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Double phase anisotropic variational problems and combined effects of reaction and absorption terms

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ABSTRACT

This paper deals with the existence of multiple solutions for the quasilinear equation $-\text{div}\,\mathbf{A}(x,\nabla u)+V(x)\,|u|^{\alpha(x)-2}\,u=f(x,u)$ in $\mathbb{R}^N$, which involves a general variable exponent elliptic operator in divergence form. The problem corresponds to double phase anisotropic phenomena, in the sense that the differential operator has behaviors like $|\xi|^{q(x)-2}\,\xi$ for small $|\xi|$ and like $|\xi|^{p(x)-2}\,\xi$ for large $|\xi|$, where $1<\alpha(\cdot)\leq p(\cdot)<q(\cdot)<N$. Our aim is to approach variationally the problem by using the tools of critical points theory in generalized Orlicz–Sobolev spaces with variable exponent. Our results extend the previous works A. Azzollini et al. (2014) [4] and N. Chorfi and V. Rădulescu (2016) [11] from cases where the exponents p and q are constant, to the case where $p(\cdot)$ and $q(\cdot)$ are functions. We also substantially weaken some of the hypotheses in these papers and we overcome the lack of compactness by using the Cerami compactness condition.

RÉSUMÉ

Dans cet article on considère l'existence de solutions multiples pour l'équation quasi-linéaire $-\text{div}\,\mathbf{A}(x,\nabla u)+V(x)\,|u|^{\alpha(x)-2}\,u=f(x,u)$ dans $\mathbb{R}^N$, qui implique un opérateur elliptique général à exposant variable sous forme de divergence. Le problème correspond à des phénomènes anisotropes à deux phases, dans le sens où l'opérateur différentiel a des comportements comme $|\xi|^{q(x)-2}\,\xi$ pour $|\xi|$ petit et comme $|\xi|^{p(x)-2}\,\xi$ pour $|\xi|$ large, où where $1<\alpha(\cdot)\leq p(\cdot)<q(\cdot)<N$. Notre objectif est d'approcher de manière variationnelle le problème en utilisant les outils de la théorie des points critiques dans les espaces de Orlicz–Sobolev généralisés à exposant variable. Nos résultats étendent les travaux précédents A. Azzollini et al. (2014) [4] et N. Chorfi and V. Rădulescu (2016) [11] des cas où le exposants p et q sont constants, au cas où $p(\cdot)$ et $q(\cdot)$ sont des functions. Nous affaiblissons

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considérablement certaines des hypothèses de ces articles et nous surmontons le manque de compacité en utilisant la condition de compacité de Cerami.

1. Introduction

In this paper, we deal with the following variable exponent elliptic equation

$$-\operatorname{div} \mathbf{A}(x, \nabla u) + V(x) |u|^{\alpha(x)-2} u = f(x, u), \tag{$\mathcal{E}$}$$

where $\mathbf{A} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ admits a potential $\mathscr{A}$, with respect to its second variable ξ, satisfying the following assumption:

($\mathcal{A}_1$) the potential $\mathscr{A} = \mathscr{A}(x, \xi)$ is a continuous function in $\mathbb{R}^N \times \mathbb{R}^N$, with continuous derivative with respect to ξ, $\mathbf{A} = \partial_\xi \mathscr{A}(x, \xi)$, and verifies:

(i) $\mathscr{A}(x, 0) = 0$ and $\mathscr{A}(x, \xi) = \mathscr{A}(x, -\xi)$, for all $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$;

(ii) $\mathscr{A}(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$;

(iii) there exist positive constants C_1, C_2 and variable exponents p and q such that for all $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\left.\begin{array}{l} C_1 |\xi|^{p(x)}, \text{ if } |\xi| \gg 1 \\ C_1 |\xi|^{q(x)}, \text{ if } |\xi| \ll 1 \end{array}\right\} \le \mathbf{A}(x, \xi) \cdot \xi \quad \text{and} \quad |\mathbf{A}(x, \xi)| \le \begin{cases} C_2 |\xi|^{p(x)-1}, \text{ if } |\xi| \gg 1 \\ C_2 |\xi|^{q(x)-1}, \text{ if } |\xi| \ll 1 \end{cases}; \tag{1}$$

(iv) $1 \ll p(\cdot) \ll q(\cdot) \ll \min\{N, p^*(\cdot)\}$, and $p(\cdot)$, $q(\cdot)$ are Lipschitz continuous in $\mathbb{R}^N$;

(v) $\mathbf{A}(x, \xi) \cdot \xi \le s(x) \mathscr{A}(x, \xi)$ for any $(x, \xi) \in \mathbb{R}^{2N}$, where s is Lipschitz continuous and satisfies $q(\cdot) \le s(\cdot) \ll p^*(\cdot)$;

($\mathcal{A}_2$) $\mathscr{A}$ is uniformly convex, that is, for any $\varepsilon \in (0, 1)$, there exists $\delta(\varepsilon) \in (0, 1)$ such that $|u - v| \le \varepsilon \max\{|u|, |v|\}$ or $\mathscr{A}(x, \frac{u+v}{2}) \le \frac{1}{2}(1 - \delta(\varepsilon))(\mathscr{A}(x, u) + \mathscr{A}(x, v))$ for any $x, u, v \in \mathbb{R}^N$.

In this paper, for any $v : \mathbb{R}^N \to \mathbb{R}$, we denote

$$v^+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} v(x), \; v^- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} v(x),$$

and we denote by $v_1 \ll v_2$ the fact that

$$\operatorname*{ess}_{x \in \mathbb{R}^N} \inf(v_2(x) - v_1(x)) > 0.$$

We point out that condition ($\mathcal{A}_1$) (iv) is fundamental in the theory of variational second order elliptic problems with variable coefficients, as well as with more general (p, q)-growth conditions. This hypothesis, which imposes a bound of q/p in terms of p and N has been used several times in [45, Chapter 3]. This assumption is relevant in the qualitative analysis of nonlinear problems with variable exponents and describes a subcritical abstract setting.

Remark 1. The typical case of $\mathbf{A}$ is

$$\mathbf{A}(x, \nabla u) = \begin{cases} |\nabla u|^{p(x)-2} \nabla u, \text{ if } |\nabla u| > 1 \\ |\nabla u|^{q(x)-2} \nabla u, \text{ if } |\nabla u| \le 1. \end{cases}$$

Then

$$-\operatorname{div} \mathbf{A}(x, \nabla u) = \begin{cases} -\operatorname{div} (|\nabla u|^{p(x)-2} \nabla u), \text{ if } |\nabla u| > 1 \\ -\operatorname{div} (|\nabla u|^{q(x)-2} \nabla u), \text{ if } |\nabla u| \le 1 \end{cases}$$

and

$$\mathscr{A}(x,\xi) = \begin{cases} \dfrac{1}{p(x)}|\xi|^{p(x)} + \dfrac{1}{q(x)} - \dfrac{1}{p(x)}, \text{ if } |\xi| > 1 \\ \dfrac{1}{q(x)}|\xi|^{q(x)}, \text{ if } |\xi| \leq 1 \end{cases}.$$

From Lemma A.2, given in Appendix A, it is clear that this typical potential $\mathscr{A}$ satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, $1 < p^- \leq p^+ < N$ and $1 < q^- \leq q^+ < N$.

We also make the following assumptions:

$(\mathcal{H}_V)$ (i) $V \in L^1_{loc}(\mathbb{R}^N)$ and $V(\cdot) > 0$ a.e. in $\mathbb{R}^N$;

(ii) $V(\cdot) \geq V_0 > 0$;

(iii) $V(x) \to +\infty$ as $|x| \to +\infty$.

$(\mathcal{H}_f^1)$ $0 \leq f(x,u)u$, $f(x,u)u = o(|u|^{\alpha(x)})$ as $u \to 0$, and $|f(x,u)| \leq C(1+|u|^{\gamma(x)})$, where $\gamma(\cdot)$ is Lipschitz continuous and $\alpha \leq \gamma(\cdot) \ll p^*(\cdot)$.

$(\mathcal{H}_f^2)$ There exist constants $M, C_1, C_2 > 0$ and a function $a \gg q$ on $\mathbb{R}^N$ such that

$$C_1 |t|^{q(x)} [\ln(e+|t|)]^{a(x)-1} \leq C_2 \frac{tf(x,t)}{\ln(e+|t|)} \leq tf(x,t) - s(x)F(x,t), \forall\, |t| \geq M, \quad \forall x \in \mathbb{R}^N,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

$(\mathcal{H}_f^3)$ $f(x,-u) = -f(x,u)$.

Remark 2. Let $f(x,t) = |t|^{s(x)-2}\, t[\ln(1+|t|)]^{a(x)}$ where $s(\cdot) \geq q(\cdot)$, $a(\cdot) \gg q(\cdot)$ on $\mathbb{R}^N$, then we observe that f satisfies hypotheses $(\mathcal{H}_f^1)$–$(\mathcal{H}_f^3)$, but it does not satisfy the Ambrosetti–Rabinowitz condition.

This paper generalizes some results contained in [4] and [11] to the case of partial differential equations with variable exponent. If $p(\cdot) \equiv p$, $q(\cdot) \equiv q$ and $\alpha(\cdot) \equiv \alpha$ are constants, then $(\mathcal{E})$ becomes the usual constant exponent differential equation in divergence form discussed in [11]. But if either $p(\cdot)$ or $q(\cdot)$ are nonconstant functions, then $(\mathcal{E})$ has a more complicated structure, due to its non-homogeneities and to the presence of several nonlinear terms.

This paper is motivated by double phase nonlinear problems with variational structure, which have been introduced by Marcellini [34] and developed by Mingione *et al.* [6,7,14,15] in the framework of nonhomogeneous problems driven by a differential operator with variable growths described by nonconstant functions $p(x)$ and $q(x)$. In the case of two different materials that involve power hardening exponents $p(x)$ and $q(x)$, the differential operator $\operatorname{div}\mathbf{A}(x,\nabla u)$ describes the geometry of a composite of these two materials. Cf. hypothesis (1), the $p(\cdot)$-material is present if $|\xi| \gg 1$. In the contrary case, the $q(\cdot)$-material is the only one describing the composite.

In recent years, the study of differential equations and variational problems with variable exponent growth conditions have been an interesting topic, which have backgrounds in image processing, nonlinear electrorheological fluids and elastic mechanics etc. We refer the readers to [1,10,31,44,45,47,57] and the references therein for more background of applications. There are many reference papers related to the study of variational problems with variable exponent growth conditions, far from being complete, we refer the readers to [3], [8], [16–29], [32], [33], [36–38], [43], [46], [48–50], [52–55].

Our main results can be stated as follows.

Theorem 1.1. *Assuming that* $1 \ll \alpha \leq p \ll q \ll \min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, $(\mathcal{A}_1)$*–*$(\mathcal{A}_2)$, $(\mathcal{H}_V)$ *and* $(\mathcal{H}_f^1)$*–*$(\mathcal{H}_f^2)$*, then problem* $(\mathcal{E})$ *has a pair of nontrivial non-negative and non-positive solution.*

Theorem 1.2. *Assuming that* $1 \ll \alpha \le p \ll q \ll \min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, $(\mathcal{A}_1)$*–*$(\mathcal{A}_2)$, $(\mathcal{H}_V)$ *and* $(\mathcal{H}_f^1)$*–*$(\mathcal{H}_f^3)$*, then problem* $(\mathcal{E})$ *has infinitely many nontrivial solutions.*

This paper is divided into six sections; section 2 contains some properties of function spaces with variable exponent; section 3 includes several basic properties of Orlicz–Sobolev spaces; in section 4 we establish some qualitative properties of the operators involved in our analysis; sections 5 and 6 give the proofs of Theorems 1.1 and 1.2, respectively.

We refer to [12] for the basic analytic tools used in this paper.

2. Variable exponent spaces

Nonlinear problems with non-homogeneous structure are motivated by numerous models in the applied sciences that are driven by partial differential equations with one or more variable exponents. In some circumstances, the standard analysis based on the theory of usual Lebesgue and Sobolev function spaces, L^p and $W^{1,p}$, is not appropriate in the framework of material that involve non-homogeneities. For instance, both electrorheological "smart fluids" fluids and phenomena arising in image processing are described in a correct way by nonlinear models in which the exponent p is not necessarily constant. The variable exponent describes the geometry of a material which is allowed to change its hardening exponent according to the point. This leads to the analysis of variable exponents Lebesgue and Sobolev function spaces (denoted by $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$), where p is a real-valued (non-constant) function.

Throughout this paper, the letters c, c_i, C, C_i, $i = 1, 2, \ldots$ denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

In order to discuss the problem $(\mathcal{E})$, we need some theories on variable exponent Lebesgue spaces and Sobolev spaces. In the following, we will give some properties of these variable exponent spaces. Let $\Omega \subset \mathbb{R}^N$ be an open domain. Let $S(\Omega)$ be the set of all measurable real valued functions defined on Ω. Let

$$C_+(\overline{\Omega}) = \left\{ v \,\middle|\, v \in C(\overline{\Omega}),\ v(x) > 1 \text{ for } x \in \overline{\Omega} \right\},$$

$$L^{p(\cdot)}(\Omega) = \left\{ u \in S(\Omega) \mid \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

The function space $L^{p(\cdot)}(\Omega)$ is equipped with the Luxemburg norm

$$|u|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Then $(L^{p(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)})$ becomes a Banach space, we call it variable exponent Lebesgue space. If $\Omega = \mathbb{R}^N$, we simply denote $(L^{p(\cdot)}(\mathbb{R}^N), |\cdot|_{L^{p(\cdot)}(\mathbb{R}^N)})$ as $(L^{p(\cdot)}, |\cdot|_{L^{p(\cdot)}})$.

Proposition 2.1. *(see [20, Theorem 1.15]) The space* $(L^{p(\cdot)}(\Omega), |u|_{L^{p(\cdot)}(\Omega)})$ *is a separable, uniformly convex Banach space, and its conjugate space is* $L^{p'(\cdot)}(\Omega)$*, where* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$*. For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*, we have the following Hölder inequality*

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{L^{p(\cdot)}(\Omega)} \, |v|_{L^{p'(\cdot)}(\Omega)} .$$

Proposition 2.2. *(see [20, Theorem 1.16]) If $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies*

$$|f(x,s)| \leq d(x) + b\,|s|^{p_1(x)/p_2(x)} \text{ for any } x \in \Omega, s \in \mathbb{R},$$

where p_1, $p_2 \in C_+(\overline{\Omega})$, $d(x) \in L^{p_2(\cdot)}(\Omega)$, $d(x) \geq 0$, $b \geq 0$, then the Nemytsky operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.3. *(see [20, Theorem 1.3]) If we denote*

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx, \ \forall u \in L^{p(\cdot)}(\Omega),$$

then

i) $|u|_{L^{p(\cdot)}(\Omega)} < 1 (= 1; > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$;

ii) $|u|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^+}_{L^{p(\cdot)}(\Omega)}$;

$|u|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \geq \rho_{p(\cdot)}(u) \geq |u|^{p^+}_{L^{p(\cdot)}(\Omega)}$;

iii) $|u|_{L^{p(\cdot)}(\Omega)} \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u) \to \infty$.

Proposition 2.4. *(see [20, Theorem 1.4]) If u, $u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \cdots$, then the following statements are equivalent:*

1) $\lim\limits_{k\to\infty} |u_k - u|_{L^{p(\cdot)}(\Omega)} = 0$;

2) $\lim\limits_{k\to\infty} \rho_{p(\cdot)}(u_k - u) = 0$;

3) $u_k \to u$ *in measure in* Ω *and* $\lim\limits_{k\to\infty} \rho_{p(\cdot)}(u_k) = \rho_{p(\cdot)}(u)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \,\middle|\, \nabla u \in [L^{p(\cdot)}(\Omega)]^N \right\},$$

and it is equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = |u|_{L^{p(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)}, \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The Lebesgue and Sobolev spaces with variable exponents coincide with the usual Lebesgue and Sobolev spaces provided that p is constant. According to [45, pp. 8–9], these function spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ have some non-usual properties, such as:

(i) Assuming that $1 < p^- \leq p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is a smooth function, then the following co-area formula

$$\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} |\{x \in \Omega;\ |u(x)| > t\}|\, dt$$

has no analogue in the framework of variable exponents.

(ii) Spaces $L^{p(\cdot)}$ do *not* satisfy the *mean continuity property*. More exactly, if p is nonconstant and continuous in an open ball B, then there is some $u \in L^{p(\cdot)}(B)$ such that $u(x+h) \notin L^{p(\cdot)}(B)$ for every $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) Function spaces with variable exponent are *never* invariant with respect to translations. The convolution is also limited. For instance, the classical Young inequality

$$|f * g|_{p(\cdot)} \le C\,|f|_{p(\cdot)}\,\|g\|_{L^1}$$

remains true if and only if p is constant.

Conditions $(\mathcal{A}_1)$-(i) and (ii) imply that

$$\mathscr{A}(x,\xi) \le \mathbf{A}(x,\xi)\cdot\xi \text{ for all } (x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N. \tag{2}$$

Furthermore, $(\mathcal{A}_1)$-(ii) is weaker than the request that $\mathscr{A}$ is uniformly convex, that is, for any $\varepsilon\in(0,1)$, there exists a constant $\delta(\varepsilon)\in(0,1)$ such that

$$\mathscr{A}(x,\frac{\xi+\eta}{2}) \le (1-\delta(\varepsilon))\frac{\mathscr{A}(x,\xi)+\mathscr{A}(x,\eta)}{2}$$

for all $x\in\mathbb{R}^N$ and $\xi,\eta\in\mathbb{R}^N\times\mathbb{R}^N$ satisfy $|u-v|\ge\varepsilon\max\{|u|,|v|\}$.

By $(\mathcal{A}_1)$-(i) and (iii), we have

$$\mathscr{A}(x,\xi)=\int_0^1 \frac{d}{dt}\mathscr{A}(x,t\xi)dt=\int_0^1\frac{1}{t}\mathbf{A}(x,t\xi)\cdot t\xi dt \ge \begin{cases} c_1\,|\xi|^{p(x)}, |\xi|>1\\ c_1\,|\xi|^{q(x)}, |\xi|\le 1\end{cases}.$$

This estimate in combination with (1) and (2) yields

$$\left.\begin{matrix} c_1\,|\xi|^{p(x)}, |\xi|>1\\ c_1\,|\xi|^{q(x)}, |\xi|\le 1\end{matrix}\right\} \le \mathscr{A}(x,\xi)\le \mathbf{A}(x,\xi)\cdot\xi \le \begin{cases} c_2\,|\xi|^{p(x)}, |\xi|>1\\ c_2\,|\xi|^{q(x)}, |\xi|\le 1\end{cases}, \quad \forall (x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N. \tag{3}$$

Denote

$$L_V^{\alpha(\cdot)}(\Omega)=\left\{u\in S(\Omega)\left|\int_\Omega V(x)\,|u(x)|^{\alpha(x)}\,dx<\infty\right.\right\},$$

with the norm

$$|u|_{L_V^{\alpha(\cdot)}(\Omega)}=\inf\left\{\lambda>0\left|\int_\Omega V(x)\left|\frac{u(x)}{\lambda}\right|^{\alpha(x)}dx\le 1\right.\right\}.$$

If $\Omega=\mathbb{R}^N$, we simply denote $(L_V^{\alpha(\cdot)}(\mathbb{R}^N),|\cdot|_{L_V^{\alpha(\cdot)}(\mathbb{R}^N)})$ as $(L_V^{\alpha(\cdot)},|\cdot|_{L_V^{\alpha(\cdot)}})$.

From now on, we denote by B_R the ball in $\mathbb{R}^N$ centered at the origin and of radius $R>0$.

Lemma 2.5. *(see [42, Lemma 2.2]) Assume $(\mathcal{H}_V)$-(i), $\alpha^->1$ and $\alpha^+<\infty$, then $L_V^{\alpha(\cdot)}$ is separable uniformly convex Banach spaces.*

Theorem 2.6 *(Interpolation Theorem). If $p(\cdot)<\alpha(\cdot)<q(\cdot)$, then for any $u\in L^{\alpha(\cdot)}(\Omega)$, there exists $\lambda=\lambda(\Omega,\alpha,p,q,u)\in[\theta^-,\theta^+]$, where $\theta(\cdot)=\frac{p(q-\alpha)}{\alpha(q-p)}$, such that*

$$|u|_{L^{\alpha(\cdot)}(\Omega)}\le 2\,|u|^{\lambda}_{L^{p(\cdot)}(\Omega)}\cdot|u|^{1-\lambda}_{L^{q(\cdot)}(\Omega)}.$$

Moreover, if $\theta^-<\theta^+$, then $\lambda\in(\theta^-,\theta^+)$.

Proof. Let $\beta := \frac{p(q-\alpha)}{(q-p)} = \alpha\theta$. We may assume that u is nontrivial. We consider

$$\int_\Omega \left| \frac{u}{2\,|u|^{\theta(x)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(x)}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx \le \frac{1}{2} \int_\Omega \left| \frac{u}{|u|^{\theta(x)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(x)}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx$$
$$= \frac{1}{2} \int_\Omega \frac{|u|^{\beta(x)}}{|u|^{\alpha(x)\theta(x)}_{L^{p(\cdot)}(\Omega)}} \cdot \frac{|u|^{\alpha(x)-\beta(x)}}{|u|^{(1-\theta(x))\alpha(x)}_{L^{q(\cdot)}(\Omega)}} dx$$
$$\le \left| \frac{|u|^{\beta(x)}}{|u|^{\alpha(x)\theta(x)}_{L^{p(\cdot)}(\Omega)}} \right|_{L^{\frac{p(\cdot)}{\beta(\cdot)}}(\Omega)} \cdot \left| \frac{|u|^{\alpha(x)-\beta(x)}}{|u|^{(1-\theta(x))\alpha(x)}_{L^{q(\cdot)}(\Omega)}} \right|_{L^{(\frac{p(\cdot)}{\beta(\cdot)})'}(\Omega)}.$$

Note that

$$\int_\Omega \left(\frac{|u|^{\beta(x)}}{|u|^{\alpha(x)\theta(x)}_{L^{p(\cdot)}(\Omega)}} \right)^{\frac{p(x)}{\beta(x)}} dx = \int_\Omega \frac{|u|^{\beta(x)\frac{p(x)}{\beta(x)}}}{|u|^{\alpha(x)\theta(x)\frac{p(x)}{\beta(x)}}_{L^{p(\cdot)}(\Omega)}} dx = \int_\Omega \frac{|u|^{p(x)}}{|u|^{p(x)}_{L^{p(\cdot)}(\Omega)}} dx = \int_\Omega \left| \frac{u}{|u|_{L^{p(\cdot)}(\Omega)}} \right|^{p(x)} dx = 1,$$

therefore $\left| \frac{|u|^{\beta(x)}}{|u|^{\alpha(x)\theta(x)}_{L^{p(\cdot)}(\Omega)}} \right|_{\frac{p(\cdot)}{\beta(\cdot)}} = 1.$

Similarly we obtain

$$\int_\Omega \left(\frac{|u|^{\alpha(x)-\beta(x)}}{|u|^{\alpha(x)(1-\theta(x))}_{L^{q(\cdot)}(\Omega)}} \right)^{(\frac{p(x)}{\beta(x)})'} dx = \int_\Omega \frac{|u|^{(\alpha(x)-\beta(x))(\frac{p(x)}{\beta(x)})'}}{|u|^{\alpha(x)(1-\theta(x))(\frac{p(x)}{\beta(x)})'}_{L^{q(\cdot)}(\Omega)}} dx = \int_\Omega \frac{|u|^{q(x)}}{|u|^{q(x)}_{L^{q(\cdot)}(\Omega)}} dx$$
$$= \int_\Omega \left| \frac{u}{|u|_{L^{q(\cdot)}(\Omega)}} \right|^{q(x)} dx = 1,$$

therefore $\left| \frac{|u|^{\alpha(x)-\beta(x)}}{|u|^{(1-\theta(x))\alpha(x)}_{L^{q(\cdot)}(\Omega)}} \right|_{L^{(\frac{p(\cdot)}{\beta(\cdot)})'}(\Omega)} = 1.$

It follows that

$$\int_\Omega \left| \frac{u}{2\,|u|^{\theta(x)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(x)}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx \le 1.$$

If $|u|_{L^{p(\cdot)}(\Omega)} = |u|_{L^{q(\cdot)}(\Omega)}$. We observe that

$$\int_\Omega \left| \frac{u}{2\,|u|^{\theta(x)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(x)}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx = \int_\Omega \left| \frac{u}{2\,|u|_{L^{p(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx \le 1.$$

Therefore

$$|u|_{L^{\alpha(\cdot)}(\Omega)} \le 2\,|u|_{L^{p(\cdot)}(\Omega)} = 2\,|u|^{\theta(\xi)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(\xi)}_{L^{q(\cdot)}(\Omega)}, \text{ for any } \xi \in \Omega.$$

Without loss of generality, we may assume that $|u|_{L^{p(\cdot)}(\Omega)} > |u|_{L^{q(\cdot)}(\Omega)}$ and $\theta^- < \theta^+$, then we have

$$\int_\Omega \left| \frac{u}{2\left(\frac{|u|_{L^{p(\cdot)}(\Omega)}}{|u|_{L^{q(\cdot)}(\Omega)}}\right)^{\theta^+} \cdot |u|_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx < \int_\Omega \left| \frac{u}{2\,|u|^{\theta(x)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(x)}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx <$$

$$\int_\Omega \left| \frac{u}{2\left(\frac{|u|_{L^{p(\cdot)}(\Omega)}}{|u|_{L^{q(\cdot)}(\Omega)}}\right)^{\theta^-} \cdot |u|_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx.$$

Thus, there exists $\lambda(p,q,\alpha,u) \in [\theta^-,\theta^+]$ such that

$$\int_\Omega \left| \frac{u}{2\,|u|^{\lambda}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\lambda}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx = \int_\Omega \left| \frac{u}{2\,|u|^{\theta(x)}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\theta(x)}_{L^{q(\cdot)}(\Omega)}} \right|^{\alpha(x)} dx \le 1,$$

which implies $|u|_{L^{\alpha(\cdot)}(\Omega)} \le 2\,|u|^{\lambda}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\lambda}_{L^{q(\cdot)}(\Omega)}$.

To conclude the proof, we observe that $\lambda \in (\theta^-,\theta^+)$, provided that $\theta^- < \theta^+$. □

3. Variable exponent Orlicz–Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be an open domain.

Definition 3.1. Assume $(\mathcal{A}_1)$-(iv). We define the following linear space

$$L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega) = \left\{ u \in \mathcal{M} \mid u = v + w, v \in L^{p(\cdot)}(\Omega), w \in L^{q(\cdot)}(\Omega) \right\},$$

which is endowed with the norm

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = \inf \left\{ |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)} \mid v \in L^{p(\cdot)}(\Omega), w \in L^{q(\cdot)}(\Omega), v + w = u \right\}. \tag{4}$$

If $\Omega = \mathbb{R}^N$, we simply denote $(L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)})$ as $(L^{p(\cdot)} + L^{q(\cdot)}, |\cdot|_{L^{p(\cdot)}+L^{q(\cdot)}})$.

We also define the linear space

$$L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega) = \left\{ u \mid u \in L^{p(\cdot)}(\Omega) \text{ and } u \in L^{q(\cdot)}(\Omega) \right\},$$

which is endowed with the norm

$$|u|_{L^{p(\cdot)}(\Omega)\cap L^{q(\cdot)}(\Omega)} = \max \left\{ |u|_{L^{p(\cdot)}(\Omega)}, |u|_{L^{q(\cdot)}(\Omega)} \right\}.$$

Throughout this paper, we denote

$$\Lambda_u = \{x \in \Omega \mid |u(x)| > 1\} \quad \text{and} \quad \Lambda_u^c = \{x \in \Omega \mid |u(x)| \le 1\}.$$

Proposition 3.2. *Assume $(\mathcal{A}_1)$-(iv). Let $\Omega \subset \mathbb{R}^N$ and $u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. Then the following properties are true:*

(i) if $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^{p(\cdot)}(\Omega')$;

(ii) if $\Omega' \subset \Omega$ *is such that* $u \in L^{\infty}(\Omega')$*, then* $u \in L^{q(\cdot)}(\Omega')$*;*
(iii) $|\Lambda_u| < +\infty$*;*
(iv) $u \in L^{p(\cdot)}(\Lambda_u) \cap L^{q(\cdot)}(\Lambda_u^c)$*;*
(v) the infimum in (4) is attained;
(vi) if $B \subset \Omega$*, then* $|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \le |u|_{L^{p(\cdot)}(B)+L^{q(\cdot)}(B)} + |u|_{L^{p(\cdot)}(\Omega/B)+L^{q(\cdot)}(\Omega/B)}$*;*
(vii) we have

$$\max\left\{\frac{1}{1+2\left|\Lambda_u\right|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}}\left|u\right|_{L^{p(\cdot)}(\Lambda_u)}, c\min\{|u|_{L^{q(\cdot)}(\Lambda_u^c)}, |u|_{L^{q(\cdot)}(\Lambda_u^c)}^{\frac{q(\xi)}{p(\xi)}}\}\right\} \le$$
$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \le |u|_{L^{p(\cdot)}(\Lambda_u)} + |u|_{L^{q(\cdot)}(\Lambda_u^c)},$$

where $\xi \in \mathbb{R}^N$ *and* c *is a small positive constant.*

Proof. The proof uses some ideas developed in [4] and [5]. For the reader's convenient, we state it here.

(i) Let $v \in L^{p(\cdot)}(\Omega)$ and $w \in L^{q(\cdot)}(\Omega)$ be such that $u = v + w$, then $v \in L^{p(\cdot)}(\Omega')$ and $w \in L^{q(\cdot)}(\Omega')$. In order to prove $u \in L^{p(\cdot)}(\Omega')$, we only need to prove $w \in L^{p(\cdot)}(\Omega')$. By Young's inequality, we have

$$\int_{\Omega'} |w|^{p(x)}\,dx \le \int_{\Omega'} |1|^{(\frac{q(x)}{p(x)})'} + \left||w|^{p(x)}\right|^{\frac{q(x)}{p(x)}} dx = \int_{\Omega'} |1| + |w|^{q(x)}\,dx = |\Omega'| + \int_{\Omega'} |w|^{q(x)}\,dx < +\infty.$$

Thus $w \in L^{p(\cdot)}(\Omega')$. Therefore $u \in L^{p(\cdot)}(\Omega')$.

(ii) Let $v \in L^{p(\cdot)}(\Omega)$ and $w \in L^{q(\cdot)}(\Omega)$ be such that $u = v + w$, then $v \in L^{p(\cdot)}(\Omega')$ and $w \in L^{q(\cdot)}(\Omega')$. In order to prove $u \in L^{q(\cdot)}(\Omega')$, we only need to prove $v \in L^{q(\cdot)}(\Omega')$. We have

$$\int_{\Omega'} |v|^{q(x)}\,dx = \int_{\Omega'} |v|^{q(x)-p(x)}\,|v|^{p(x)}\,dx \le (1 + |\sup v|)^{q^+-p^-} \int_{\Omega'} |v|^{p(x)}\,dx < +\infty.$$

Thus, $v \in L^{q(\cdot)}(\Omega')$. Therefore $u \in L^{q(\cdot)}(\Omega')$.

(iii) Let $v \in L^{p(\cdot)}(\Omega)$ and $w \in L^{q(\cdot)}(\Omega)$ be such that $u = v + w$. Since $1 < |u| \le |v| + |w|$ implies $|v| \ge \frac{1}{2}$ or $|w| \ge \frac{1}{2}$ for any $x \in \Omega$. We get

$$+\infty > \int_{\Omega} |v|^{p(x)} + |w|^{q(x)}\,dx \ge \int_{\Lambda_u} |v|^{p(x)} + |w|^{q(x)}\,dx \ge \int_{\Lambda_u} \left|\frac{1}{2}\right|^{p^++q^+} dx = \left|\frac{1}{2}\right|^{p^++q^+} |\Lambda_u|.$$

Thus, $|\Lambda_u| < +\infty$.

(iv) It is easy to see from (i)–(iii).

(v) For any $u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$, and consider a minimizing sequence for u, namely $v_n \in L^{p(\cdot)}(\Omega)$ and $w_n \in L^{q(\cdot)}(\Omega)$ be such that $u = v_n + w_n$, and $\lim_{n\to+\infty} |v_n|_{L^{p(\cdot)}(\Omega)} + |w_n|_{L^{q(\cdot)}(\Omega)} = |u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}$. Without loss of generality, we may assume that $|v_n|_{L^{p(\cdot)}(\Omega)} + |w_n|_{L^{q(\cdot)}(\Omega)} \le 1 + |u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}$ for $n = 1, 2 \ldots$, and $\lim_{n\to\infty} |v_n|_{L^{p(\cdot)}(\Omega)}$ exists. Thus, $\{v_n\}$ is bounded in $L^{p(\cdot)}(\Omega)$, then the reflexibility of $L^{p(\cdot)}(\Omega)$ implies that up to a subsequence, there exists $v_0 \in L^{p(\cdot)}(\Omega)$ such that $v_n \rightharpoonup v_0$ in $L^{p(\cdot)}(\Omega)$. Similarly, we may assume that there exists $w_0 \in L^{q(\cdot)}(\Omega)$ such that $w_n \rightharpoonup w_0$ in $L^{q(\cdot)}(\Omega)$. Therefore $(v_n, w_n) \rightharpoonup (v_0, w_0)$ in $L^{p(\cdot)}(\Omega) \times L^{q(\cdot)}(\Omega)$, and then $v_n + w_n \rightharpoonup v_0 + w_0$ in $L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. Thus, $u = v_0 + w_0$.

By lower semicontinuity we have

$$|v_0|_{L^{p(\cdot)}(\Omega)} \le \liminf_{n\to\infty} |v_n|_{L^{p(\cdot)}(\Omega)} \quad \text{and} \quad |w_0|_{L^{q(\cdot)}(\Omega)} \le \liminf_{n\to\infty} |w_n|_{L^{q(\cdot)}(\Omega)}.$$

It follows that

$$\begin{aligned}|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} &= \lim_{n\to\infty}\left(|v_n|_{L^{p(\cdot)}(\Omega)} + |w_n|_{L^{q(\cdot)}(\Omega)}\right)\\ &\geq \liminf_{n\to\infty}|v_n|_{L^{p(\cdot)}(\Omega)} + \liminf_{n\to\infty}|w_n|_{L^{q(\cdot)}(\Omega)}\\ &= |v_0|_{L^{p(\cdot)}(\Omega)} + |w_0|_{L^{q(\cdot)}(\Omega)}\,.\end{aligned}$$

According to the definition of $|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}$, we have

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v_0|_{L^{p(\cdot)}(\Omega)} + |w_0|_{L^{q(\cdot)}(\Omega)}\,.$$

(vi) The proof of property (vi) we refer to Proposition 2.2 of [4].

(vii) From (iv) and the definition of the norm $|\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}$, we observe that

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \leq |u|_{L^{p(\cdot)}(\Lambda_u)} + |u|_{L^{q(\cdot)}(\Lambda_u^c)}\,.$$

From (v), there exists $v \in L^{p(\cdot)}(\Omega), w \in L^{q(\cdot)}(\Omega), v + w = u$ such that $|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)}$. We have

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)} \geq |v|_{L^{p(\cdot)}(\Lambda_u)} + |w|_{L^{q(\cdot)}(\Lambda_u)} \geq |u|_{L^{p(\cdot)}(\Lambda_u)+L^{q(\cdot)}(\Lambda_u)}\,,$$

and

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)} \geq |v|_{L^{p(\cdot)}(\Lambda_u^c)} + |w|_{L^{q(\cdot)}(\Lambda_u^c)} \geq |u|_{L^{p(\cdot)}(\Lambda_u^c)+L^{q(\cdot)}(\Lambda_u^c)}\,.$$

By Proposition 2.1, we get $|w|_{L^{p(\cdot)}(\Lambda_u)} \leq 2\,|\Lambda_u|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}\,|w|_{L^{q(\cdot)}(\Lambda_u)}$ for all $\xi \in \mathbb{R}^N$. It follows that

$$\begin{aligned}|u|_{L^{p(\cdot)}(\Lambda_u)} &\leq |v|_{L^{p(\cdot)}(\Lambda_u)} + |w|_{L^{p(\cdot)}(\Lambda_u)} \leq |v|_{L^{p(\cdot)}(\Lambda_u)} + 2\,|\Lambda_u|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}\,|w|_{L^{q(\cdot)}(\Lambda_u)}\\ &\leq \left(1 + 2\,|\Lambda_u|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}\right)\left(|v|_{L^{p(\cdot)}(\Lambda_u)} + |w|_{L^{q(\cdot)}(\Lambda_u)}\right)\\ &= \left(1 + 2\,|\Lambda_u|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}\right)|u|_{L^{p(\cdot)}(\Lambda_u)+L^{q(\cdot)}(\Lambda_u)}\\ &\leq \left(1 + 2\,|\Lambda_u|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}\right)|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}\,.\end{aligned}$$

Without loss of generality, we may assume that u is nonnegative, $0 \leq v \in L^{p(\cdot)}(\Omega), 0 \leq w \in L^{q(\cdot)}(\Omega), v + w = u$ such that $|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)}$. Obviously, $0 \leq w \leq 1$ on Λ_u^c.

Denote $\lambda_p = |w|_{L^{p(\cdot)}(\Lambda_u^c)}$ and $\lambda_q = |w|_{L^{q(\cdot)}(\Lambda_u^c)}$. Since $|w| \leq 1$ on Λ_u^c, we have

$$1 = \int_{\Lambda_u^c}\left|\frac{w}{\lambda_p}\right|^{p(x)}dx \geq \int_{\Lambda_u^c}\left|\frac{w}{\lambda_q}\right|^{q(x)}\frac{\lambda_q^{q(x)}}{\lambda_p^{p(x)}}dx = \frac{\lambda_q^{q(\xi)}}{\lambda_p^{p(\xi)}}\int_{\Lambda_u^c}\left|\frac{w}{\lambda_q}\right|^{q(x)}dx = \frac{\lambda_q^{q(\xi)}}{\lambda_p^{p(\xi)}},$$

where $\xi \in \mathbb{R}^N$. Thus $\lambda_p \geq \lambda_q^{\frac{q(\xi)}{p(\xi)}}$. Similarly, we have

$$\begin{aligned}&|u|_{L^{p(\cdot)}(\Lambda_u^c)+L^{q(\cdot)}(\Lambda_u^c)} = |v|_{L^{p(\cdot)}(\Lambda_u^c)} + |w|_{L^{q(\cdot)}(\Lambda_u^c)} \geq |v|^{\frac{q(\xi)}{p(\xi)}}_{L^{q(\cdot)}(\Lambda_u^c)} + |w|_{L^{q(\cdot)}(\Lambda_u^c)} \geq\\ &c\min\left\{|u|_{L^{q(\cdot)}(\Lambda_u^c)}\,,|u|^{\frac{q(\xi)}{p(\xi)}}_{L^{q(\cdot)}(\Lambda_u^c)}\right\}\end{aligned}$$

Summarizing, we get the result. □

We define the following norm on $L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$:

$$\|u\|_t = \inf\left\{\left(|v|^t_{L^{p(\cdot)}(\Omega)} + |w|^t_{L^{q(\cdot)}(\Omega)}\right)^{1/t} \mid v \in L^{p(\cdot)}(\Omega), w \in L^{q(\cdot)}(\Omega), v + w = u\right\}$$

if $1 \leq t < +\infty$ and

$$\|u\|_\infty = \inf\left\{\max\{|v|_{L^{p(\cdot)}(\Omega)}, |w|_{L^{q(\cdot)}(\Omega)}\} \mid v \in L^{p(\cdot)}(\Omega), w \in L^{q(\cdot)}(\Omega), v + w = u\right\}.$$

Proposition 3.3. *Assume* $(\mathcal{A}_1)$*-(iv). Then* $\{\|u\|_t\}_{1\leq t\leq\infty}$ *is a family of equivalent norms on* $L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)$*. Moreover,* $\|u\|_t = \||u|\|_t$ *for every* $u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$ *and* $1 \leq t \leq \infty$.

Proposition 3.4. *(see [13, Theorem 1]) The uniformly convex product of a finite number of uniformly convex Banach spaces is uniformly convex.*

Proposition 3.5. *The norm* $\|\cdot\|_t$ *is uniformly convex for* $1 < t < \infty$.

Proposition 3.6. *(see [30, Theorem 2]) If* $(X, \|\cdot\|)$ *is a Banach space, then the following two statements are equivalent:*

(i) $(X, \|\cdot\|)$ *is reflexive;*

(ii) any bounded sequence of $(X, \|\cdot\|)$ *has a weak convergent subsequence.*

From Proposition 3.6, we obtain the following property.

Proposition 3.7. *If* $(X, \|\cdot\|)$ *is a reflexive Banach space, and* $\|\cdot\|_*$ *is an equivalent norm of* $\|\cdot\|$ *on* X*, then* $(X, \|\cdot\|_*)$ *is also a reflexive Banach space.*

Proposition 3.8. *Assume* $(\mathcal{A}_1)$*-(iv). Then* $(L^{p'(\cdot)}(\Omega) \cap L^{q'(\cdot)}(\Omega))' = L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$.

Proof. Similar to the proof of Theorem 2.10 of [5]. We omit it here. □

Proposition 3.9. *Assume* $(\mathcal{A}_1)$*-(iv). Then* $(L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)})$ *is a reflexive Banach space.*

Proof. By Proposition 3.8, we obtain that $(L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)})$ is a Banach space.

Because $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on $L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)$, and $(L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega), \|\cdot\|_2)$ is reflexive from Proposition 3.5, we deduce that $(L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega), \|\cdot\|_1)$ is a reflexive space. Since $|\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = \|\cdot\|_1$, we deduce that $(L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)})$ is also a reflexive space. □

Define $X(\Omega)=\left\{u \in L^{\alpha(\cdot)}_V(\Omega) \mid \nabla u \in (L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega))^N\right\}$ with the following norm

$$\|u\|_\Omega = |u|_{L^{\alpha(\cdot)}_V(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}.$$

If $\Omega = \mathbb{R}^N$, we simply denote $(X(\Omega), \|u\|_\Omega)$ as $(X, \|u\|)$.

Proposition 3.10. *Assume* $(\mathcal{A}_1)$*-(iv) and* $(\mathcal{H}_V)$*-(i). Then* $(X(\Omega), \|u\|_\Omega)$ *is a Banach space.*

Proof. The proof is similar to Proposition 2.4 of [4], we omit it here. □

Proposition 3.11. *Assume* $(\mathcal{A}_1)$*-(iv) and* $(\mathcal{H}_V)$*-(i). Then* $(X(\Omega), \|u\|_\Omega)$ *is reflexive.*

Proof. On $L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)$, we can consider the equivalent norm $\|\cdot\|_2$, it is uniformly convex.

We consider the following norm on X:

$$\|u\|_{\#}=\left(|u|^2_{L_V^{\alpha(\cdot)}(\Omega)}+\|\cdot\|_2^2\right)^{1/2}.$$

It is easy to see that $\|\cdot\|_{\#}$ is an equivalent norm of $\|\cdot\|_\Omega$ on X. By Proposition 3.4, we deduce that $\|\cdot\|_{\#}$ is uniformly convex, then $(X(\Omega),\|\cdot\|_{\#})$ is reflexive. By Proposition 3.7, we conclude that $(X(\Omega),\|\cdot\|_\Omega)$ is a reflexive space. □

Theorem 3.12. *Assume $(\mathcal{A}_1)$-(iv) and $(\mathcal{H}_V)$-(i) and (ii), $1\ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, α satisfies $1\ll\alpha(\cdot)\ll p^*(\cdot)\frac{N-1}{N}$ and $1\ll\alpha(\cdot)\leq p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$. Then the space $X(\Omega)$ is continuously embedded into $L^{p^*(\cdot)}(\Omega)$.*

Proof. For any $u\in X(\Omega)\backslash\{0\}$, we only need to prove that $u\in L^{p^*(\cdot)}(\Omega)$. Write

$$X_{loc}(\Omega)=\{u\in X(\Omega):\operatorname{supp}u \text{ is compact.}\}$$

We distinguish the following three cases.

Case 1: We assume $u\in X_{loc}(\Omega)\cap L^\infty(\Omega)$, set $|u|_{p^*(x)}=\lambda$, then we have

$$\int_\Omega\left|\frac{u}{\lambda}\right|^{p^*(x)}dx=1. \tag{5}$$

Let $f(x)=\left|\frac{u(x)}{\lambda}\right|^{p^*(x)\frac{N-1}{N}}$, then it is easy to see that $f\in W^{1,1}(\Omega)$ following the proof below. By the Sobolev embedding theorem we have (see [2]) $W^{1,1}(\Omega)\hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, then there is a positive constant $C_1=C_1(\Omega,N)$ such that

$$|v|_{\frac{N}{N-1}}\leq C_1\int_\Omega(|\nabla v|+|v|)dx,\ \forall v\in W^{1,1}(\Omega). \tag{6}$$

Obviously, $f\in L^{\frac{N}{N-1}}(\Omega)$. It follows that

$$|f|_{L^{\frac{N}{N-1}}(\Omega)}\leq C_1\int_\Omega(|\nabla f|+|f|)dx. \tag{7}$$

We have

$$|f|_{\frac{N}{N-1}}=(\int_\Omega|f|^{\frac{N}{N-1}}dx)^{\frac{N-1}{N}}=(\int_\Omega\left|\frac{u}{\lambda}\right|^{p^*(x)}dx)^{\frac{N-1}{N}}=1, \tag{8}$$

$$\begin{aligned}|\nabla f(x)|&\leq p^*(x)\frac{N-1}{N}\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1}\left|\frac{\nabla u}{\lambda}\right|+|\nabla p^*(x)|\frac{N-1}{N}\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}}\left|\ln\left|\frac{u}{\lambda}\right|\right|\\&\leq\frac{C_2}{\lambda}\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1}|\nabla u|+C_3\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}}\left|\ln\left|\frac{u}{\lambda}\right|\right|.\end{aligned} \tag{9}$$

From (6), (7), (8), and (9) we get

$$1\leq\frac{C_*}{\lambda}\int_\Omega\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1}|\nabla u|\,dx+C_*\int_\Omega\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}}\left|\ln\left|\frac{u}{\lambda}\right|\right|dx+C_*\int_\Omega\left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}}dx. \tag{10}$$

We have

$$J_1 = \frac{1}{\lambda} \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1} |\nabla u|\, dx,$$

$$J_2 = \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx,$$

$$J_3 = \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}} dx.$$

In the following, we will estimate J_1, J_2, J_3 respectively.

By (v) of Proposition 3.2, there exist $v \in (L^{p(\cdot)}(\Omega))^N, w \in (L^{q(\cdot)}(\Omega))^N, v + w = \nabla u$ such that

$$|\nabla u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)}$$

From the Young inequality, we have

$$\begin{aligned} J_1 &= \frac{1}{\lambda} \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1} |v+w|\, dx \\ &\le \frac{1}{\lambda} \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1} |v|\, dx + \frac{1}{\lambda} \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-1} |w|\, dx \\ &\le \frac{C_4}{\lambda} \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{(p^*(x)\frac{N-1}{N}-1)p'(x)} dx + \frac{C_4}{\lambda} \int\limits_{\Omega} |v|^{p(x)}\, dx \\ &\quad + \frac{C_4}{\lambda} \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{(p^*(x)\frac{N-1}{N}-1)q'(x)} dx + \frac{C_4}{\lambda} \int\limits_{\Omega} |w|^{q(x)}\, dx. \end{aligned} \tag{11}$$

We observe that

$$\alpha(x) \le p^*(x)\frac{q'(x)}{p'(x)} = (p^*(x)\frac{N-1}{N} - 1)q'(x) \le (p^*(x)\frac{N-1}{N} - 1)p'(x) = p^*(x),\ \forall x \in \overline{\Omega}. \tag{12}$$

From (11), (12) and (5), we deduce that

$$\begin{aligned} J_1 &\le \frac{C_5}{\lambda} \int\limits_{\Omega} (\frac{V(x)}{\alpha(x)} \left|\frac{u}{\lambda}\right|^{\alpha(x)} + \left|\frac{u}{\lambda}\right|^{p^*(x)}) dx + \frac{C_5}{\lambda} \int\limits_{\Omega} |v|^{p(x)} + |w|^{q(x)}\, dx \\ &\le \frac{C_5}{\lambda} \int\limits_{\Omega} (\frac{V(x)}{\alpha(x)} \left|\frac{u}{\lambda}\right|^{\alpha(x)} + \left|\frac{u}{\lambda}\right|^{p^*(x)}) dx + \frac{C_5}{\lambda} (|v|^{p^+}_{L^{p(\cdot)}(\Omega)} + |w|^{q^+}_{L^{q(\cdot)}(\Omega)} + 2) \\ &\le \frac{C_5}{\lambda} \int\limits_{\Omega} \frac{V(x)}{\alpha(x)} \left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{C_5}{\lambda} + \frac{2C_5}{\lambda} (|\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 3). \end{aligned} \tag{13}$$

We have

$$J_2 = \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx = \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-\varepsilon} \left|\frac{u}{\lambda}\right|^{\varepsilon} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx,$$

where ε is a small enough positive constant such that

$$p^*(x)\frac{N-1}{N} + 2\varepsilon \leq p^*(x), \qquad \forall x \in \overline{\Omega}, \tag{14}$$

and

$$p^*(x)\frac{N-1}{N} - 2\varepsilon \geq \alpha(x), \qquad \forall x \in \overline{\Omega}. \tag{15}$$

By (14), there exists $t_0 > 0$ such that

$$t^{p^*(x)\frac{N-1}{N}+\varepsilon} \leq \frac{1}{3C_*} t^{p^*(x)}, \ \forall t \geq t_0, \ x \in \overline{\Omega}. \tag{16}$$

Let

$$C_6 = \sup_{0<t\leq t_0} t^{\varepsilon} \left|\ln t\right|.$$

Since $\lim_{t\to 0^+} t^{\varepsilon} |\ln t| = 0$, we have

$$0 < C_6 < +\infty.$$

Set

$$C_7 = \sup_{\substack{0<t\leq t_0 \\ x\in\overline{\Omega}}} t^{p^*(x)\frac{N-1}{N}-\varepsilon-\alpha(x)}.$$

Then relation (15) implies

$$0 < C_7 < +\infty.$$

Let

$$\Omega_1 = \{x \in \Omega \mid \left|\frac{u(x)}{\lambda}\right| \leq t_0\}, \ \Omega_2 = \{x \in \Omega \mid \left|\frac{u(x)}{\lambda}\right| > t_0\}, \tag{17}$$

then

$$J_2 = \int_{\Omega_1} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-\varepsilon} \left|\frac{u}{\lambda}\right|^{\varepsilon} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx + \int_{\Omega_2} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-\varepsilon} \left|\frac{u}{\lambda}\right|^{\varepsilon} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx.$$

Since

$$\int_{\Omega_1} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-\varepsilon} \left|\frac{u}{\lambda}\right|^{\varepsilon} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx \leq C_6 \int_{\Omega_1} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-\varepsilon} dx$$

$$\leq C_6 C_7 \int_{\Omega_1} \left|\frac{u}{\lambda}\right|^{\alpha(x)} dx \leq \frac{C_6 C_7 \alpha^+}{V_0} \int_{\Omega} \frac{V(x)}{\alpha(x)} \left|\frac{u}{\lambda}\right|^{\alpha(x)} dx,$$

and

$$\int_{\Omega_2} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}-\varepsilon} \left|\frac{u}{\lambda}\right|^{\varepsilon} \left|\ln\left|\frac{u}{\lambda}\right|\right| dx \leq \int_{\Omega_2} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}+\varepsilon} dx$$

$$\leq \frac{1}{3C_*}\int_{\Omega_2} \left|\frac{u}{\lambda}\right|^{p^*(x)} dx \leq \frac{1}{3C_*}\int_{\Omega} \left|\frac{u}{\lambda}\right|^{p^*(x)} dx \leq \frac{1}{3C_*},$$

we have

$$J_2 \leq \frac{C_6C_7}{V_0}\int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{1}{3C_*}. \tag{18}$$

In order to estimate J_3, we pick $t_0 > 1$ such that (16) is satisfied. Therefore

$$t^{p^*(x)\frac{N-1}{N}} \leq \frac{1}{3C_*}t^{p^*(x)},\ \forall t \geq t_0,\ x \in \overline{\Omega}. \tag{19}$$

Let

$$C_8 = \sup_{\substack{0<t\leq t_0\\ x\in\overline{\Omega}}} t^{p^*(x)\frac{N-1}{N}-\alpha(x)}. \tag{20}$$

Note that $\alpha(\cdot) \ll p^*(\cdot)\frac{N-1}{N}$, hence

$$0 < C_8 < +\infty.$$

Assume that Ω_1 and Ω_2 are defined as in (17). By (19) and (20) we obtain

$$\begin{aligned} J_3 &= \int_{\Omega_1} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}} dx + \int_{\Omega_2} \left|\frac{u}{\lambda}\right|^{p^*(x)\frac{N-1}{N}} dx \\ &\leq C_8\int_{\Omega_1} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{1}{3C_*}\int_{\Omega_2} \left|\frac{u}{\lambda}\right|^{p^*(x)} dx \\ &\leq C_8\int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{1}{3C_*}. \end{aligned} \tag{21}$$

From (10), (13), (18) and (21) we have

$$\begin{aligned} 1 \leq{}& C_*[\frac{C_5}{\lambda}\int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{C_5}{\lambda} + \frac{2C_5}{\lambda}(|\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 3)] \\ &+[C_*\frac{C_6C_7\alpha^+}{V_0}\int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{1}{3}] + [C_*C_8\int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{1}{3}]. \end{aligned} \tag{22}$$

Thus, by (22)

$$1 \leq C[\frac{1}{\lambda}\int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \int_{\Omega} \frac{V(x)}{\alpha(x)}\left|\frac{u}{\lambda}\right|^{\alpha(x)} dx + \frac{1}{\lambda}|\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + \frac{1}{\lambda}]. \tag{23}$$

If $\lambda \geq 1$, we deduce from (23) that

$$\begin{aligned}\lambda &\leq C[\frac{1}{\lambda^{\alpha^+}}\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + \frac{\lambda}{\lambda^{\alpha^+}}\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^++q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 1]\\ &\leq C[2\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^++q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 1] \qquad (24)\\ &\leq C_0[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^++q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 1].\end{aligned}$$

Without loss of generality, we can assume $C_0 > 1$.

If $\lambda < 1$, relation (24) is naturally satisfied, so we have proved that there is a constant $C_0 > 1$ which is independent of u such that

$$|u|_{p^*(\cdot)} \leq C_0[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^++q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 1],\ \forall u \in X_{loc} \cap L^\infty(\Omega). \qquad (25)$$

Furthermore, for any $u \in X_{loc}(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega |u|^{p^*(x)}\,dx \leq C_o^{(p*)^+}[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^++q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 1]^{(p*)^+}. \qquad (26)$$

Case 2: For any $u \in X(\Omega) \cap L^\infty(\Omega)$, we prove that (26) is satisfied.

Let $\{\psi\} \subset C^\infty(\mathbb{R}^N, \mathbb{R})$ satisfy

$$\psi(x) = 1,\ \forall |x| \leq 1; \quad \psi_n(x) = 0,\ \forall |x| \geq 3;$$
$$\psi(x) \in [0,1],\ |\nabla\psi(x)| \leq 1,\ \forall x \in \mathbb{R}^N.$$

Let $\psi_n(x) = \psi(\frac{x}{n})$, and set $A_n = \{x \in \mathbb{R}^n \mid n \leq |x| \leq 3n\} \cap \Omega$, $A_n^c = \Omega/A_n$, $B_n = \{x \in \mathbb{R}^n \mid |x| \leq n\} \cap \Omega$, and $B_n^c = \{x \in \mathbb{R}^n \mid |x| > n\} \cap \Omega$.

Define $u_n = \psi_n u$, thus $u_n \in L^\infty(\Omega)$ and $|u_n(x)| \leq |u(x)|$. Certainly, u_n has a compact support and it is in $L^\alpha_V(\Omega)$. Since $\nabla u_n = \psi_n \nabla u + u\nabla\psi_n$, we have that $\nabla u_n \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$ if both of the terms of the sum are in $L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. Since $\psi_n \in L^\infty(\mathbb{R}^N)$ and $\nabla u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$, we deduce that $\psi_n\nabla u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. Since $\nabla\psi_n$ vanishes in A_n^c, $|A_n| < +\infty, \nabla\psi_n \in (L^\infty(\mathbb{R}^N))$ and $u \in L^\infty(\Omega)$, then $u\nabla\psi_n \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. We conclude that $u_n \in X_{loc}(\Omega) \cap L^\infty(\Omega)$.

We also observe that

$$\int_\Omega \frac{V(x)}{\alpha(x)}|u - u_n|^{\alpha(x)}\,dx \leq \int_{B_n^c} \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx = o_n(1),$$

where $o_n(1)$ denotes vanishing function as $n \to 0$. Thus, $|u - u_n|_{L_V^{\alpha(\cdot)}(\Omega)} \to 0$ as $n \to +\infty$.

We only need to prove that

$$|\nabla u - \nabla u_n|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = o_n(1).$$

Using (iv) of Proposition 3.2, we deduce that $\nabla u \in L^{p(\cdot)}(\Lambda_{\nabla u}) \cap L^{q(\cdot)}(\Lambda^c_{\nabla u})$.

We observe that

$$
\begin{aligned}
|(1-\psi_n)\nabla u|_{L^{p(\cdot)}(B_n^c)+L^{q(\cdot)}(B_n^c)} &\le |\nabla u|_{L^{p(\cdot)}(B_n^c)+L^{q(\cdot)}(B_n^c)} \\
&\le |\nabla u|_{L^{p(\cdot)}(\Lambda_{\nabla u}\cap B_n^c)} + |\nabla u|_{L^{q(\cdot)}(\Lambda_{\nabla u}^c\cap B_n^c)} = o_n(1),
\end{aligned} \tag{27}
$$

and

$$
|u|_{L_V^{\alpha(\cdot)}(A_n)} = o_n(1).
$$

By (27) and (vi) of Proposition 3.2, we have

$$
\begin{aligned}
|\nabla u - \nabla u_n|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} &\le |\nabla u - \nabla u_n|_{L^{p(\cdot)}(B_n)+L^{q(\cdot)}(B_n)} + |\nabla u - \nabla u_n|_{L^{p(\cdot)}(B_n^c)+L^{q(\cdot)}(B_n^c)} \\
&= |\nabla u - \nabla u_n|_{L^{p(\cdot)}(B_n^c)+L^{q(\cdot)}(B_n^c)} \\
&\le |(1-\psi_n)\nabla u|_{L^{p(\cdot)}(B_n^c)+L^{q(\cdot)}(B_n^c)} + |u\nabla\psi_n|_{L^{p(\cdot)}(B_n^c)+L^{q(\cdot)}(B_n^c)} \\
&= |u\nabla\psi_n|_{L^{p(\cdot)}(A_n)+L^{q(\cdot)}(A_n)} + o_n(1) \\
&\le \frac{1}{n}|u|_{L^{p(\cdot)}(A_n)+L^{q(\cdot)}(A_n)} + o_n(1) \\
&\le \frac{1}{n}|u|_{L^{p(\cdot)}(\Lambda_u\cap A_n)} + \frac{1}{n}|u|_{L^{q(\cdot)}(\Lambda_u^c\cap A_n)} + o_n(1) \\
&\le \frac{1}{n}|u|_{L^{p(\cdot)}(A_n)} + \frac{1}{n}|u|_{L^{q(\cdot)}(A_n)} + o_n(1).
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{1}{n}|u|_{L^{p(\cdot)}(A_n)} &\le \frac{1}{n}|u|_{L^{p(\cdot)}(A_n\cap\Lambda_u)} + \frac{1}{n}|u|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c)} \\
&\le \frac{1}{n}|\Lambda_u|^{\frac{1}{p(\xi)}}\operatorname{ess\,sup}|u| + \frac{1}{n}|u|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c)} \\
&\le \frac{1}{n}|u|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c\cap[p\ge\alpha])} + \frac{1}{n}|u|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c\cap[p<\alpha])} + o_n(1) \\
&\le \frac{1}{n}|u|^{\frac{\alpha(\xi)}{p(\xi)}}_{L^{\alpha(\cdot)}(A_n\cap\Lambda_u^c\cap[p\ge\alpha])} + \frac{1}{n}|u|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c\cap[p<\alpha])} + o_n(1), \text{ where } \xi\in\mathbb{R}^N \\
&= \left|\frac{1}{n}u\right|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c\cap[p<\alpha])} + o_n(1).
\end{aligned}
$$

By Hölder's inequality we obtain

$$
\begin{aligned}
\int_{A_n\cap\Lambda_u^c\cap[p<\alpha]} \left|\frac{1}{n}u\right|^{p(x)} dx &\le 2\left||u|^{p(x)}\right|_{L^{\frac{\alpha}{p}}(A_n\cap\Lambda_u^c\cap[p<\alpha])}\cdot\left|\frac{1}{n^{p(x)}}\right|_{L^{(\frac{\alpha}{p})'}(A_n\cap\Lambda_u^c\cap[p<\alpha])} \\
&\le 2|u|^{p(\xi)}_{L^{\alpha}(A_n\cap\Lambda_u^c\cap[p<\alpha])}\cdot\frac{1}{n^{p(\xi)}}|A_n|^{1/(\frac{\alpha}{p})'(\xi)} \\
&\le C|u|^{p(\xi)}_{L^{\alpha}(A_n\cap\Lambda_u^c\cap[p<\alpha])} \\
&= o_n(1).
\end{aligned}
$$

Therefore

$$
\left|\frac{1}{n}u\right|_{L^{p(\cdot)}(A_n\cap\Lambda_u^c\cap[p<\alpha])} = o_n(1),
$$

hence

$$\frac{1}{n}|u|_{L^{p(\cdot)}(A_n)} = o_n(1).$$

Analogously, if $p < q < p^*$, we have

$$\frac{1}{n}|u|_{L^{q(\cdot)}(A_n)} \le \left|\frac{1}{n}u\right|_{L^{q(\cdot)}(A_n \cap \Lambda_u^c \cap [q<\alpha])} + o_n(1) = o_n(1).$$

Therefore

$$|\nabla u - \nabla u_n|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = o_n(1).$$

Therefore, we conclude that $u_n \to u$ in X.

By (26), we have

$$\int_\Omega |u_n|^{p^*(x)}\,dx \le C_o^{(p*)^+}[\int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}\,dx + |\nabla u_n|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 1]^{(p*)^+}$$

$$\le C[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 2]^{(p*)^+}.$$

Since $u_n(x) \to u(x)$ *a.e.* $x \in \Omega$, by Fatou's lemma we have

$$\int_\Omega |u|^{p^*(x)}\,dx \le C[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 2]^{(p*)^+},\ \forall u \in X(\Omega) \cap L^\infty(\Omega), \tag{28}$$

where $C > 1$ is a constant independent of u.

Case 3: For any $u \in X(\Omega)$, we will prove that (26) is satisfied.

For $n = 1, 2, \cdots$, let

$$u_n(x) = \begin{cases} u(x), & \text{if } |u(x)| \le n, \\ n \operatorname{sgn} u(x), & \text{if } |u(x)| > n. \end{cases}$$

Then $u_n \in X(\Omega) \cap L^\infty(\Omega)$. Notice that

$$\int_\Omega \frac{V(x)}{\alpha(x)}|u_n(x)|^{\alpha(x)}\,dx \le \int_\Omega \frac{V(x)}{\alpha(x)}|u(x)|^{\alpha(x)}\,dx \text{ and } \int_\Omega |\nabla u_n(x)|^{p(x)}\,dx \le \int_\Omega |\nabla u(x)|^{p(x)}\,dx.$$

By (28) we get

$$\int_\Omega |u_n|^{p^*(x)}\,dx \le C[\int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}\,dx + |\nabla u_n|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 2]^{(p*)^+}$$

$$\le C[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 2]^{(p*)^+}.$$

Since $u_n(x) \to u(x)$ *a.e.* $x \in \Omega$, by the Fatou lemma we have

$$\int_\Omega |u|^{p^*(x)}\,dx \le C[\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx + |\nabla u|^{p^+ + q^+}_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} + 2]^{(p*)^+} < +\infty.$$

We conclude that $u \in L^{p^*(\cdot)}(\Omega)$, which means that $X \subset L^{p^*(\cdot)}(\Omega)$. □

Corollary 3.13. *Assume conditions of Theorem 3.12. We have the following properties:*

(i) for any $u \in X(\Omega)$, $\psi_n u \to u$ in $X(\Omega)$;

(ii) for any $u \in X$, we have $u_\varepsilon = u * \mathbf{j}_\varepsilon \to u$ in X (where $\mathbf{j}_\varepsilon(x) = \varepsilon^{-N}\mathbf{j}(\frac{x}{\varepsilon})$ and $\mathbf{j} : \mathbb{R}^N \to \mathbb{R}^+$ is in $C_c^\infty(\mathbb{R}^N)$, a function inducing a probability measure);

(iii) for any $u \in X$, there exists a sequence $\{u_n\} \subset C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in X.

Proof. (i) The arguments are similar to the proof of Theorem 3.12 (ii). For any $u \in X$, by Theorem 3.12, we have $u \in L^{p^*(\cdot)}(\Omega)$. Similar to the proof of (ii) of Theorem 3.12, since $\nabla\psi_n$ vanishes in A_n^c, $|A_n| < +\infty, \nabla\psi_n \in (L^\infty(\mathbb{R}^N))^N$ and $u \in L^{p^*(\cdot)}(\Omega)$, we conclude that $u\nabla\psi_n \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$.

We have

$$\frac{1}{n}|u|_{L^{p(\cdot)}(A_n)} \le \frac{2}{n}|u|_{L^{p^*(\cdot)}(A_n)}|A_n|^{\frac{1}{N}} \le C|u|_{L^{p^*(\cdot)}(A_n)} = o_n(1).$$

Analogously, if $p < q < p^*$,

$$\frac{1}{n}|u|_{L^{q(\cdot)}(A_n)} \le \frac{2}{n}|u|_{L^{p^*(\cdot)}(A_n)}|A_n|^{\frac{1}{q(\xi)}-\frac{1}{p^*(\xi)}} \le C|u|_{L^{p^*(\cdot)}(A_n)} = o_n(1).$$

Similar to the proof of case (ii) of Theorem 3.12, we conclude that $u_n \to u$ in $X(\Omega)$.

(ii) By the method of mollifiers, we have $u_\varepsilon \to u$ in $L^{\alpha(\cdot)}(\mathbb{R}^N)$ as $\varepsilon \to 0$. Moreover, if we write $\nabla u = \mathbf{a}+\mathbf{b}$, with $\mathbf{a} \in (L^{p(\cdot)}(\mathbb{R}^N))^N$ and $\mathbf{b} \in (L^{q(\cdot)}(\mathbb{R}^N))^N$, we have $\nabla u_\varepsilon = \nabla u * j_\varepsilon = \mathbf{a}*\mathbf{j}_\varepsilon + \mathbf{b}*\mathbf{j}_\varepsilon$, with $\mathbf{a}*\mathbf{j}_\varepsilon \in (L^{p(\cdot)}(\mathbb{R}^N))^N$ and $\mathbf{b}*\mathbf{j}_\varepsilon \in (L^{q(\cdot)}(\mathbb{R}^N))^N$. Therefore

$$|\nabla u_\varepsilon - \nabla u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \le |\mathbf{a}*\mathbf{j}_\varepsilon - \mathbf{a}|_{L^{p(\cdot)}(\Omega)} + |\mathbf{b}*\mathbf{j}_\varepsilon - \mathbf{b}|_{L^{q(\cdot)}(\Omega)} \to 0.$$

Thus $u_\varepsilon \to u$ in X.

(iii)This follows from (i) and (ii). □

Theorem 3.14. *Assume conditions of Theorem 3.12.*

(i) For any $\alpha \le s \le p^*$, the space $X(\Omega)$ is continuously embedded into $L^{s(\cdot)}(\Omega)$.

(ii) For any bounded subset $\Omega \subset \mathbb{R}^N$, there is a compact embedding $X(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$.

(iii) We also assume $(\mathcal{H}_V)$-(iii) and $s(\cdot) \in C(\mathbb{R}^N)$ is Lipschitz continuous and

$$\alpha(\cdot) \le s(\cdot) \ll p^*(\cdot), \text{ in } \mathbb{R}^N. \tag{29}$$

Then there is a compact embedding $X \hookrightarrow L^{s(\cdot)}$.

Proof. The proofs of (i) and (ii) are trivial. We only need to prove (iii).

From (i) and Theorem 2.6 (interpolation theorem), it remains to prove that $|u_n|_{\alpha(\cdot)} \to 0$ if $u_n \rightharpoonup 0$ in X.

Obviously, we have

$$A = \sup_{n\to\infty}\|u_n\| < \infty.$$

For any $\varepsilon > 0$, there exists $R > 0$ such that

$$V(x) \geq \frac{\alpha^+}{\varepsilon}, \forall |x| \geq R.$$

Set

$$B = B_R(0) .$$

The following mapping is linear and bounded

$$P_B : X(\Omega) \to X(B) : u \to u^B.$$

It follows that $u_n |_B \rightharpoonup 0$ in $X(B)$. From the compactness of the embedding $W^{1,p(\cdot)}(B) \hookrightarrow L^{\alpha(\cdot)}(B)$, we have

$$\lim_{n\to\infty} \left|u_n^B\right|_{L^{\alpha(\cdot)}(B)} = 0,$$

which implies that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\int_B |u_n|^{\alpha(x)} dx \leq C \int_B \frac{V(x)}{\alpha(x)} \left|u_n^B\right|^{\alpha(x)} dx < \varepsilon, \forall\, n > N. \tag{30}$$

From $(\mathcal{H}_V)$-(iii), we obtain

$$\int_{\Omega\cap[|x|\geq R]} |u_n|^{\alpha(x)} dx \leq \varepsilon \int_\Omega \frac{V(x)}{\alpha(x)} |u_n|^{\alpha(x)} dx \leq \varepsilon(A+1)^{\alpha^+}, \forall\, n \geq 1. \tag{31}$$

Combining relations (30) and (31), we deduce that

$$\int_\Omega |u_n|^{\alpha(x)} dx \leq 2\varepsilon(A+1)^{\alpha^+},\ \forall n \geq N.$$

Therefore

$$\lim_{n\to\infty} |\, u_n \,|_{L^{\alpha(\cdot)}(\Omega)} \to 0,$$

thus the embedding $X \hookrightarrow L^{\alpha(\cdot)}$ is compact. This completes the proof. □

4. Properties of functionals and operators

By (vii) of Proposition 3.2, we deduce that $\mathscr{A}(x, \nabla u)$ is integrable on $\mathbb{R}^N$ for all $u \in X$. Thus, $\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)dx$ is well defined. For $u \in X$, it follows by (3) that

$$\begin{aligned} &\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla u dx + \int_{\mathbb{R}^N} V(x) |u|^{\alpha(x)} dx \\ &\geq c_1 \left(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)} dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)} dx + \int_{\mathbb{R}^N} V(x) |u|^{\alpha(x)} dx \right), \end{aligned} \tag{32}$$

and

$$\int_{\mathbb{R}^N} \mathbf{A}(x,\nabla u)\cdot\nabla u dx + \int_{\mathbb{R}^N} V(x)\,|u|^{\alpha(x)}\,dx$$
$$\leq c_2\left(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} V(x)\,|u|^{\alpha(x)}\,dx\right)$$

where c_1 and c_2 are positive constants.

Similarly, using (3), we get for all $u \in X$

$$\int_{\mathbb{R}^N} \mathscr{A}(x,\nabla u)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)}\,dx$$
$$\geq c_1\left(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)}\,dx\right), \tag{33}$$

and

$$\int_{\mathbb{R}^N} \mathscr{A}(x,\nabla u)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)}\,dx$$
$$\leq c_2\left(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)}\,dx\right).$$

We say that $u \in X$ is a solution of problem $(\mathcal{E})$ if

$$\int_{\mathbb{R}^N} \mathbf{A}(x,\nabla u)\cdot\nabla v dx + \int_{\mathbb{R}^N} V(x)\,|u|^{\alpha(x)-2}\,uv dx = \int_{\mathbb{R}^N} f(x,u)v dx, \quad \forall v \in X.$$

It follows that solutions of $(\mathcal{E})$ correspond to the critical points of the Euler–Lagrange energy functional $\Phi : X \to \mathbb{R}$, defined by

$$\Phi = \int_{\mathbb{R}^N} \mathscr{A}(x,\nabla u)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)}\,dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Define functionals $\Phi_{\mathscr{A}}, \Phi_\alpha, \Phi_f : X \to \mathbb{R}$ by

$$\Phi_{\mathscr{A}}(u) = \int_{\mathbb{R}^N} \mathscr{A}(x,\nabla u)dx,\ \Phi_\alpha(u) = \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)}\,dx,\ \Phi_f(u) = \int_{\mathbb{R}^N} F(x,u)dx.$$

Lemma 4.1. *Assume the structure conditions $(\mathcal{A}_1)$ and $(\mathcal{H}_V)$-(i). Then the functional $\Phi_{\mathscr{A}}$ is convex, of class C^1, and sequentially weakly lower semicontinuous in X. Moreover, $\Phi'_{\mathscr{A}} : X \to X^*$ is bounded.*

Proof. If $u_n \to u$ in X, then $|\nabla u_n - \nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}} \to 0$. Thus $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. By Lemma A.4,

$$\Phi_{\mathscr{A}}(u_n) \leq \int_{\mathbb{R}^N} C(\mathscr{A}(x,\nabla u) + \mathscr{A}(x,\nabla u_n - \nabla u))dx.$$

From (vii) of Proposition 3.2, we observe that $\{\mathscr{A}(x, \nabla u_n)\}$ is uniformly integrable. By the Vitali theorem (Theorem A.3 of [42]), we have $\lim_{n\to+\infty} \Phi_{\mathscr{A}}(u_n) = \Phi_{\mathscr{A}}(u)$. This means that $\Phi_{\mathscr{A}}$ is continuous.

Next, we prove that $\Phi'_{\mathscr{A}} : X \to X^*$ is bounded.

For any $\varphi \in X$, there exists $v \in (L^{p(\cdot)})^N$ and $w \in (L^{q(\cdot)})^N$ such that $\varphi = v + w$ and $|\nabla\varphi|_{L^{p(\cdot)}+L^{q(\cdot)}} = |v|_{L^{p(\cdot)}} + |w|_{L^{q(\cdot)}}$. Since $p \ll q$, we have $(q(x) - 1)\frac{p(x)}{p(x)-1} \geq q(x)$ and $(p(x) - 1)\frac{q(x)}{q(x)-1} \leq p(x)$. By Young's inequality, we have

$$
\begin{aligned}
\|\Phi'_{\mathscr{A}}(u)\| &= \sup_{\varphi\in X,\|\varphi\|=1} \left| \int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u)\nabla\varphi dx \right| \\
&\leq \sup_{\varphi\in X,\|\varphi\|=1} \int_{\mathbb{R}^N} |\mathbf{A}(x, \nabla u)|\,|v|\,dx + \sup_{\varphi\in X,\|\varphi\|=1} \int_{\mathbb{R}^N} |\mathbf{A}(x, \nabla u)|\,|w|\,dx \\
&\leq \sup_{\varphi\in X,\|\varphi\|=1} C(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)-1}\,|v|\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)-1}\,|v|\,dx \\
&\quad + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)-1}\,|w|\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)-1}\,|w|\,dx) \\
&\leq \sup_{\varphi\in X,\|\varphi\|=1} 2C(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)} + |v|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{(q(x)-1)\frac{p(x)}{p(x)-1}} + |v|^{p(x)}\,dx \\
&\quad + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{(p(x)-1)\frac{q(x)}{q(x)-1}} + |w|^{q(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)} + |w|^{q(x)}\,dx) \\
&\leq \sup_{\varphi\in X,\|\varphi\|=1} 4C(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx \\
&\quad + \int_{\mathbb{R}^N} |v|^{p(x)}\,dx + \int_{\mathbb{R}^N} |w|^{q(x)}\,dx) \\
&\leq 4C(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + 1) \\
&\leq 8C(|\nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}} + 1)^{q^+} + 8C.
\end{aligned}
$$

Thus, $\Phi'_{\mathscr{A}} : X \to X^*$ is bounded.

By (vii) of Proposition 3.2, $|\nabla u_n - \nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}} \to 0$ implies

$$
\int_{\mathbb{R}^N} |\mathbf{A}(x, \nabla u_n - \nabla u)|^{p'(x)}\,dx \to 0.
$$

By Lemma A.4, we have

$$
\begin{aligned}
|\mathbf{A}(x, \nabla u_n)|^{p'(x)} &\leq C(|\mathbf{A}(x, \nabla u)| + |\mathbf{A}(x, \nabla u_n - \nabla u)|)^{p'(x)} \\
&\leq C(|\mathbf{A}(x, \nabla u)|^{p'(x)} + |\mathbf{A}(x, \nabla u_n - \nabla u)|^{p'(x)}).
\end{aligned}
$$

This relation combined with (vii) of Proposition 3.2 and $|\nabla u_n - \nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}} \to 0$ implies that $\{|\mathbf{A}(x,\nabla u_n)|^{p'(x)}\}$ is uniformly integrable. Similarly, $\{|\mathbf{A}(x,\nabla u_n)|^{q'(x)}\}$ is also uniformly integrable. Note that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Thus, by the Vitali theorem, that

$$\|\Phi'_{\mathscr{A}}(u_n) - \Phi'_{\mathscr{A}}(u)\| = \sup_{\varphi\in X,\|\varphi\|=1}\left|\int_{\mathbb{R}^N}(\mathbf{A}(x,\nabla u_n) - \mathbf{A}(x,\nabla u))\varphi dx\right|$$
$$\leq |\mathbf{A}(x,\nabla u_n) - \mathbf{A}(x,\nabla u)|_{L^{p'(\cdot)}\cap L^{q'(\cdot)}} \to 0 \text{ as } n \to 0.$$

We deduce that $\Phi'_{\mathscr{A}}$ is also continuous. Thus, $\Phi_{\mathscr{A}}$ is C^1.

Note that $\mathscr{A}$ is convex. Obviously, $\Phi_{\mathscr{A}}$ is convex. By Theorem A.2 of [42], we deduce that $\Phi_{\mathscr{A}}$ is sequentially weakly lower semicontinuous in X. □

Lemma 4.2. *Assume the structure conditions ($\mathcal{A}_1$)-(iv) and ($\mathcal{H}_V$)-(i). Then the functional Φ_α is convex, of class C^1 and sequentially weakly lower semicontinuous. Moreover, if $u_n, u \in X$ and $u_n \rightharpoonup u$ in X, then $\Phi'_\alpha(u_n) \overset{*}{\rightharpoonup} \Phi'_\alpha(u)$ in X^*.*

Proof. Similarly with the arguments in the proof of Lemma 4.1, Φ_α is convex, of class C^1 and sequentially weakly lower semicontinuous.

By (ii) of Theorem 3.14, we have $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Obviously, the integrals of the family $\left\{V(x)\left||u_n(x)|^{\alpha(x)-2}u_n(x) - |u(x)|^{\alpha(x)-2}u(x)\right||v(x)|\right\}$ are uniformly integrable in $\mathbb{R}^N$. By Vitali's theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\left||u_n(x)|^{\alpha(x)-2}u_n(x) - |u(x)|^{\alpha(x)-2}u(x)\right||v(x)|\,dx$$
$$= \int_{\mathbb{R}^N}\lim_{n\to\infty} V(x)\left||u_n(x)|^{\alpha(x)-2}u_n(x) - |u(x)|^{\alpha(x)-2}u(x)\right||v(x)|\,dx = 0.$$

Thus, $\Phi'_\alpha(u_n) \overset{*}{\rightharpoonup} \Phi'_\alpha(u)$ in X'. □

Lemma 4.3. *Assume conditions of Theorem 3.14-(iii) and ($\mathcal{H}_f^1$), γ is Lipschitz continuous and $\alpha \ll \gamma \ll p^*(\cdot)$. Then the Nemytsky operator $(N_f u)(x) = f(x,u(x))$ is weak continuous from X to X^*, that is, for any $u_n \rightharpoonup u$, $N_f u_n \to N_f u$ in X^*.*

Proof. Since $u_n \rightharpoonup u$ in X, we have $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, and $u_n \to u$ in $L^{\gamma(\cdot)}(\mathbb{R}^N)$ for any $\alpha \leq \gamma(\cdot) \ll p^*(\cdot)$. We have

$$\|N_f u_n - N_f u\|_{X^*} = \sup_{\varphi\in X,\|\varphi\|=1}\left|\int_{\mathbb{R}^N}(f(x,u_n(x)) - f(x,u(x)))\varphi dx\right|$$
$$= \sup_{\varphi\in X,\|\varphi\|=1}\left|\int_{B_R}(f(x,u_n(x)) - f(x,u(x)))\varphi dx\right|$$
$$+ \sup_{\varphi\in X,\|\varphi\|=1}\left|\int_{B_R^c}(f(x,u_n(x)) - f(x,u(x)))\varphi dx\right|$$
$$\leq \sup_{\varphi\in X,\|\varphi\|=1} C\{(|u_n(x)|_{L^{\gamma'(\cdot)}(B_R^c)} + |u(x)|_{L^{\gamma'(\cdot)}(B_R^c)}) \cdot |\varphi|_{L^{\gamma(\cdot)}}$$

$$
\begin{aligned}
&+(|u_n(x)|_{L^{\alpha'(\cdot)}(B_R^c)} + |u(x)|_{L^{\alpha'(\cdot)}(B_R^c)}) \cdot |\varphi|_{L^{\alpha(\cdot)}}\} + o_n(1) \\
&\leq \sup_{\varphi \in X, \|\varphi\|=1} 2C\{(|u_n(x) - u|_{L^{\gamma'(\cdot)}(B_R^c)} + |u(x)|_{L^{\gamma'(\cdot)}(B_R^c)}) \cdot |\varphi|_{L^{\gamma(\cdot)}} \\
&\quad +(|u_n(x) - u|_{L^{\alpha'(\cdot)}(B_R^c)} + |u(x)|_{L^{\alpha'(\cdot)}(B_R^c)}) \cdot |\varphi|_{L^{\alpha(\cdot)}}\} + o_n(1) \\
&\leq 2C\{|u(x)|_{L^{\gamma'(\cdot)}(B_R^c)} + |u(x)|_{L^{\alpha'(\cdot)}(B_R^c)}\} + o_n(1).
\end{aligned}
$$

Since R and n are arbitrary, we deduce that $\|N_f u_n - N_f u\|_{X^*} \to 0$ as $n \to +\infty$. □

Lemma 4.4. *Assume conditions of Theorem 3.14-(iii) and ($\mathcal{H}_f^1$). Then Φ_f is of class C^1 and sequentially weakly continuous, that is, if $u_n \rightharpoonup u$ in X then $\Phi_f(u_n) \to \Phi_f(u)$ and $\Phi'_f(u_n) \to \Phi'_f(u)$ in X^*.*

Proof. Similar to the proof of Lemma 4.3, we only need to prove the sequentially weakly continuity of Φ_f. Assume that $u_n \rightharpoonup u$ in X. By Theorem 3.14, $u_n \to u$ in $L^{\gamma(\cdot)}$ for any $\alpha(\cdot) \leq \gamma(\cdot) \ll p^*(\cdot)$. From Proposition 2.2, we have $\Phi_f(u_n) \to \Phi_f(u)$. □

Lemma 4.5. *Assume conditions of Theorem 3.14-(iii) and ($\mathcal{H}_f^1$). Then the functional Φ is of class C^1 and sequentially weakly lower semicontinuous in X, that is, if $u_n \rightharpoonup u_0$ in X, then*

$$\Phi(u_0) \leq \liminf_{n\to\infty} \Phi(u_n).$$

Proof. According to Lemmas 4.1–4.4, we deduce the C^1 continuity of Φ. Next, we will prove that Φ is the sequentially weakly lower semicontinuous in X.

Assume that $(\mathcal{A}_1)$, $(\mathcal{H}_V^1)$ and $(\mathcal{H}_f^1)$ are satisfied. By Lemma 4.4, $\Phi_f(u)$ is weakly continuous. Obviously

$$
\begin{aligned}
\liminf_{n\to\infty} \Phi(u_n) &\geq \liminf_{n\to\infty} (\Phi_{\mathscr{A}}(u_n) + \Phi_a(u_n)) - \limsup_{n\to\infty} \Phi_f(u_n) \\
&\geq \Phi_{\mathscr{A}}(u_0) + \Phi_\alpha(u_0) - \Phi_f(u_0) \\
&= \Phi(u_0).
\end{aligned}
$$

Thus, Φ is sequentially weakly lower semicontinuous in X. □

Lemma 4.6. *Suppose that $\mathscr{A}$ satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ (namely $\mathscr{A}(x,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is a uniformly convex function), that is, for any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that $|u - v| \leq \varepsilon \max\{|u|, |v|\}$ or $\mathscr{A}(y, \frac{u+v}{2}) \leq \frac{(1-\delta(\varepsilon))}{2}(\mathscr{A}(x,u) + \mathscr{A}(x,v))$. Then we have*

(i) $\Phi_{\mathscr{A}}(\cdot) : X \to \mathbb{R}$ is uniformly convex, that is, for any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that for all $u, v \in X$

$$\Phi_{\mathscr{A}}(\frac{u-v}{2}) \leq \epsilon \frac{\Phi_{\mathscr{A}}(u) + \Phi_{\mathscr{A}}(v)}{2} \text{ or } \Phi_{\mathscr{A}}(\frac{u+v}{2}) \leq (1-\delta(\varepsilon))\frac{\Phi_{\mathscr{A}}(u) + \Phi_{\mathscr{A}}(v)}{2};$$

(ii) if $u_n \rightharpoonup u$ in X and $\overline{\lim_{n\to\infty}} (\Phi'_{\mathscr{A}}(u_n) - \Phi'_{\mathscr{A}}(u), u_n - u) \leq 0$, then $\Phi_{\mathscr{A}}(u_n - u) \to 0$ and $|\nabla u_n - \nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}} \to 0$.

Proof. (i) See Lemma A.3 (Theorem 2.4.11 of [18]). We omit the details.

(ii) For the proof, we refer to Proposition 2.1 in [40]. For the readers' convenience, we state it here.

Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty} \langle \Phi'_{\mathscr{A}}(u_n), u_n - u \rangle \leq 0.$$

Since the sequence $\{u_n\}$ is weakly convergent, then it is bounded, hence $\|u_n\| \le R$ for all n. Since $\mathscr{A}$ is locally bounded, we deduce that $\mathscr{A}(u_n)$ is bounded. For a subsequence, we may assume that $\Phi_{\mathscr{A}}(u_n) \to c$. By weak lower semicontinuity,

$$\Phi_{\mathscr{A}}(u) \le \liminf_{n\to\infty} \Phi_{\mathscr{A}}(u_n) = c.$$

On the other hand, since $\Phi_{\mathscr{A}}$ is convex, we have

$$\Phi_{\mathscr{A}}(u) \ge \Phi_{\mathscr{A}}(u_n) + \langle \Phi'_{\mathscr{A}}(u_n), u - u_n \rangle .$$

Using $\limsup_{n\to\infty} \langle \Phi'_{\mathscr{A}}(u_n), u_n - u \rangle \ge 0$, we deduce that $\Phi_{\mathscr{A}}(u) \ge c$. Thus, $\mathscr{A}(u) = c$.

Also we have that $(u_n + u)/2 \rightharpoonup u$, and again by lower weakly semicontinuity,

$$c = \Phi_{\mathscr{A}}(u) \le \liminf_{n\to\infty} \Phi_{\mathscr{A}}(\frac{u_n + u}{2}). \tag{34}$$

If we suppose the contrary, then there exist $\epsilon \in (0, 1)$ and a subsequence $\{u_{n_k}\}$ such that

$$\Phi_{\mathscr{A}}(u - u_{n_k}) \ge \epsilon \frac{\Phi_{\mathscr{A}}(u) + \Phi_{\mathscr{A}}(u_{n_k})}{2}.$$

Let $\delta > 0$ corresponding to the uniform convexity of $\Phi_{\mathscr{A}}$ over the ball $B(0; R) \subset X$. Then

$$\Phi_{\mathscr{A}}(\frac{u_{n_k} + u}{2}) \le (1 - \delta) \frac{\Phi_{\mathscr{A}}(u) + \Phi_{\mathscr{A}}(u_{n_k})}{2} \to (1 - \delta)c \text{ as } k \to \infty,$$

which contradicts (34). Thus, $\Phi_{\mathscr{A}}(u_n - u) \to 0$ in X.

By (vii) of Proposition 3.2, we have $|\nabla u_n - \nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}} \to 0$. $\square$

Define $\rho(\cdot) : X \to \mathbb{R}$ as

$$\rho(u) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)} |u|^{\alpha(x)} dx,$$

and we denote the derivative operator by L, that is, $L = \rho' : X \to X^*$ with

$$(L(u), v) = \int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u)\nabla v dx + \int_{\mathbb{R}^N} V(x) |u|^{\alpha(x)-2} uv dx \quad \forall u, v \in X.$$

Lemma 4.7. *Under the structure conditions ($\mathcal{A}_1$) and ($\mathcal{H}_V$)-(i), we have the following properties.*

(i) $L : X \to X^*$ is a continuous, bounded strictly monotone operator.

If ($\mathcal{A}_2$) is also satisfied, we have

(ii) L is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in X and $\overline{\lim_{n\to\infty}}\ (L(u_n) - L(u), u_n - u) \le 0$, then $u_n \to u$ in X;

(iii) $L : X \to X^*$ is a homeomorphism.

Proof. (i) It follows by Lemmas 4.1–4.4 that L is continuous and bounded.

Denote $g(t) = \Phi_{\mathscr{A}}(t(u - v) + v) + \Phi_\alpha(t(u - v) + v)$. Note that since $\mathscr{A}$ is strictly convex, then $t \mapsto \Phi_{\mathscr{A}}(t(u - v) + v)$ is convex. Combining this information with the inequality

$$\left[(|\xi|^{p(x)-2} \xi - |\eta|^{p(x)-2} \eta)(\xi - \eta)\right] \cdot (|\xi| + |\eta|)^{2-p} \ge (p - 1) |\xi - \eta|^2 , \quad \text{if } 1 < p < 2,$$

$$(|\xi|^{p(x)-2}\,\xi - |\eta|^{p(x)-2}\,\eta)(\xi-\eta) \geq (\frac{1}{2})^p\,|\xi-\eta|^p\,, \quad \text{if } p\geq 2,$$

we deduce that

$$0 < g'(1) - g'(0) = (L(u) - L(v), u - v), \text{ for any } u \neq v \text{ in } X.$$

Thus, $g'(t)$ is strictly increasing, hence L is strictly monotone.

(ii) By Lemma 4.6, we deduce that L is of type (S_+).

(iii) By the strict monotonicity, L is an injection. We observe that

$$\lim_{\|u\|\to+\infty}\frac{(Lu,u)}{\|u\|} = \lim_{\|u\|\to+\infty}\frac{\int_\Omega \mathbf{A}(x,\nabla u)\nabla u dx + \int_\Omega V(x)\,|u|^{\alpha(x)}\,dx}{\|u\|} = +\infty,$$

hence L is coercive. Thus, by the Minty–Browder theorem, L is a surjection (see [51, Theorem 26A]). Hence L has an inverse mapping $L^{-1} : X^* \to X$. Thus, in order to show the continuity of L^{-1}, it is sufficient to establish that L is a homeomorphism.

If $f_n,\, f \in X^*$, $f_n \to f$, let $u_n = L^{-1}(f_n)$, $u = L^{-1}(f)$, then $L(u_n) = f_n$, $L(u) = f$. So $\{u_n\}$ is bounded in X. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$. Since $f_n \to f$, then

$$\lim_{n\to+\infty}(L(u_n) - L(u_0), u_n - u_0) = \lim_{n\to+\infty}(f_n - f, u_n - u_0) = 0. \tag{35}$$

Since L is of type (S_+), $u_n \to u_0$ in X. Note that $f_n \to f$ and L is continuous in X, then we conclude that $L(u_0) = \lim_{n\to+\infty} L(u_n) = \lim_{n\to+\infty} f_n = f$. Note that L is a surjection, so $u_0 = u$, then L^{-1} is continuous. The proof of Lemma 4.7 is complete. □

Lemma 4.8. *We assume the structure conditions $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, $(\mathcal{H}_V)$, $(\mathcal{H}_f^1)$–$(\mathcal{H}_f^2)$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$ and $\alpha \leq p$. Then Φ satisfies the Cerami compactness condition, that is, if $\{u_n\} \subset X$ satisfies $\Phi(u_n) \to c$ and $\|\Phi'(u_n)\|_{X^*}\,(1 + \|u_n\|) \to 0$, then $\{u_n\}$ has a convergent subsequence.*

Proof. Assume that $\{u_n\}$ is bounded. By Theorem 3.14, the embedding $X \hookrightarrow\hookrightarrow L^{\gamma(\cdot)}$ is compact. Thus, up to a subsequence, we have $\Phi'_f(u_n) \to \Phi'_f(u_0)$ in X^*. By Lemma 4.7, L^{-1} is continuous from X^* to X, then $u_n \to L^{-1} \circ \Phi'_f(u_0)$ in X.

We only need to prove that $\{u_n\}$ is bounded in X.

We argue by contradiction. Suppose not, then there exist $c \in \mathbb{R}$ and $\{u_n\} \subset X$ satisfying:

$$\Phi(u_n) \to c,\ \|\Phi'(u_n)\|_{X^*}\,(1 + \|u_n\|) \to 0,\ \|u_n\| \to +\infty.$$

We observe that

$$\left|\frac{1}{s(x)}u_n\right|_{L^{s(\cdot)}} \leq \frac{1}{s^-}\,|u_n|_{L^{s(\cdot)}}\,, \left|\nabla\frac{1}{s(x)}u_n\right|_{L^{p(\cdot)}} \leq \frac{1}{s^-}\,|\nabla u_n|_{L^{p(\cdot)}} + C\,|u_n|_{L^{p(\cdot)}}\,.$$

Thus, $\left\|\frac{1}{s(x)}u_n\right\| \leq C\,\|u_n\|$. Therefore $(\Phi'(u_n), \frac{1}{s(x)}u_n) \to 0$. We may assume that

$$\begin{aligned} c + 1 &\geq \Phi(u_n) - (\Phi'(u_n), \frac{1}{s(x)}u_n) \\ &= \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u_n)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}\,|u_n|^{\alpha(x)}\,dx - \int_{\mathbb{R}^N} F(x, u_n)dx \end{aligned}$$

$$
\begin{aligned}
&-\{\int_{\mathbb{R}^N} \frac{1}{s(x)}\mathbf{A}(x,\nabla u_n)\nabla u_n dx + \int_{\mathbb{R}^N} \frac{V(x)}{s(x)} |u_n|^{\alpha(x)} dx \\
&- \int_{\mathbb{R}^N} \frac{1}{s(x)} f(x,u_n)u_n dx - \int_{\mathbb{R}^N} \frac{1}{s^2(x)} u_n \mathbf{A}(x,\nabla u_n)\nabla s dx\} \\
\geq &\int_{\mathbb{R}^N} \frac{1}{s^2(x)} u_n \mathbf{A}(x,\nabla u_n)\nabla s dx + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)}) V(x) |u_n|^{\alpha(x)} dx \\
&+ \int_{\mathbb{R}^N} \{\frac{1}{s(x)} f(x,u_n)u_n - F(x,u_n)\} dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \{\frac{f(x,u_n)u_n}{s(x)} - F(x,u_n)\} dx + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)}) V(x) |u_n|^{\alpha(x)} dx \\
&\leq C_1 (\int_{\mathbb{R}^N} |u_n| |\mathbf{A}(x,\nabla u_n)| dx + 1) \\
&\leq \sigma \int_{\mathbb{R}^N \cap \Lambda_{\nabla u_n}} \frac{|\mathbf{A}(x,\nabla u_n)|^{\frac{p(x)}{p(x)-1}}}{\ln(e+|u_n|)} dx + C(\sigma) \int_{\mathbb{R}^N \cap \Lambda_{\nabla u_n}} |u_n|^{p(x)} [\ln(e+|u_n|)]^{p(x)-1} dx \\
&\quad + \sigma \int_{\mathbb{R}^N \cap \Lambda^c_{\nabla u_n}} \frac{|\mathbf{A}(x,\nabla u_n)|^{\frac{q(x)}{q(x)-1}}}{\ln(e+|u_n|)} dx + C(\sigma) \int_{\mathbb{R}^N \cap \Lambda^c_{\nabla u_n}} |u_n|^{q(x)} [\ln(e+|u_n|)]^{q(x)-1} dx + C_1 \\
&\leq \sigma C \int_{\mathbb{R}^N} \frac{\mathbf{A}(x,\nabla u_n)\nabla u_n}{\ln(e+|u_n|)} dx + C(\sigma) \int_{\mathbb{R}^N \cap \Lambda_{\nabla u_n}} |u_n|^{p(x)} [\ln(e+|u_n|)]^{p(x)-1} dx \\
&\quad + C(\sigma) \int_{\mathbb{R}^N \cap \Lambda^c_{\nabla u_n}} |u_n|^{q(x)} [\ln(e+|u_n|)]^{q(x)-1} dx + C_1, \qquad (36)
\end{aligned}
$$

where σ is a small positive constant.

Note that $\frac{u_n}{\ln(e+|u_n|)} \in X$, and $\left\| \frac{u_n}{\ln(e+|u_n|)} \right\| \leq C_2 \|u_n\|$. Let $\frac{u_n}{\ln(e+|u_n|)}$ be a test function. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u_n) \frac{u_n}{\ln(e+|u_n|)} dx \\
&= \int_{\mathbb{R}^N} \mathbf{A}(x,\nabla u_n) \nabla \frac{u_n}{\ln(e+|u_n|)} dx + \int_{\mathbb{R}^N} \frac{V(x)|u_n|^{\alpha(x)}}{\ln(e+|u_n|)} dx + o(1) \\
&= \int_{\mathbb{R}^N} \frac{\mathbf{A}(x,\nabla u_n)\nabla u_n}{\ln(e+|u_n|)} dx - \int_{\mathbb{R}^N} \frac{|u_n| \mathbf{A}(x,\nabla u_n)\nabla u_n}{(e+|u_n|)[\ln(e+|u_n|)]^2} dx + \int_{\mathbb{R}^N} \frac{V(x)|u_n|^{\alpha(x)}}{\ln(e+|u_n|)} dx + o(1).
\end{aligned}
$$

We observe that

$$
\frac{|u_n|}{(e+|u_n|)[\ln(e+|u_n|)]^2} \leq \frac{1}{2} \frac{1}{\ln(e+|u_n|)}.
$$

It follows that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} \frac{\mathbf{A}(x,\nabla u_n)\nabla u_n}{\ln(e+|u_n|)}dx + \int_{\mathbb{R}^N} \frac{V(x)\,|u_n|^{\alpha(x)}}{\ln(e+|u_n|)}dx - C_4 \\
&\leq \int_{\mathbb{R}^N} f(x,u_n)\frac{u_n}{\ln(e+|u_n|)}dx \qquad (37)\\
&\leq \frac{3}{2}\int_{\mathbb{R}^N} \frac{\mathbf{A}(x,\nabla u_n)\nabla u_n}{\ln(e+|u_n|)}dx + \int_{\mathbb{R}^N} \frac{V(x)\,|u_n|^{\alpha(x)}}{\ln(e+|u_n|)}dx + C_6.
\end{aligned}$$

By (36), (37) and condition $(\mathcal{H}_f^2)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u_n)\frac{u_n}{\ln(e+|u_n|)}dx + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x)\,|u_n|^{\alpha(x)}\,dx \\
&-C\int_{|u_n|\leq M} (|u_n|^{\alpha(x)} + |u_n|^{\gamma(x)})dx \\
&\overset{(\mathcal{H}_f^2)}{\leq} C_7\int_{\mathbb{R}^N} \{\frac{f(x,u_n)u_n}{s(x)} - F(x,u_n)\}dx + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x)\,|u_n|^{\alpha(x)}\,dx \\
&\leq C_7\{\sigma\int_{\mathbb{R}^N} \frac{\mathbf{A}(x,\nabla u_n)\nabla u_n}{\ln(e+|u_n|)}dx + C(\sigma)\int_{\mathbb{R}^N\cap\Lambda_{\nabla u_n}} |u_n|^{p(x)}\,[\ln(e+|u_n|)]^{p(x)-1}dx \\
&+C(\sigma)\int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u_n}} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{q(x)-1}dx + C_8\} \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N} f(x,u_n)\frac{u_n}{\ln(e+|u_n|)}dx - \int_{\mathbb{R}^N} \frac{V(x)\,|u_n|^{\alpha(x)}}{\ln(e+|u_n|)}dx \\
&+C(\sigma)\int_{\mathbb{R}^N\cap\Lambda_{\nabla u_n}} |u_n|^{p(x)}\,[\ln(e+|u_n|)]^{p(x)-1}dx \\
&+C(\sigma)\int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u_n}} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{q(x)-1}dx + C_{10}.
\end{aligned}$$

Thus, from condition $(\mathcal{H}_f^2)$ and the above inequality, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{a(x)-1}dx + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x)\,|u_n|^{\alpha(x)}\,dx \\
&-C\int_{|u_n|\leq M} (|u_n|^{\alpha(x)} + |u_n|^{\gamma(x)})dx \\
&\leq C_{11}\int_{\mathbb{R}^N} f(x,u_n)\frac{u_n}{\ln(e+|u_n|)}dx + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x)\,|u_n|^{\alpha(x)}\,dx
\end{aligned}$$

$$\leq C_{12} \int\limits_{\mathbb{R}^N \cap \Lambda_{\nabla u_n}} |u_n|^{p(x)} [\ln(e+|u_n|)]^{p(x)-1} dx + \int\limits_{\mathbb{R}^N \cap \Lambda^c_{\nabla u_n}} |u_n|^{q(x)} [\ln(e+|u_n|)]^{q(x)-1} dx + C_{12}$$

$$\leq C_{12} \int\limits_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1} dx + C_{12}.$$

Notice that $V(x) \to \infty$ as $|x| \to \infty$, then

$$\int\limits_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)}) V(x) |u_n|^{\alpha(x)} dx - C \int\limits_{|u_n| \leq M} (|u_n|^{\alpha(x)} + |u_n|^{\gamma(x)}) dx$$

$$\geq \frac{1}{2} \int\limits_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)}) V(x) |u_n|^{\alpha(x)} dx - C_0$$

Therefore

$$\int\limits_{\mathbb{R}^N} |u_n|^{q(x)} [\ln(e+|u_n|)]^{\alpha(x)-1} dx + \int\limits_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)}) V(x) |u_n|^{\alpha(x)} dx$$

$$\leq C_{11} \int\limits_{\mathbb{R}^N} f(x, u_n) \frac{u_n}{\ln(e+|u_n|)} dx + \int\limits_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{s(x)}) V(x) |u_n|^{\alpha(x)} dx$$

$$\leq C_{12} \int\limits_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1} dx + C_{12}. \tag{38}$$

Claim 1. *The sequence* $\{\int_{\mathbb{R}^N} f(x, u_n) \frac{u_n}{\ln(e+|u_n|)} dx\}$ *is unbounded.*

We suppose the contrary. Up to a sequence, we obtain that $\{\int_{\mathbb{R}^N} f(x, u_n) \frac{u_n}{\ln(e+|u_n|)} dx\}$ is bounded. Let $\varepsilon > 0$ satisfy $\varepsilon < \min\{1, \alpha^- - 1, \frac{1}{p^{*+}}\}$. Since $\|\Phi'(u_n)\|_{X^*} \|u_n\| \to 0$, we have

$$\int\limits_{\mathbb{R}^N} \mathbf{A}(x, \nabla u_n) \nabla u_n dx + \int\limits_{\mathbb{R}^N} V(x) |u_n|^{\alpha(x)} dx$$

$$= \int\limits_{\mathbb{R}^N} f(x, u_n) u_n dx + o(1)$$

$$= \int\limits_{\mathbb{R}^N} [f(x, u_n) u_n]^{\varepsilon} [\ln(e+|u_n|)]^{1-\varepsilon} \left[f(x, u_n) \frac{u_n}{\ln(e+|u_n|)} \right]^{1-\varepsilon} dx + o(1)$$

$$\leq C_9 (1 + \|u_n\|)^{1+\varepsilon} \int\limits_{\mathbb{R}^N} \left[\frac{[f(x, u_n) u_n]^{1+\varepsilon} + C_{10}}{(1 + \|u_n\|)^{\frac{1+\varepsilon}{\varepsilon}}} \right]^{\varepsilon} \left[f(x, u_n) \frac{u_n}{\ln(e+|u_n|)} \right]^{1-\varepsilon} dx + o(1)$$

$$\leq C_{11} (1 + \|u_n\|)^{1+\varepsilon} + C_{12}.$$

On the other hand, we have

$$\int\limits_{\mathbb{R}^N} \mathbf{A}(x, \nabla u_n) \nabla u_n dx + \int\limits_{\mathbb{R}^N} V(x) |u_n|^{\alpha(x)} dx$$

$$\geq C(\int\limits_{\Lambda_{\nabla u_n}} |\nabla u_n|^{p(x)} dx + \int\limits_{\Lambda^c_{\nabla u_n}} |\nabla u_n|^{q(x)} dx) + \int\limits_{\mathbb{R}^N} V(x) |u_n|^{\alpha(x)} dx$$

$$\geq C(|\nabla u_n|^{p^-}_{L^{p(\cdot)}+L^{q(\cdot)}} + |u_n|^{\alpha^-}_{L_V^{\alpha(\cdot)}} - 2)$$
$$\geq C\,\|u_n\|^{\alpha^-} - 3C.$$

It follows that the sequence $\{u_n\}$ is bounded, a contradiction. Thus, Claim 1 is valid. Therefore

$$\{\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx\} \text{ is unbounded.} \tag{39}$$

Note that $a \gg q$ in $\mathbb{R}^N$, then there is a positive constant $M^\# > 1 + M$ (where M is defined in $(\mathcal{H}_f^2)$) such that

$$[\ln(e+|t|)]^{a(x)-q(x)} \geq 14C_{12}, \ \forall \ |t| \geq M^\#, \quad \forall x \in \mathbb{R}^N.$$

Claim 2. *We have*

$$\overline{\lim_{n\to\infty}} \int_{|u_n|\geq M^\#} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx < \frac{1}{2}\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx.$$

We argue again by contradiction. Up to a sequence, we have

$$\int_{|u_n|\geq M^\#} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx \geq \frac{1}{3}\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx. \tag{40}$$

Combining (39) and (40), we obtain

$$\int_{|u_n|\geq M^\#} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx \to +\infty \text{ as } n \to +\infty. \tag{41}$$

According to the definition of $M^\#$ and (40), we deduce that

$$\begin{aligned}
&14C_{12} \int_{|u_n|\geq M^\#} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{q(x)-1}dx \\
&\leq \int_{|u_n|\geq M^\#} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{a(x)-1}dx \\
&\leq \int_{\mathbb{R}^N} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{a(x)-1}dx \\
&\leq 2C_{12}\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx + C_{12} \\
&\leq 12C_{12} \int_{|u_n|\geq M^\#} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{q(x)-1}dx + C_{12}.
\end{aligned}$$

Therefore, $\int_{|u_n|\geq M^\#} |u_n|^{q(x)}\,[\ln(e+|u_n|)]^{q(x)-1}dx \leq C_{12}$, which contradicts (41). We conclude that Claim 2 is valid. From Claim 2, for large enough n, we have

$$\int_{|u_n|<M^\#} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx \geq \frac{1}{2}\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx. \quad (42)$$

Relations (38) and (42) yield

$$\begin{aligned}
&\int_{\mathbb{R}^N\cap[|u_n|<M^\#]} (\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x)\left|\frac{u_n}{M^\#}\right|^{p(x)} dx \\
&\leq \int_{\mathbb{R}^N\cap[|u_n|<M^\#]} (\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x)\left|\frac{u_n}{M^\#}\right|^{\alpha(x)} dx \\
&\leq 2C_{12}\int_{\mathbb{R}^N\cap[|u_n|<M^\#]} ((M^\#)^{p(x)}\left|\frac{u_n}{M^\#}\right|^{p(x)} + (M^\#)^{q(x)}\left|\frac{u_n}{M^\#}\right|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx + C_{12} \\
&\leq 2C_{12}\int_{\mathbb{R}^N\cap[|u_n|<M^\#]} ((M^\#)^{p^+}\left|\frac{u_n}{M^\#}\right|^{p(x)} + (M^\#)^{q^+}\left|\frac{u_n}{M^\#}\right|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx + C_{12} \\
&\leq 4(M^\#)^{2q^+ +1}C_{12}\int_{\mathbb{R}^N\cap[|u_n|<M^\#]} \left|\frac{u_n}{M^\#}\right|^{p(x)} dx + C_{12}.
\end{aligned}$$

Notice that $V(x) \to +\infty$ as $|x| \to +\infty$. Then there exists $R > 0$ such that

$$(\frac{1}{\alpha(x)} - \frac{1}{s(x)})V(x) > 5(M^\#)^{2q^+}C_{12}.$$

Therefore

$$(M^\#)^{2q^+}C_{12}\int_{\mathbb{R}^N\cap[|u_n|<M^\#]} (M^\#)^{q^+}\left|\frac{u_n}{M^\#}\right|^{p(x)} dx \leq C_{12}.$$

Obviously, $\{\int_{\mathbb{R}^N\cap[|u_n|<M^\#]} (M^\#)^{q^+}\left|\frac{u_n}{M^\#}\right|^{q(x)} dx\}$ is bounded. We deduce that the sequence $\int_{|u_n|<M^\#} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx$ is bounded.

Thus, $\{\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |u_n|^{q(x)})[\ln(e+|u_n|)]^{q(x)-1}dx\}$ is bounded, which contradicts Claim 2. The proof of Lemma 4.8 is complete. □

Denote

$$B(x_0, \varepsilon, \delta, \theta) = \{x \in \mathbb{R}^N \mid \delta \leq |x - x_0| \leq \varepsilon, \frac{x - x_0}{|x - x_0|} \cdot \frac{\nabla p(x_0)}{|\nabla p(x_0)|} \geq \cos\theta\},$$

where $\theta \in (0, \frac{\pi}{2})$. Then we have the following property.

Lemma 4.9. *(see [42, Lemma 2.8]) If $p \in C^1(\overline{\Omega})$, $x_0 \in \Omega$ satisfy $\nabla p(x_0) \neq 0$, then there exists a small positive ε such that*

$$(x - x_0) \cdot \nabla p(x) > 0, \forall x \in B(x_0, \varepsilon, \delta, \theta), \quad (43)$$

and

$$\max\{p(x) \mid x \in \overline{B(x_0, \varepsilon)}\} = \max\{p(x) \mid x \in B(x_0, \varepsilon, \varepsilon, \theta)\}. \quad (44)$$

Lemma 4.10. *Suppose that $F(x,u)$ satisfies*

$$C_1 |u|^{q(x)} [\ln(e+|u|)]^{a(x)} \leq F(x,u), \quad \forall |u| \geq M, \quad \forall x \in \Omega,$$

where $a(x) > q(x)$ on $\overline{\Omega}$, and $x_0 \in \Omega$ with $\nabla p(x_0) \neq 0$. Let

$$h(x) = \begin{cases} 0, & |x-x_0| > \varepsilon \\ \varepsilon - |x-x_0|, & |x-x_0| \leq \varepsilon \end{cases},$$

where ε is defined in Lemma 4.9. Then we have

$$\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla th)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)} |th|^{\alpha(x)} dx - \int_{\mathbb{R}^N} F(x,th)dx \to -\infty \text{ as } t \to +\infty.$$

Proof. Note that

$$\int_{\Omega} \mathscr{A}(x, \nabla th)dx \leq C \int_{B(x_0,\varepsilon)} |t|^{p(x)} dx \text{ for } t \geq 1.$$

We observe that

$$C \int_{B(x_0,\varepsilon)} |t|^{p(x)} dx + \int_{B(x_0,\varepsilon)} \frac{V(x)}{\alpha(x)} |th|^{\alpha(x)} dx - \int_{B(x_0,\varepsilon)} C_1 |th|^{q(x)} [\ln(e+|th|)]^{a(x)} dx \to -\infty \text{ as } t \to +\infty.$$

The proof of Lemma 4.10 is complete. □

5. Proof of Theorem 1.1

Let us consider the following auxiliary problem:

$$-\text{div}\, \mathbf{A}(x, \nabla u) + V(x) |u|^{\alpha(x)-2} u = f^+(x,u) \quad , \tag{$\mathcal{E}^+$}$$

where

$$f^+(x,u) = \begin{cases} f(x,u), \text{ if } f(x,u) \geq 0 \\ 0, \text{ if } f(x,u) < 0. \end{cases}$$

The corresponding Euler–Lagrange functional is

$$\Phi^+(u) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)} |u|^{\alpha(x)} dx - \int_{\mathbb{R}^N} F^+(x,u)dx,$$

where $F^+(x,u) = \int_0^u f^+(x,t)dt$.

Similar to the proof of Lemma 4.8, we deduce that Φ^+ satisfies Cerami condition.

Next, we prove that $\Phi^+(u)$ satisfies the conditions of mountain pass lemma.

Obviously, $\Phi^+(0) = 0$. By Lemma 4.10, we have $\Phi^+(th) \to -\infty$ as $t \to +\infty$, where h is defined as in the proof of Lemma 4.10.

We only need to prove that there exist $r > 0$ and $\delta > 0$ such that $\varphi(u) \geq \delta > 0$ for every $u \in X$ and $\|u\| = r$.

Since $\alpha(x) \leq \gamma(x) \ll p^*(x)$ and $\gamma(\cdot) \in C(\mathbb{R}^N)$ is Lipschitz continuous, the embedding $X \hookrightarrow L^{\gamma(\cdot)}$ is compact, then there exists $C_0 > 0$ such that

$$|u|_{\gamma(\cdot)} \leq C_0 \|u\| \quad \forall u \in X.$$

By the assumption $(\mathcal{H}_f^1)$, we have

$$F^+(x,t) \leq \sigma \frac{1}{p(x)} |t|^{\alpha(x)} + C(\sigma) |t|^{\gamma(x)}, \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Let $\sigma \in (0, \frac{1}{4\alpha^+} V_0)$, where V_0 is defined in $(\mathcal{H}_V)$-(ii). We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u) dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)} |u|^{\alpha(x)} dx - \sigma \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{\alpha(x)} dx \\
&\geq \frac{3}{4} \int_{\mathbb{R}^N} (\mathscr{A}(x, \nabla u) + \frac{V(x)}{\alpha(x)} |u|^{p(x)}) dx.
\end{aligned}$$

Since $\gamma(\cdot) \in C(\mathbb{R}^N)$ is Lipschitz continuous and $\alpha(x) \ll \gamma(x) \ll p^*(x)$, we can divide the domain $\mathbb{R}^N$ into a sequence of disjoint equal small cubes Ω_i $(i = 1, \cdots, \infty)$ such that $\mathbb{R}^N = \overset{\infty}{\underset{i=1}{\cup}} \overline{\Omega_i}$ and

$$\sup_{\Omega_i} q(x) < \inf_{\Omega_i} \gamma(x) \leq \sup_{\Omega_i} \gamma(x) < \inf_{\Omega_i} p^*(x).$$

Let

$$\epsilon := \inf_{1 \leq i \leq \infty} \{\inf_{\Omega_i} \gamma(x) - \sup_{\Omega_i} q(x)\}.$$

By our assumptions, we observe that $\epsilon > 0$ as long as the side length of Ω_i is made sufficiently small.

Let $\|u\|_{\Omega_i}$ denote the Orlicz–Sobolev norm of u on Ω_i, that is,

$$\|u\|_{\Omega_i} = |\nabla u|_{L^{p(\cdot)}(\Omega_i) + L^{q(\cdot)}(\Omega_i)} + |u|_{L_V^{\alpha(\cdot)}(\Omega_i)}.$$

We observe that $\|u\|_{\Omega_i} \leq \|u\|$, and there exists $\xi_i \in \overline{\Omega_i}$ such that

$$|u|^{\gamma(\xi_i)}_{L^{p(\cdot)}(\Omega_i)} = \int_{\Omega_i} |u|^{\gamma(x)} dx,$$

$$\|u\|^{q^+}_{\Omega_i} \leq \int_{\Omega_i} (\mathscr{A}(x, \nabla u) + \frac{V(x)}{\alpha(x)} |u|^{\alpha(x)}) dx.$$

When $\|u\|$ is small enough, we have

$$\begin{aligned}
C(\sigma) \int_{\Omega} |u|^{\gamma(x)} dx &= C(\sigma) \sum_{i=1}^{\infty} \int_{\Omega_i} |u|^{\gamma(x)} dx \\
&= C(\sigma) \sum_{i=1}^{\infty} |u|^{\gamma(\xi_i)}_{L^{\gamma(\cdot)}(\Omega_i)} \ \text{(where } \xi_i \in \overline{\Omega_i})
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{i=1}^{\infty} \|u\|_{\Omega_i}^{\gamma(\xi_i)} \text{ (by Corollary 8.3.2 of [18])} \\
&\leq C\,\|u\|^{\epsilon} \sum_{i=1}^{\infty} \|u\|_{\Omega_i}^{q^+} \\
&= C\,\|u\|^{\epsilon} \sum_{i=1}^{\infty} \int_{\Omega_i} (\mathscr{A}(x,\nabla u) + \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)})dx \\
&= C\,\|u\|^{\epsilon} \int_{\mathbb{R}^N} (\mathscr{A}(x,\nabla u) + \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)})dx \\
&\leq \frac{1}{4} \int_{\mathbb{R}^N} (\mathscr{A}(x,\nabla u) + \frac{V(x)}{p(x)}\,|u|^{p(x)})dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Phi^+(u) &\geq \int_{\mathbb{R}^N} (\mathscr{A}(x,\nabla u) + \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)})dx - \sigma \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}\,|u|^{\alpha(x)}\,dx - C(\sigma)\int_{\mathbb{R}^N} |u|^{\gamma(x)}\,dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} (\mathscr{A}(x,\nabla u) + \frac{V(x)}{\alpha(x)}\,|u|^{\alpha(x)})dx \text{ when } \|u\| \text{ is small enough.}
\end{aligned}
$$

Thus, there exist $r > 0$ and $\delta > 0$ such that $\Phi^+(u) \geq \delta > 0$ for every $u \in X$ and $\|u\| = r$. Then $(\mathcal{E}^+)$ has a solution u, and it is easy to see that $u \geq 0$, so u is a solution of $(\mathcal{E})$.

Similarly, we establish the existence of a non-positive solution. □

6. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need to recall some preliminary results. Since X is a reflexive and separable Banach space (see [56, Section 17, Theorems 2–3]), there exist sequences $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$
X = \overline{\text{span}}\,\{e_j,\ j = 1,2,\cdots\}, \quad X^* = \overline{\text{span}}^{w^*}\{e_j^*,\ j = 1,2,\cdots\},
$$

and

$$
< e_j^*, e_j >= \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}
$$

For convenience, we write

$$
X_j = \text{span}\,\{e_j\},\ Y_k = \bigoplus_{j=1}^{k} X_j,\ Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{45}
$$

Lemma 6.1. *(see [42, Lemma 5.1]) Assume that $\Theta : X \to \mathbb{R}$ is weakly-strongly continuous and $\Theta(0) = 0$, $\gamma > 0$ is a given number. Let*

$$
\beta_k = \beta_k(\gamma) = \sup\left\{\Theta(u) \mid \|u\| \leq \gamma, u \in Z_k\right\}.
$$

Then $\beta_k \to 0$ as $k \to \infty$.

To complete the proof of Theorem 1.2, we recall the following critical point lemma (see, e.g., [58, Theorem 4.7]). If the Cerami condition is replaced by the well known $(P.S.)$-condition, we refer to [9, p. 221, Theorem 3.6] for the corresponding version of the critical point theorem.

Lemma 6.2. *Suppose that $\Phi \in C^1(X,R)$ is even and satisfies the Cerami condition. Let V^+, $V^- \subset X$ be closed subspaces of X with* $\operatorname{codim} V^+ + 1 = \dim V^-$, *and suppose that the following conditions are fulfilled:*
(1^0) $\Phi(0) = 0$;
(2^0) $\exists \tau > 0$, $\gamma > 0$ such that $\forall u \in V^+$: $\|u\| = \gamma \Rightarrow \Phi(u) \geq \tau$;
(3^0) $\exists \rho > 0$ such that $\forall u \in V^-$: $\|u\| \geq \rho \Rightarrow \Phi(u) \leq 0$.
Consider the following set:

$$\Gamma = \{g \in C^0(X,X) \mid g \text{ is odd, } g(u) = u \text{ if } u \in V^- \text{ and } \|u\| \geq \rho\}.$$

Then
(a) $\forall \delta > 0$, $g \in \Gamma$, $S_\delta^+ \cap g(V^-) \neq \varnothing$, here $S_\delta^+ = \{u \in V^+ \mid \|u\| = \delta\}$;
(b) the number $\varpi := \inf\limits_{g\in\Gamma} \sup\limits_{u\in V^-} \Phi(g(u)) \geq \tau > 0$ is a critical value for Φ.

Proof of Theorem 1.2. According to our assumptions, Φ is an even functional and satisfies the Cerami compactness condition. Let $V_k^+ = Z_k$, which is a closed linear subspace of X and $V_k^+ \oplus Y_{k-1} = X$.

We may assume that there exists $x_n \in \Omega$ such that $\nabla p(x_n) \neq 0$.

Define $h_n \in C_0(\overline{B(x_n, \varepsilon_n)})$ as

$$h_n(x) = \begin{cases} 0, & |x - x_n| \geq \varepsilon_n \\ \varepsilon_n - |x - x_n|, & |x - x_n| < \varepsilon_n. \end{cases}$$

By Lemma 4.10, we may let small enough $\varepsilon_n > 0$ such that

$$\Phi(th_n) \to -\infty \text{ as } t \to +\infty.$$

Without loss of generality, we may assume that

$$\operatorname{supp} h_i \cap \operatorname{supp} h_j = \emptyset \quad , \forall i \neq j.$$

Set $V_k^- = \operatorname{span}\{h_1, \cdots, h_k\}$. We will prove that there are infinitely many pairs of V_k^+ and V_k^-, such that φ satisfies the conditions of Lemma 6.2. We also show that the corresponding critical value $\varpi_k := \inf\limits_{g\in\Gamma} \sup\limits_{u\in V_k^-} \Phi(g(u))$ tends to $+\infty$ when $k \to \infty$, which implies that there are infinitely many pairs of solutions to the problem $(\mathcal{E})$.

For any $k = 1, 2, \cdots$, we prove that there exist $\rho_k > \tau_k > 0$ and large enough k such that

$$(A_1) \qquad b_k := \inf\left\{\Phi(u) \mid u \in V_k^+, \|u\| = \tau_k\right\} \to +\infty \quad \text{as } k \to +\infty;$$

$$(A_2) \qquad a_k := \max\{\Phi(u) |\ u \in V_k^-, \|u\| = \rho_k\} \leq 0.$$

We first show that (A_1) holds. Let $\sigma \in (0, V_0)$ be small enough, where V_0 is defined in (V). By $(\mathcal{H}_f^1)$, there exists $C(\sigma) > 0$ such that

$$F(x,u) \leq \sigma |u|^{\alpha(x)} + C(\sigma) |u|^{\gamma(x)}, \quad \forall x \in \mathbb{R}^N, \ \forall u \in \mathbb{R}.$$

By computation, for any $u \in Z_k$ with $\|u\| = \tau_k = (2C(\sigma)\frac{1}{c_1}\beta_k^{\gamma^+})^{1/(\alpha^- - \gamma^+)}$, we have

$$\begin{aligned}\Phi(u) &= \int_{\mathbb{R}^N} \mathscr{A}(x,\nabla u)dx + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx - \int_{\mathbb{R}^N} F(x,u)dx \\
&\geq 2c_1\Big(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx\Big) + \int_{\mathbb{R}^N} \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx \\
&\quad -\sigma\int_{\mathbb{R}^N}|u|^{\alpha(x)}\,dx - C(\sigma)\int_{\mathbb{R}^N}|u|^{\gamma(x)}\,dx \\
&\geq c_1\|u\|^{\alpha^-} - C(\sigma)\,|u|^{\gamma(\xi)}_{L^{\gamma(\cdot)}} - C \text{ (where } \xi\in\mathbb{R}^N) \\
&\geq \begin{cases} c_1\|u\|^{\alpha^-} - C(\sigma), \text{ if } |u|_{L^{\alpha(\cdot)}}\leq 1, \\ c_1\|u\|^{\alpha^-} - C(\sigma)\beta_k^{\gamma^+}\|u\|^{\gamma^+}, \text{ if } |u|_{L^{\alpha(\cdot)}} > 1, \end{cases} \\
&\geq c_1\|u\|^{\alpha^-} - C(\sigma)\beta_k^{\gamma^+}\|u\|^{\gamma^+} - C(\sigma) \\
&= c_1\Big(2C(\sigma)\frac{1}{c_1}\beta_k^{\gamma^+}\Big)^{\alpha^-/(\alpha^- - \gamma^+)} - C(\sigma)\beta_k^{\gamma^+}\Big(2C(\sigma)\frac{1}{c_1}\beta_k^{\gamma^+}\Big)^{\gamma^+/(\alpha^- - \gamma^+)} - C(\sigma) \\
&= \frac{c_1}{2}\Big(2C(\sigma)\frac{1}{c_1}\beta_k^{\gamma^+}\Big)^{\gamma^+/(\alpha^- - \gamma^+)} - C(\sigma) \to +\infty \text{ (as } k\to\infty),\end{aligned}$$

because $\alpha^- < \gamma^+$ and $\beta_k \to 0^+$ as $k\to\infty$. Therefore, $b_k \to +\infty$ as $k\to\infty$.

Now, we show that (A_2) holds. By Lemma 4.10, we deduce that

$$\Phi(th) \to -\infty \text{ as } t\to+\infty, \quad \forall h\in V_k^- = \operatorname{span}\{h_1,\cdots,h_k\} \text{ with } \| h \| = 1,$$

which implies that (A_2) holds.

We conclude that the proof of Theorem 1.2 is complete. □

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Appendix A

In this section we present some auxiliary results.

Define

$$H(x,t) = \begin{cases} \dfrac{1}{p(x)}t^{p(x)} + \dfrac{1}{q(x)} - \dfrac{1}{p(x)}, & \text{if } t > 1 \\ \dfrac{1}{p(x)}t^{p(x)}, & \text{if } t \leq 1. \end{cases}$$

Lemma A.1. *Suppose that $1 \ll p(\cdot) \ll q(\cdot) \ll N$ or $1 \ll q(\cdot) \ll p(\cdot) \ll N$. Then $H(x,\cdot)$ is uniformly convex, that is, for any $\epsilon\in(0,1)$ there exists $\delta=\delta(\epsilon)\in(0,1)$ such that either $|t-s|\leq\varepsilon\max\{t,s\}$, or*

$$H(x,(t+s)/2) \leq \frac{1}{2}(1-\delta)[H(x,t)+H(x,s)].$$

Proof. We denote

$$\varphi_{q(x)}(s) = \frac{1}{q(x)} s^{q(x)}, \quad \varphi_{p(x)}(t) = \frac{1}{p(x)} t^{p(x)} \quad \text{and} \quad \varphi^*_{p(x)}(t) = \frac{1}{p(x)} t^{p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)}\,.$$

It follows that

$$\varphi^*_{p(x)}(t) = \varphi_{p(x)}(t) - (\frac{1}{p(x)} - \frac{1}{q(x)})\,.$$

We have

$$H(x,t) = \varphi_{q(x)}(t) \text{ for any } t \le 1$$

and

$$H(x,t) = \varphi^*_{p(x)}(t) \text{ for any } t \ge 1.$$

We divide the proof of the lemma in two steps.

Step 1. We assume that $1 \ll p(\cdot) \ll q(\cdot) \ll N$.

We distinguish the following four cases. Without loss of generality, we may assume that $t \ge s$ and $|t-s| > \varepsilon t = \varepsilon \max\{t,s\}$.

Case (i): $s \le t \le 1$.

By Lemma 1.9 of [20], we know that $\varphi_{q(x)}(\cdot)$ is uniformly convex, then for the former $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) \in (0,1)$ such that either $|t-s| \le \varepsilon \max\{t,s\}$, or $\varphi_{q(x)}(x,(t+s)/2) \le \frac{1}{2}(1-\delta_1)[\varphi_{q(x)}(x,t) + \varphi_{q(x)}(x,s)]$. We deduce that

$$H(x,(t+s)/2) \le \frac{1}{2}(1-\delta_1)[H(x,t) + H(x,s)].$$

Case (ii): $1 \le s \le t$.

Note that $\varphi_{p(x)}(\cdot)$ is uniformly convex, then for the former $\varepsilon > 0$, there exists $\delta_2 = \delta_2(\epsilon) \in (0,1)$ such that either $|t-s| \le \varepsilon \max\{t,s\}$, or $\varphi_{p(x)}(x,(t+s)/2) \le \frac{1}{2}(1-\delta_2)[\varphi_{p(x)}(x,t) + \varphi_{p(x)}(x,s)]$. Notice that $(\frac{1}{p(x)} - \frac{1}{q(x)}) > 0$. Therefore

$$\begin{aligned}
&\varphi_{p(x)}((t+s)/2) - (\frac{1}{p(x)} - \frac{1}{q(x)}) \\
&\le \frac{1}{2}(1-\delta_2)[\varphi_{p(x)}(t) + \varphi_{p(x)}(s)] - (\frac{1}{p(x)} - \frac{1}{q(x)}) \\
&\le \frac{1}{2}(1-\delta_2)\{[\varphi_{p(x)}(t) - (\frac{1}{p(x)} - \frac{1}{q(x)})] + [\varphi_{p(x)}(s) - (\frac{1}{p(x)} - \frac{1}{q(x)})]\} - \delta_2(\frac{1}{p(x)} - \frac{1}{q(x)}) \\
&\le \frac{1}{2}(1-\delta_2)\{[\varphi_{p(x)}(t) - (\frac{1}{p(x)} - \frac{1}{q(x)})] + [\varphi_{p(x)}(s) - (\frac{1}{p(x)} - \frac{1}{q(x)})]\}.
\end{aligned}$$

It follows that

$$\varphi^*_{p(x)}((t+s)/2) \le \frac{1}{2}(1-\delta_2)[\varphi^*_{p(x)}(t) + \varphi^*_{p(x)}(s)]$$

Therefore

$$H(x,(t+s)/2) \le \frac{1}{2}(1-\delta_2)[H(x,t) + H(x,s)].$$

Case (iii): $s \le 1 \le (t+s)/2 \le t$.

We first observe that $\varphi^*_{p(x)}(s) \le \varphi_{q(x)}(s)$ for any $s \le 1$. It follows that

$$\begin{aligned} H(x,(t+s)/2) &= \varphi^*_{p(x)}((t+s)/2) \\ &\le \frac{1}{2}(1-\delta_2)[\varphi^*_{p(x)}(t)+\varphi^*_{p(x)}(s)] \\ &\le \frac{1}{2}(1-\delta_2)[\varphi^*_{p(x)}(t)+\varphi_{q(x)}(s)] \\ &= \frac{1}{2}(1-\delta_2)[H(x,t)+H(x,s)]. \end{aligned}$$

Case (iv): $s \le (t+s)/2 < 1 \le t$.

We only need to prove that exists $\delta_3 = \delta_3(\epsilon) \in (0,1)$ such that either $|t-s| \le \varepsilon \max\{t,s\}$, or

$$\frac{1}{q(x)}\left(\frac{t+s}{2}\right)^{q(x)} \le \frac{1}{2}(1-\delta_3)\left(\frac{1}{p(x)}t^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}s^{q(x)}\right). \tag{46}$$

Fix $\frac{t+s}{2}$. We may let $\frac{t+s}{2}=c$, then $s=2c-t$. Let t vary. Denote

$$g(t)=\frac{1}{p(x)}t^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}(2c-t)^{q(x)},$$

hence

$$g'(t)=t^{p(x)-1}-(2c-t)^{q(x)-1}>0 \text{ for any } t>c.$$

Thus $g(t)$ is strictly increasing, and then $g(t) \ge g(c+\frac{\varepsilon_0}{2})$, where $\varepsilon_0 = \frac{\varepsilon c}{1-\varepsilon/2}$ satisfies

$$|t-s| = \varepsilon_0 \ge \varepsilon(c+\frac{\varepsilon_0}{2}) = \varepsilon t.$$

Since $c+\frac{\varepsilon_0}{2}=t\ge 1$, we have $1-\frac{\varepsilon_0}{2} \le c \le 1$.

Consider

$$h(c)=\frac{\frac{1}{q(x)}(c)^{q(x)}}{\frac{1}{p(x)}(c+\frac{\varepsilon_0}{2})^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}(c-\frac{\varepsilon_0}{2})^{q(x)}}.$$

By computation,

$$\begin{aligned} h'(c) &= \frac{(c)^{q(x)-1}[\frac{1}{p(x)}(c+\frac{\varepsilon_0}{2})^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}(c-\frac{\varepsilon_0}{2})^{q(x)}]}{[\frac{1}{p(x)}(c+\frac{\varepsilon_0}{2})^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}(c-\frac{\varepsilon_0}{2})^{q(x)}]^2} \\ &\quad -\frac{\frac{1}{q(x)}(c)^{q(x)}[(c+\frac{\varepsilon_0}{2})^{p(x)-1}+(c-\frac{\varepsilon_0}{2})^{q(x)-1}]}{[\frac{1}{p(x)}(c+\frac{\varepsilon_0}{2})^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}(c-\frac{\varepsilon_0}{2})^{q(x)}]^2}. \end{aligned}$$

We claim that $h'(c) \ge 0$ for any $c \in [1-\frac{\varepsilon_0}{2},1]$.

Denote

$$\begin{aligned} f(c) &= \frac{1}{p(x)}(c+\frac{\varepsilon_0}{2})^{p(x)}+\frac{1}{q(x)}-\frac{1}{p(x)}+\frac{1}{q(x)}(c-\frac{\varepsilon_0}{2})^{q(x)} \\ &\quad -\frac{1}{q(x)}c[(c+\frac{\varepsilon_0}{2})^{p(x)-1}+(c-\frac{\varepsilon_0}{2})^{q(x)-1}]. \end{aligned}$$

We can rewrite f in order to obtain

$$f(c) = (c + \frac{\varepsilon_0}{2})^{p(x)-1} c[\frac{1}{p(x)} - \frac{1}{q(x)}] + \frac{\varepsilon_0}{2p(x)}(c + \frac{\varepsilon_0}{2})^{p(x)-1} + \frac{1}{q(x)} - \frac{1}{p(x)} - \frac{\varepsilon_0}{2q(x)}(c - \frac{\varepsilon_0}{2})^{q(x)-1}.$$

We may assume that $c > \frac{1}{2}$ and $\frac{\varepsilon_0}{2} < \frac{1}{4}$. By computation, we observe that $f'(c) > 0$ as $\varepsilon > 0$ is small enough. Obviously,

$$\begin{aligned} f(c) \geq f(1 - \frac{\varepsilon_0}{2}) &= (1 - \frac{\varepsilon_0}{2})[\frac{1}{p(x)} - \frac{1}{q(x)}] + \frac{\varepsilon_0}{2p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)} - \frac{\varepsilon_0}{2q(x)}(1 - \varepsilon_0)^{q(x)-1} \\ &= \frac{\varepsilon_0}{2q(x)} - \frac{\varepsilon_0}{2q(x)}(1 - \varepsilon_0)^{q(x)-1} > 0. \end{aligned}$$

Therefore $f(c) > 0$ on $[1 - \frac{\varepsilon_0}{2}, 1]$ when $\varepsilon > 0$ is small enough. Thus

$$h'(c) = \frac{(c)^{q(x)-1}}{[\frac{1}{p(x)}(c + \frac{\varepsilon_0}{2})^{p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)} + \frac{1}{q(x)}(c - \frac{\varepsilon_0}{2})^{q(x)}]^2} f(c) > 0$$

for any $c \in [1 - \frac{\varepsilon_0}{2}, 1]$. Therefore

$$h(c) \leq h(1).$$

By Case (iii), we have $h(1) \leq \frac{1}{2}(1 - \delta_2)$.

Thus, relation (46) is valid for $\delta_3 = \delta_2$.

Summarizing Cases (i)–(iv), we conclude that $H(x, \cdot)$ is uniformly convex.

Step 2. We assume that $1 \ll q(\cdot) \ll p(\cdot) \ll N$.

We will discuss this result in four cases. Without loss of generality, we may assume that $t \geq s$ and $|t - s| > \varepsilon t = \varepsilon \max\{t, s\}$.

Case (1^0): $s \leq t \leq 1$.

By Lemma 1.9. of [20], we know that $\varphi_{q(x)}(\cdot)$ is uniformly convex, then for the former $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) \in (0, 1)$ such that either $|t - s| \leq \varepsilon \max\{t, s\}$, or $\varphi_{q(x)}(x, (t + s)/2) \leq \frac{1}{2}(1 - \delta_1)[\varphi_{q(x)}(x, t) + \varphi_{q(x)}(x, s)]$. We deduce that

$$H(x, (t + s)/2) \leq \frac{1}{2}(1 - \delta_1)[H(x, t) + H(x, s)].$$

Case (2^0): $1 \leq s \leq t$.

For the former $\varepsilon > 0$, we need to find out a $\delta_2 = \delta_2(\epsilon) \in (0, 1)$ such that either $|t - s| \leq \varepsilon \max\{t, s\}$, or

$$H(x, (t + s)/2) \leq \frac{1}{2}(1 - \delta_2)[H(x, t) + H(x, s)].$$

By Lemma 1.9. of [20], we know that $\varphi_{p(x)}(\cdot)$ is uniformly convex, then for the former $\varepsilon > 0$, there exists $\delta_* = \delta_*(\epsilon) \in (0, 1)$ such that either $|t - s| \leq \varepsilon \max\{t, s\}$, or $\varphi_{p(x)}(x, (t + s)/2) \leq \frac{1}{2}(1 - \delta_*)[\varphi_{p(x)}(x, t) + \varphi_{p(x)}(x, s)]$.

It follows that

$$\begin{aligned} H(x, (t + s)/2) &= \varphi_{p(x)}(x, (t + s)/2) + \frac{1}{q(x)} - \frac{1}{p(x)} \\ &\leq \frac{1}{2}(1 - \delta_*)[\varphi_{p(x)}(x, t) + \varphi_{p(x)}(x, s)] + \frac{1}{q(x)} - \frac{1}{p(x)} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(1-\delta)[\varphi_{p(x)}(x,t)+\varphi_{p(x)}(x,s)] \\
&\quad + \frac{1}{2}(\delta-\delta_*)[\varphi_{p(x)}(x,t)+\varphi_{p(x)}(x,s)] + \frac{1}{q(x)} - \frac{1}{p(x)} \\
&= \frac{1}{2}(1-\delta)[H(x,t)+H(x,s)] - (1-\delta)(\frac{1}{q(x)} - \frac{1}{p(x)}) + \frac{1}{q(x)} - \frac{1}{p(x)} \\
&\quad + \frac{1}{2}(\delta-\delta_*)[\varphi_{p(x)}(x,t)+\varphi_{p(x)}(x,s)] \\
&= \frac{1}{2}(1-\delta)[H(x,t)+H(x,s)] \\
&\quad + \frac{1}{2}(\delta-\delta_*)[\varphi_{p(x)}(x,t)+\varphi_{p(x)}(x,s)] + \delta(\frac{1}{q(x)} - \frac{1}{p(x)}) \\
&\le \frac{1}{2}(1-\delta)[H(x,t)+H(x,s)] + \frac{1}{2}(\delta-\delta_*)(\frac{1}{q(x)} - \frac{1}{p(x)}) + \delta(\frac{1}{q(x)} - \frac{1}{p(x)}) \\
&\le \frac{1}{2}(1-\delta)[H(x,t)+H(x,s)] \text{ when } \delta = \frac{\delta_*}{4}.
\end{aligned}$$

Case (3^0): $s \le (t+s)/2 \le 1 \le t$.

Notice that $\varphi^*_{p(x)}(x,t) \ge \varphi_{q(x)}(x,t)$. By Lemma 1.9. of [20], we know that $\varphi_{q(x)}(\cdot)$ is uniformly convex. Thus, for the former $\varepsilon > 0$, there exists $\delta_* = \delta_*(\epsilon) \in (0,1)$ such that either $|t-s| \le \varepsilon \max\{t,s\}$, or

$$\begin{aligned}
H(x,(t+s)/2) &= \varphi_{q(x)}(x,(t+s)/2) \\
&\le \frac{1}{2}(1-\delta_*)[\varphi_{q(x)}(x,t)+\varphi_{q(x)}(x,s)] \\
&\le \frac{1}{2}(1-\delta_*)[\varphi^*_{p(x)}(x,t)+\varphi_{q(x)}(x,s)] \\
&= \frac{1}{2}(1-\delta_*)[H(x,t)+H(x,s)].
\end{aligned}$$

Case (4^0): $s \le 1 < (t+s)/2 \le t$.

Fix $\frac{t+s}{2}$. We let $\frac{t+s}{2} = c$, hence $s = 2c - t$. Let t vary. Denote

$$g(t) = \frac{1}{p(x)} t^{p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)} + \frac{1}{q(x)}(2c-t)^{q(x)},$$

then

$$g'(t) = t^{p(x)-1} - (2c-t)^{q(x)-1} > 0 \text{ for any } t > c.$$

Thus $g(t)$ is strictly increasing, and then $g(t) \ge g(c + \frac{\varepsilon_0}{2})$, where $\varepsilon_0 = \frac{\varepsilon c}{1-\varepsilon/2}$ satisfies

$$|t-s| = \varepsilon_0 \ge \varepsilon(c + \frac{\varepsilon_0}{2}) = \varepsilon t.$$

If $s = c - \frac{\varepsilon_0}{2} \ge 1$, then this corresponds to Case (iii).

If $s = c - \frac{\varepsilon_0}{2} < 1$, we have $1 \le c \le 1 + \frac{\varepsilon_0}{2}$. Consider

$$h(c) = \frac{\frac{1}{p(x)}(c)^{p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)}}{\frac{1}{p(x)}(c+\frac{\varepsilon_0}{2})^{p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)} + \frac{1}{q(x)}(c-\frac{\varepsilon_0}{2})^{q(x)}}.$$

We only need to prove that $h(1) < \frac{1}{2}$. Then we will get a $\delta_3 = \delta_3(\varepsilon_0) = \delta_3(\varepsilon) \in (0,1)$ such that

$$h(c) \leq \frac{1-\delta_3}{2} \text{ for } c \in [1, 1+\frac{\varepsilon_0}{2}] \text{ as } \varepsilon \text{ is small enough.}$$

Denote

$$f(\varepsilon_0) = \frac{1}{p(x)}(1+\frac{\varepsilon_0}{2})^{p(x)} + \frac{1}{q(x)}(1-\frac{\varepsilon_0}{2})^{q(x)} + \frac{1}{q(x)} - \frac{1}{p(x)}.$$

We observe that $f'(\varepsilon_0) > 0$ for $\varepsilon_0 \in (0,1)$, thus

$$f(\varepsilon_0) > f(0) = \frac{2}{q(x)} \text{ on } (0,1].$$

It follows that $h(1) < \frac{1}{2}$ for $c \in [1, 1+\frac{\varepsilon_0}{2}]$ as ε is small enough.

Therefore, for the former $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon) \in (0,1)$ such that either $|t-s| \leq \varepsilon \max\{t,s\}$, or

$$H(x,(t+s)/2) \leq \frac{1}{2}(1-\delta_1)[H(x,t) + H(x,s)].$$

Summarizing Cases (1^0)–(4^0), we conclude that $H(x,\cdot)$ is uniformly convex. □

Lemma A.2. *The function $H(x,|\cdot|)$ is uniformly convex in $\mathbb{R}^N$.*

Proof. From [18, Lemma 2.4.7], $H(x,|\cdot|)$ is uniformly convex in $\mathbb{R}^N$. □

From [18, Theorem 2.4.11], we have the following property.

Lemma A.3. *Assume $(\mathcal{A}_1)$–$(\mathcal{A}_2)$. Then $\rho_{\mathscr{A}}(u) = \int_\Omega \mathscr{A}(x,u)dx$ is uniformly convex.*

Lemma A.4. *Assume $(\mathcal{A}_1)$-(iii). For any $x,\xi,\eta \in \mathbb{R}^N$, there exists a positive constant C such that $|\mathbf{A}(x,\xi+\eta)| \leq C(|\mathbf{A}(x,\xi)| + |\mathbf{A}(x,\eta)|)$.*

Proof. We may assume that $|\eta| \geq |\xi|$, hence $|2\eta| \geq |\xi+\eta|$.

If $|\xi+\eta| \geq 1$ then $|2\eta| \geq 1$, and we have

$$\begin{aligned} |\mathbf{A}(x,\xi+\eta)| &\leq C_2 |\xi+\eta|^{p(x)-1} \\ &\leq \begin{cases} C_2 |2\eta|^{p(x)-1} \leq \frac{C_2}{C_1} 2^{p^+} C_1 |\eta|^{p(x)-1} \leq \frac{C_2}{C_1} 2^{p^+} |\mathbf{A}(x,\eta)|, |\eta| \geq 1, \\ C_2 |2\eta|^{q(x)-1} \leq \frac{C_2}{C_1} 2^{q^+} C_1 |\eta|^{q(x)-1} \leq \frac{C_2}{C_1} 2^{q^+} |\mathbf{A}(x,\eta)|, |\eta| < 1. \end{cases} \end{aligned}$$

If $|\xi+\eta| < 1$, notice that $q(x) > p(x)$. It follows that

$$\begin{aligned} |\mathbf{A}(x,\xi+\eta)| &\leq C_2 |\xi+\eta|^{q(x)-1} \\ &\leq \begin{cases} C_2 |\xi+\eta|^{p(x)-1} \leq C_2 |2\eta|^{p(x)-1} \leq \frac{C_2}{C_1} 2^{p^+} C_1 |\eta|^{p(x)-1} \leq \frac{C_2}{C_1} 2^{p^+} |\mathbf{A}(x,\eta)|, |\eta| \geq 1, \\ C_2 |2\eta|^{q(x)-1} \leq \frac{C_2}{C_1} 2^{q^+} C_1 |\eta|^{q(x)-1} \leq \frac{C_2}{C_1} 2^{q^+} |\mathbf{A}(x,\eta)|, |\eta| < 1. \end{cases} \end{aligned}$$

Therefore $|\mathbf{A}(x,\xi+\eta)| \leq C(|\mathbf{A}(x,\xi)| + |\mathbf{A}(x,\eta)|)$. □

Appendix B

In this section, we address to the readers several comments, perspectives, and open problems.

(i) Hypothesis $(\mathcal{A}_1)$(iv) establishes that problem $(\mathcal{E})$ is described in a *subcritical* setting. To the best of our knowledge, there is no result in the literature corresponding to the following *almost critical* framework described in what follows. Assume that condition $q(\cdot) \ll \min\{N, p^*(\cdot)\}$ in $(\mathcal{A}_1)$(iv) is replaced with the following hypothesis: there exists a finite set $A \subset \mathbb{R}^N$ such that $q(a) = \min\{N, p^*(a)\}$ for all $a \in A$ and $q(x) < \min\{N, p^*(x)\}$ for all $x \in \mathbb{R}^N \setminus A$.

Open problem. Study if Theorems 1.1 and 1.2 established in this paper still remain true in the above almost critical abstract setting.

(ii) Another very interesting research direction is to extend the approach developed in this paper to the case of *double phase* problems studied by Mingione *et al.* [14,15]. This corresponds to the following non-homogeneous potential

$$\mathscr{A}(x,\xi) = \frac{a(x)}{p(x)}\,|\xi|^{p(x)} + \frac{b(x)}{q(x)}\,|\xi|^{q(x)},$$

where the coefficients $a(x)$ and $b(x)$ are non-negative and at least one is strictly positive for all $x \in \mathbb{R}^N$. At this stage, we do not know any multiplicity results for double phase problems of this type.

We also refer to the pioneering papers by Marcellini [34,35] on (p,q)-growth conditions, which involve integral functionals of the type

$$W^{1,1} \ni u \mapsto \int_\Omega f(x, \nabla u)dx,$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set. The integrand $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfied unbalanced polynomial growth conditions of the type

$$|\xi|^p \lesssim f(x,\xi) \lesssim |\xi|^q + 1 \quad \text{with } 1 < p < q,$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

(iii) The differential operator $\mathscr{A}(x,\xi)$ considered in problem $(\mathcal{E})$ falls in the realm of those related to the so-called Musielak–Orlicz spaces (see [39,41]), more in general, of the operators having non-standard growth conditions (which are widely considered in the calculus of variations). These function spaces are Orlicz spaces whose defining Young function exhibits an additional dependence on the x variable. Indeed, classical Orlicz spaces L^Φ are defined requiring that a member function f satisfies

$$\int_\Omega \Phi(|f|)dx < \infty,$$

where $\Phi(t)$ is a Young function (convex, non-decreasing, $\Phi(0) = 0$). In the new case of Musielak–Orlicz spaces, the above condition becomes

$$\int_\Omega \Phi(x, |f|)dx < \infty.$$

The problems considered in this paper are indeed driven by the function

$$\Phi(x, |\xi|) := \begin{cases} |\xi|^{p(x)} & \text{if } |\xi| \le 1 \\ |\xi|^{q(x)} & \text{if } |\xi| \ge 1. \end{cases} \tag{47}$$

When $p(x) = q(x)$ we find the so-called variable exponent spaces, which are defined by

$$\Phi(x, |\xi|) := |\xi|^{p(x)}.$$

We conclude these comments by saying that the present paper is concerned with a double phase variant of the operators stemming from the energy generated by the function defined in (47).

(iv) As it has been kindly pointed out by one of the referees of this paper, an interesting double phase type operator considered in the papers of Baroni, Colombo and Mingione [6,7,14,15], addresses functionals of the type

$$w \mapsto \int_\Omega (|\nabla w|^p + a(x)|\nabla w|^q)dx, \tag{48}$$

where $a(x) \geq 0$. The meaning of this functional is also to give a sharper version of the following energy

$$w \mapsto \int_\Omega |\nabla w|^{p(x)}dx,$$

thereby describing sharper phase transitions. Composite materials with locally different hardening exponents p and q can be described using the energy defined in (48). Problems of this type are also motivated by applications to elasticity, homogenization, modelling of strongly anisotropic materials, Lavrentiev phenomenon, etc.

Accordingly, a new double phase model can be given by

$$\Phi_d(x, |\xi|) := \begin{cases} |\xi|^p + a(x)|\xi|^q & \text{if } |\xi| \leq 1 \\ |\xi|^{p_1} + a(x)|\xi|^{q_1} & \text{if } |\xi| \geq 1, \end{cases}$$

with $a(x) \geq 0$.

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