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A Multiplicity Theorem for Locally Lipschitz Periodic Functionals

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We prove in this paper a multiplicity theorem of the Ljusternik–Schnirelmann type for locally Lipschitz periodic funcitonals and related results. The key argument in our proofs is Ekeland's variational principle and a non-smooth pseudo-gradient lemma. As an application of these abstract results we solve a non-linear set-valued elliptic problem.

INTRODUCTION

In PDE, two important tools for proving existence of solutions are the *mountain-pass theorem* of Ambrosetti and Rabinowitz (and its various generalizations) and the Ljusternik–Schnirelmann theorem. These results apply to the case when the solutions of the given problem are critical points of an appropriate energy functional f, which is supposed to be real and C^1, or even differentiable, on a real Banach space X. One may ask what happens if f, which often is associated to the original equation in a canonical way, fails to be C^1 or differentiable. In this case the gradient of f must be replaced by a generalized one, in a sense which is to be defined.

The first approach is due to Chang [8] and Aubin and Clarke [2], who considered the case of a locally Lipschitz function f. For such functions, Clarke [11] defined a generalized gradient, which coincides to the usual ones if f is C^1 or convex. Still denoting this generalized gradient by ∂f, critical points of f are all points x such that $0 \in \partial f(x)$. In this setting, Chang

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[8] proved a version of the mountain pass lemma, in the case when X is reflexive. For this aim, he used a "Lipschitz version" of the deformation lemma. The same result was used for the proof of the Ljusternik–Schnirelmann theorem in the Lipschitz case. As observed by Brézis, the reflexivity assumption on X is not necessary.

Our main result is a multiplicity theorem for locally Lipschitz periodic functionals, their set of periods being a discrete subgroup of the space where they are defined. This result can be regarded as a Ljusternik–Schnirelmann type theorem for non-differentiable functionals.

Following [8], authors usually impose measurability conditions to some *a priori* unknown functions in order to be able to find ∂f. We first show that these conditions are automatically fulfilled and then we prove the existence of critical points, which are shown to be solutions of a multivalued PDE.

1. The Theoretical Setting

Throughout, X will be a real Banach space. Let X^* be its dual and $\langle x^*, x\rangle$, for $x \in X$, $x \in X^*$, denote the duality pairing between X^* and X. We say that a function $f: X \to \mathbb{R}$ is locally Lipschitz ($f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$) if, for each $x \in X$, there is a neighbourhood V of x and a constant $k = k(V)$ depending on V such that

$$|f(y) - f(z)| \leq k\|y - z\|$$

for each $y, z \in V$.

We recall in what follows the definition of the Clarke subdifferential and some of its most important properties (see [10] for details).

For each $x, v \in X$, we define the generalized directional derivative at x in the direction v of a given $f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Then $f^0(x, v)$ is a finite number of $|f^0(x, v)| \leq k\|v\|$. The mapping $v \mapsto f^0(x, v)$ is positively homogeneous and subadditive, hence convex continuous. The generalized gradient (the Clarke subdifferential) of f at x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*; f^0(x, v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\}.$$

If f is convex, $\partial f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis. The fundamental properties of the Clarke subdifferential are:

(a) For each $x \in X$, $\partial f(x)$ is a nonempty convex weak-$*$ compact subset of X^*.

(b) For each $x, v \in X$, we have

$$f^0(x, v) = \max\{\langle x^*, v\rangle; x^* \in \partial f(x)\}.$$

(c) The set-valued mapping $x \mapsto \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X$, $\varepsilon > 0$, $v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\|x - x_0\| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

(d) The function $f^0(\cdot, \cdot)$ is upper semi-continuous.

(e) If f achieves a local minimum or maximum at x, then $0 \in \partial f(x)$.

(f) The function

$$\lambda(x) = \min_{x^* \in \partial f(x)} \|x^*\|$$

exists and is lower semi-continuous.

(g) Lebourg's mean value theorem. If x and y are distinct points in X, then there is a point z in the open segment between x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x\rangle.$$

Definition 1. *A point $u \in X$ is said to be a critical point of $f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ if $0 \in \partial f(u)$, namely $f^0(u, v) \geq 0$ for every $v \in X$. A real number c is called a critical value of f if there is a critical point $u \in X$ such that $f(u) = c$.*

Definition 2. *If $f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ and c is a real number, we say that f satisfies the Palais–Smale condition at the level c (in short $(\mathrm{PS})_c$) if any sequence (x_n) in X with the properties $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence. The function f is said to satisfy the Palais–Smale condition (in short (PS)) if each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lim_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence.*

Let Z be a discrete subgroup of X, that is,

$$\inf_{z \in Z \setminus \{0\}} \|z\| > 0.$$

A function $f: X \to \mathbb{R}$ is said to be *Z-periodic* if $f(x + z) = f(x)$, for every $x \in X$ and $z \in Z$.

If $f \in \text{Lip}_{\text{loc}}(X, \mathbb{R})$ is Z-periodic, then $x \mapsto f^0(x, v)$ is Z-periodic for all $v \in X$ and ∂f is Z-invariant; that is, $\partial f(x + z) = \partial f(x)$ for every $x \in X$ and $z \in Z$. This implies that λ inherits the Z-periodicity property.

If $\pi: X \to X/Z$ is the canonical surjection and x is a critical point of f, then $\pi^{-1}(\pi(x))$ contains only critical points. Such a set is called a *critical orbit* of f. Note that X/Z is a complete metric space endowed with the metric

$$d(\pi(x), \pi(y)) = \inf_{z \in Z} \|x - y - z\|.$$

DEFINITION 3. *A locally Lipschitz Z-periodic function $f: X \to \mathbb{R}$ is said to satisfy the* $(\text{PS})_Z$*-condition provided that, for each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, then $(\pi(x_n))$ is relatively compact in X/Z. If c is a real number, then f is said to satisfy the* $(\text{PS})_{Z,c}$*-condition if, for any sequence (x_n) in X such that $f(x_n) \to c$ and $\lambda(x_n) \to 0$, there is a convergent subsequence of $(\pi(x_n))$.*

Denote $\text{Cr}(f, c)$ the set of critical points of the locally Lipschitz function $f: X \to \mathbb{R}$ at the level $c \in \mathbb{R}$; that is,

$$\text{Cr}(f, c) = \{x \in X; f(x) = c \quad \text{and} \quad \lambda(x) = 0\}.$$

2. THE MAIN RESULT

THEOREM 1. *Let $f: X \to \mathbb{R}$ be a bounded below locally Lipschitz Z-periodic function with the* $(\text{PS})_Z$*-property. Then f has at least $n + 1$ distinct critical orbits, where n is the dimension of the vector space generated by the discrete subgroup Z.*

Before beginning the proof, we shall recall the notion of *category* and some of its properties, which will be required by the proof of the main result.

A topological space X is said to be *contractible* if the identity of X is homotopical to a constant map; that is, there exist $u_0 \in X$ and a continuous map $F: [0, 1] \times X \to X$ such that

$$F(0, \cdot) = \text{id}_X \qquad \text{and} \qquad F(1, \cdot) = u_0.$$

A subset M of X is said to be *contractible in X* if there exist $u_0 \in X$ and a continuous map $F: [0, 1] \times M \to X$ such that

$$F(0, \cdot) = \text{id}_M \qquad \text{and} \qquad F(1, \cdot) = u_0.$$

If A is a subset of X, we define the category of A in X as

$\mathrm{Cat}_X(A) = 0$, if $A = \varnothing$.

$\mathrm{Cat}_X(A) = n$, if n is the smallest integer such that A can be covered by n closed sets which are contractible in X.

$\mathrm{Cat}_X(A) = \infty$, otherwise.

LEMMA 1. *Let A and B be subsets of X. Then the following hold:*

(i) *If $A \subset B$, then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

(ii) $\mathrm{Cat}_X(A \cup B) \le \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$.

(iii) *Let $h: [0, 1] \times A \to X$ be a continuous mapping such that $h(0, x) = x$ for every $x \in A$. If A is closed and $b = h(1, A)$, then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

(iv) *If n is the dimension of the vector space generated by the discrete group Z, then, for each $1 \le i \le n + 1$, the set*

$$\mathcal{A}_i = \{A \subset X;\ A \text{ is compact and } \mathrm{Cat}_{\pi(X)}\pi(A) \ge i\}$$

is nonempty. Obviously, $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_{n+1}$.

The only nontrivial part is (iv), which can be found in [19].

The following two lemmas are proved in [26].

LEMMA 2. *For each $1 \le j \le n + 1$, the space $\mathcal{A}_i$ endowed with the Hausdorff metric*

$$\delta(A, B) = \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\}$$

is a complete metric space.

LEMMA 3. *If $1 \le i \le n + 1$ and $f \in C(X, \mathbb{R})$, then the function $\eta: \mathcal{A}_i \to \mathbb{R}$ defined by*

$$\eta(A) = \max_{x \in A} f(x)$$

is lower semi-continuous.

If n is the dimension of the vector space generated by the discrete group Z, one sets for each $1 \le i \le n + 1$,

$$c_i = \inf_{A \in \mathcal{A}_i} \eta(A).$$

For each $c \in \mathbb{R}$ we denote $[f \le c] = \{x \in X; f(x) \le c\}$.

3. Proof of Theorem 1

It follows from Lemma 1 (iv) and the lower boundedness of f that

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_{n+1} < +\infty.$$

It is sufficient to show that, if $1 \leq i \leq j \leq n + 1$ and $c_i = c_j = c$, then the set $\mathrm{Cr}(f, c)$ contains at least $j - i + 1$ distinct critical orbits. We argue by contradiction and suppose that, for some $i \leq j$ such that $c_i = c_j = c$, the set $\mathrm{Cr}(f, c)$ has $k \leq j - i$ distinct critical orbits, generated by $x_1, ..., x_k \in X$. We construct first an open neighbourhood of $\mathrm{Cr}(f, c)$ of the form

$$V_r = \bigcup_{l=1}^{k} \bigcup_{z \in Z} B(x_l + z, r).$$

Moreover, we may suppose that $r > 0$ is chosen such that π is one-to-one on $\overline{B}(x_l, 2r)$. This condition ensures that $\mathrm{Cat}_{\pi(X)}(\pi(\overline{B}(x_l, 2r))) = 1$, for each $l = 1, ..., k$. Here $V_r = \emptyset$ if $k = 0$.

Step 1. We prove that there exists $0 < \varepsilon < \min\{\frac{1}{4}, r\}$ such that, for each $x \in [c - \varepsilon \leq f \leq c + \varepsilon]\backslash V_r$, one has

$$\lambda(x) > \sqrt{\varepsilon}. \tag{1}$$

Indeed, if not, there is a sequence (x_m) in $X\backslash V_r$ such that, for each $m \geq 1$,

$$c - \frac{1}{m} \leq f(x_m) \leq c + \frac{1}{m}, \qquad \lambda(x_m) \leq \frac{1}{\sqrt{m}}.$$

Since f satisfies $(\mathrm{PS})_Z$, it follows that, up to a subsequence, $\pi(x_m) \to \pi(x)$ as $m \to \infty$, for some $x \in X\backslash V_r$. By the Z-periodicity of f and λ, we can assume that $x_m \to x$ as $m \to \infty$. The continuity of f and the lower semi-continuity of λ imply $f(x) = c$ and $\lambda(x) = 0$, which is a contradiction, since $x \in X\backslash V_r$.

Step 2. For ε found above and according to the definition of c_j, there exists $\mathcal{A} \in \mathcal{A}_j$ such that

$$\max_{x \in A} f(x) < c + \varepsilon^2.$$

Setting $B = A \setminus V_{2r}$, we get by Lemma 1 that

$$\begin{aligned} j \le \operatorname{Cat}_{\pi(X)}(\pi(A)) &\le \operatorname{Cat}_{\pi(X)}(\pi(B) \cup \pi(\overline{V}_{2r})) \\ &\le \operatorname{Cat}_{\pi(X)}(\pi(B)) + \operatorname{Cat}_{\pi(X)}(\pi(\overline{V}_{2r})) \\ &\le \operatorname{Cat}_{\pi(X)}(\pi(B)) + k \le \operatorname{Cat}_{\pi(X)}(\pi(B)) + j - i. \end{aligned}$$

Hence, $\operatorname{Cat}_{\pi(X)}(\pi(B)) \ge i$; that is $B \in \mathcal{A}_i$.

Step 3. For ε and B as above we apply Ekeland's principle to the functional η defined in Lemma 3. It follows that there exists $C \in \mathcal{A}_i$ such that, for each $D \in \mathcal{A}_i$, $D \neq C$,

$$\begin{aligned} \eta(C) \le \eta(B) \le \eta(A) \le c + \varepsilon^2, \\ \delta(B, C) \le \varepsilon, \\ \eta(D) > \eta(C) - \varepsilon\delta(C, D). \end{aligned} \tag{2}$$

Since $B \cap V_{2r} = \varnothing$ and $\delta(B, C) \le \varepsilon < r$, it follows that $C \cap V_r = \varnothing$. In particular, the set $F = [c - \varepsilon \le f] \cap C$ is contained in $[c - \varepsilon \le f \le c + \varepsilon]$ and $F \cap V_r = \varnothing$.

LEMMA 4. *Let M be a compact metric space and let $\varphi: M \to 2^{X^*}$ be a set-valued mapping which is upper semi-continuous (in the sense of* (c)) *and with weak-* compact convex values. For $t \in M$ denote*

$$\gamma(t) = \inf\{\|x^*\|;\ x^* \in \varphi(t)\}$$

and

$$\gamma = \inf_{t \in M} \gamma(t).$$

Then, given $\varepsilon > 0$, there exists a continuous function $v: M \to X$ such that for all $t \in M$ and $x^* \in \varphi(t)$,

$$\|v(t)\| \le 1, \qquad \langle x^*, v(t)\rangle \ge \gamma - \varepsilon.$$

Proof of Lemma. We may suppose $\gamma > 0$ and $0 < \varepsilon < \gamma$. If B_r denotes the open ball in X^* centered at 0 with radius r, then, for each $t \in M$, one has

$$B_{\gamma-\varepsilon/2} \cap \varphi(t) = \varnothing.$$

Since $\varphi(t)$ and $B_{\gamma-\varepsilon/2}$ are convex, weak-* compact, and disjoint, it follows from the Theorem 3.4 in [24], applied to the space $(X^*, \sigma(X^*, X))$ and from the fact that the dual space of the above one is X: for every $t \in M$, there is some $v_t \in X$, $\|v_t\| = 1$, such that

$$\langle \xi, v_t \rangle \leq \langle x^*, v_t \rangle,$$

for each $\xi \in B_{\gamma-\varepsilon/2}$ and $x^* \in \varphi(t)$. Therefore, for each $x^* \in \varphi(t)$,

$$\langle x^*, v_t \rangle \geq \sup_{\xi \in B_{\gamma-\varepsilon/2}} \langle \xi, v_t \rangle = \gamma - \varepsilon/2.$$

Because of the upper semi-continuity of φ, there is an open neighbourhood $V(t)$ of t such that, for each $t' \in V(t)$ and each $x^* \in \varphi(t')$,

$$\langle x^*, v_t \rangle > \gamma - \varepsilon.$$

Since M is compact and $M = \bigcup_{t \in M} V(t)$, we can find a finite subcovering $\{V_1, \ldots, V_n\}$ of M. Let $v_1, \ldots, v_n$ be on the unit sphere of X such that $\langle x^*, v_i \rangle > \gamma - \varepsilon$, for all $1 \leq i \leq n$, $t \in V_i$ and $x^* \in \varphi(t)$.

If $\rho_i(t) = \operatorname{dist}(t, \partial V_i)$, define

$$\zeta_i(t) = \frac{\rho_i(t)}{\sum_{j=1}^{n} \rho_j(t)}, \qquad v(t) = \sum_{i=1}^{n} \zeta_i(t) v_i.$$

The function v is the desired mapping. ∎

Applying Lemma 4 to $\varphi = \partial f$ on F, we find a continuous map $v: F \to X$ such that, for all $x \in F$ and $x^* \in \partial f(x)$,

$$\|v(x)\| \leq 1, \qquad \langle x^*, v(x) \rangle \geq \inf_{x \in F} \lambda(x) - \varepsilon \geq \inf_{x \in C} \lambda(x) - \varepsilon \geq \sqrt{\varepsilon} - \varepsilon,$$

where the last inequality is justified by (1).

It follows that, for each $x \in F$ and $x^* \in \partial f(x)$,

$$f^0(x, -v(x)) = \max_{x^* \in \partial f(x)} \langle x^*, -v(x) \rangle = -\min_{x^* \in \partial f(x)} \langle x^*, v(x) \rangle \leq \varepsilon - \sqrt{\varepsilon} < -\varepsilon,$$

from our choice of ε.

From the upper semi-continuity of f^0 and the compactness of F, there exists $\delta > 0$ such that if $x \in F$, $y \in X$, $\|y - x\| \leq \delta$, then

$$f^0(y, -v(x)) < -\varepsilon. \tag{3}$$

Since $C \cap \mathrm{Cr}(f, c) = \varnothing$ and C is compact, while $\mathrm{Cr}(f, c)$ is closed, there is a continuous extension $w: X \to X$ of v such that $w|_{\mathrm{Cr}(f,c)} = 0$ and $\|w(x)\| \leq 1$, for all $x \in X$.

Let $\alpha: X \to [0, 1]$ be a continuous Z-periodic function such that $\alpha = 1$ on $[f \geq c]$ and $\alpha = 0$ on $[f \leq c - \varepsilon]$. Let $h:[0, 1] \times X \to X$ be the continuous mapping defined by

$$h(t, x) = x - t\delta\alpha(x)w(x).$$

If $D = h(1, C)$, it follows from Lemma 1 that

$$\mathrm{Cat}_{\pi(X)}(\pi(D)) \geq \mathrm{Cat}_{\pi(X)}(\pi(C)) \geq i$$

which shows that $D \in \mathcal{A}_i$, since D is compact.

Step 4. By Lebourg's mean value theorem we get that, for each $x \in X$, there exists $\theta \in (0, 1)$ such that

$$f(h(1, x)) - f(h(0, x)) \in \langle \partial f(h(\theta, x)), -\delta\alpha(x)w(x)\rangle.$$

Hence, there is some $x^* \in \partial f(h(\theta, x))$ such that

$$f(h(1, x)) - f(h(0, x)) = \alpha(x)\langle x^*, -\delta\omega(x)\rangle.$$

It follows by (3) that, if $x \in F$, then

$$\begin{aligned} f(h(1,x)) - f(h(0,x)) &= \delta\alpha(x)\langle x^*, -\omega(x)\rangle \\ &\leq \delta\alpha(x)f^0(x - \theta\delta\alpha(x)w(x), -v(x)\rangle \leq -\varepsilon\delta\alpha(x). \end{aligned} \tag{4}$$

It follows that, for each $x \in C$,

$$f(h(1, x)) \leq f(x).$$

Let $x_0 \in C$ be such that $f(h(1, x_0)) = \eta(D)$. Hence,

$$c \leq f(h(1, x_0)) \leq f(x_0).$$

By the definition of α and F, it follows that $\alpha(x_0) = 1$ and $x_0 \in F$. Therefore, by (4), we get

$$f(h(1, x_0)) - f(x_0) \leq -\varepsilon\delta.$$

Thus,

$$\eta(D) + \varepsilon\delta \leq f(x_0) \leq \eta(C). \tag{5}$$

Taking into account the definition of D, it follows that

$$\delta(C, D) \leq \delta.$$

Therefore,

$$\eta(D) + \varepsilon\delta(C, D) \leq \eta(C),$$

so that (2) implies $C = D$, which contradicts (5). ▌

4. A Multivalued Generalized Version of the Forced-Pendulum Problem

As an application of the above results, we shall study the periodic multivalued problem of the forced-pendulum

$$\begin{aligned} &x''(t) + f(t) \in [\underline{g}(x(t)), \overline{g}(x(t))], \qquad \text{a.e. } t \in (0,1) \\ &x(0) = x(1), \end{aligned} \tag{6}$$

where

$$f \in L^p(0, 1) \qquad \text{for some } p \geq 1, \tag{7}$$

$$g \in L^\infty(\mathbb{R}), \qquad g(u + T) = g(u) \qquad \text{for some } T > 0, \text{ a.e. } u \in \mathbb{R}, \tag{8}$$

$$\underline{g}(u) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{g(v); |u - v| < \varepsilon\} \; \overline{g}(u) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup}\{g(v); |u - v| < \varepsilon\},$$

$$\int_0^T g(u)\, du = \int_0^1 f(t)\, dt = 0. \tag{9}$$

We shall prove the following.

THEOREM 2. *If f, g are as above, then the problem* (6) *has at least two solutions in*

$$X := H^1_{\text{per}}(0, 1) = \{x \in H^1(0, 1); x(0) = x(1)\},$$

which are distinct in the sense that their difference is not an integer multiple of T.

Define the function ψ in $L^\infty(0, 1)$ by

$$\psi(x) = \int_0^1 \left(\int_0^{x(s)} g(u)\, du \right) ds.$$

It is obvious that ψ is a Lipschitz map on $L^\infty(0, 1)$. Let $G(u) = \int_0^u g(v)\, dv$. The following results show that the description of $\partial\psi$ given in [8] holds without further assumptions on g.

LEMMA 5. *Let g be a locally bounded measurable function defined on* $\mathbb{R}$ *and* $\underline{g}, \overline{g}$ *as above. Then the Clarke subdifferential of G is given by*

$$\partial G(u) = [\underline{g}(u), \overline{g}(u)] \qquad \textit{for every } u \in \mathbb{R}.$$

Proof. The required equality is equivalent to $G^0(u, 1) = \overline{g}(u)$ and $G^0(u, -1) = \underline{g}(u)$. As a matter of fact, examining the definitions of G^0, $\overline{g}$, and $\underline{g}$, it follows that $\underline{g}(u) = -(\overline{-g})(u)$ and $G^0(u, -1) = -(-G)^0(u, 1)$, so that the second required equality is equivalent to the first one.

Now the inequality $G^0(u, 1) \le \overline{g}(u)$ can be found in [8], so we have only to prove that $G^0(u, 1) \ge \overline{g}(u)$. Suppose that $G^0(u, 1) = \overline{g}(u) - \varepsilon$ for some $\varepsilon > 0$. Let $\delta > 0$ be such that

$$\frac{G(\tau + \lambda) - G(\tau)}{\lambda} < \overline{g}(u) - \frac{\varepsilon}{2},$$

if $|\tau - u| < \delta$ and $0 < \lambda < \delta$. Then

$$\frac{1}{\lambda} \int_\tau^{\tau+\lambda} g(s)\, ds < \overline{g}(u) - \frac{\varepsilon}{2} \qquad \text{if } |\tau - u| < \delta, \lambda > 0. \tag{10}$$

We claim that there exist $\lambda_n \searrow 0$ such that

$$\frac{1}{\lambda_n} \int_\tau^{\tau+\lambda_n} g(s)\, ds \to g(\tau) \qquad \text{a.e. } \tau \in (u - \delta, u + \delta). \tag{11}$$

Suppose for the moment that (11) has already been proved. Now (10) and (11) show that

$$g(\tau) \leq \overline{g}(u) - \frac{\varepsilon}{2} \qquad \text{if } \tau \in (u - \delta, u + \delta),$$

so we obtain the contradictory inequalities

$$\overline{g}(u) \leq \operatorname{ess\,sup}\{g(s); \qquad s \in [u - \delta, u + \delta]\} \leq \overline{g}(u) - \varepsilon/2.$$

All that remains to be proved is (11). Note that we may cut g in order to suppose that $g \in L^\infty \cap L^1$. Then (11) is nothing but the classical fact that for each $\varphi \in L^1(\mathbb{R})$,

$$T_\lambda(\varphi) \to \varphi \qquad \text{as } \lambda \searrow 0, \tag{12}$$

where

$$T_\lambda\varphi(u) = \frac{1}{\lambda}\int_u^{u+\lambda} \varphi(s)\, ds \qquad \text{for } \lambda > 0, u \in \mathbb{R}, \varphi \in L^1(\mathbb{R}).$$

Indeed, it can be easily seen that T_λ is linear and continuous in $L^1(\mathbb{R})$ and $\lim_{\lambda \searrow 0} T_\lambda\varphi = \varphi$ in $\mathcal{D}(\mathbb{R})$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Now (12) follows by a density argument. ∎

Returning to our problem, it follows by Theorem 2.1 in [8] that

$$\partial\psi_{|H_0^1(\Omega)}(x) \subset \partial\psi(x). \tag{13}$$

In order to obtain information on $\partial\psi$, we shall need an improvement of the Theorem 2.1 in [8].

THEOREM 3. *If* $x \in L^\infty(0, 1)$, *then*

$$\partial\psi(x)(t) \subset [\underline{g}(x(t)), \overline{g}(x(t))] \qquad a.e.\ t \in (0, 1),$$

in the sense that if $w \in \partial\psi(x)$ *then*

$$\underline{g}(x(t)) \leq w(t) \leq \overline{g}(x(t)) \qquad a.e.\ t \in (0, 1) \tag{14}$$

Proof. Let h be a Borel function such that $h = g$ a.e. on $\mathbb{R}$. It follows that the set

$$A = \{t \in (0, 1); \underline{g}(x(t)) \neq \underline{h}(x(t))\}$$

is a null set. (A similar reasoning can be done for $\overline{g}$ and $\overline{h}$).

Therefore we may suppose that g is a Borel function. We would like to deal with $\int_0^1 \overline{g}(x(t))\,dt$, so we have to prove that $\overline{g}$ is a Borel function.

LEMMA 6. *Let $g: \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function. Then $\overline{g}$ is a Borel function.*

Proof. Since the requirement is local, we may suppose that g is bounded by 1, for example, and it is nonnegative. Since

$$\overline{g} = \lim_{n\to\infty} \lim_{m\to\infty} g_{m,n},$$

where

$$g_{m,n}(u) = \left(\int_{u-1/n}^{u+1/n} |g^m(s)|\,ds\right)^{1/m},$$

it suffices to prove that $g_{m,n}$ is Borel. Let

$$\mathcal{M} = \{g: \mathbb{R} \to \mathbb{R}; |g| \leq 1 \text{ and } g \text{ is a Borel function}\}$$
$$\mathcal{N} = \{g \in \mathcal{M}; g_{m,n} \text{ is a Borel function}\}.$$

It is known (see [3, p. 178]) that $\mathcal{M}$ is the smallest set of functions having the following properties:

(i) $\{g \in C(\mathbb{R}, \mathbb{R}); |g| \leq 1\} \subset \mathcal{M}$.

(ii) $g^{(k)} \in \mathcal{M}$ and $g^{(k)} \xrightarrow{k} g$ imply $g \in \mathcal{M}$. Note that here we have an "each point" convergence.

Since $\mathcal{N}$ contains obviously the continuous functions and (ii) is also true for $\mathcal{N}$, by the dominated convergence theorem, it follows that $\mathcal{M} = \mathcal{N}$. ∎

Proof of Theorem 3 *Continued.* Let $v \in L^\infty(0, 1)$, $v \geq 0$. Then, for suitable $\lambda_i \searrow 0$ and $h_i \to 0$ in $L^\infty(0, 1)$ one has

$$\psi^0(x, v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_0 \int_{x(t)+h_i(t)}^{x(t)+h_i(t)+\lambda_i v(t)} g(s)\,ds\,dt$$

We may suppose that $h_i \to 0$ a.e., so that

$$\begin{aligned}\psi^0(x, v) &= \lim_{i\to\infty} \frac{1}{\lambda_i} \int_{[v>0]} \int_{x(t)+h_i(t)}^{x(t)+h_i(t)+\lambda_i v(t)} g(s)\, ds\, dt \\ &\le \int_{[v>0]} \left(\limsup_{i\to\infty} \frac{1}{\lambda_i} \int_{x(t)+h_i(t)}^{x(t)+h_i(t)+\lambda_i v(t)} g(s)\, ds \right) dt \\ &\le \int_{[v>0]} \overline{g}(x(t)) v(t)\, dt,\end{aligned}$$

so that

$$\psi^0(x, v) \le \int_{[v>0]} \overline{g}(x(t)) v(t)\, dt \tag{15}$$

for such v.

Suppose now that (14) is false, that is, for example, there exist $\varepsilon > 0$, a set E with $|E| > 0$, and $w \in \partial\psi(x)$ such that

$$w(t) \ge \overline{g}(x(t)) + \varepsilon \qquad \text{on } E. \tag{16}$$

Now (15) with $v = \mathbf{1}_E$ shows that

$$\langle w, v\rangle = \int_E w \le \psi^0(x, v) \le \int_E \overline{g}(x(t))\, dt,$$

which contradicts (16). ∎

Proof of Theorem 2. Define on the space $X = H^1_{\mathrm{per}}(0, 1)$ the locally Lipschitz functional

$$\varphi(x) = \frac{1}{2} \int_0^1 x'^2(t)\, dt - \int_0^1 f(t) x(t)\, dt + \int_0^1 G(x(t))\, dt.$$

The critical points of φ are solutions of (6). Indeed, it is obvious that

$$\partial\varphi(x) = x'' + f - \partial\psi_{|H^1_{\mathrm{per}}(0,1)}(x) \qquad \text{in } H^{-1}(0, 1).$$

If x_0 is a critical point of φ, then there exists $w \in \partial\psi_{|H^1_{\mathrm{per}}(0,1)}(x_0)$ such that

$$x'' + f = w \qquad \text{in } H^{-1}(0, 1).$$

Since $\varphi(x+T) = \varphi(x)$, we are going to use Theorem 1. All we have to do is to verify the $(PS)_{Z,c}$ condition, for each c, and to prove that (6) has a solution x_0 that minimizes φ on $H^1_{\text{per}}(0,1)$. To do this, it suffices to show that φ is coercive. Note first that every $x \in H^1_{\text{per}}(0,1)$ can be written

$$x(t) = \int_0^1 x(s)\,ds + \bar{x}(t) \qquad \text{with } \bar{x} \in H^1_0(0,1).$$

Hence, by the Poincaré's inequality,

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\int_2^1 \bar{x}'^2(t)\,dt - \int_0^1 f(t)\bar{x}(t)\,dt + \int_0^1 G(x(t))\,dt \\ &\geq \frac{1}{2}\|\bar{x}'^2\|^2_{L^2} - \|f\|_{L^{p'}}\cdot\|\bar{x}\|_{L^p} - \|G\|_{L^\infty} \\ &\geq \frac{1}{2}\|\bar{x}'\|^2_{L^2} - C\|f\|_{L^{p'}}\cdot\|\bar{x}'\|_{L^2} - \|G\|_{L^\infty} \to \infty \qquad \text{as } \|\bar{x}\|_{H^1} \to \infty,\end{aligned}$$

where p' denotes the conjugated exponent of p.

We verify in what follows the $(PS)_{Z,c}$ condition, for each c. Let $(x_n) \subset X$ be such that

$$\varphi(x_n) \to c \tag{17}$$

$$\lambda(x_n) \to 0. \tag{18}$$

Let $w_n \in \partial\varphi(x_n) \subset L^\infty(0,1)$ (because $\underline{g} \circ x_n \leq w_n \leq \bar{g} \circ x_n$ and $\underline{g}, \bar{g} \in L^\infty(\mathbb{R})$) be such that

$$\lambda(x_n) = -x_n'' - f + w_n \to 0 \qquad \text{in } H^{-1}(0,1).$$

Then, multiplying (18) by x_n we get

$$\int_0^1 (x_n')^2 - \int_0^1 fx_n + \int_0^1 w_n x_n = o(1)\|x_n\|_{H^1_p}$$

and, by (17),

$$\frac{1}{2}\int_0^1 (x_n')^2 - \int_0^1 fx_n + \int_0^1 G(x_n) \to c,$$

so that there exist positive constants C_1, C_2 such that

$$\int_0^1 (x_n')^2 \le C_1 + C_2 \|x_n\|_{H_p^1}.$$

Note that G is also T-periodic; hence it is bounded.

Replacing x_n by $x_n + kT$ for a suitable integer k, we may suppose that

$$x_n(0) \in [0, T]$$

so that (x_n) is bounded in H^1_{per}.

Let $x \in H_p^1$ be such that, up to a subsequence, $x_n \rightharpoonup x$ and $x_n(0) \to x(0)$. Then

$$\int_0^1 (x_n')^2 = \langle -x_n'' - f + w_n, x_n - x\rangle + \int_0^1 w_n(x_n - x) - \int_0^1 f(x_n - x) + \int_0^1 x_n' x' \to \int_0^1 x'^2$$

because $x_n \to x$ in $L^{p'}$, where p' is the conjugated exponent of p. It follows that $x_n \to x$ in H_p^1. ∎

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