcal{C}}{2\nu\theta} + o_n(1). \end{aligned}$$

This contradicts the fact that $I_\lambda(t_n u_n) \to +\infty$ as $n \to \infty$. Thus, $\{u_n\}_n$ is bounded in E. □

Lemma 3.2. Let $s, t \in (0,1)$ *satisfy* $2t + 4s > 3$. *Assume that* (M_1) *and* (M_2), (V_1) and (V_2), (F_1)–(F_3), *and* (F_5) *hold. Then the functional* I_λ *satisfies the* $(C)_c$*-condition for any* $\lambda > 0$.

Proof. Let $\{u_n\}_n$ be a $(C)_c$-sequence in E. As in the proof of Lemma 3.1, we only need to prove that $\{u_n\}_n$ is bounded in E. Suppose that $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$. Then $\|v_n\|_E = 1$ and $\|v_n\|_{L^r(\mathbb{R}^3)} \le C_r \|v_n\|_E = C_r$ for $2 \le r \le 2_s^*$ by the continuous embedding in Lemma 2.2. Passing to a subsequence, again by Lemma 2.2, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ a.e. in $\mathbb{R}^3$, $v_n \to v$ in $L^r(\mathbb{R}^3)$ for $2 \le r < 2_s^*$. By the assumptions (M_1), (M_2), and (F_5), one has

$$\begin{aligned}
c+1 &\ge I_\lambda(u_n) - \tfrac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&= \tfrac{1}{2}\mathcal{M}([u_n]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u_n)dx \\
&\quad -\tfrac{1}{\mu} M([u_n]_s^2)[u_n]_s^2 - \tfrac{1}{\mu}\int_{\mathbb{R}^3} V(x)|u_n|^2 dx - \tfrac{1}{\mu}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx + \tfrac{\lambda}{\mu}\int_{\mathbb{R}^3} f(x,u_n)u_n dx \\
&\ge \left(\tfrac{1}{2\theta} - \tfrac{1}{\mu}\right) M([u_n]_s^2)[u_n]_s^2 + \left(\tfrac{1}{2} - \tfrac{1}{\mu}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \left(\tfrac{1}{4} - \tfrac{1}{\mu}\right)\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \\
&\qquad +\lambda \int_{\mathbb{R}^3} \tfrac{1}{\mu} f(x,u_n)u_n - F(x,u_n) dx \\
&\ge \left(\tfrac{1}{2\theta} - \tfrac{1}{\mu}\right)\min\{1, m_0\}\|u_n\|_E^2 - \tfrac{1}{\mu}\lambda\varsigma \int_{\mathbb{R}^3} |u_n|^2 dx,
\end{aligned} \tag{3.11}$$

which implies that

$$1 \le \frac{\lambda\varsigma 2\theta}{\min\{1,m_0\}(\mu-2\theta)} \limsup_{n\to\infty} \|v_n\|_{L^2(\mathbb{R}^3)}^2 = \frac{\lambda\varsigma 2\theta}{\min\{1,m_0\}(\mu-2\theta)} \|v\|_{L^2(\mathbb{R}^3)}^2. \tag{3.12}$$

Hence, it follows from (3.12) that $v \ne 0$. From the same argument as that in Lemma 3.1, we can check the relations (3.4), (3.6), and (3.7) and hence yield the relation (3.8). Therefore we arrive at a contradiction. Thus, $\{u_n\}_n$ is bounded in E. □

Next, using the oddity on f and applying the fountain theorem of Bartsch,[5] we demonstrate the existence of infinitely many weak solutions for problem (1.1). To do this, let X be a separable and reflexive Banach space. It is well known that there are $\{e_n\}_n \subseteq X$ and $\{f_n^*\}_n \subseteq X^*$ such that

$$X = \overline{\mathrm{span}\{e_n : n = 1, 2, \cdots\}}, \quad X^* = \overline{\mathrm{span}\{f_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \ne j \end{cases}.$$

Let us denote $X_n = \mathrm{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^k X_n$ and $Z_k = \overline{\bigoplus_{n=k}^\infty X_n}$. Then we can state the fountain theorem under the Cerami condition as follows.

Proposition 3.1. Let X be a real reflexive Banach space, $I \in C^1(X, \mathbb{R})$ *satisfies the* $(C)_c$*-condition for any* $c > 0$*, and I is even. If for each sufficiently large* $k \in \mathbb{N}$*, there exist* $\rho_k > r_k > 0$ *such that the following conditions hold:*

(1) $a_k := \max\{I(u) \colon u \in Y_k, \|u\|_X = \rho_k\} \le 0$ *and*
(2) $b_k := \inf\{I(u) \colon u \in Z_k, \|u\|_X = r_k\} \to \infty$ *as* $k \to \infty$,

then the functional I has an unbounded sequence of critical values, i.e., *there exists a sequence* $\{u_n\}_n \subset X$ *such that* $I'(u_n) = 0$ *and* $I(u_n) \to \infty$ *as* $n \to \infty$.

Proof of Theorem 1.1. Obviously, I_λ is an even functional and satisfies the $(C)_c$-condition by Lemma 3.1. Note that E is a separable and reflexive Banach space. According to Proposition 3.1 with X = E, and the same notation about Y_k and Z_k (see Ref. 33, Appendix), it suffices to show that there exist $\rho_k > r_k > 0$ such that (1) and (2) in Proposition 3.1 are satisfied.

Denote

$$\alpha_k := \sup_{u\in Z_k; \|u\|_E=1} \left(\int_{\mathbb{R}^3} |u(x)|^q dx\right)^{\frac{1}{q}}, 1 < q < 2_s^*.$$

Then we have $\alpha_k \to 0$ as $k \to \infty$. In fact, suppose to the contrary that there exist $\epsilon_0 > 0$ and the sequence $\{u_k\}_k$ in Z_k such that

$$\|u_k\|_E = 1, \quad \int_{\mathbb{R}^3} |u_k(x)|^q dx \ge \epsilon_0$$

for all $k \ge k_0$. Since the sequence $\{u_k\}_k$ is bounded in E, there exists $u \in E$ such that $u_k \rightharpoonup u$ in E as $k \to \infty$ and

$$\langle f_i^*, u\rangle = \lim_{k\to\infty} \langle f_i^*, u_k\rangle = 0$$

for $i = 1, 2, \ldots$. Hence we get $u = 0$ since $f_i^* \neq 0$. Then we obtain

$$0 < \frac{1}{q}\epsilon_0 \le \lim_{k\to\infty} \int_{\mathbb{R}^3} \frac{1}{q}|u_k(x)|^q dx = \int_{\mathbb{R}^3} \frac{1}{q}|u(x)|^q dx = 0,$$

which is a contradiction.

For any $u \in Z_k$, it follows from condition (M_2), (F_2), and the Hölder inequality that

$$\begin{aligned}
I_\lambda(u) &= \tfrac{1}{2}\mathcal{M}([u]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \lambda \int_{\mathbb{R}^3} F(x,u) dx \\
&\ge \tfrac{1}{2}\mathcal{M}([u]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \lambda \int_{\mathbb{R}^3} |\rho(x)||u| dx - \lambda \int_{\mathbb{R}^3} \tfrac{\sigma(x)}{q}|u(x)|^q dx \\
&\ge \tfrac{1}{2}\mathcal{M}([u]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx \\
&\quad -\lambda\|\rho\|_{L^2(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^3)} - \tfrac{\lambda}{q}\|\sigma(x)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} |u(x)|^q dx \\
&\ge \tfrac{1}{2}\mathcal{M}([u]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \lambda C_2\|u\|_E - \tfrac{\lambda}{q}C_3\int_{\mathbb{R}^3} |u(x)|^q dx \\
&\ge \tfrac{1}{2}\min\left\{1, \tfrac{m_0}{\theta}\right\}\|u\|_E^2 - \lambda C_2\|u\|_E - \lambda C_3\alpha_k^q\|u\|_E^q.
\end{aligned} \tag{3.13}$$

Set $r_k = (4\lambda C_3\alpha_k^q/\min\{1, m_0/\theta\})^{1/(2-q)}$. Since $2\theta < q$ and $\alpha_k \to 0$ as $k \to \infty$, we assert that $r_k \to \infty$ as $k \to \infty$. Hence, if $u \in Z_k$ and $\|u\|_E = r_k$, then we deduce that

$$\begin{aligned}
I_\lambda(u) &\ge \tfrac{1}{2}\min\left\{1, \tfrac{m_0}{\theta}\right\}\|u\|_E^2 - \lambda C_3\alpha_k^q\|u\|_E^q - \lambda C_2\|u\|_E \\
&= \tfrac{1}{4}\min\left\{1, \tfrac{m_0}{\theta}\right\}r_k^2 - \lambda C_2 r_k \to \infty \text{ as } k \to \infty,
\end{aligned}$$

which implies (2) of Proposition 3.1.

Assume that condition (1) in Proposition 3.1 does not hold for some k. Then there exists a sequence $\{u_n\}_n$ in Y_k such that

$$\|u_n\|_E > 1 \text{ and } \|u_n\|_E \to \infty \text{ as } n \to \infty \text{ and } I_\lambda(u_n) > 0. \tag{3.14}$$

Let $v_n = \frac{u_n}{\|u_n\|_E}$. Then $\|v_n\|_E = 1$. Since $\dim Y_k < \infty$, there exists $v \in Y_k\backslash\{0\}$ such that, up to a subsequence,

$$\|v_n - v\|_E \to 0 \text{ and } v_n(x) \to v(x) \text{ for almost all } x \in \mathbb{R}^3 \text{ as } n \to \infty.$$

For $x \in \Omega := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, we get $|u_n(x)| \to \infty$ as $n \to \infty$. Hence it follows from (M_1), (M_2), and (F_3) that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{F(x,u_n)}{\mathcal{M}([u_n]_s^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx} \\
&\ge \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|_E^{2\theta} + \frac{1}{2}\|u_n\|_E^4} \\
&\ge \lim_{n\to\infty} \frac{F(x,u_n)}{(\mathcal{M}(1)+1)\|u_n\|_E^{2\theta} + \frac{1}{2}\|u_n\|_E^{2\theta}} \\
&= \lim_{n\to\infty} \frac{1}{(\mathcal{M}(1)+3/2)} \cdot \frac{F(x,u_n)}{|u_n|^{2\theta}} \cdot |w_n|^{2\theta} = \infty.
\end{aligned} \tag{3.15}$$

Since $\mathrm{meas}(\Omega) \neq 0$, we get

$$\lim_{n\to\infty} \int_\Omega \frac{F(x,u_n)}{\mathcal{M}([u_n]_s^2) + \int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx} dx \ge \infty. \tag{3.16}$$

Therefore, we have

$$\begin{aligned} I_\lambda(u_n) &= \tfrac{1}{2}\mathcal{M}([u_n]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|u_n|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx - \lambda\int_\Omega F(x,u_n)dx \\ &\le \frac{\mathcal{M}([u_n]_s^2)+\int_{\mathbb{R}^3}V(x)|u_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx}{2}\cdot \\ &\quad\left[1-2\lambda\int_\Omega \frac{F(x,u_n)}{\mathcal{M}([u_n]_s^2)+\int_{\mathbb{R}^3}V(x)|u|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx}dx\right] \\ &\to -\infty \quad \text{as } n\to\infty, \end{aligned}$$

which contradicts (3.14). This completes the proof. □

Proof of Theorem 1.2. By Lemma 3.2 and Proposition 3.1, although we replace (F_4) with (F_5) in the assumption of Theorem 1.2, we also get that problem (1.1) possesses an unbounded sequence of nontrivial weak solutions $\{u_n\}_n$ in E such that $I_\lambda(u_n) \to \infty$ as $n \to \infty$. □

IV. THE CRITICAL CASE

In order to find nontrivial critical points of I_λ in the case that $\lambda f(x,u) = \lambda h(x)|u|^{p-2}u + |u|^{2_s^*-2}u, 1 < p < 2$, we will apply the symmetric mountain pass lemma due to Kajikiya to prove Theorem 1.3. Let X be a Banach space and Σ be the class of subsets of $X\backslash\{0\}$ which are closed and symmetric with respect to the origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \inf\{n \in \mathbb{N} : \exists\phi \in C(A, \mathbb{R}^n \setminus \{0\}), \phi(z) = -\phi(-z)\},$$

$$N_\delta(A) = \{x \in X : \mathrm{dist}(x - A) \le \delta\}, \ \text{ here } \mathrm{dist}(x - A) = \inf\{\|x - y\|_X : y \in A\}.$$

If there is no mapping as above for any $n \in \mathbb{N}$, then $\gamma(A) = +\infty$. Let Σ_n denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \ge n$. We summarize the property of genus, which will be used in the proof of Theorem 1.3. We refer the readers to Ref. 34 for the proof of the next lemma.

Proposition 4.1. Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following conditions hold.

(1) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \le \gamma(B)$.*
(2) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*
(3) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \ge \gamma(A) - \gamma(B)$.*
(4) *Then n-dimensional sphere S^n has a genus of $n + 1$ by the Borsuk-Ulam Theorem.*
(5) *If A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$, with $N_\delta(A) = \{x \in X : \mathrm{dist}(x, A) \le \delta\}$.*

The following version of the symmetric mountain-pass lemma was proposed by Kajikiya.[22]

Proposition 4.2. Let E be an infinite-dimensional space and $J \in C^1(E, \mathbb{R})$ and suppose the following conditions hold.

(J_1) *J(u) is even, bounded from below, J(0) = 0 and J(u) satisfies the local Palais-Smale condition, i.e., for some $\bar{c} > 0$, in the case when every sequence $\{u_n\}_n$ in E satisfying $\lim_{n\to\infty} J(u_n) = c < \bar{c}$ and $\lim_{n\to\infty} \|J'(u_n)\|_{E'} = 0$ has a convergent subsequence.*
(J_2) *For each $n \in \mathbb{N}$, there exists an $A_n \in \Sigma_n$ such that $\sup_{u\in A_n} J(u) < 0$.*

Then either (i) or (ii) below holds.

(i) *There exists a sequence $\{u_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) < 0$, and $\{u_n\}_n$ converges to zero.*
(ii) *There exist two sequences $\{u_n\}_n$ and $\{v_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) = 0$, $u_n \ne 0$, $\lim_{n\to\infty} u_n = 0$; $J'(v_n) = 0$, $J(v_n) < 0$, $\lim_{n\to\infty} J(v_n) = 0$; and $\{v_n\}_n$ converges to a non-zero limit.*

Remark 4.1. In view of Proposition 4.2, we know that a sequence $\{u_n\}_n$ of critical points satisfies $I_\lambda(u_n) \le 0$, $u_n \ne 0$, and $\lim_{n\to\infty} u_n = 0$.

In order to get infinitely many solutions, we need to verify the compact condition.

Lemma 4.1. Let $0 < s, t < 1$, $s \geq 3/4$. Assume that (V_1) *and* (V_2), $(\widetilde{M_1}), (M_2)$, *and* $\lambda f(x,u) = \lambda h(x)|u|^{p-2}u + |u|^{2_s^*-2}u$ *hold.* $h(x) \in L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^3)$, $1 < p < 2$. *Then there exists $\lambda_0 > 0$ such that the functional I_λ satisfies the $(PS)_c$-condition for any $\lambda \in (0, \lambda_0)$.*

Proof. For $c \in \mathbb{R}$, fix any sequence $\{u_n\}_n$ which is a $(PS)_c$-sequence in E, that is,

$$I_\lambda(u_n) \to c \ \text{ and } \ I'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.1}$$

In the following, we divide the proof into two parts.

• *Case* $\inf_{n\in\mathbb{N}} \|u_n\|_E = d > 0$. We first show that $\{u_n\}_n$ is bounded. By $(\widetilde{M_1})$, and the assumptions that $\theta < 2_s^*/2$ and $1 < p < 2$, we get

$$\begin{aligned} c + o_n(1)\|u_n\|_E &\geq I_\lambda(u_n) - \tfrac{1}{2_s^*}\langle I'_\lambda(u_n), u_n\rangle \\ &\geq \left(\tfrac{1}{2\theta} - \tfrac{1}{2_s^*}\right)M([u_n]_s^2)[u_n]_s^2 + \left(\tfrac{1}{2} - \tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3} V(x)|u_n|^2 dx \\ &\quad + \left(\tfrac{1}{4} - \tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \lambda\left(\tfrac{1}{p} - \tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3} h(x)|u_n|^p dx \\ &\geq \left(\tfrac{1}{2\theta} - \tfrac{1}{2_s^*}\right)\min\{\kappa, 1\}\|u_n\|_E^2 - \lambda\left(\tfrac{1}{p} - \tfrac{1}{2_s^*}\right)\|h(x)\|_{L^{p'}(\mathbb{R}^3)}\|u_n\|_E^p S_s^{-2/p}. \end{aligned} \tag{4.2}$$

This yields at once that $\{u_n\}_n$ is bounded in E.

Next we show that I_λ satisfies the (PS) condition. Since $\{u_n\}_n$ is bounded in E, then by Lemma 2.1, there exist $u_\lambda \in E$ and a subsequence, still denoted by $\{u_n\}_n$, such that

$$\begin{aligned} &u_n \rightharpoonup u_\lambda \text{ weakly in } E, \quad \|u_n\|_E \to \alpha_\lambda, \\ &u_n \rightharpoonup u_\lambda \text{ weakly in } L^{2_s^*}(\mathbb{R}^3), \\ &\phi_{u_n} \rightharpoonup \phi_{u_\lambda} \text{ weakly in } L^{2_s^*}(\mathbb{R}^3), \\ &u_n \to u_\lambda \text{ a.e. in } \mathbb{R}^3. \end{aligned}$$

Thus, using Lemma 2.2, (2.4), and (2.5), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_{u_\lambda}^t u_\lambda)(u_n - u_\lambda)dx \\ &\leq \left(\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_{u_\lambda}^t u_\lambda)^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\left(|u_n - u_\lambda|^2 dx\right)^{\frac{1}{2}}\right. \\ &\leq \left[2\int_{\mathbb{R}^3}(|\phi_{u_n}^t u_n|^2 + |\phi_{u_\lambda}^t u_\lambda|^2)dx\right]^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\left(|u_n - u_\lambda|^2 dx\right)^{\frac{1}{2}}\right. \\ &\leq C\left(\|\phi_{u_n}^t\|_{L^6(\mathbb{R}^3)}^2\|u_n\|_{L^6(\mathbb{R}^3)}^2 + \|\phi_{u_\lambda}^t\|_{L^6(\mathbb{R}^3)}^2\|u_\lambda\|_{L^3(\mathbb{R}^3)}^2\right)^{\frac{1}{2}}\|u_n - u_\lambda\|_{L^2(\mathbb{R}^3)} \\ &\leq C\left(\|u_n\|_{H^t}(\mathbb{R}^3)^4 + \|u_\lambda\|_{H^t(\mathbb{R}^3)}^4\right)^{\frac{1}{2}}\|u_n - u_\lambda\|_{L^2(\mathbb{R}^3)} \to 0 \end{aligned} \tag{4.3}$$

as $n \to \infty$.

Since $h \in L^{\frac{2_s^*}{2_s^*-p}}(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^3\setminus B_{R_\varepsilon}} h^{\frac{2_s^*}{2_s^*-p}}(x)dx \leq \varepsilon.$$

Also, for any measurable subset $U \subset B_{R_\varepsilon}$, we have

$$\int_U h(x)|u_n|^p dx \leq \left(\int_U h^{\frac{2_s^*}{2_s^*-p}}(x)dx\right)^{\frac{2_s^*-p}{2_s^*}}\left(\int_U |u_n|^{2_s^*}dx\right)^{\frac{p}{2_s^*}} \leq C\int_U h^{\frac{2_s^*}{2_s^*-p}}(x)dx\Big)^{\frac{2_s^*-p}{2_s^*}},$$

which implies that $\{h(x)|u_n|^p\}_n$ is equi-integrable in B_{R_ε}. By $u_n \to u_\lambda$ a.e. in $\mathbb{R}^3$, we have $h(x)|u_n|^p \to h(x)|u_\lambda|^p$ a.e. in $\mathbb{R}^3$. Then the Vitali convergence theorem yields

$$\int_{B_{R_\varepsilon}} h(x)||u_n|^p - |u_\lambda|^p|dx = 0.$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^3} h(x)||u_n|^p - |u_\lambda|^p|dx\\ &\le \int_{B_{R_\varepsilon}} h(x)||u_n|^p - |u_\lambda|^p|dx + \int_{\mathbb{R}^3\setminus B_{R_\varepsilon}} h(x)||u_n|^p - |u_\lambda|^p|dx\\ &\le \int_{B_{R_\varepsilon}} h(x)||u_n|^p - |u_\lambda|^p|dx + C\varepsilon^{\frac{2_s^*-p}{2_s^*}}.\end{aligned}$$

Letting $n \to \infty$ and using the arbitrary of ε, one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h(x)||u_n|^p - |u_\lambda|^p|dx = 0. \tag{4.4}$$

This together with the following Brézis–Lieb lemma

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (|u_n|^p - |u_n - u_\lambda|^p)dx = \int_{\mathbb{R}^3} |u_\lambda|^p dx \tag{4.5}$$

yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h(x)|u_n - u_\lambda|^p dx = 0. \tag{4.6}$$

Note that for any measurable subset $U \subset \mathbb{R}^3$, we have

$$\int_U |u_n|^{2_s^*-2} u_n u_\lambda dx \le \|u_n\|_{2_s^*}^{\frac{2_s^*-1}{2_s^*}} \|u_\lambda\|_{L^{2_s^*}(U)} \le C\|u_\lambda\|_{L^{2_s^*}(U)},$$

which implies that $\{|u_n|^{2_s^*-2}u_n u_\lambda\}_n$ is equi-integrable in $\mathbb{R}^3$. Observe that

$$|u_n|^{2_s^*-2}u_n u_\lambda \to |u_\lambda|^{2_s^*} \ a.e. \text{ in } \mathbb{R}^3,$$

then the Vitali convergence theorem yields that as $n \to \infty$,

$$|u_n|^{2_s^*-2}u_n u_\lambda \to |u_\lambda|^{2_s^*} \text{ in } L^1(\mathbb{R}^3),$$

that is,

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*-2}u_n u_\lambda dx = \int_{\mathbb{R}^3} |u_\lambda|^{2_s^*} dx + o_n(1). \tag{4.7}$$

The weak convergence of $\{u_n\}_n$ in E gives that as $n \to \infty$

$$\langle u_\lambda, u_n - u_\lambda\rangle_E \to 0. \tag{4.8}$$

Since $\{u_n\}_n$ is bounded in E, we have for any $v \in E$

$$\langle u_n, v\rangle_E \to \langle u_\lambda, v\rangle_E \tag{4.9}$$

as $n \to \infty$. Since $I'_\lambda(u_n) \to 0$ as $n \to \infty$, we have $\langle I'_\lambda(u_n), u_\lambda\rangle \to 0$. Then by (4.3), (4.6), and (4.8), we obtain

$$M(\alpha_\lambda^2)[u_\lambda]_s^2 + \int_{\mathbb{R}^3} V(x)|u_\lambda|^2 dx + \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda^2 dx - \lambda \int_{\mathbb{R}^3} h(x)|u_\lambda|^p dx - \int_{\mathbb{R}^3} |u_\lambda|^{2_s^*} dx = 0,$$

which means that $\langle I'_{\alpha_\lambda}(u_\lambda), u_\lambda\rangle = 0$. Here I_{α_λ} is defined as follows:

$$I_{\alpha_\lambda}(v) = \frac{1}{2}M(\alpha_\lambda^2)[v]_s^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|v|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v^t v^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} h(x)|v|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |v|^{2_s^*} dx$$

for any $v \in E$. By the Brézis–Lieb lemma, one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (|u_n|^{2_s^*} - |u_n - u_\lambda|^{2_s^*})dx = \int_{\mathbb{R}^3} |u_\lambda|^{2_s^*} dx, \tag{4.10}$$

$$\|u_n\|_E^2 = \|u_n - u_\lambda\|_E^2 + \|u_\lambda\|_E^2 + o_n(1). \tag{4.11}$$

Since $\{u_n\}_n$ is a (PS) sequence, we deduce from (4.3), (4.6)–(4.8), (4.10), and (4.11) that

$$\begin{aligned} o_n(1) &= \langle I'_\lambda(u_n) - I'_{\alpha_\lambda}(u_\lambda), u_n - u_\lambda\rangle_E \\ &= M([u_n]_s^2)[u_n]_s^2 - M([u_n]_s^2)\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}u_\lambda dx - M(\alpha_\lambda^2)\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n-u_\lambda)dx \\ &+ \int_{\mathbb{R}^3}V(x)(u_n-u_\lambda)^2dx + \int_{\mathbb{R}^3}(\phi_{u_n}^t u_n - \phi_{u_\lambda}^t u_\lambda)(u_n-u_\lambda)dx \\ &-\lambda\int_{\mathbb{R}^3}h(x)(|u_n|^{p-2}u_n - |u_\lambda|^{p-2}u_\lambda)(u_n-u_\lambda)dx \\ &- \int_{\mathbb{R}^3}(|u_n|^{2_s^*-2}u_n - |u_\lambda|^{2_s^*-2}u_\lambda)(u_n-u_\lambda)dx \\ &\geq \min\{M(\alpha_\lambda^2),1\}\|u_n-u_\lambda\|_E^2 - \int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx + o_n(1). \end{aligned} \tag{4.12}$$

It follows from (4.12) that

$$\lim_{n\to\infty}\min\{M(\alpha_\lambda^2),1\}\|u_n-u_\lambda\|_E^2 = \lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx. \tag{4.13}$$

On the other hand, it follows from Lemma 2.1 (5) that

$$\int_{\mathbb{R}^3}\phi_{u_n}^t(x)u_n^2dx \to \int_{\mathbb{R}^3}\phi_u^t(x)u^2dx. \tag{4.14}$$

Therefore, by (M_2), (4.4), (4.10), (4.11), and (4.14), we have

$$\begin{aligned} c + o_n(1) &= I_\lambda(u_n) - \tfrac{1}{2\theta}\langle I'_{\alpha_\lambda}(u_\lambda), u_\lambda\rangle \\ &\geq \tfrac{1}{2\theta}M(\alpha_\lambda^2)[u_n]_s^2 + \tfrac{1}{2\theta}\int_{\mathbb{R}^3}V(x)u_n^2dx - \tfrac{1}{2\theta}M(\alpha_\lambda^2)[u_\lambda]_s^2 + \tfrac{1}{2\theta}\int_{\mathbb{R}^3}V(x)u_\lambda^2dx \\ &+\left(\tfrac{1}{4}-\tfrac{1}{2\theta}\right)\int_{\mathbb{R}^3}\phi_{u_\lambda}^t u_\lambda^2dx - \lambda\left(\tfrac{1}{p}-\tfrac{1}{2\theta}\right)\int_{\mathbb{R}^3}h(x)|u_\lambda|^pdx \\ &-\tfrac{1}{2_s^*}\int_{\mathbb{R}^3}(|u_n-u_\lambda|^{2_s^*}+|u_\lambda|^{2_s^*})dx - \tfrac{1}{2\theta}\int_{\mathbb{R}^3}|u_\lambda|^{2_s^*}dx + o_n(1) \\ &\geq \tfrac{1}{2\theta}M(\alpha_\lambda^2)([u_n]_s^2 - [u_\lambda]_s^2) + \tfrac{1}{2\theta}\int_{\mathbb{R}^3}V(x)(u_n^2-u_\lambda^2)dx \\ &-\lambda\left(\tfrac{1}{p}-\tfrac{1}{2\theta}\right)\int_{\mathbb{R}^3}h(x)|u_\lambda|^pdx - \tfrac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx + \left(\tfrac{1}{2\theta}-\tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_\lambda|^{2_s^*}dx + o_n(1) \\ &\geq \tfrac{1}{2\theta}\min\{M(\alpha_\lambda^2),1\}\|u_n-u_\lambda\|_E^2 \quad -\lambda\left(\tfrac{1}{p}-\tfrac{1}{2\theta}\right)\int_{\mathbb{R}^3}h(x)|u_\lambda|^pdx \\ &-\tfrac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx + \left(\tfrac{1}{2\theta}-\tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_\lambda|^{2_s^*}dx + o_n(1). \end{aligned}$$

Thanks to the assumption $\theta < 2_s^*/2$, we obtain by (4.13)

$$\begin{aligned} &c + o_n(1) \\ &\geq \left(\frac{1}{2\theta}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}(|u_n-u_\lambda|^{2_s^*}+|u_\lambda|^{2_s^*})dx - \lambda\left(\frac{1}{p}-\frac{1}{2\theta}\right)\int_{\mathbb{R}^3}h(x)|u_\lambda|^pdx. \end{aligned} \tag{4.15}$$

Combining the Hölder inequality and the Young inequality, since $1 < p < 2 < \theta$, we have for any $\varepsilon > 0$

$$\begin{aligned} \lambda\left(\tfrac{1}{p}-\tfrac{1}{2\theta}\right)\int_{\mathbb{R}^3}h(x)|u_\lambda|^pdx &\leq \left(\tfrac{1}{\varepsilon}\right)^{\frac{p}{2_s^*}}\left(\tfrac{\lambda}{p}-\tfrac{\lambda}{2\theta}\right)\|h(x)\|_{\frac{2_s^*}{2_s^*-p}}\cdot\varepsilon^{\frac{p}{2_s^*}}\|u_\lambda\|_{2_s^*}^p \\ &\leq \varepsilon\|u_\lambda\|_{2_s^*}^{2_s^*} + \varepsilon^{-\frac{p}{2_s^*-p}}\left(\left(\tfrac{\lambda}{p}-\tfrac{\lambda}{2\theta}\right)\|h(x)\|_{\frac{2_s^*}{2_s^*-p}}\right)^{\frac{2_s^*}{2_s^*-p}}. \end{aligned}$$

Taking $\varepsilon = 1/(2\theta) - 1/2_s^*$ in the above inequality and putting it in (4.15), we arrive at

$$\begin{aligned} c + o_n(1) &\geq \left(\tfrac{1}{2\theta}-\tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx \\ &-\left(\tfrac{1}{2\theta}-\tfrac{1}{2_s^*}\right)^{-\frac{p}{2_s^*-p}}\left(\left(\tfrac{1}{p}-\tfrac{1}{2\theta}\right)\lambda\|h(x)\|_{\frac{2_s^*}{2_s^*-p}}\right)^{\frac{2_s^*}{2_s^*-p}}. \end{aligned} \tag{4.16}$$

Thus,

$$\left(\frac{1}{2\theta}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx$$
$$\le\left(\frac{1}{2\theta}-\frac{1}{2_s^*}\right)^{-\frac{p}{2_s^*-p}}\left(\left(\frac{1}{p}-\frac{1}{2\theta}\right)\lambda\|h(x)\|_{\frac{2_s^*}{2_s^*-p}}\right)^{\frac{2_s^*}{2_s^*-p}}+c+o_n(1),$$

which together with $1<p<2$ and $c<0$ yields that

$$\left(\tfrac{1}{2\theta}-\tfrac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx\le\left(\tfrac{1}{2\theta}-\tfrac{1}{2_s^*}\right)^{-\frac{p}{2_s^*-p}}\left(\left(\tfrac{1}{p}-\tfrac{1}{2\theta}\right)\|h(x)\|_{\frac{2_s^*}{2_s^*-p}}\right)^{\frac{2_s^*}{2_s^*-p}}\lambda^{\frac{2_s^*}{2_s^*-p}}.$$

Thus, we conclude that

$$\lim_{\lambda\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n-u_\lambda|^{2_s^*}dx=0. \tag{4.17}$$

Then by (4.13), we get

$$\lim_{\lambda\to0}\lim_{n\to\infty}\|u_n-u_\lambda\|_E=0. \tag{4.18}$$

This yields that $u_n\to u_\lambda$ strongly in E as $n\to\infty$ and $\lambda\to0$, i.e., there exists $\lambda_0>0$ such that the functional I_λ satisfies the $(PS)_c$-condition for any $\lambda\in(0,\lambda_0)$.

• *Case* $\inf_{n\in\mathbb{N}}\|u_n\|_E=d>0$. If 0 is an isolated point for the real sequence $\{\|u_n\|_E\}_n$, then there is a subsequence $\{u_{n_k}\}_k$ such that $\inf_{k\in\mathbb{N}}\|u_{n_k}\|_E=d>0$, and we can proceed as before. Otherwise, 0 is an accumulation point of the sequence $\{\|u_n\|_E\}_n$, and so there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ such that $u_{n_k}\to0$ strongly in E as $n\to\infty$.

In conclusion, I_λ satisfies the $(PS)_c$-condition for any $\lambda\in(0,\lambda_0)$ in E. □

We also need some technical lemmas. Let $I_\lambda(u)$ be the functional defined as before, $1<p<2$. Then, one has

$$\begin{aligned}I_\lambda(u)&=\tfrac12\mathcal{M}([u]_s^2)+\tfrac12\int_{\mathbb{R}^3}V(x)|u|^2dx+\tfrac14\int_{\mathbb{R}^3}\phi_u^t(x)|u|^2dx\\&\quad-\tfrac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx-\tfrac{\lambda}{2}\int_{\mathbb{R}^3}h(x)|u|^pdx\\&\ge\tfrac{1}{2\theta}M([u]_s^2)[u]_s^2+\tfrac12\int_{\mathbb{R}^3}V(x)|u|^2dx-\tfrac{1}{2_s^*}\left(S^{-1}\|u\|_E^2\right)^{\frac{2_s^*}{2}}-\tfrac{\lambda}{p}\|h(x)\|_{\frac{2_s^*}{2_s^*-2}}\|u\|_{2_s^*}^{\frac{2}{2_s^*}}\\&\ge\tfrac{1}{2\theta}\min\{\kappa,1\}\|u\|_E^2-\tfrac{1}{2_s^*}S^{-\frac{2_s^*}{2}}\|u\|_E^{2_s^*}-\tfrac{\lambda}{p}\|h(x)\|_{\frac{2_s^*}{2_s^*-2}}S_s^{-\frac{p}{2}}\|u\|_E^p\\&\ge C_1\|u\|_E^2-C_2\|u\|_E^{2_s^*}-\lambda C_3\|u\|_E^p.\end{aligned}$$

Define

$$g(t)=C_1t^2-C_2t^{2_s^*}-\lambda C_3t^p.$$

Then, since $1<p<2$, it is easy to see that there exists $\lambda^*>0$ so small that if $0<\lambda<\lambda^*$, there exists $0<t_0<t_1$ such that $g(t)<0$ for $0<t<t_0$; $g(t)>0$ for $t_0<t<t_1$; and $g(t)<0$ for $t>t_1$.

Clearly, $g(t_0)=0=g(t_1)$. Following the same idea as in Ref. 4, we consider the truncated functional

$$\begin{aligned}\tilde{I}_\lambda(u)&=\tfrac12\mathcal{M}([u]_s^2)+\tfrac12\int_{\mathbb{R}^3}V(x)|u|^2dx+\tfrac14\int_{\mathbb{R}^3}\phi_u^t(x)|u|^2dx\\&\quad-\tfrac{1}{2_s^*}\psi(u)\int_{\mathbb{R}^3}|u|^{2_s^*}dx-\tfrac{\lambda}{p}\int_{\mathbb{R}^3}h(x)|u|^pdx,\end{aligned}$$

where $\psi(u)=\tau(\|u\|_E)$ and $\tau:\mathbb{R}^+\to[0,1]$ is a non-increasing C^∞ function such that $\tau(t)=1$ if $t\le t_0$ and $\tau(t)=0$ if $t\ge t_1$. Obviously, $\tilde{I}_\lambda(u)$ is even. Thus, it follows from Lemma 4.1, and we can get the following result.

Lemma 4.2. Let $c<0$ and $1<p<2$. Then

(1) *$\tilde{I}_\lambda\in C^1$ and $\tilde{I}_\lambda$ is bounded from below.*
(2) *If $\tilde{I}_\lambda(u)<0$, then $\|u\|_E<t_0$ and $\tilde{I}_\lambda(u)=I_\lambda(u)$.*
(3) *There exists $\tilde{\lambda}^*$ such that if $0<\lambda<\tilde{\lambda}^*$, then $\tilde{I}_\lambda$ satisfies $(PS)_c$.*

Proof. Obviously, (1) and (2) are immediate. To prove (3), observe that all $(PS)_c$ sequences for $\tilde{I}_\lambda$ with $c < 0$ must be bounded, similar to the proof of Lemma 4.1, there exists a strong convergent subsequence in E. □

Remark 4.2. Denote $K_c = \{u \in E; \tilde{I}'_\lambda(u) = 0, \tilde{I}_\lambda(u) = c\}$. *If* λ *are as in* (3) *above, then it follows from* $(PS)_c$ *that* $K_c(c < 0)$ *is compact.*

Lemma 4.3. Denote $\tilde{I}^c_\lambda := \{u \in E; \tilde{I}'_\lambda(u) = 0, \tilde{I}_\lambda(u) \le c\}$. *Given* $n \in \mathbb{N}$, *there exists* $\epsilon_n < 0$ *such that*

$$\gamma(\tilde{I}^{\epsilon_n}_\lambda) := \gamma(\{u \in E : \tilde{I}_\lambda(u) \le \epsilon_n\}) \ge n.$$

Proof. Let X_n be an n-dimensional subspace of E. For any $u \in X_n, u \ne 0$, write $u = r_n w$ with $w \in X_n$, $\|w\|_E = 1$ and then $r_n = \|u\|_E$. From the assumptions $h(x)$, it is easy to see that, for every $w \in X_n$ with $\|w\|_E = 1$, there exists $d_n > 0$ such that $\int_{\mathbb{R}^3} h(x)|w|^p dx \ge d_n$. Thus for $0 < r_n < t_0$, by the continuity of M, we have

$$\begin{aligned}
\tilde{I}_\lambda(w) &= \tfrac{1}{2}\mathcal{M}([w]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|w|^2 dx + \tfrac{1}{4}\int_{\mathbb{R}^3} \phi^t_w(x)|w|^2 dx \\
&\quad - \tfrac{1}{2_s^*}\psi(w)\int_{\mathbb{R}^3} |w|^{2_s^*} dx - \tfrac{\lambda}{p}\int_{\mathbb{R}^3} h(x)|w|^p dx \\
&\le \tfrac{1}{2} r_n^2 \mathcal{M}([w]_s^2) + \tfrac{1}{2}\int_{\mathbb{R}^3} V(x)|w|^2 dx - \tfrac{1}{2_s^*} r_n^{2_s^*}\int_{\mathbb{R}^3} |w|^{2_s^*} dx - \tfrac{\lambda}{p} r_n^p \int_{\mathbb{R}^3} h(x)|w|^p dx \\
&\le \tfrac{C_1}{2} r_n^2 - \tfrac{1}{2_s^*} r_n^{2_s^*}\int_{\mathbb{R}^3} |w|^{2_s^*} dx - \tfrac{\lambda}{2} d_n r_n^p \\
&= \epsilon_n.
\end{aligned}$$

Therefore we can choose $r_n \in (0, t_0)$ so small that $\tilde{I}_\lambda(u) \le \epsilon_n < 0$. Let

$$S_{r_n} = \{u \in X_n : \|u\|_E = r_n\}. \tag{4.19}$$

Then $S_{r_n} \cap X_n \subset \tilde{I}^{\epsilon_n}_\lambda$. Hence by Proposition 4.1,

$$\gamma(\tilde{I}^{\epsilon_n}_\lambda) \ge \gamma(S_{r_n} \cap X_n) = n$$

as desired. □

According to Lemma 4.2, we denote $\Sigma_n = \{A \in \Sigma : \gamma(A) \ge n\}$ and let

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \tilde{I}_\lambda(u). \tag{4.20}$$

Then

$$-\infty < c_n \le \epsilon_n < 0 \tag{4.21}$$

because $\tilde{I}^{\epsilon_n}_\lambda \in \Sigma_n$ and $\tilde{I}_\lambda$ is bounded from below.

Lemma 4.4. Let λ *be as in* (3) *of Lemma 4.2. Then all* c_n *[given by* (4.20)*] are critical values of* $\tilde{I}_\lambda$ *and* $c_n \to 0$.

Proof. Since $\Sigma_{n+1} \subset \Sigma_n$, it is clear that $c_n \le c_{n+1}$. By (4.21), we have $c_n < 0$. Hence there is a $\bar{c} \le 0$ such that $c_n \to \bar{c} \le 0$. Moreover, since that all c_n are critical values of $\tilde{I}_\lambda$ (see Ref. 34), we claim that $\bar{c} = 0$. If $\bar{c} < 0$, then by Remark 4.2, $K_{\bar{c}} = \{u \in E; \tilde{I}'_\lambda(u) = 0, \tilde{I}_\lambda(u) = \bar{c}\}$ is compact and $K_{\bar{c}} \in \Sigma$, then $\gamma(K_{\bar{c}}) = n_0 < +\infty$, and there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_\delta(K_{\bar{c}})) = n_0$, where $N_\delta(K_{\bar{c}}) = \{x \in X; \|x - K_{\bar{c}}\| \le \delta\}$. By the deformation lemma (see Ref. 6, Theorem 3.4), there exist $\epsilon > 0$ $(\bar{c} + \epsilon < 0)$ and an odd homeomorphism η: $E \to E$ such that

$$\eta(\tilde{I}^{\bar{c}+\epsilon}_\lambda \setminus N_\delta(K_{\bar{c}})) \subset \tilde{I}^{\bar{c}-\epsilon}_\lambda.$$

Since c_n is increasing and converges to $\bar{c}$, there exists $n \in \mathbb{N}$ such that $c_n > \bar{c} - \epsilon$ and $c_{n+n_0} \le \bar{c}$. Choose $A \in \Sigma_{n+n_0}$ such that $\sup_{u \in A} \tilde{I}_\lambda(u) < \bar{c} + \epsilon$, that is, $A \subset \tilde{I}^{\bar{c}+\epsilon}_\lambda$. By the properties of γ, we have

$$\gamma(\overline{A \setminus N_\delta(K_{\bar{c}})}) \ge \gamma(A) - \gamma(N_\delta(K_{\bar{c}}))\big) \ge n, \quad \gamma(\overline{\eta(A \setminus N_\delta(K_{\bar{c}}))}) \ge n.$$

Hence, we have $\overline{\eta(A \setminus N_\delta(K_{\bar{c}}))} \in \Sigma_n$. Consequently, $\sup_{u \in \overline{\eta(A\setminus N_\delta(K_{\bar{c}}))}} \tilde{I}_\lambda(u) \geq c_n > \bar{c} - \epsilon$, which is a contradiction; hence, $c_n \to 0$. □

Proof of Theorem 1.3. By Lemma 4.2 (2), $\tilde{I}_\lambda(u) = I_\lambda(u)$ if $\tilde{I}_\lambda(u) < 0$. Then, by Lemmas 4.2–4.4, one can see that all the assumptions of the new version of symmetric mountain pass lemma proposed by Kajikiya[22] are satisfied. Hence, the proof is complete. □

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