Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 333 (2007) 265–273
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Ground state solutions for the singular Lane–Emden–Fowler equation with sublinear convection term

Marius Ghergu [a], Vicenţiu Rădulescu [b,*]

[a] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, PO Box 1-764, RO-014700 Bucharest, Romania*
[b] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*

Received 23 September 2006
Available online 27 October 2006
Submitted by R.P. Agarwal

Dedicated to Professor William F. Ames on his 80th birthday

Abstract

We are concerned with singular elliptic equations of the form $-\Delta u = p(x)(g(u) + f(u) + |\nabla u|^a)$ in $\mathbb{R}^N$ ($N \geqslant 3$), where p is a positive weight and $0 < a < 1$. Under the hypothesis that f is a nondecreasing function with sublinear growth and g is decreasing and unbounded around the origin, we establish the existence of a ground state solution vanishing at infinity. Our arguments rely essentially on the maximum principle.

Keywords: Ground state solution; Singular elliptic equation; Convection term; Maximum principle

1. Introduction and the main result

The Lane–Emden–Fowler equation originated from earlier theories concerning gaseous dynamics in astrophysics around the turn of the last century (see, e.g., Fowler [11]). It also arises in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemical

* Corresponding author.
E-mail addresses: marius.ghergu@imar.ro (M. Ghergu), vicentiu.radulescu@math.cnrs.fr, vicentiu.radulescu@u-picardie.fr (V. Rădulescu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.09.074

reaction systems, one can see the survey article by Wong [24] for detailed background of the generalized Emden–Fowler equation. The Lane–Emden–Fowler equation has been studied by many authors using various methods and techniques. For example, critical point theory, fixed point theory, topological degree theory and coincidence degree theory are widely used to study the existence of the BVP for ordinary and partial differential equations (see, e.g., Agarwal and O'Regan [1], Arcoya [3], Hale and Mawhin [17], Mawhin and Willem [19], Nato and Tanaka [20], Wang [23], etc.).

We are concerned in this paper with the following singular Lane–Emden–Fowler type problem

$$\begin{cases} -\Delta u = p(x)\big(g(u) + f(u) + |\nabla u|^a\big) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $N \geqslant 3$, $0 < a < 1$, and $p : \mathbb{R}^N \to (0, \infty)$ is a Hölder continuous function of exponent $\gamma \in (0, 1)$. We assume that $g \in C^1(0, \infty)$ is a positive decreasing function such that

(g1) $\lim_{t\to 0^+} g(t) = +\infty$.

Throughout this paper we suppose that $f : [0, \infty) \to [0, \infty)$ is a Hölder continuous function of exponent $0 < \gamma < 1$ which is nondecreasing with respect to the second variable and such that f is positive on $\overline{\Omega} \times (0, \infty)$. The analysis we develop in this paper concerns the case where f is sublinear, that is,

(f1) the mapping $(0, \infty) \ni t \mapsto \frac{f(t)}{t}$ is nonincreasing;
(f2) $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$ and $\lim_{t\to\infty} \frac{f(t)}{t} = 0$.

We are concerned in this paper with ground state solutions, that is, positive solutions defined in the whole space and decaying to zero at infinity. It is worth pointing out here that we do not assume any blow-up rate of decay on g around the origin (as in Ghergu and Rădulescu [12]) that may imply the property of compact support of Bénilan, Brezis and Crandall (see [4]).

There is a large number of works dealing with singular elliptic equations in bounded domains. In this sense, we refer the reader to Callegari and Nachman [5], Cîrstea, Ghergu and Rădulescu [6], Coclite and Palmieri [8], Crandall, Rabinowitz and Tartar [9], Díaz, Morel, and Oswald [10], Ghergu and Rădulescu [13], Shi and Yao [21]. The influence of the convection term has been emphasized in Ghergu and Rădulescu [14,15] and Zhang [25]. Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids. We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

Concerning the ground state solutions for singular elliptic equations, we mention here the works of Lair and Shaker [18], Sun and Li [22]. In [18] it is considered the following singular boundary value problem

$$\begin{cases} -\Delta u = p(x)g(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $g:(0,\infty)\to(0,\infty)$ is a smooth decreasing function (possibly, unbounded around the origin) such that $\int_0^1 g(t)\,dt<\infty$. The results have been extended by Cîrstea and Rădulescu [7] to the case where g satisfies the weaker assumptions $\lim_{t\searrow 0} g(t)/t=+\infty$ and the mapping $t\mapsto g(u)/(u+\beta)$ is decreasing, for some $\beta>0$.

It is proved in Lair and Shaker [18] (see also Cîrstea and Rădulescu [7]) that a necessary condition in order to have a solution for (1.2) is

$$\int_1^\infty t\psi(t)\,dt<\infty, \tag{1.3}$$

where $\psi(r)=\min_{|x|=r} p(x)$, $r\geqslant 0$. Note that condition (1.3) is also necessary for our problem (1.1), since any solution of (1.1) is a super-solution of (1.2). The sufficient condition for existence supplied in Lair and Shaker [18] is

$$\int_1^\infty t\phi(t)\,dt<\infty, \tag{1.4}$$

where $\phi(r)=\max_{|x|=r} p(x)$, $r\geqslant 0$. Hence, when p is radially symmetric, then the problem (1.2) has solutions if and only if $\int_1^\infty tp(t)\,dt<\infty$.

The main feature here is the presence of the convection term $|\nabla u|^a$. In this sense we prove the following result.

Theorem 1.1. *Assume that* (f1)–(f2)*,* (g1) *and* (1.4) *are fulfilled. Then problem* (1.1) *has at least one solution.*

We point out that the uniqueness of the solution to (1.1) is a delicate matter even in case of bounded domains (see, e.g., Ghergu and Rădulescu [14,15]), due to the lack of an adequate comparison principle. We also notice that the growth decay of the potential $p(x)$ described in our hypothesis (1.4) implies that p is in a certain Kato class $K^N_{\mathrm{loc}}(\mathbb{R}^N)$. This theory was introduced by Aizenman and Simon in [2] to describe wide classes of functions arising in potential theory.

2. Proof of Theorem 1.1

The solution of problem (1.1) is obtained as a limit in $C^{2,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ of a monotone sequence of solutions associated to (1.1) in smooth bounded domains. A basic ingredient in our approach is the following auxiliary result.

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^N$ be a smooth bounded domain. Assume that f and g satisfy* (f1)–(f2) *and* (g1)*, respectively. Then the boundary value problem*

$$\begin{cases} -\Delta u=p(x)\big(g(u)+f(u)+|\nabla u|^a\big) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a unique solution $u\in C^{2,\gamma}(\Omega)\cap C(\overline{\Omega})$.

Proof. The proof relies on sub- and super-solution method. The assumptions on f and g imply that $m := \inf_{t>0}\{g(t) + f(t)\} > 0$. So, the unique solution $\underline{u}$ of the problem

$$\begin{cases} -\Delta u = mp(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

is a sub-solution of (2.1). The main point is to find a super-solution $\bar{u}$ of problem (2.1) such that $\underline{u} \leqslant \bar{u}$ in Ω. Then, by classical results (see, e.g., Gilbarg and Trudinger [16]) we deduce that the problem (2.1) has at least one solution.

Let $h : [0, \eta] \to [0, \infty)$ be such that

$$\begin{cases} h''(t) = -g\big(h(t)\big) & \text{for all } 0 < t < \eta, \\ h(0) = 0, \\ h > 0 \quad \text{in } (0, \eta]. \end{cases} \tag{2.3}$$

The existence of h follows from the results in Agarwal and O'Regan [1]. Since h is concave, there exists $h'(0+) \in (0, +\infty]$. Taking $\eta > 0$ small enough, we can assume that $h' > 0$ on $(0, \eta]$, that is, h is increasing on $[0, \eta]$. Multiplying by $h'(t)$ in (2.3) and integrating on $[t, \eta]$, we obtain

$$\begin{aligned} (h')^2(t) &= 2\int_t^{\eta} g\big(h(s)\big)h'(s)\,ds + (h')^2(\eta) \\ &= 2\int_{h(t)}^{h(\eta)} g(\tau)\,d\tau + (h')^2(\eta), \quad \text{for all } 0 < t < \eta. \end{aligned} \tag{2.4}$$

Using the monotonicity of g in (2.4) we get

$$(h')^2(t) \leqslant 2h(\eta)g\big(h(t)\big) + (h')^2(\eta), \quad \text{for all } 0 < t < \eta.$$

Since $s^a \leqslant s^2 + 1$, for all $s \geqslant 0$, the last inequality yields

$$(h')^a(t) \leqslant Cg\big(h(t)\big), \quad \text{for all } 0 < t < \eta, \tag{2.5}$$

for some $C > 0$. Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$ in $H_0^1(\Omega)$. We fix $c > 0$ such that $c\|\varphi_1\|_\infty < \eta$.

By Hopf's maximum principle, there exist $\omega \Subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \quad \text{in } \Omega \setminus \omega \quad \text{and} \quad \varphi_1 > \delta \quad \text{in } \omega. \tag{2.6}$$

Let $M > 1$ be such that

$$(Mc)^{1-a}\lambda_1(h')^{1-a}(\eta) > 3\max_{x\in\overline{\Omega}} p(x)\|\nabla\varphi_1\|_\infty^a, \tag{2.7}$$

$$Mc\lambda_1 h'(\eta) > 3\max_{x\in\overline{\Omega}} p(x)g\Big(h\Big(c\min_{x\in\overline{\omega}}\varphi_1\Big)\Big) \tag{2.8}$$

and

$$\min\big\{M(c\delta)^2, M^{1-a}C^{-1}(c\delta)^{2-a}\big\} > 3\max_{x\in\overline{\Omega}} p(x), \tag{2.9}$$

where C is the constant from (2.5). Since

$$\lim_{t\to 0^+}\Big((c\delta)^2 g\big(h(t)\big) - 3\max_{x\in\overline{\Omega}} p(x)f\big(h(t)\big)\Big) = \infty,$$

we can assume that

$$(c\delta)^2 g\big(h(c\varphi_1)\big) > 3 \max_{x\in\overline{\Omega}} p(x) f\big(h(c\varphi_1)\big) \quad \text{in } \Omega \setminus \omega. \tag{2.10}$$

Finally, from the assumption (f2) on f we have $\lim_{t\to\infty} \frac{f(th(c\|\varphi_1\|_\infty)}{t} = 0$, so that we can choose $M > 1$ large enough with the property

$$c\lambda_1 \inf_{x\in\overline{\omega}} \varphi_1 h'(\eta) > \max_{x\in\overline{\Omega}} p(x) \frac{f(Mh(c\|\varphi_1\|_\infty)}{M}.$$

The last inequality combined with the fact that h' is decreasing yields

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant 3f\big(Mh(c\varphi_1)\big) \quad \text{in } \omega. \tag{2.11}$$

We claim that $\bar{u} = Mh(c\varphi_1)$ is a super-solution of the problem (2.1) provided that M satisfies (2.7)–(2.9). We have

$$-\Delta\bar{u} = Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega.$$

From (2.6), (2.9) and the monotonicity of g we obtain

$$\frac{1}{3}Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant p(x)g\big(h(c\varphi_1)\big) \geqslant p(x)g\big(Mh(c\varphi_1)\big) = p(x)g(\bar{u}) \quad \text{in } \Omega \setminus \omega. \tag{2.12}$$

From (2.6), (2.10) and our hypothesis (f1), we obtain

$$\frac{1}{3}Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant Mp(x)f\big(h(c\varphi_1)\big) \geqslant p(x)f\big(Mh(c\varphi_1)\big) = p(x)f(\bar{u}) \quad \text{in } \Omega \setminus \omega. \tag{2.13}$$

From (2.5) and (2.9) we have

$$\frac{1}{3}Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant p(x)\big(Mch'(c\varphi_1)|\nabla\varphi_1|\big)^a = p(x)|\nabla\bar{u}|^a \quad \text{in } \Omega \setminus \omega. \tag{2.14}$$

Now, relations (2.12)–(2.14) yield

$$-\Delta\bar{v} \geqslant Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant p(x)\big(g(\bar{u}) + f(\bar{u}) + |\nabla\bar{u}|^a\big) \quad \text{in } \Omega \setminus \omega. \tag{2.15}$$

Similarly, from (2.7)–(2.11) we deduce that

$$-\Delta\bar{u} \geqslant Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant p(x)\big(g(\bar{u}) + f(\bar{u}) + |\nabla\bar{u}|^a\big) \quad \text{in } \omega. \tag{2.16}$$

Using (2.15) and (2.16), it follows that $\bar{u}$ is a super-solution of problem (2.1). The maximum principle implies that $\underline{u} \leqslant \bar{u}$ in Ω. Thus, the problem (2.1) has at least one classical solution. This concludes the proof of our lemma. □

In what follows we apply Lemma 2.1 for $B_n := \{x \in \mathbb{R}^N; |x| < n\}$. Hence, for all $n \geqslant 1$ there exists $u_n \in C^{2,\gamma}(B_n) \cap C(\overline{B_n})$ such that

$$\begin{cases} -\Delta u_n = p(x)\big(g(u_n) + f(u_n) + |\nabla u_n|^a\big) & \text{in } B_n, \\ u_n > 0 & \text{in } B_n, \\ u_n = 0 & \text{on } \partial B_n. \end{cases} \tag{2.17}$$

We extend u_n by zero outside of B_n. We claim that

$$u_n \leqslant u_{n+1} \quad \text{in } B_n.$$

Assume by contradiction that the inequality $u_n \leqslant u_{n+1}$ does not hold throughout B_n and let

$$\zeta(x) = \frac{u_n(x)}{u_{n+1}(x)}, \quad x \in B_n.$$

Clearly $\zeta = 0$ on ∂B_n, so that ζ achieves its maximum in a point $x_0 \in B_n$. At this point we have $\nabla\zeta(x_0) = 0$ and $\Delta\zeta(x_0) \leqslant 0$. This yields

$$-\operatorname{div}\big(u_{n+1}^2 \nabla\zeta\big)(x_0) = -\big(\operatorname{div}\big(u_{n+1}^2\big)\nabla\zeta + u_{n+1}^2 \Delta\zeta\big)(x_0) \geqslant 0.$$

A straightforward computation shows that

$$-\operatorname{div}\big(u_{n+1}^2 \nabla\zeta\big) = -u_{n+1}\Delta u_n + u_n \Delta u_{n+1}.$$

Hence

$$(-u_{n+1}\Delta u_n + u_n\Delta u_{n+1})(x_0) \geqslant 0.$$

The above relation produces

$$\left(\frac{g(u_n) + f(u_n)}{u_n} - \frac{g(u_{n+1}) + f(u_{n+1})}{u_{n+1}}\right)(x_0) + \left(\frac{|\nabla u_n|^a}{u_n} - \frac{|\nabla u_{n+1}|^a}{u_{n+1}}\right)(x_0) \geqslant 0. \tag{2.18}$$

Since $t \mapsto \frac{g(t)+f(t)}{t}$ is decreasing on $(0, \infty)$ and $u_n(x_0) > u_{n+1}(x_0)$, from (2.18) we obtain

$$\left(\frac{|\nabla u_n|^a}{u_n} - \frac{|\nabla u_{n+1}|^a}{u_{n+1}}\right)(x_0) > 0. \tag{2.19}$$

On the other hand, $\nabla\zeta(x_0) = 0$ implies

$$u_{n+1}(x_0)\nabla u_n(x_0) = u_n(x_0)\nabla u_{n+1}(x_0).$$

Furthermore, relation (2.19) leads us to

$$u_n^{a-1}(x_0) - u_{n+1}^{a-1}(x_0) > 0,$$

which is a contradiction since $0 < a < 1$. Hence $u_n \leqslant u_{n+1}$ in B_n which means that

$$0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant u_{n+1} \leqslant \cdots \quad \text{in } \mathbb{R}^N.$$

The main point is to find an upper bound for the sequence $(u_n)_{n\geqslant 1}$. This is provided in the following result.

Lemma 2.2. *The inequality problem*

$$\begin{cases} -\Delta v \geqslant p(x)\big(g(v) + f(v) + |\nabla v|^a\big) & \text{in } \mathbb{R}^N, \\ v > 0 & \text{in } \mathbb{R}^N, \\ v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{2.20}$$

has at least one solution in $C^2(\mathbb{R}^N)$.

Proof. Set

$$\Phi(r) = r^{1-N} \int_0^r t^{N-1}\phi(t)\,dt, \quad \text{for all } r > 0.$$

Using the assumption (1.4) and l'Hôpital's rule, we get $\lim_{r\to\infty}\Phi(r)=\lim_{r\searrow 0}\Phi(r)=0$. Thus, Φ is bounded on $(0,\infty)$ and it can be extended in the origin by taking $\Phi(0)=0$. On the other hand, we have

$$\begin{aligned}\int_0^r \Phi(t)\,dt &= -\frac{1}{N-2}\int_0^r \frac{d}{dt}\big(t^{2-N}\big)\int_0^t s^{N-1}\phi(s)\,ds\,dt\\ &= -\frac{1}{N-2}\left[r^{N-2}\int_0^r s^{N-1}\phi(s)\,ds-\int_0^r t\phi(t)\,dt\right]\\ &= \frac{1}{N-2}\,\frac{-\int_0^r s^{N-1}\phi(s)\,ds+r^{N-2}\int_0^r t\phi(t)\,dt}{r^{N-2}}.\end{aligned}$$

By l'Hôpital's rule we obtain

$$\lim_{r\to\infty}\frac{-\int_0^r s^{N-1}\phi(s)\,ds+r^{N-2}\int_0^r t\phi(t)\,dt}{r^{N-2}}=\int_0^\infty r\phi(r)\,dr. \tag{2.21}$$

The last two relations imply

$$\int_0^\infty \Phi(r)\,dr=\lim_{r\to\infty}\int_0^r \Phi(t)\,dt=\frac{1}{N-2}\int_0^\infty r\phi(r)\,dr<\infty.$$

Let $k>2$ be such that

$$k^{1-a}\geqslant 2\max_{r\geqslant 0}\Phi^a(r). \tag{2.22}$$

In view of (2.21) we can define

$$\xi(x)=k\int_{|x|}^\infty \Phi(t)\,dt,\quad \text{for all } x\in\mathbb{R}^N.$$

Then ξ satisfies

$$\begin{cases}-\Delta\xi=k\phi\big(|x|\big) & \text{in } \mathbb{R}^N,\\ \xi>0 & \text{in } \mathbb{R}^N,\\ \xi(x)\to 0 & \text{as } |x|\to\infty.\end{cases}$$

Since the mapping $[0,\infty)\ni t\mapsto \int_0^t \frac{1}{g(s)+1}\,ds\in[0,\infty)$ is bijective, we can implicitly define $w:\mathbb{R}^N\to(0,\infty)$ by

$$\int_0^{w(x)}\frac{1}{g(t)+1}\,dt=\xi(x),\quad \text{for all } x\in\mathbb{R}^N.$$

It is easy to see that $w\in C^2(\mathbb{R}^N)$ and $w(x)\to 0$ as $|x|\to\infty$. Furthermore, we have

$$|\nabla w|=|\nabla\xi|\big(g(w)+1\big)=k\Phi\big(|x|\big)\big(g(w)+1\big)\quad \text{in } \mathbb{R}^N, \tag{2.23}$$

and

$$
\begin{aligned}
-\Delta w &= -\big(g(w)+1\big)\Delta\xi - g'(w)\big(g(w)+1\big)|\nabla\xi|^2 \geqslant k\phi\big(|x|\big)\big(g(w)+1\big) \\
&\geqslant \phi\big(|x|\big)\big(g(w)+1\big) + \frac{1}{2}k\phi\big(|x|\big)\big(g(w)+1\big).
\end{aligned}
$$

By (2.22) and (2.23) we deduce

$$
\frac{k}{2}\phi\big(|x|\big)\big(g(w)+1\big) \geqslant \phi\big(|x|\big)k^a\big(g(w)+1\big)^a\Phi^a\big(|x|\big) \geqslant p(x)|\nabla w|^a \quad \text{in } \mathbb{R}^N.
$$

Hence

$$
\begin{cases}
-\Delta w \geqslant p(x)\big(g(w)+1+|\nabla w|^a\big) & \text{in } \mathbb{R}^N,\\
w > 0 & \text{in } \mathbb{R}^N,\\
w(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\tag{2.24}
$$

Using the assumption (f1), we can find $M > 1$ large enough such that $M > f(Mw)$ in $\mathbb{R}^N$. Multiplying by M in (2.24) we deduce that $v := Mw$ satisfies (2.20) and the proof of Lemma 2.2 is now complete. □

Proof of Theorem 1.1 (concluded). With the same proof as above we deduce that $u_n \leqslant v$ in B_n, for all $n \geqslant 1$. This implies

$$
0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant v \quad \text{in } \mathbb{R}^N.
$$

Thus, there exists $u(x) = \lim_{n\to\infty} u_n(x)$, for all $x \in \mathbb{R}^N$ and $u_n \leqslant u \leqslant v$ in $\mathbb{R}^N$. Since $v(x) \to 0$ as $|x| \to \infty$, we deduce that $u(x) \to 0$ as $|x| \to \infty$. A standard bootstrap argument (see Gilbarg and Trudinger [16]) implies that $u_n \to u$ in $C^{2,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ and that u is a solution of problem (1.1).

This finishes the proof of Theorem 1.1. □

Acknowledgments

The authors have been partially supported by grants CNCSIS 308/2006 and GAR 12/2006.

References

[1] R. Agarwal, D. O'Regan, Existence theory for single and multiple solutions to singular positone boundary value problems, J. Differential Equations 175 (2001) 393–414.
[2] M. Aizenman, B. Simon, Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure Appl. Math. 35 (1982) 209–273.
[3] D. Arcoya, Positive solutions for semilinear Dirichlet problems in an annulus, J. Differential Equations 94 (1991) 217–227.
[4] P. Bénilan, H. Brezis, M. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, Ann. Sc. Norm. Super. Pisa 4 (1975) 523–555.
[5] A. Callegari, A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math. 38 (1980) 275–281.
[6] F. Cîrstea, M. Ghergu, V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type, J. Math. Pures Appl. 84 (2005) 493–508.
[7] F. Cîrstea, V. Rădulescu, Existence and uniqueness of positive solutions to a semilinear elliptic problem in $\mathbb{R}^N$, J. Math. Anal. Appl. 229 (1999) 417–425.
[8] M.M. Coclite, G. Palmieri, On a singular nonlinear Dirichlet problem, Comm. Partial Differential Equations 14 (1989) 1315–1327.
[9] M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (1977) 193–222.

[10] J.I. Díaz, J.M. Morel, L. Oswald, An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations 12 (1987) 1333–1344.
[11] R.H. Fowler, The solution of Emden's and similar differential equations, Monthly Notices Roy. Astro. Soc. 91 (1930) 63–91.
[12] M. Ghergu, V. Rădulescu, Sublinear singular elliptic problems with two parameters, J. Differential Equations 195 (2003) 520–536.
[13] M. Ghergu, V. Rădulescu, Bifurcation for a class of singular elliptic problems with quadratic convection term, C. R. Acad. Sci. Paris Ser. I 338 (2004) 831–836.
[14] M. Ghergu, V. Rădulescu, Multiparameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with convection term, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 61–84.
[15] M. Ghergu, V. Rădulescu, On a class of sublinear singular elliptic problems with convection term, J. Math. Anal. Appl. 311 (2005) 635–646.
[16] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, Berlin, 1983.
[17] J.K. Hale, J. Mawhin, Coincidence degree and periodic solutions of neutral equations, J. Differential Equations 15 (1974) 295–307.
[18] A.V. Lair, A.W. Shaker, Classical and weak solutions of a singular semilinear elliptic problem, J. Math. Anal. Appl. 211 (1997) 371–385.
[19] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, New York, 1989.
[20] Y. Nato, S. Tanaka, On the existence of multiple solutions of the boundary value problems for nonlinear second order differential equations, Nonlinear Anal. 56 (2004) 919–935.
[21] J. Shi, M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998) 1389–1401.
[22] Y. Sun, S. Li, Structure of ground state solutions of singular semilinear elliptic equation, Nonlinear Anal. 55 (2003) 399–417.
[23] H. Wang, On the existence of positive solutions for semilinear elliptic equations in annulus, J. Differential Equations 109 (1994) 1–7.
[24] J.S.W. Wong, On the generalized Emden–Fowler equation, SIAM Rev. 17 (1975) 339–360.
[25] Z. Zhang, Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term, Nonlinear Anal. 27 (1996) 957–961.