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Existence and Non-existence Results for a Quasilinear Problem with Nonlinear Boundary Condition

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We study the problem

$$-\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = \lambda(1+|x|)^{\alpha_1}|u|^{q-2}u - h(x)|u|^{r-2}u \qquad \text{in } \Omega \subset \mathbb{R}^N,$$
$$a(x)|\nabla u|^{p-2}\nabla u\cdot n + b(x)\cdot|u|^{p-2}u = \theta g(x,u) \qquad \text{on } \Gamma,$$
$$u \geq 0 \qquad \text{in } \Omega,$$

where Ω is an unbounded domain with smooth boundary Γ, n denotes the unit outward normal vector on Γ, and $\lambda > 0$, θ are real parameters. We assume throughout that $p < q < r < p^* = \frac{pN}{N-p}$, $1 < p < N$, $-N < \alpha_1 < q\cdot\frac{N-p}{p} - N$, while a, b, and h are positive functions. We show that there exist an open interval I and $\lambda^* > 0$ such that the problem has no solution if $\theta \in I$ and $\lambda \in (0, \lambda^*)$. Furthermore, there exist an open interval $J \subset I$ and $\lambda_0 > 0$ such that, for any $\theta \in J$, the above problem has at least a solution if $\lambda \geq \lambda_0$, but it has no solution provided that $\lambda \in (0, \lambda_0)$. Our paper extends previous results obtained by J. Chabrowski and K. Pflüger.

1. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ. We assume throughout this paper that p, q, r, and α_1 are real numbers satisfying

$$1 < p < N, \qquad p < q < r < p^* := \frac{pN}{N-p},$$
$$-N < \alpha_1 < q\cdot\frac{N-p}{p} - N. \tag{1}$$

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Denote by $C^\infty_\delta(\Omega)$ the space of $C^\infty_0(\mathbb{R}^N)$-functions restricted to Ω. We define the weighted Sobolev space E as the completion of $C^\infty_\delta(\Omega)$ in the norm

$$\|u\|_E = \left(\int_\Omega |\nabla u(x)|^p + \frac{1}{(1+|x|)^p}|u(x)|^p\,dx\right)^{1/p}.$$

Denote by $L^q(\Omega; w_1)$ and $L^m(\Gamma; w_2)$ the weighted Lebesgue spaces with weight functions

$$w_i(x) = (1+|x|)^{\alpha_i}, \qquad i = 1,2, \qquad \alpha_i \in \mathbb{R} \tag{2}$$

and norms defined by

$$\|u\|^q_{q,w_1} = \int_\Omega w_1|u(x)|^q\,dx \quad \text{and} \quad \|u\|^m_{m,w_2} = \int_\Gamma w_2|u(x)|^m\,d\Gamma.$$

The following embedding and trace result holds.

PROPOSITION 1. *Assume* (1) *holds. Then the embedding* $E \subset L^q(\Omega; w_1)$ *is compact. If*

$$p \le m \le p\cdot\frac{N-1}{N-p} \quad \textit{and} \quad -N < \alpha_2 \le m\cdot\frac{N-p}{p} - N + 1, \tag{3}$$

then the trace operator $E \to L^m(\Gamma; w_2)$ *is continuous. If the upper bounds for* m *in* (3) *are strict, then the trace is compact.*

This proposition is a consequence of Theorem 2 and Corollary 6 of [4].
We assume throughout that $a \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma)$ such that

$$a(x) \ge a_0 > 0 \qquad \text{for a.e. } x \in \Omega \tag{4}$$

and

$$\frac{c}{(1+|x|)^{p-1}} \le b(x) \le \frac{C}{(1+|x|)^{p-1}},$$

$$\text{for a.e. } x \in \Gamma, \text{ where } c, C > 0. \tag{5}$$

LEMMA 1. *The quantity*

$$\|u\|^p_b = \int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,d\Gamma$$

defines an equivalent norm on E.

For the proof of this result we refer to [3, Lemma 2].

Let $h\colon \Omega \to \mathbb{R}$ be a positive and continuous function satisfying

$$\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx < \infty. \tag{6}$$

We assume that $g\colon \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the following conditions:

(g1) $g(\cdot, 0) = 0$, $g(x,s) + g(x,-s) \geq 0$ for a.e. $x \in \Gamma$ and for any $s \in \mathbb{R}$;

(g2) $|g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1}$, $p \leq m < p \cdot \frac{N-1}{N-p}$, where g_i are nonnegative, measurable functions such that

$$0 \leq g_i(x) \leq C_g w_2 \quad \text{a.e.,} \qquad g_0 \in L^{m/(m-1)}\big(\Gamma; w_2^{1/(1-m)}\big),$$

where $-N < \alpha_2 < m \cdot \frac{N-p}{p} - N + 1$ and w_2 is defined as in (2).

Let G be the primitive function of g with respect to the second variable. We denote by N_g, N_G the corresponding Nemytskii operators.

LEMMA 2. *The operators*

$$N_g\colon L^m(\Gamma; w_2) \to L^{m/(m-1)}\big(\Gamma; w_2^{1/(1-m)}\big), \qquad N_G\colon L^m(\Gamma; w_2) \to L^1(\Gamma)$$

are bounded and continuous.

Proof. Let $m' = m/(m-1)$ and $u \in L^m(\Gamma; w_2)$. Then, by (g2),

$$\begin{aligned}
&\int_\Gamma \big|N_g(u)\big|^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma \\
&\qquad \leq 2^{m'-1}\left(\int_\Gamma g_0^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma + \int_\Gamma g_1^{m'}|u|^m \cdot w_2^{1/(1-m)}\, d\Gamma\right) \\
&\qquad \leq 2^{m'-1}\left(C + C_g \cdot \int_\Gamma |u|^m \cdot w_2\, d\Gamma\right),
\end{aligned}$$

which shows that N_g is bounded. In a similar way we obtain

$$\begin{aligned}
\int_\Gamma |N_G(u)|\, d\Gamma &\leq \int_\Gamma g_0|u|\, d\Gamma + \int_\Gamma g_1|u|^m\, d\Gamma \\
&\leq \left(\int_\Gamma g_0^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma\right)^{1/m'} \cdot \left(\int_\Gamma |u|^m \cdot w_2\, d\Gamma\right)^{1/m} \\
&\qquad + C_g \cdot \int_\Gamma |u|^m \cdot w_2\, d\Gamma
\end{aligned}$$

and we claim that N_G is bounded.

From the usual properties of Nemytskii operators we deduce the continuity of these operators. ∎

Set

$$X = \left\{ u \in E : \int_\Omega h(x)|u|^r \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_X^p = \|u\|_b^p + \left(\int_\Omega h(x) |u(x)|^r \, dx \right)^{p/r}.$$

We observe that X is a Banach space.

Consider the problem

$$(1_{\lambda,\theta}) \begin{cases} -\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = \lambda(1+|x|)^{\alpha_1}|u|^{q-2}u - h(x)|u|^{r-2}u \\ \qquad \text{in } \Omega \subset \mathbb{R}^N, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)\cdot|u|^{p-2}u = \theta g(x,u) \\ \qquad \text{on } \Gamma, \\ u \geq 0 \qquad \text{in } \Omega. \end{cases}$$

The energy functional corresponding to $(1_{\lambda,\theta})$ is given by $\Phi\colon X \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u|^p \, dx + \frac{1}{p}\int_\Gamma b(x)|u|^p \, d\Gamma - \frac{\lambda}{q}\int_\Omega w_1|u|^q \, dx + \frac{1}{r}\int_\Omega h(x)|u|^r \, dx - \theta\int_\Gamma G(x,u) \, d\Gamma.$$

Proposition 1 shows that the embedding $E \subset L^q(\Omega; w_1)$ is continuous. This implies that the functional Φ is well defined. Solutions to problem $(1_{\lambda,\theta})$ will be found as critical points of Φ. Therefore, a function $u \in X$ is a solution of the problem $(1_{\lambda,\theta})$ provided that, for any $v \in X$,

$$\int_\Omega a|\nabla u|^{p-2}\nabla u \cdot \nabla v + \int_\Gamma b|u|^{p-2}uv = \lambda\int_\Omega w_1|u|^{q-2}uv - \int_\Omega h|u|^{r-2}uv + \theta\int_\Gamma gv.$$

2. MAIN RESULTS

THEOREM 1. *Assume hypotheses* (1), (4), (5), (6), (g1), *and* (g2) *hold. Then there exist real numbers* θ_*, θ^*, *and* $\lambda^* > 0$ *such that the problem*

$(1_{\lambda, \theta})$ *does not have a nontrivial solution, for any* $\theta_* < \theta < \theta^*$ *and* $0 < \lambda < \lambda^*$.

Proof. Suppose that u is a solution in X of $(1_{\lambda, \theta})$. Then u satisfies

$$\int_\Omega a(x)|\nabla u|^p \, dx + \int_\Gamma b(x)|u|^p \, d\Gamma - \theta \int_\Gamma g(x,u)u \, d\Gamma + \int_\Omega h(x)|u|^r \, dx = \lambda \int_\Omega w_1 |u|^q \, dx. \tag{7}$$

It follows from the Young inequality that

$$\lambda \int_\Omega w_1 |u|^q \, dx = \int_\Omega \frac{\lambda w_1}{h^{q/r}} \cdot h^{q/r} |u|^q \, dx \le \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx + \frac{q}{r} \int_\Omega h|u|^r \, dx.$$

This combined with (7) gives

$$\|u\|_b^p - \theta \int_\Gamma g(x,u)u \, d\Gamma \le \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx + \frac{q-r}{r} \int_\Omega h|u|^r \, dx \le \frac{r-q}{r} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx. \tag{8}$$

Set

$$A = \left\{ u \in X : \int_\Gamma g(x,u)u \, d\Gamma < 0 \right\},$$
$$B = \left\{ u \in X : \int_\Gamma g(x,u)u \, d\Gamma > 0 \right\} \tag{9}$$
$$\theta_* = \sup_{u \in A} \frac{\|u\|_b^p}{\int_\Gamma g(x,u)u \, d\Gamma}, \qquad \theta^* = \inf_{u \in B} \frac{\|u\|_b^p}{\int_\Gamma g(x,u)u \, d\Gamma}.$$

We introduce the convention that if $A = \emptyset$ then $\theta_* = -\infty$ and if $B = \emptyset$ then $\theta^* = +\infty$.

We show that if we take $\theta_* < \theta < \theta^*$ then there exists $C_0 > 0$ such that

$$C_0 \|u\|_b^p \le \|u\|_b^p - \theta \int_\Gamma g(x,u)u \, d\Gamma \qquad \text{for all } u \in X. \tag{10}$$

If $\theta < \theta^*$ then there exists a constant $C_1 \in (0,1)$ such that

$$\theta \le (1 - C_1)\theta^* \le (1 - C_1)\frac{\|u\|_b^p}{\int_\Gamma g(x,u)u\,d\Gamma} \qquad \text{for all } u \in B$$

which implies that

$$\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma \ge C_1\|u\|_b^p \qquad \text{for all } u \in B. \tag{11}$$

If $\theta_* < \theta$ then there exists a constant $C_2 \in (0,1)$ such that

$$(1 - C_2)\frac{\|u\|_b^p}{\int_\Gamma g(x,u)u\,d\Gamma} \le (1 - C_2)\theta_* \le \theta \qquad \text{for all } u \in A$$

which yields

$$\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma \ge C_2\|u\|_b^p \qquad \text{for all } u \in A. \tag{12}$$

From (11) and (12) we conclude that

$$\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma \ge \min\{C_1, C_2\}\|u\|_b^p \qquad \text{for all } u \in X$$

and taking $C_0 = \min\{C_1, C_2\}$ we obtain (10).

By (7), (10), and Proposition 1 we have

$$C_0\overline{C}\left(\int_\Omega w_1|u|^q\,dx\right)^{p/q} \le C_0\|u\|_b^p \le \lambda\int_\Omega w_1|u|^q\,dx, \tag{13}$$

for some constant $\overline{C} > 0$. This inequality implies

$$\left(\overline{C}\lambda^{-1}C_0\right)^{q/(q-p)} \le \int_\Omega w_1|u|^q\,dx$$

which combined with (13) leads to the inequality

$$C_0\overline{C}\left(\overline{C}\lambda^{-1}C_0\right)^{p/(q-p)} \le C_0\|u\|_b^p.$$

Combining this with (8) and (10) we obtain that

$$C_0\overline{C}\left(\overline{C}\lambda^{-1}C_0\right)^{p/(q-p)} \le \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx.$$

If we take

$$\lambda^* = \left(\left(C_0 \overline{C} \right)^{q/(q-p)} \frac{r}{r-q} \left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx \right)^{-1} \right)^{(r-q)(q-p)/q(r-p)}$$

the result follows. ∎

Set

$$U = \left\{ u \in X : \int_\Gamma G(x,u)\, d\Gamma < 0 \right\}, \qquad V = \left\{ u \in X : \int_\Gamma G(x,u)\, d\Gamma > 0 \right\}$$

$$\theta_- = \sup_{u \in U} \frac{\|u\|_b^p}{p \int_\Gamma G(x,u)\, d\Gamma}, \qquad \theta^+ = \inf_{u \in V} \frac{\|u\|_b^p}{p \int_\Gamma G(x,u)\, d\Gamma}. \tag{14}$$

If $U = \emptyset$ (resp. $V = \emptyset$) then we set $\theta_- = -\infty$ (resp. $\theta^+ = +\infty$). Proceeding in the same manner as we did for proving (10) we can show that if we take $\theta_- < \theta < \theta^+$ then there exists $c > 0$ such that

$$\frac{1}{p}\|u\|_b^p - \theta \int_\Gamma G(x,u)\, d\Gamma \geq c\|u\|_b^p \qquad \text{for all } u \in X. \tag{15}$$

In what follows, we shall employ the following elementary inequality: for every $h > 0$, $k > 0$, and $0 < \beta < \gamma$ we have

$$k|u|^\beta - h|u|^\gamma \leq C_{\beta,\gamma} k \left(\frac{k}{h} \right)^{\beta/(\gamma-\beta)} \tag{16}$$

for all $u \in \mathbb{R}$, where $C_{\beta,\gamma} > 0$ is a constant depending on β and γ.

PROPOSITION 2. *If $\theta_- < \theta < \theta^+$ then the functional Φ is coercive.*

Proof. By virtue of (16) we write the estimate

$$\begin{aligned} \int_\Omega \left(\frac{\lambda}{q}|u|^q w_1 - \frac{h}{2r}|u|^r \right) dx &\leq C_{r,q} \int_\Omega \lambda w_1 \left(\frac{\lambda w_1}{h} \right)^{q/(r-q)} dx \\ &= C_{r,q} \lambda^{r/(r-q)} \int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx. \end{aligned}$$

Using (15) it follows that

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|_b^p - \theta\int_\Gamma G(x,u)\,d\Gamma - \int_\Omega\left(\frac{\lambda}{q}|u|^q w_1 - \frac{h}{2r}|u|^r\right)dx \\ &\quad + \frac{1}{2r}\int_\Omega h|u|^r\,dx \\ &\geq c\|u\|_b^p + \frac{1}{2r}\int_\Omega h|u|^r\,dx - C_1\end{aligned}$$

and the coercivity follows. ∎

PROPOSITION 3. *Suppose $\theta_- < \theta < \theta^+$ and let $\{u_n\}$ be a sequence in X such that $\Phi(u_n)$ is bounded. Then there exists a subsequence of $\{u_n\}$, relabelled again by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in X and*

$$\Phi(u_0) \leq \liminf_{n\to\infty}\Phi(u_n).$$

Proof. Since Φ is coercive in X we see that the boundedness of $\Phi(u_n)$ implies that $\|u_n\|_b$ and $\int_\Omega h|u_n|^r\,dx$ are bounded. From Proposition 1 we have that the embedding $E \subset L^q(\Omega; w_1)$ is compact and using the fact that $\{u_n\}$ is bounded in E we may assume that $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ in $L^q(\Omega; w_1)$.

Set $F(x,u) = \frac{\lambda}{q}|u|^q w_1 - \frac{1}{r}h|u|^r$ and $f(x,u) = F_u(x,u)$.

A simple computation yields

$$\begin{aligned}f_u(x,u) &= (q-1)\lambda|u|^{q-2}w_1 - (r-1)h|u|^{r-2} \\ &\leq C_{r,q}\lambda w_1\left(\frac{\lambda w_1}{h}\right)^{(q-2)/(r-q)},\end{aligned} \tag{17}$$

where the last inequality follows from (16) and $C_{r,q} > 0$ is a constant depending only on r and q. We now use (17) to derive the estimate for $\Phi(u_0) - \Phi(u_n)$,

$$\begin{aligned}&\Phi(u_0) - \Phi(u_n) \\ &\quad = \frac{1}{p}\int_\Omega a(x)|\nabla u_0|^p\,dx + \frac{1}{p}\int_\Gamma b(x)|u_0|^p\,d\Gamma \\ &\qquad - \frac{1}{p}\int_\Omega a(x)|\nabla u_n|^p\,dx - \frac{1}{p}\int_\Gamma b(x)|u_n|^p\,d\Gamma\end{aligned}$$

$$
\begin{aligned}
&- \theta\int_\Gamma G(x,u_0)\,d\Gamma + \theta\int_\Gamma G(x,u_n)\,d\Gamma \\
&+ \int_\Omega (F(x,u_n) - F(x,u_0))\,dx \\
= {} & \frac{1}{p}(\|u_0\|_b^p - \|u_n\|_b^p) \\
&+ \theta\left(\int_\Gamma G(x,u_n)\,d\Gamma - \int_\Gamma G(x,u_0)\,d\Gamma\right) \\
&+ \int_\Omega \left(\int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0))\,dt\,ds\right) \\
&\quad \times (u_n - u_0)^2\,dx \\
\le {} & \frac{1}{p}(\|u_0\|_b^p - \|u_n\|_b^p) + \theta\left(\int_\Gamma G(x,u_n)\,d\Gamma - \int_\Gamma G(x,u_0)\,d\Gamma\right) \\
&+ C_2\int_\Omega (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}\,dx,
\end{aligned}
$$

where $C_2 = \frac{1}{2}C_{r,q}\lambda^{(r-2)/(r-q)}$. We show that the last integral tends to 0 as $n \to \infty$. Indeed, applying Hölder's inequality we obtain

$$
\int_\Omega (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}\,dx \le \left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx\right)^{(q-2)/q} \cdot \left(\int_\Omega w_1 |u_n - u_0|^q dx\right)^{2/q}.
$$

Since $u_n \to u_0$ in $L^q(\Omega; w_1)$ we obtain

$$
\lim_{n\to\infty} \int_\Omega (u_n - u_0)^2 \frac{w_1^{(r-2)/(r-q)}}{h^{(q-2)/(r-q)}}\,dx = 0. \tag{18}
$$

The compactness of the trace operator $E \to L^m(\Gamma; w_2)$ and the continuity of the Nemytskii operator $N_G\colon L^m(\Gamma; w_2) \to L^1(\Gamma)$ imply $N_G(u_n) \to N_G(u_0)$ in $L^1(\Gamma)$, i.e., $\int_\Gamma |N_G(u_n) - N_G(u_0)|\,d\Gamma \to 0$ as $n \to \infty$. It follows that

$$
\lim_{n\to\infty} \int_\Gamma G(x,u_n)\,d\Gamma = \int_\Gamma G(x,u_0)\,d\Gamma. \tag{19}
$$

Since the norm in E is lower semicontinuous with respect to the weak topology we deduce from (18) and (19) that

$$\Phi(u_0) \leq \liminf_{n\to\infty} \Phi(u_n).$$

▮

PROPOSITION 4. *If $\theta_* < \theta < \theta^*$ and u is a solution of problem $(1_{\lambda,\theta})$, then*

$$C_0\|u\|_b^p + \frac{r-q}{r}\int_\Omega h|u|^r\,dx \leq \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx$$

and

$$\|u\|_b \geq K\lambda^{-1/(q-p)},$$

where $K > 0$ is a constant independent of u.

Proof. If u is a solution of $(1_{\lambda,\theta})$ then

$$\|u\|_b^p - \theta\int_\Gamma g(x,u)u\,d\Gamma + \int_\Omega h|u|^r\,dx$$

$$= \lambda\int_\Omega w_1|u|^q\,dx$$

$$\leq \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\,dx + \frac{q}{r}\int_\Omega h|u|^r\,dx.$$

Using (10) we obtain the first part of the assertion.

From Proposition 1 we have that there exists $C_q > 0$ such that

$$\|u\|_{L^q(\Omega;w_1)}^q \leq C_1\|u\|_b^q, \qquad \text{for all } u \in E.$$

This inequality and (10) imply that

$$\|u\|_b \geq C_0^{1/(q-p)}C_q^{-1/(q-p)}\lambda^{-1/(q-p)}$$

and taking $K = C_0^{1/(q-p)}C_q^{-1/(q-p)}$ the second part follows. ▮

THEOREM 2. *Assume hypotheses* (1), (4), (5), (6), (g1), *and* (g2) *hold. Set* $\underline{\theta} = \max\{\theta_*, \theta_-\}$, $\overline{\theta} = \min\{\theta^*, \theta^+\}$, *and* $J = (\underline{\theta}, \overline{\theta})$. *There exists $\lambda_0 > 0$ such that the following hold*:

(i) *the problem $(1_{\lambda,\theta})$ admits a nontrivial solution, for any $\lambda \geq \lambda_0$ and every $\theta \in J$*;

(ii) *the problem $(1_{\lambda,\theta})$ does not have any nontrivial solution, provided that $0 < \lambda < \lambda_0$ and $\theta \in J$.*

Proof. According to Propositions 2 and 3, Φ is coercive and lower semicontinuous. Therefore there exists $\tilde{u} \in X$ such that $\Phi(\tilde{u}) = \inf_X \Phi(u)$. To ensure that $\tilde{u} \not\equiv 0$ we shall prove that $\inf_X \Phi < 0$. We set

$$\tilde{\lambda} := \inf\left\{ \frac{q}{p}\|u\|_b^p - q\theta\int_\Gamma G(x,u)\, d\Gamma + \frac{q}{r}\int_\Omega h|u|^r\, dx : u \in X, \right. \\ \left. \int_\Omega w_1|u|^q\, dx = 1 \right\}.$$

First we check that $\tilde{\lambda} > 0$. In order to prove that we consider the constrained minimization problem

$$M := \inf\left\{ \int_\Omega a(x)|\nabla u|^p\, dx + \int_\Gamma b(x)|u|^p\, d\Gamma : u \in E, \int_\Omega w_1|u|^q\, dx = 1 \right\}.$$

Clearly, $M > 0$. Since X is embedded in E, we have

$$\int_\Omega a(x)|\nabla u|^p\, dx + \int_\Gamma b(x)|u|^p\, d\Gamma \geq M$$

for all $u \in X$ with $\int_\Omega w_1|u|^q\, dx = 1$. Now, applying the Hölder inequality we find

$$1 = \int_\Omega w_1|u|^q\, dx = \int_\Omega \frac{w_1}{h^{q/r}} h^{q/r}|u|^q\, dx \\ \leq \left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx \right)^{(r-q)/r} \cdot \left(\int_\Omega h|u|^r\, dx \right)^{q/r}. \tag{20}$$

Relation (15) implies that

$$\frac{q}{p}\|u\|_b^p - q\theta\int_\Gamma G(x,u)\, d\Gamma \geq qc\|u\|_b^p.$$

By virtue of (20) we have

$$\frac{q}{p}\|u\|_b^p - q\theta\int_\Gamma G(x,u)\, d\Gamma + \frac{q}{r}\int_\Omega h|u|^r\, dx \\ \geq qc\|u\|_b^p + \frac{q}{r}\int_\Omega h|u|^r\, dx \\ \geq qcM + \frac{q}{r}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx \right)^{-(r-q)/q}$$

for all $u \in X$ with $\int_\Omega w_1 |u|^q \, dx = 1$. It follows that

$$\tilde{\lambda} \geq qcM + \frac{q}{r} \left(\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}} \, dx \right)^{-(r-q)/q}$$

and our claim follows.

Let $\lambda > \tilde{\lambda}$. Then there exists a function $u \in X$ with $\int_\Omega w_1 |u|^q \, dx = 1$ such that

$$\lambda > \frac{q}{p} \|u\|_b^p - q\theta \int_\Gamma G(x,u) \, d\Gamma + \frac{q}{r} \int_\Omega h|u|^r \, dx.$$

This can be rewritten as

$$\Phi(u) = \frac{1}{p} \|u\|_b^p - \theta \int_\Gamma G(x,u) \, d\Gamma + \frac{1}{r} \int_\Omega h|u|^r \, dx - \frac{\lambda}{q} \int_\Omega w_1 |u|^q \, dx < 0$$

and consequently $\inf_{u \in X} \Phi(u) < 0$. By Propositions 2 and 3 it follows that the problem $(1_{\lambda, \theta})$ has a solution.

We set

$$\lambda_0 = \inf\{\lambda > 0 : (1_{\lambda, \theta}) \text{ admits a solution}\}.$$

Suppose $\lambda_0 = 0$. Then taking $\lambda_1 \in (0, \lambda^*)$ (where λ^* is given by Theorem 1) we have that there is $\overline{\lambda}$ such that the problem $(1_{\overline{\lambda}, \theta})$ admits a solution. But this is a contradiction, according to Theorem 1. Consequently, $\lambda_0 > 0$.

We now show that for each $\lambda > \lambda_0$ problem $(1_{\lambda, \theta})$ admits a solution. Indeed, for every $\lambda > \lambda_0$ there exists $\rho \in (\lambda_0, \lambda)$ such that the problem $(1_{\rho, \theta})$ has a solution u_ρ which is a subsolution of problem $(1_{\lambda, \theta})$. We consider the variational problem

$$\inf\{\Phi(u) : u \in X \text{ and } u \geq u_\rho\}.$$

By Propositions 2 and 3 this problem admits a solution $\overline{u}$. This minimizer $\overline{u}$ is a solution of problem $(1_{\lambda, \theta})$. Since the hypothesis $g(x,s) + g(x,-s) \geq 0$ for a.e. $x \in \Gamma$ and for all $s \in \mathbb{R}$ implies that $G(x, |\overline{u}|) \geq G(x, \overline{u})$ (that is, $\Phi(|\overline{u}|) \leq \Phi(\overline{u})$) we may assume that $\overline{u} \geq 0$ on Ω. It remains to show that problem $(1_{\lambda_0, \theta})$ also has a solution. Let $\lambda_n \to \lambda_0$ and $\lambda_n > \lambda_0$ for each n. Problem $(1_{\lambda_n, \theta})$ has a solution u_n for each n. By Proposition 4 the sequence $\{u_n\}$ is bounded in X. Therefore we may assume that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^q(\Omega; w_1)$. We have that u_0 is a solution of $(1_{\lambda_0, \theta})$.

Since u_n and u_0 are solutions of $(1_{\lambda_n, \theta})$ and $(1_{\lambda_0, \theta})$, respectively, we have

$$\begin{aligned}
&\int_\Omega a(x)\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0\big)(\nabla u_n - \nabla u_0)\,dx \\
&\quad + \int_\Gamma b(x)\big(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\big)(u_n - u_0)\,d\Gamma \\
&\quad + \int_\Omega h\big(|u_n|^{r-2}u_n - |u_0|^{r-2}u_0\big)(u_n - u_0)\,dx \\
&\quad = \lambda_n \int_\Omega w_1\big(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0\big)(u_n - u_0)\,dx \\
&\qquad + (\lambda_n - \lambda_0)\int_\Omega w_1|u_0|^{q-2}u_0(u_n - u_0)\,dx \\
&\qquad + \theta\int_\Gamma (g(x,u_n) - g(x,u_0))(u_n - u_0)\,d\Gamma \\
&\quad = J_{1,n} + J_{2,n} + J_{3,n},
\end{aligned}$$

where

$$\begin{aligned}
J_{1,n} &= \lambda_n \int_\Omega w_1\big(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0\big)(u_n - u_0)\,dx, \\
J_{2,n} &= (\lambda_n - \lambda_0)\int_\Omega w_1|u_0|^{q-2}u_0(u_n - u_0)\,dx, \\
J_{3,n} &= \theta\int_\Gamma (g(x,u_n) - g(x,u_0))(u_n - u_0)\,d\Gamma.
\end{aligned}$$

We have

$$|J_{1,n}| \le \sup_{n\ge 1}\lambda_n\left(\int_\Omega w_1|u_n|^{q-1}|u_n - u_0|\,dx + \int_\Omega w_1|u_0|^{q-1}|u_n - u_0|\,dx\right)$$

and it follows from the Hölder inequality that

$$\begin{aligned}
|J_{1,n}| \le \sup_{n\ge 1}\lambda_n\Bigg[&\left(\int_\Omega w_1|u_n|^q\,dx\right)^{(q-1)/q}\cdot\left(\int_\Omega w_1|u_n - u_0|^q\,dx\right)^{1/q} \\
&+ \left(\int_\Omega w_1|u_0|^q\,dx\right)^{(q-1)/q}\cdot\left(\int_\Omega w_1|u_n - u_0|^q\,dx\right)^{1/q}\Bigg].
\end{aligned}$$

We easily observe that $J_{1,n} \to 0$ as $n \to \infty$.

From the estimate

$$|J_{2,n}| \le |\lambda_n - \lambda_0| \left(\int_\Omega w_1 |u_0|^q \, dx \right)^{(q-1)/q} \cdot \left(\int_\Omega w_1 |u_n - u_0|^q \, dx \right)^{1/q}$$

we obtain that $J_{2,n} \to 0$ as $n \to \infty$.

Using the compactness of the trace operator $E \to L^m(\Gamma; w_2)$, the continuity of Nemytskii operator N_g: $L^m(\Gamma; w_2) \to L^{m/(m-1)}(\Gamma; w_2^{1/(1-m)})$, and the estimate

$$\begin{aligned} &\int_\Gamma |g(x,u_n) - g(x,u_0)| \cdot |u_n - u_0| \, d\Gamma \\ &\quad \le \left(\int_\Gamma |g(x,u_n) - g(x,u_0)|^{m/(m-1)} w_2^{1/(1-m)} \, d\Gamma \right)^{(m-1)/m} \\ &\quad\quad \cdot \left(\int_\Gamma w_2 |u_n - u_0|^m \, d\Gamma \right)^{1/m} \end{aligned}$$

we see that $J_{3,n} \to 0$ as $n \to \infty$.

We have so proved that

$$\begin{aligned} \lim_{n\to\infty} \bigg(&\int_\Omega a(x) \big(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0 \big) (\nabla u_n - \nabla u_0) \, dx \\ &+ \int_\Gamma b(x) \big(|u_n|^{p-2} u_n - |u_0|^{p-2} u_0 \big)(u_n - u_0) \, d\Gamma \bigg) = 0. \end{aligned}$$

Now we apply the following inequality for $\xi, \zeta \in \mathbb{R}^N$ (see [2, Lemma 4.10])

$$|\xi - \zeta|^p \le C\big(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta \big)(\xi - \zeta), \qquad \text{for } p \ge 2.$$

Then we obtain

$$\begin{aligned} \|u_n - u_0\|_b^p &= \int_\Omega a(x) |\nabla u_n - \nabla u_0|^p \, dx + \int_\Gamma b(x) |u_n - u_0|^p \, dx \\ &\le C \bigg(\int_\Omega a(x) \big(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0 \big)(\nabla u_n - \nabla u_0) \, dx \\ &\quad + \int_\Gamma b(x) \big(|u_n|^{p-2} u_n - |u_0|^{p-2} u_0 \big)(u_n - u_0) \, d\Gamma \bigg) \to 0 \\ &\qquad \text{as } n \to \infty \end{aligned}$$

which shows that $\|u_n\|_b \to \|u_0\|_b$ and, by Proposition 4, $u_0 \not\equiv 0$. In the case $1 < p < 2$ we obtain the same conclusion, by using the corresponding inequality (see [2, Lemma 4.10])

$$|\xi - \zeta|^2 \leq C\big(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\big)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p},$$

for any $\xi, \zeta \in \mathbb{R}^N$. This concludes our proof. ∎

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