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Another look at planar Schrödinger-Newton systems ☆

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Abstract

In this paper, we focus on the existence of positive solutions to the following planar Schrödinger-Newton system with general subcritical growth

$$\begin{cases} -\Delta u + u + \phi u = f(u) & \text{in } \mathbb{R}^2, \\ \Delta \phi = u^2 & \text{in } \mathbb{R}^2, \end{cases}$$

where f is a smooth reaction. We introduce a new variational approach, which enables us to study the above problem in the Sobolev space $H^1(\mathbb{R}^2)$. The analysis developed in this paper also allows to investigate the relationship between a Schrödinger-Newton system of Riesz-type and a Schrödinger-Newton system of logarithmic-type. Furthermore, this new approach can provide a new look at the planar Schrödinger-Newton system and may it have some potential applications in various related problems.

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1. Introduction and results

1.1. Overview

The Schrödinger-Newton system was intensively studied by Penrose [41] and it can be used to describe the quantum mechanics of a polaron at rest. Penrose also derived these equations as a model of self-gravitating matter, in which quantum state reduction is understood as a gravitational phenomenon. The underlying idea is that a linear superposition of two quantum states would give rise to two space-time geometries, which poses serious conceptual problems from the viewpoint of general relativity; see Penrose [40]. Penrose thus suggested that the collapse of the wave function might be related to gravitational effects, and proposed the (stationary) Schrödinger-Newton system as a possible candidate for such gravitationally-induced collapse. The Schrödinger-Newton system was also used by Choquard to describe an electron trapped in its own hole in a certain approximating to Hartree-Fock theory of one component plasma; see Lieb [29].

Consider the following nonlinear Schrödinger-Newton system

$$\begin{cases} i\hbar\dfrac{\partial\psi}{\partial t} = \dfrac{\hbar^2}{2m}\Delta\psi - \phi\psi + |\psi|^{p-2}\psi & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ -\Delta\phi = \lambda|\psi|^2 & \text{in } \mathbb{R}^d \times \mathbb{R}, \end{cases} \tag{1.1}$$

where $\lambda = 1$ if $d \geq 3$, $\lambda = -1$ if $d = 2$, i is the imaginary unit, $\hbar$ is the Planck constant. For $d = 3$, $m > 0$ stands for the mass of the particle, $\psi : \mathbb{R}^3 \times [0, T] \to \mathbb{C}$ is a wave function and such a system often appears in quantum mechanics models and semiconductor theory (see [31,32]) and also arises, for example, as a model of the interaction of a charged particle with the electrostatic field (see [5,8]).

The Schrödinger-Newton system is a usual Schrödinger equation coupled with a Newtonian potential, satisfying the Poisson equation and representing the interaction of the particle with its own gravitational field. In the literature, the Schrödinger-Newton system is also referred to as the Schrödinger-Poisson system. One of most interesting questions about problem (1.1) concerns the existence of stationary solutions $\psi(x,t) = u(x)e^{-\frac{iEt}{\hbar}}$ (for any $(x,t) \in \mathbb{R}^d \times \mathbb{R}$), where $u : \mathbb{R}^d \to \mathbb{R}$ is a real function to be found. Thus, u must solve

$$\begin{cases} -\dfrac{\hbar^2}{2m}\Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^d, \\ -\Delta\phi = \lambda u^2 & \text{in } \mathbb{R}^d. \end{cases} \tag{1.2}$$

The second equation in system (1.2) is called the Poisson equation, which can be solved by

$$\phi(x) = \lambda\Gamma_d(x) * u^2(x) = \lambda\int_{\mathbb{R}^d} \Gamma_d(x-y)u^2(y)\mathrm{d}y,$$

where Γ_d is the Newtonian kernel in dimension d, and is expressed by

$$\Gamma_d(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x|}, & d = 2, \\ \frac{1}{d(d-2)\omega_d} |x|^{2-d}, & d \geq 3. \end{cases}$$

Here, ω_d is the volume of the unit d-ball.

Under such a formal inversion of the second equation in system (1.2), we obtain the following non-local equation

$$-\Delta u + V(x)u + \lambda(\Gamma_d * |u|^2)u = f(u) \quad \text{in } \mathbb{R}^d. \tag{1.3}$$

Formally, problem (1.3) has a variational structure with the associated energy functional

$$I_d(u) = \frac{1}{2} \int_{\mathbb{R}^d} \left(|\nabla u|^2 + V(x)u^2 \right) \mathrm{d}x + \frac{\lambda}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_d \left(|x-y|^2 \right) u^2(x)u^2(y) \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^d} F(u)\mathrm{d}x,$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

In the case $d = 3$, I_d is well defined and of class C^1 in $H^1(\mathbb{R}^d)$, provided that $V \in L^\infty(\mathbb{R}^d)$. In the literature, by applying variational and topological methods, the existence, nonexistence, multiplicity and concentrating results of (1.3) have been investigated when f and V satisfy various assumptions, see e.g. [4,7,8,17,26,28,33,34,42,45] and the references therein. However, compared with the higher dimensional case $d \geq 3$, in dimension two, the fundamental solution $\Gamma_2(x) = \frac{1}{2\pi} \ln |x|$ is sign-changing and presents singularities at zero and infinity. So, the two-dimensional case seems much more delicate. In particular, unlike the higher dimensional cases, in the planar case $d = 2$, the corresponding energy functional is not well-defined on $H^1(\mathbb{R}^2)$. Precisely, the energy functional I_2 involves a convolution term

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,$$

which is not well defined for all $u \in H^1(\mathbb{R}^2)$. Therefore, the approaches dealing with higher dimensional cases seem difficult to be adapted to the case $d = 2$. So, the rigorous study of the planar Schrödinger-Newton system had remained open for a long time. This is why much less is known in the case $d = 2$. Recall that Choquard, Stubbe and Vuffray [22] proved the existence of a unique positive radially symmetric solution to (1.3) with $d = 2$ and $f(x, u) = 0$ by applying a shooting method. In [43], Stubbe introduced a variational framework for (1.3) with $d = 2$ and $V(x) \equiv 1$ by setting a weighted Sobolev space

$$X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(1 + |x|)|u(x)|^2 \mathrm{d}x < +\infty \right\},$$

endowed with the norm

$$\|u\|_X^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x + \int_{\mathbb{R}^2} \ln(1+|x|)|u(x)|^2 \mathrm{d}x,$$

which yields that the associated energy functional is well-defined and continuously differentiable on the space X. One of the main ingredients in [43] is to set a suitable working space for the planar Schrödinger-Newton system. Formally, one has the following decomposition

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality [30], for any $u \in H^1(\mathbb{R}^2)$, the quantity

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y$$

is finite if

$$\int_{\mathbb{R}^2} \ln(1+|x|)u^2(x)\mathrm{d}x$$

is finite. When $f(u) = 0$ and $d = 2$ in (1.3), Stubbe used strict rearrangement inequalities to prove that (1.3) has a unique ground state solution which is a positive and spherically symmetric decreasing function. In studying the planar Schrödinger-Newton system in the underlying space X, one of main obstacles is that the norm $\|\cdot\|_X$ lacks translation invariance. This makes problems tough in verifying the compactness via the concentration-compactness principle. This difficulty can be overcome via the symmetric bilinear form

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\, u(x)v(y)\mathrm{d}x\mathrm{d}y.$$

When $f(u) = |u|^{p-2}u$ and $d = 2$ in problem (1.3), a sequence of higher energy solutions are obtained by Cingolani and Weth [23] for $p \geq 4$ in a periodic setting, where the corresponding energy functional is invariant under $\mathbb{Z}^2$-translations. For $p \in (2, 4)$, we refer to Du and Weth [24]. Under the above variational framework in [43], Chen and Tang in [20] considered the planar Schrödinger-Newton system in the axially symmetric setting. By using Jeanjean's monotonicity trick [27] and a Nehari-Pohozaev manifold, they proved that there exists at least a ground state solution to (1.3), see also [19]. When $f(u)$ satisfies some suitable super-quadratic growth assumptions, Chen, Shi and Tang [18] proved that (1.3) has a nontrivial solution of mountain-pass type and a ground state solution of Nehari-Pohozaev type. Very recently, Cao, Dai and Zhang [14] considered the following Schrödinger-Newton equations in the two dimensional case

$$-\Delta u + a(x)u + \frac{\gamma}{2\pi}\left(\ln(|\cdot|) * |u|^p\right)|u|^{p-2}u = b|u|^{q-2}u \quad \text{in } \mathbb{R}^2,$$

where $\inf_{\mathbb{R}^2} a > 0$, $\gamma > 0$, $b \geq 0$, $p \geq 2$ and $q \geq 2$. They introduced the following working space

$$X := \left\{ u \in H^1\left(\mathbb{R}^2\right) : \int_{\mathbb{R}^2} \ln(1+|x|)|u|^p(x)dx < \infty \right\}, \tag{1.4}$$

and obtained the existence of ground state solutions and mountain pass solutions to the above equations for $p \geq 2$ and $q \geq 2p-2$ via variational methods. For some other related works to the two dimensional case, see [2,3,6,10,12,21,44] and so on.

In all the results mentioned above for the planar Schrödinger-Newton system, we emphasize that the weighted function space X plays a fundamental role in ensuring that the energy functional is well defined and continuously differentiable. A natural question for us is

Whether does there exist another variational framework to deal with problem (1.3) with $d = 2$?

Particularly, of much interest is that ones can investigate the non-local equation (1.3) by applying the variational approaches in the standard Sobolev space $H^1(\mathbb{R}^2)$. The main purpose of this paper is to give this question an affirmative answer. Our study can be seen as a new approach different from the works mentioned above for the two dimensional case.

1.2. Main results

In this paper, we aim to establish a novel variational approach to study the existence of positive solutions to the following Schrödinger-Newton equation with general nonlinear growth

$$-\Delta u + u + (\ln(|\cdot|) * |u|^2)u = f(u) \quad \text{in } \mathbb{R}^2. \tag{1.5}$$

Throughout this paper, in order to find positive solutions of equation (1.5), we assume that $f \in C^1(\mathbb{R}, \mathbb{R}^+)$, $f(s) \equiv 0$ for $s \leq 0$ and $f(s) > 0$ for $s > 0$. Assume that f satisfies the following hypotheses.

(f_1) For every $\theta > 0$, there exists $C_\theta > 0$ such that $|f(s)| \leq C_\theta \min\{1, s\}e^{\theta s^2}$ for any $s > 0$.
(f_2) The function $\frac{f(s)}{s^3}$ is nondecreasing for $s > 0$.

Remark 1.1. It follows from conditions (f_2) that $0 < 4F(s) \leq f(s)s$ for $s > 0$, where $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$. Moreover, (f_2) implies that $f(s) = o(s)$ as $s \to 0^+$.

Remark 1.2. As a prototype of (f_1)-(f_2), $f(s) = |s|^{p-2}s$, $p > 4$.

The first main result in this paper establishes the following property.

Theorem 1.3. *Assume (f_1)-(f_2) hold, equation (1.5) has at least a positive solution $u \in H^1(\mathbb{R}^2)$ satisfying*

$$\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y \right| < +\infty. \tag{1.6}$$

In the following, we also consider the case containing $p \in [3, 4)$ as well as $p \in (2, 3)$. Precisely,

Theorem 1.4. *Assume that the following hypotheses hold*

(f_3) $f(s) = o(s)$ *as* $s \to 0$.
(f_4) $|f(s)| \le C(1 + |s|^{p-1})$ *for* $p \in (3, +\infty)$.
(f_5) *There exists* $\mu \in (3, 4)$ *such that* $0 < \mu F(s) \le f(s)s$ *for* $s > 0$.

Then equation (1.5) has at least a positive solution $u \in H_r^1(\mathbb{R}^2)$ *satisfying (1.6), where* $H_r^1(\mathbb{R}^2)$ *denotes the subspace of radially symmetric functions in* $H^1(\mathbb{R}^2)$.

Theorem 1.5. *Equation (1.5) with* $f(u) = \varrho u^2$ *has at least a positive solution* $u \in H_r^1(\mathbb{R}^2)$ *satisfying (1.6).*

Theorem 1.6. *Assume* (f_3) *and the following hypothesis hold*

(f_6) $|f(s)| \le C(1 + |s|^{p-1})$ *for* $p \in (2, 3)$.
(f_7) *There exists* $\mu > 2$ *such that* $0 < \mu F(s) \le f(s)s$ *for* $s > 0$.

Then equation (1.5) has at least a positive solution $u \in H_r^1(\mathbb{R}^2)$ *satisfying (1.6).*

1.3. Main difficulty and strategy

In this present paper, our main aim is to investigate problem (1.5) in the standard Sobolev space $H^1(\mathbb{R}^2)$ by variational methods. The main difficulty is due to the sign-changing property of the Newtonian kernel $\Gamma_d(x) = \frac{1}{2\pi} \ln |x|$, which leads to failure in setting the variational framework in $H^1(\mathbb{R}^2)$. In order to overcome this difficulty, motivated by the fact that

$$\lim_{\alpha \to 0^+} G_\alpha(x) := \lim_{\alpha \to 0^+} \frac{|x|^{-\alpha} - 1}{\alpha} = -\ln |x|$$

for $x \in \mathbb{R}^2 \setminus \{0\}$, we modified equation (1.5) as follows

$$-\Delta u + u - (G_\alpha(|\cdot|) * u^2)u = f(u) \quad \text{in } \mathbb{R}^2, \tag{1.7}$$

where $\alpha \in (0, 1)$ is a parameter. The corresponding energy functional to (1.7) becomes to be well defined in $H^1(\mathbb{R}^2)$ for fixed $\alpha \in (0, 1)$, which enables us to use critical point theory (such as the well-known mountain pass theorem) to study the existence of positive solutions for (1.7) in $H^1(\mathbb{R}^2)$. By passing to the limit, a convergence argument within $H^1(\mathbb{R}^2)$ allows us to get positive solutions of the original equation (1.5).

Remark 1.7. Similar modification was also used in [47], where Z.-Q. Wang and C. Zhang reveal an interesting relation between power-law nonlinear scalar field equations

$$-\Delta u + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{1.8}$$

and logarithmic-law scalar field equations

$$-\Delta u = \lambda u \ln(|u|) \quad \text{in } \mathbb{R}^N. \tag{1.9}$$

They show that as $p \downarrow 2$, the unique positive solution of (1.8), after a unique re-scaling, converges to the unique (up to translations) positive solution $U(x) = e^{N/2}e^{-\lambda|x|^2/4}$ of (1.9).

In the limit process above as $\alpha \to 0^+$, one of main difficulties in the current paper is the lack of compactness and the appearance of singularity at $\alpha = 0$. Here we point out that in [47], term $\int_{\mathbb{R}^N} u^2 \ln|u| \mathrm{d}x$ can be estimated by the associated logarithmic Sobolev inequality (see [30]). Besides, uniqueness of positive solutions of (1.8) and (1.9) also plays an crucial role in getting compactness as $p \downarrow 2$. However, as $\alpha \to 0^+$, it is completely different from [47] that, on the one hand, compared with the logarithmic Schrödinger equation (1.9), it seems much difficult to use the associated logarithmic Sobolev inequality to give an upper bound estimate for

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.$$

Hence, energy estimates of the associated functional become more complicated. On the other hand, (1.5) and (1.7) are nothing but the original problem and the perturbation problem respectively investigated in the present paper, and uniqueness of their solutions (even for existence) is unknown. So it seems impossible that one can get the compactness with the help of the logarithmic Sobolev inequality or the information of uniqueness to the perturbation and limit problems similarly as in [47].

In order to overcome their obstacles, in the proof of Theorem 1.3, we firstly use the concentration-compactness principle to prove the modified equation (1.7) has a positive mountain pass solution u_α. Moreover, the mountain pass value c_α is uniformly bounded from below and above as $\alpha \to 0^+$. We then use the moving plane method to show that u_α is radially symmetric and strictly decreasing with respect to the origin. With these conclusions at hand, we employ Moser's iteration argument to verify u_α is uniformly bounded in the L^∞-norm and exponentially decay uniformly with respect to the parameter α, which, together with the Lebesgue dominated convergence theorem, enables us to get the Frechet derivative of the corresponding energy functional is weakly sequence continuous and then get compactness.

As for the proofs of Theorem 1.4-1.6, apart from the lack of compactness and the singularity on the parameter α, another difficulty arises in verifying the boundedness of the Palais-Smale sequences, even though the well-known *Ambrosetti-Rabinowitz* condition (f_7) is involved. Similarly to Schrödinger-Newton systems in the three dimensional case, due to the convolution term $G_\alpha(|\cdot|) * u^2$, it is not easy to show Palais-Smale sequences are bounded if $\mu < 4$ in (f_7). In the literature, Jeanjean's monotonicity trick [27] is a powerful tool, which was frequently used to obtain bounded Palais-Smale sequences. However, it seems not valid in this case. Because of the singularity at $\alpha = 0$, it seems no hopeful to give a uniform upper bound of the mountain pass value as $\alpha \to 0^+$. To bypass this obstacle, we adopt a perturbation argument introduced in [34] (see also [36]) by inserting a nonlocal perturbation term

$$\lambda\left(\int_{\mathbb{R}^2} u^2 \mathrm{d}x\right)^{\frac{1}{4}} u$$

in the left side of the modified equation (1.7) and a higher order perturbation term $\lambda|u|^{r-2}u$ in the right side of (1.7). These two additional perturbation terms can guarantee the boundedness of Palais-Smale sequences. Then as $\lambda, \alpha \to 0^+$, a limit argument yields one solution of the original problem (1.5). We note that, in these cases, we need to restrict the energy functional in the radial function space $H_r^1(\mathbb{R}^2)$, which is useful in proving uniformly exponential decay of solutions on the parameter α.

This paper is organized as follows. Firstly, some preliminaries are given in Section 2, and Section 3 is devoted to the existence of mountain pass type solutions to the modified equation. Then in Section 4, we use a variant of the moving plane method to prove the symmetry of positive solutions to the modified equation. In Section 5–8, we complete the proofs of Theorem 1.3, 1.4-1.6.

2. Preliminary results

Let us fix some notations. The letter C will be repeatedly used to denote various positive constants, whose exact values are irrelevant. For every $1 \le s \le +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^2)$. The function space

$$H^1(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2)\}$$

is the usual Sobolev space endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

The formal variational functional associated with (1.5) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^4} \ln(|x-y|)u^2(y)u^2(x)\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^2} F(u)\mathrm{d}x. \tag{2.1}$$

In what follows, we recall the Hardy-Littlewood-Sobolev inequality, see [30], which will be frequently used throughout this paper.

Lemma 2.1. *(Hardy-Littlewood-Sobolev inequality [30]) Let $s, r > 1$ and $\alpha \in (0, d)$ with $1/s + \alpha/d + 1/r = 2$, $f \in L^s(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$. There exists a sharp constant $C_{s,N,\alpha,r}$ independent of f, h, such that*

$$\int_{\mathbb{R}^d} \left[\frac{1}{|x|^\alpha} * f(x)\right] h(x)\mathrm{d}x \le C_{s,d,\alpha,r}\|f\|_s\|h\|_r.$$

If $r = s = \frac{2d}{2d-\alpha}$, then

$$C_{s,d,\alpha,r} = C_{d,\alpha} = \pi^{\alpha/2} \frac{\Gamma(\frac{d}{2} - \frac{\alpha}{2})}{\Gamma(d - \frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \right\}^{-1+\frac{\alpha}{d}},$$

and if $d = 2, \alpha \in (0, 1]$*, then* $C_{2,\alpha} \le 2\sqrt{\pi}$.

Lemma 2.2. *(Moser-Trudinger inequality [1,13]) For any* $\beta \in (0, 4\pi)$*, there exists* $C = C_\beta > 0$ *such that for every* $u \in H^1(\mathbb{R}^2)$ *satisfying* $\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x \le 1$*, one has*

$$\int_{\mathbb{R}^2} \min\{1, u^2\} e^{\beta|u|^2} \mathrm{d}x \le C_\beta \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x.$$

Lemma 2.3. *The following conclusions hold.*

(1) *Assume* $u \in H^1(\mathbb{R}^2)$*, then for fixed* $\lambda \in (0, 1]$*, we have*

$$\int_{\mathbb{R}^2} \frac{u^2(y)}{|x-y|^\lambda} \mathrm{d}y \to 0, \quad \text{as } |x| \to +\infty.$$

(2) *There exists* $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ *such that* $u_n \rightharpoonup u_0$ *in* $H_r^1(\mathbb{R}^2)$*, then as* $|x| \to \infty$*, one has*

$$\int_{|x-y|\le 1} \frac{u_n^2(y)}{|x-y|} \mathrm{d}y \to 0$$

uniformly for n.

Proof. (1) Consider $|x| \ge 1$. We note that

$$\int_{\mathbb{R}^2} \frac{u^2(y)}{|x-y|^\lambda} \mathrm{d}y - \frac{\|u\|_2^2}{|x|^\lambda} = \int_{\mathbb{R}^2} \left(\frac{1}{|x-y|^\lambda} - \frac{1}{|x|^\lambda} \right) u^2(y) \mathrm{d}y.$$

Define $f(x, y) := \frac{1}{|x-y|^\lambda} - \frac{1}{|x|^\lambda}$, $x, y \in \mathbb{R}^2$ and $x \ne y, x \ne 0$. Observe that $f(x, y) \to 0$ as $|x| \to \infty$ for every $y \in \mathbb{R}^2$. Moreover,

$$-1 \le f(x, y) 1_{|x-y|\ge\frac{1}{2}}(y) \le 2^\lambda \quad \text{for } x, y \in \mathbb{R}^2 \text{ with } |x| \ge 1.$$

Since $u \in H^1(\mathbb{R}^2)$, it follows from Lebesgue's theorem that

$$\int_{|y-x|\ge\frac{1}{2}} f(x, y) u^2(y) \mathrm{d}y \to 0 \quad \text{as } |x| \to \infty.$$

Moreover, by $u \in H^1(\mathbb{R}^2)$ we have

$$0 \leq \frac{1}{|x|^{\lambda}} \int\limits_{|x-y|\leq\frac{1}{2}} u^2(y)\mathrm{d}y \leq \frac{1}{|x|^{\lambda}}\|u\|_2^2 \to 0, \quad \text{as } |x| \to \infty,$$

and

$$0 \leq \int\limits_{|x-y|\leq\frac{1}{2}} \frac{1}{|x-y|^{\lambda}} u^2(y)\mathrm{d}y \leq C_\lambda \|u\|^2_{L^6(B_{\frac{1}{2}}(x))} \to 0, \quad \text{as } |x| \to \infty,$$

for some constant $C_\lambda > 0$. Based on the above estimates, we infer that

$$\int\limits_{\mathbb{R}^2} f(x,y)u^2(y)\mathrm{d}y \to 0 \quad \text{as } |x| \to \infty.$$

(2) Note that $u_n \to u_0$ in $L^6(\mathbb{R}^2)$, then using the Hölder inequality we have

$$0 \leq \int\limits_{|x-y|\leq 1} \frac{1}{|x-y|} u_n^2(y)\mathrm{d}y \leq C\|u_n\|^2_{L^6(B_{\frac{1}{2}}(x))} \to 0, \quad \text{as } |x| \to \infty.$$

The proof is complete. □

3. The modified problem

Since the fact that I is not well defined on $H^1(\mathbb{R}^2)$, we introduce a perturbation technique to overcome this difficulty by modifying Schrödinger-Newton systems. We state the following modified problem

$$-\Delta u + u - (G_\alpha(|\cdot|) * u^2)u = f(u), \tag{3.1}$$

where $\alpha \in (0,1)$ is a parameter and $G_\alpha(x) = \frac{|x|^{-\alpha}-1}{\alpha}$, $x \in \mathbb{R}^2 \setminus \{0\}$. Its associated functional is

$$I_\alpha(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int\limits_{\mathbb{R}^2} (G_\alpha(x) * u^2)u^2\mathrm{d}x - \int\limits_{\mathbb{R}^2} F(u)\mathrm{d}x.$$

According to the definition of G_α, using the Hardy-Littlewood-Sobolev inequality, it is not hard to show that for any given α, the perturbation functional I_α is well-defined on $H^1(\mathbb{R}^2)$ and belongs to $C^1(H^1(\mathbb{R}^2), \mathbb{R})$,

$$I'_\alpha(u)v = \int\limits_{\mathbb{R}^2} (\nabla u \nabla v + uv)\mathrm{d}x - \int\limits_{\mathbb{R}^2} (G_\alpha(x) * u^2)uv\mathrm{d}x - \int\limits_{\mathbb{R}^2} f(u)v\mathrm{d}x$$

for $u, v \in H^1(\mathbb{R}^2)$. We call u is a weak solution of (3.1) if $u \in H^1(\mathbb{R}^2)$ is a critical point of I_α.

Now we provide a Pohozaev type identity for equation (3.1). The strategy of the proof is similar as in [9].

Lemma 3.1. *Suppose that $u \in H^1(\mathbb{R}^2)$ is a weak solution to equation (3.1). Then we have the following identity*

$$P_\alpha(u) := \int\limits_{\mathbb{R}^2} u^2 \mathrm{d}x + \frac{1}{\alpha}\|u\|_2^4 - \frac{4-\alpha}{4\alpha} Q(u) - 2\int\limits_{\mathbb{R}^2} F(u)\mathrm{d}x = 0,$$

where $Q(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u^2(x)u^2(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y$.

In order to study (3.1) as $\alpha \to 0^+$, we need some preliminaries. The following lemma is to describe the behavior of G_α.

Lemma 3.2. *[47, Lemma 2.1] For any $\beta \in (0, \infty)$, there exists $C_\beta > 0$ such that*

$$\frac{s^{-\alpha} - 1}{\alpha} \le C_\beta s^{-\beta}, \ s > 0$$

holds for all $\alpha \in (0, \beta)$.

Lemma 3.3. *Suppose (f_1) -(f_2) hold, then*
(i) there exist $\rho, \delta_0 > 0$ (independent of α) such that $I_\alpha|_{S_\rho}(u) \ge \delta_0$ for every $u \in S_\rho = \{u \in H^1(\mathbb{R}^2) : \|u\| = \rho\}$;
(ii) there is $e \in C_0^\infty(\mathbb{R}^2)$ with $\|e\| > \rho$ such that $I_\alpha(e) < 0$, where e does not depend on α.

Proof. (i) For every $\theta \in (0, 4\pi)$, it follows from (f_1)-(f_2) that, for any $\varepsilon > 0$, there exists $C_{\varepsilon,\theta} > 0$ such that

$$|f(s)| \le \varepsilon \min\{1, |s|\} e^{\theta|s|^2} + C_{\varepsilon,\theta}|s|^{p-1} \quad \text{for some } p > 2. \tag{3.2}$$

We assume $u \in H^1(\mathbb{R}^2)$ and $\|u\|^2 < 1$. Obviously, $\int_{\mathbb{R}^2} |\nabla u|^2 < 1$. So by Moser-Trudinger's inequality, one has

$$\begin{aligned} \int\limits_{\mathbb{R}^2} F(u)\mathrm{d}x &\le C_{\varepsilon,\theta} \int\limits_{\mathbb{R}^2} |u|^p \mathrm{d}x + \varepsilon \int\limits_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\theta|u|^2} \mathrm{d}x \\ &\le C_{\varepsilon,\theta} \int\limits_{\mathbb{R}^2} |u|^p \mathrm{d}x + \varepsilon C_\theta \int\limits_{\mathbb{R}^2} |u|^2 \mathrm{d}x. \end{aligned} \tag{3.3}$$

From (3.3), Lemma 3.2 and Hardy-Littlewood-Sobolev's inequality we deduce that

$$
\begin{aligned}
I_\alpha(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha}-1}{\alpha} u^2(y)u^2(x)\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \\
&\geq \frac{1-2\varepsilon C_\theta}{2}\|u\|^2 - \frac{1}{4}\int \int_{|x-y|\leq 1} \frac{|x-y|^{-\alpha}-1}{\alpha} u^2(y)u^2(x)\mathrm{d}y\mathrm{d}x - C_{\varepsilon,\theta}\int_{\mathbb{R}^2} |u|^p\mathrm{d}x \\
&\geq \frac{1-2\varepsilon C_\theta}{2}\|u\|^2 - \frac{1}{4}\int \int_{|x-y|\leq 1} \frac{u^2(y)u^2(x)}{|x-y|}\mathrm{d}x\mathrm{d}y - C_{\varepsilon,\theta}C\|u\|^p \\
&\geq \frac{1-2\varepsilon C_\theta}{2}\|u\|^2 - \frac{C}{4}\|u\|^4 - C_{\varepsilon,\theta}C\|u\|^p.
\end{aligned}
\tag{3.4}
$$

So, by fixed $\varepsilon \in (0, \frac{1}{2C_\theta})$ and letting $\|u\| = \rho > 0$ small enough, it is easy to check that there exists $\delta_0 > 0$ such that $I_\alpha(u) \geq \delta_0$ for every $u \in S_\rho$.
(ii) Take $e_0 \in C_0^\infty(\mathbb{R}^2)$ such that $e_0(x) \equiv 1$ for $x \in B_{\frac{1}{8}}(0)$, $e_0(x) \equiv 0$ for $x \in \mathbb{R}^2 \setminus B_{\frac{1}{4}}(0)$ and $|\nabla e_0(x)| \leq C$. Note that

$$
\frac{s^{-\alpha}-1}{\alpha} \geq \ln\frac{1}{s}, \quad \text{for } s \in (0,1].
$$

It then follows from the definition of I_α that

$$
\begin{aligned}
I_\alpha(se_0) &= \frac{s^2}{2}\|e_0\|^2 - \frac{s^4}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha}-1}{\alpha} e_0^2(y)e_0^2(x)\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^2} F(se_0)\mathrm{d}x \\
&\leq \frac{s^2}{2}\|e_0\|^2 - \frac{s^4}{4}\int_{|x|\leq\frac{1}{4}}\int_{|y|\leq\frac{1}{4}} \frac{|x-y|^{-\alpha}-1}{\alpha} e_0^2(y)e_0^2(x)\mathrm{d}y\mathrm{d}x \\
&\leq \frac{s^2}{2}\|e_0\|^2 - \frac{s^4}{4}\int_{|x|\leq\frac{1}{4}}\int_{|y|\leq\frac{1}{4}} \ln\frac{1}{|x-y|} e_0^2(y)e_0^2(x)\mathrm{d}y\mathrm{d}x \\
&\leq \frac{s^2}{2}\|e_0\|^2 - \frac{s^4\ln 2}{4}\left(\int_{\mathbb{R}^2} e_0^2(x)\mathrm{d}x\right)^2,
\end{aligned}
\tag{3.5}
$$

which implies that there exists $t_0 > 0$ large enough such that $I_\alpha(t_0e_0) < 0$. □

Based on the mountain pass theorem without the Palais-Smale condition (see [46]), there exists a $(PS)_{c_\alpha}$ sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$, that is,

$$
I_\alpha(u_n) \to c_\alpha \quad \text{and} \quad I'_\alpha(u_n) \to 0. \tag{3.6}
$$

Here c_α is the mountain pass level characterized by

$$
c_\alpha = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\alpha(\gamma(t)) \tag{3.7}
$$

with

$$\Gamma := \{\gamma \in C^1([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\},$$

where e has been given in Lemma 3.3.

Remark 3.4. Observe from Lemma 3.3 that there exist two constants $a, b > 0$ independent of α such that $a < c_\alpha < b$.

We state the following splitting lemma to the Palais-Smale sequences of I_α.

Lemma 3.5. *Assume that $\{u_n\}$ is a bounded $(PS)_{c_\alpha}$ sequence of I_α for fixed $\alpha \in (0,1)$, then there exist $B \in \mathbb{R}$, a number $k \in \mathbb{N} \cup \{0\}$, and a finite sequence*

$$(u_0, w^1, ..., w^k) \subset H^1(\mathbb{R}^2), \quad w^j \not\equiv 0, \quad \textit{for } j = 1, ..., k$$

of critical points for the following functional

$$J_{B,\alpha}(u) := \frac{1}{2}\|u\|^2 + \frac{B}{2\alpha}\int_{\mathbb{R}^2} |u_n|^2 \mathrm{d}x - \frac{1}{4\alpha}\int_{\mathbb{R}^2}\left(\frac{1}{|x|^\alpha} * u^2\right)u^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \tag{3.8}$$

and k sequences of points $\{y_n^j\} \subset \mathbb{R}^2$, $1 \le j \le k$, such that if $k \ge 1$,

(i) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ *if* $i \ne j, n \to +\infty$,

(ii) $\|u_n - u_0 - \sum_{j=1}^k w^j(\cdot - y_n^j)\| \to 0$, $c_\alpha + \frac{B^2}{4\alpha} = J_{B,\alpha}(u_0) + \sum_{j=1}^k J_{B,\alpha}(w^j)$

(iii) $B = \|u_0\|_2^2 + \sum_{j=1}^k \|w^j\|_2^2$.

Otherwise, if $k = 0$, then $u_n \to u_0$ in $H^1(\mathbb{R}^2)$.

Proof. Since $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^2)$, up to subsequence, there exist $u_0 \in H^1(\mathbb{R}^2)$ and $B \in \mathbb{R}$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} |u_n|^2 \mathrm{d}x \to B$ as $n \to \infty$. By (f_1)-(f_2) and Moser-Trudinger's inequality, we deduce that for $n \to +\infty$,

$$\int_{\mathbb{R}^2} f(u_n)\varphi \mathrm{d}x \to \int_{\mathbb{R}^2} f(u_0)\varphi \mathrm{d}x, \ \varphi \in C_0^\infty(\mathbb{R}^2). \tag{3.9}$$

Then it follows from $I'_\alpha(u_n) \to 0$ in H^{-1} that $J'_{B,\alpha}(u_0) = 0$. Observe that

$$I_\alpha(u_n) = J_{B,\alpha}(u_n) - \frac{B^2}{4\alpha} + o(1),$$

and so $J_{B,\alpha}(u_n) \to c_\alpha + \frac{B^2}{4\alpha}$ and $J'_{B,\alpha}(u_n) \to 0$ in H^{-1} as $n \to \infty$. Moreover, $v_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ if we define $v_n^1 := u_n - u_0$. By the Brezis-Lieb lemma [11], it follows that

$$\begin{aligned} &\|u_n\|^2 = \|v_n^1\|^2 + \|u_0\|^2 + o(1), \\ &\int_{\mathbb{R}^2} \Big(\frac{1}{|x|^\alpha} * u_n^2\Big)u_n^2 \mathrm{d}x = \int_{\mathbb{R}^2} \Big(\frac{1}{|x|^\alpha} * u_0^2\Big)u_0^2 \mathrm{d}x + \int_{\mathbb{R}^2} \Big(\frac{1}{|x|^\alpha} * (v_n^1)^2\Big)(v_n^1)^2 \mathrm{d}x + o(1). \end{aligned} \tag{3.10}$$

Moreover, similarly to Lemma 5.2 in [35], we have for any $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned} &\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} F(v_n^1)\mathrm{d}x + \int_{\mathbb{R}^2} F(u_0)\mathrm{d}x + o(1), \\ &\int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^2} f(v_n^1)v_n^1\mathrm{d}x + \int_{\mathbb{R}^2} f(u_0)u_0\mathrm{d}x + o(1), \\ &\int_{\mathbb{R}^2} f(u_n)\varphi\mathrm{d}x = \int_{\mathbb{R}^2} f(v_n^1)\varphi\mathrm{d}x + \int_{\mathbb{R}^2} f(u_0)\varphi\mathrm{d}x + o(1). \end{aligned} \tag{3.11}$$

Using (3.11) and (3.10), we immediately obtain

$$\begin{aligned} &J_{B,\alpha}(v_n^1) = J_{B,\alpha}(u_n) - J_{B,\alpha}(u_0) + o(1), \\ &J'_{B,\alpha}(v_n^1)v_n^1 = J'_{B,\alpha}(u_n)u_n - J'_{B,\alpha}(u_0)u_0 + o(1) = o(1) \\ &J'_{B,\alpha}(v_n^1)\varphi = J'_{B,\alpha}(u_n)\varphi - J'_{B,\alpha}(u_0)\varphi + o(1) = o(1). \end{aligned} \tag{3.12}$$

Consider sequence $\{v_n^1\}$. One of the following conclusions holds.

(v1) $v_n^1 \to 0$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$, or
(v2) there exist $r', m > 0$ and $\{y_n^1\} \subset \mathbb{R}^2$ such that for $q \in (2, +\infty)$,

$$\liminf_{n\to\infty} \int_{B_{r'}(y_n^1)} |v_n^1|^q \mathrm{d}x \geq m > 0. \tag{3.13}$$

Indeed, if (v2) is false, then for any $r > 0$, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |v_n^1|^q \mathrm{d}x = 0. \tag{3.14}$$

Recalling the well-known Lions' lemma, (3.14) implies that $v_n^1 \to 0$ in $L^q(\mathbb{R}^2)$ for $q \in (2, +\infty)$. And so, using the Hardy-Littlewood-Sobolev inequality, we have $\int_{\mathbb{R}^2} \big(\frac{1}{|x|^\alpha} * (v_n^1)^2\big)(v_n^1)^2\mathrm{d}x = o(1)$. Thanks to Moser-Trudinger's inequality and (f_1) and (f_2), we have $\int_{\mathbb{R}^2} f(v_n^1)v_n^1\mathrm{d}x = o(1)$. From the above estimates and $J'_{B,\alpha}(v_n^1)v_n^1 = o(1)$ in (3.12), we deduce that

$$v_n^1 \to 0 \quad \text{in } H^1(\mathbb{R}^2),$$

and (v1) holds for $\{v_n^1\}$, Lemma 3.5 holds with $k=0$. Let us now suppose that (v2) holds, that is, (3.13) is true. Then there exists $w^1 \in H^1(\mathbb{R}^2)$ such that $v_n^1(\cdot + y_n^1) \rightharpoonup w^1$ in $H^1(\mathbb{R}^2)$. Using conclusion (v2), we have

$$\int_{B_{r'}(0)} |v_n^1(x+y_n^1)|^q \mathrm{d}x \geq m$$

for n large. From the Rellich theorem it follows that

$$\int_{B_{r'}(0)} |w^1(x)|^q \mathrm{d}x \geq m,$$

and thus, $w^1 \neq 0$. Recalling that $v_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$, we find that $\{y_n^1\}$ must be unbounded. That is, $|y_n^1| \to +\infty$. Let us now show that $J'_{B,\alpha}(w^1)=0$. Indeed, it suffices to show that $J'_{B,\alpha}(v_n^1(\cdot + y_n^1))\varphi \to 0$ for fixed $\varphi \in C_0^\infty(\mathbb{R}^2)$. Since $v_n^1 = u_n - u_0 \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$, it follows that $J'_{B,\alpha}(v_n^1) \to 0$ in H^{-1}, and then $J'_{B,\alpha}(v_n^1)\varphi(\cdot - y_n^1) \to 0$. Thus, it follows that as $n \to \infty$,

$$\begin{aligned} &J'_{B,\alpha}(v_n^1(\cdot + y_n^1))\varphi \\ &= \int_{\mathbb{R}^2} \nabla v_n^1(x+y_n^1)\nabla\varphi + v_n^1(x+y_n^1)\varphi \mathrm{d}x + \frac{B}{\alpha}\int_{\mathbb{R}^2} v_n^1(x+y_n^1)\varphi \mathrm{d}x \\ &\quad - \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * (v_n^1(\cdot + y_n^1))^2\right] v_n^1(x+y_n^1)\varphi \mathrm{d}x - \int_{\mathbb{R}^2} f(v_n^1(x+y_n^1))\varphi \mathrm{d}x \to 0. \end{aligned} \tag{3.15}$$

So, $J'_{B,\alpha}(w^1)=0$. Set

$$v_n^2(x) = v_n^1(x) - w^1(x - y_n^1), \tag{3.16}$$

then using the fact that $v_n^2(\cdot + y_n^1) \rightharpoonup 0$, we have $v_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$. Using again the Brezis-Lieb lemma we obtain

$$\begin{aligned} &\|u_n\|^2 = \|w^1\|^2 + \|u_0\|^2 + \|v_n^2\|^2 + o(1), \\ &\int_{\mathbb{R}^2} \big(\frac{1}{|x|^\alpha} * u_n^2\big) u_n^2 \mathrm{d}x = \int_{\mathbb{R}^2} \big(\frac{1}{|x|^\alpha} * u_0^2\big) u_0^2 \mathrm{d}x + \int_{\mathbb{R}^2} \big(\frac{1}{|x|^\alpha} * (w^1)^2\big)(w^1)^2 \mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^2} \big(\frac{1}{|x|^\alpha} * (v_n^2)^2\big)(v_n^2)^2 \mathrm{d}x + o(1). \end{aligned} \tag{3.17}$$

Similarly to (3.11), one has for any $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$
\begin{aligned}
&\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} F(w^1)\mathrm{d}x + \int_{\mathbb{R}^2} F(u_0)\mathrm{d}x + \int_{\mathbb{R}^2} F(v_n^2)\mathrm{d}x + o(1),\\
&\int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^2} f(w^1)w^1\mathrm{d}x + \int_{\mathbb{R}^2} f(u_0)u_0\mathrm{d}x + \int_{\mathbb{R}^2} f(v_n^2)v_n^2\mathrm{d}x + o(1)\\
&\int_{\mathbb{R}^2} f(u_n)\varphi\mathrm{d}x = \int_{\mathbb{R}^2} f(w^1)\varphi\mathrm{d}x + \int_{\mathbb{R}^2} f(v_n^1)\varphi\mathrm{d}x + \int_{\mathbb{R}^2} f(u_0)\varphi\mathrm{d}x + o(1).
\end{aligned}
\tag{3.18}
$$

By virtue of the above estimates, we deduce that

$$
\begin{aligned}
&(1)\quad J_{B,\alpha}(v_n^2) = J_{B,\alpha}(u_n) - J_{B,\alpha}(u_0) - J_{B,\alpha}(w^1) + o(1),\\
&(2)\quad J'_{B,\alpha}(v_n^2)v_n^2 = J'_{B,\alpha}(u_n)u_n - J'_{B,\alpha}(u_0)u_0 - J'_{B,\alpha}(w^1)w^1 + o(1) = o(1)\\
&(3)\quad J'_{B,\alpha}(v_n^1)\varphi = J'_{B,\alpha}(u_n)\varphi - J'_{B,\alpha}(u_0)\varphi - J'_{B,\alpha}(w^1)\varphi + o(1) = o(1).
\end{aligned}
\tag{3.19}
$$

Let us now study $\{v_n^2\}$. Since $\{v_n^2\}$ is bounded in $H^1(\mathbb{R}^2)$, one of (v1) and (v2) holds for $\{v_n^2\}$. The same arguments used before imply that Lemma 3.5 holds with $k = 1$ if $v_n^2 \to 0$ in $H^1(\mathbb{R}^2)$. Otherwise, (v2) holds for $\{v_n^2\}$. We repeat the arguments above. Iterating this procedure, there exists $y_n^j \in \mathbb{R}^2$ such that $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ if $i \neq j$ as $n \to +\infty$ and $v_n^j = v_n^{j-1} - w^{j-1}(x - y_n^{j-1})$ (like (3.16)), $v_n^j(\cdot + y_n^j) \rightharpoonup w^j \neq 0$ with $j \geq 2$ such that

$$
v_n^j \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^2), \qquad J'_{B,\alpha}(w^j) = 0.
$$

Moreover, by the properties of the weak convergence, we get

$$
\begin{aligned}
&(a)\quad \|u_n\|^2 = \|u_0\|^2 + \sum_{i=1}^{j}\|w^i\|^2 + \|u_n - u_0 - \sum_{i=1}^{j} w^i(\cdot - y_n^i)\|^2 + o(1),\\
&(b)\quad c_\alpha + \frac{B^2}{4\alpha} = J_{B,\alpha}(u_0) + \sum_{i=1}^{j-1} J_{B,\alpha}(w^i) + J_{B,\alpha}(v_n^j) + o(1),\\
&(c)\quad B = \|u_0\|_2^2 + \sum_{i=1}^{j}\|w^i\|_2^2 + \|u_n - u_0 - \sum_{i=1}^{j} w^i(\cdot - y_n^i)\|_2^2 + o(1).
\end{aligned}
\tag{3.20}
$$

Due to $J'_{B,\alpha}(w^i)w^i = 0$, (3.2), Hardy-Littlewood-Sobolev's inequality, Moser-Trudinger's inequality and Lemma 3.2, one finds that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
\|w^i\|^2 &\leq \int\int_{|x-y|\leq 1} \frac{|x-y|^{-\alpha} - 1}{\alpha}|w^i(x)|^2|w^i(y)|^2\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^2} f(w^i)w^i\mathrm{d}x\\
&\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|w^i(x)|^2|w^i(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y + C_\varepsilon\|w^i\|_p^p + \varepsilon\|w^i\|_2^2\\
&\leq C\|w^i\|^4 + \varepsilon C\|w^i\|^2 + C_\varepsilon\|w^i\|^p, \quad p \in (2 + \infty).
\end{aligned}
$$

Taking $\varepsilon = \frac{1}{2C}$ in the above estimate, we conclude that there exists $C > 0$ such that $\|w^i\|^2 \geq C$. Recall that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$, from (3.20)(a) we deduce that the iteration must stop at some finite index k. And so $v_n^{k+1} \to 0$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$. The proof is complete. $\square$

Lemma 3.6. *Assume that conditions (f_1)-(f_2) hold. Let $\{u_n\} \subset H^1(\mathbb{R}^2)$ be a $(PS)_{c_\alpha}$ sequence of I_α for fixed $\alpha \in (0,1)$, then there exists $u_0 \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that $I'_\alpha(u_0) = 0$.*

Proof. By assumption (f_2), one has

$$c_\alpha = \lim_{n\to\infty}\left(I_\alpha(u_n) - \frac{1}{4}I'_\alpha(u_n)u_n\right) = \frac{1}{4}\|u_n\|^2 - \int_{\mathbb{R}^2}\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right]\mathrm{d}x,$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. So there is $u_0 \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^2)$. There also exists $B \geq 0$ such that

$$\|u_n\|_2^2 \to B, \quad \text{as } n \to \infty, \tag{3.21}$$

from which we deduce that $J'_{B,\alpha}(u_n) \to 0$ in H^{-1} and $J'_{B,\alpha}(u_0) = 0$. In view of Lemma 3.5, for each nontrivial critical point $u \in \{u_0, w^j\,(j = 1, ..., k)\}$ of $J_{B,\alpha}$, the following relation holds

$$\|u\|^2 + \frac{1}{\alpha}\|u\|_2^4 - \frac{1}{\alpha}\int_{\mathbb{R}^2}\left(\frac{1}{|x|^\alpha} * u^2\right)u^2\mathrm{d}x - \int_{\mathbb{R}^2} f(u)u\mathrm{d}x \leq 0, \tag{3.22}$$

which means $I'_\alpha(u)u \leq 0$. By assumptions (f_1)-(f_2), we conclude that $g(t) := I'_\alpha(u_t)u_t > 0$ for small $t > 0$. Here, $u_t = tu(x)$. Thus, there exists $t_0 \in (0, 1]$ such that $I_\alpha(u_{t_0}) = \max_{t\in[0,1]} I_\alpha(u_t)$. Using again (f_2) one has for $t > 1$

$$g(t) = t^4\left(\frac{1}{t^2}\|\nabla u\|_2^2 + \frac{1}{t^2}\|u\|_2^2 + \frac{1}{\alpha}\|u\|_2^4 - \frac{1}{\alpha}\int_{\mathbb{R}^2}\left(\frac{1}{|x|^\alpha} * u^2\right)u^2\mathrm{d}x - \int_{\mathbb{R}^2}\frac{f(tu)}{t^3u^3}u^4\mathrm{d}x\right) < 0, \tag{3.23}$$

which implies that there exists $t_1 > 1$ such that $I_\alpha(u_{t_1}) < 0$ and $I_\alpha(u_{t_0}) = \max_{t\in[0,t_1]} I_\alpha(u_t)$. It is easy to see that $\gamma(t) := u_{tt_1} \in \Gamma$, which has been defined in (3.7). As a result, $c_\alpha \leq I_\alpha(u_{t_0})$ and a direct calculation yields

$$
\begin{aligned}
J_{B,\alpha}(u) =& J_{B,\alpha}(u) - \frac{1}{4}J'_{B,\alpha}(u)u \\
=& \frac{1}{4}\|u\|_2^2 + \frac{B}{4\alpha}\|u\|_2^2 + \int_{\mathbb{R}^2} \frac{1}{4}f(u)u - F(u)\mathrm{d}x \\
\geq& \frac{t_0^2}{4}\|u\|_2^2 + \frac{B}{4\alpha}\|u\|_2^2 + \int_{\mathbb{R}^2} \frac{1}{4}f(t_0u)t_0u - F(t_0u)\mathrm{d}x \\
=& I_\alpha(u_{t_0}) - \frac{1}{4}I'_\alpha(u_{t_0})u_{t_0} + \frac{B}{4\alpha}\|u\|_2^2 \\
\geq& c_\alpha + \frac{B}{4\alpha}\|u\|_2^2,
\end{aligned}
\tag{3.24}
$$

where we have used the fact that $\frac{1}{4}f(s)s - F(s)$ is nondecreasing at $s > 0$. Then from Lemma 3.5 we conclude that

$$
\begin{aligned}
c_\alpha + \frac{B^2}{4\alpha} &= J_{B,\alpha}(u_0) + \sum_{j=1}^{k} J_{B,\alpha}(w^j) \\
&\geq kc_\alpha + \frac{B}{4\alpha}\int_{\mathbb{R}^2} |u_0|^2 dx + \frac{B}{4\alpha}\sum_{j=1}^{k}\int_{\mathbb{R}^2} |w^j|^2 dx \\
&\geq kc_\alpha + \frac{B^2}{4\alpha},
\end{aligned}
\tag{3.25}
$$

where $w^j \neq 0$ for $j = 1, ..., k$. Observe that $k > 1$ is impossible.

• If $k = 0$, we are done. Then it follows that $J_{B,\alpha}(u_0) = I_\alpha(u_0) + \frac{B^2}{4\alpha}$ and $u_n \to u_0$ strongly in $H^1(\mathbb{R}^2)$.

• If $k = 1$ and $u_0 \neq 0$, then we have

$$
J_{B,\alpha}(u_0) \geq \frac{1}{4}\left[\frac{B}{\alpha}\int_{\mathbb{R}^2} |u_0|^2 dx + \|\nabla u_0\|_2^2\right]
$$

and then the first inequality in (3.25) strictly holds. This yields a contradiction.

• If $k = 1$ and $u_0 = 0$, then by conclusion (iii) of Lemma 3.5, we get $B = \|w^1\|_2^2$ and $I'_\alpha(w^1) = 0$ in $H^1(\mathbb{R}^2)$. The proof is complete. □

Remark 3.7. We emphasize that it is not difficult to get a positive mountain pass type solution for equation (3.1). Indeed, we can modify properly equation (3.1) to be the following equation

$$
-\Delta u + u + \frac{1}{\alpha}\|u\|_2^2 u - \frac{1}{\alpha}[|\cdot|^{-\alpha} * (u^+)^2]u^+ = f(u),
\tag{3.26}
$$

whose energy functional is

$$I_\alpha^+(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2)\mathrm{d}x + \frac{1}{4\alpha}\int_{\mathbb{R}^2}|u|^4\mathrm{d}x - \frac{1}{4\alpha}\int_{\mathbb{R}^4}\frac{(u^+)^2(y)(u^+)^2(x)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^2}F(u)\mathrm{d}x.$$

It is easy to check that I^+ is of C^1 class and

$$I_\alpha^+(u)v = \int_{\mathbb{R}^2}(\nabla u\nabla v + uv)\mathrm{d}x + \frac{1}{\alpha}\|u\|_2^2\int_{\mathbb{R}^2}uv\mathrm{d}x - \frac{1}{\alpha}\int_{\mathbb{R}^4}\frac{(u^+)^2(y)\mathrm{d}y}{|x-y|^\alpha}u^+v\mathrm{d}x - \int_{\mathbb{R}^2}f(u)v\mathrm{d}x,$$

where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$ and $v \in H^1(\mathbb{R}^2)$, then all the above calculations can be repeated word by word. So I_α^+ has a nontrivial mountain pass type critical point $u \in H^1(\mathbb{R}^2)$ for fixed $\alpha \in (0, 1)$. Using u^- as a test function to equation (3.26), we have immediately $u \geq 0$ and then the maximum principle yields $u > 0$.

4. Symmetry of positive solutions

In this section, we are concerned with the symmetry of positive mountain pass solutions of the modified equation. More precisely, we will use a variant of the moving plane method (see [15,16,25,37]) to prove a symmetry result for the following modified equation.

$$-\Delta u + u = (G_\alpha(|\cdot|) * u^2)u + f(u),\ x \in \mathbb{R}^2. \tag{4.1}$$

Observe that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with $f(0) = 0$ due to the fact that $f \in C^1(\mathbb{R}, \mathbb{R})$. Assume u_α is a positive solution of (4.1) such that $I_\alpha(u_\alpha) = c_\alpha$ for fixed $\alpha \in (0, 1)$. Using the well-known Nash-Moser's iteration arguments, one can prove $u_\alpha \in L^\infty(\mathbb{R}^2)$ and

$$u_\alpha(x) \leq C_\alpha e^{-c_\alpha|x|}, \quad x \in \mathbb{R}^2, \tag{4.2}$$

where $C_\alpha, c_\alpha > 0$ depend on the parameter α.

Lemma 4.1. *There exist positive numbers $\omega, \alpha_0 \in (0, 1)$ such that*

$$\|u_\alpha\|_2^2 \geq \omega \quad \textit{for} \quad \alpha \in (0, \alpha_0), \tag{4.3}$$

where ω is independent of α.

Proof. Assume by contradiction that there exists $\{\alpha_n\}$ with $\alpha_n \to 0^+$ such that $\|u_{\alpha_n}\|_2^2 \to 0$. By assumption (f_2) and $c_{\alpha_n} \in (a, b)$ in Remark 3.4, we have immediately $\|u_{\alpha_n}\| \leq C$ for some C independent of α. As a consequence, recalling (3.3) together with the Sobolev interpolation inequality yields

$$\begin{aligned}\int_{\mathbb{R}^2}F(u_{\alpha_n})\mathrm{d}x &\leq C_{\varepsilon,\theta}\int_{\mathbb{R}^2}|u_{\alpha_n}|^p\mathrm{d}x + \varepsilon\int_{\mathbb{R}^2}\min\{1, |u_{\alpha_n}|^2\}e^{\theta|u_{\alpha_n}|^2}\mathrm{d}x \\ &\leq C_{\varepsilon,\theta}\int_{\mathbb{R}^2}|u_{\alpha_n}|^p\mathrm{d}x + \varepsilon C_\theta\int_{\mathbb{R}^2}|u_{\alpha_n}|^2\mathrm{d}x \\ &= o(1),\ (p > 2),\end{aligned} \tag{4.4}$$

where θ is choosen small enough. We can also similarly obtain

$$\int_{\mathbb{R}^2} f(u_{\alpha_n})u_{\alpha_n}\mathrm{d}x = o(1). \tag{4.5}$$

Moreover, using the Hardy-Littlewood-Sobolev inequality and the Sobolev interpolation inequality, we have

$$\begin{aligned} Q(u_{\alpha_n}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u^2_{\alpha_n}(x)u^2_{\alpha_n}(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &= \int\int_{|x-y|\le 1} \frac{u^2_{\alpha_n}(x)u^2_{\alpha_n}(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y + \int\int_{|x-y|\ge 1} \frac{u^2_{\alpha_n}(x)u^2_{\alpha_n}(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &\le \int\int_{|x-y|\le 1} \frac{u^2_{\alpha_n}(x)u^2_{\alpha_n}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y + \int\int_{|x-y|\ge 1} u^2_{\alpha_n}(x)u^2_{\alpha_n}(y)\mathrm{d}x\mathrm{d}y \\ &\le C\|u_{\alpha_n}\|^4_{8/3} + \|u_{\alpha_n}\|^4_2 = o(1). \end{aligned} \tag{4.6}$$

Recalling the Pohozaev type identity in Lemma 3.1, by (4.4) and (4.6), we have

$$\int_{\mathbb{R}^2} (G_{\alpha_n}(x) * u^2_{\alpha_n})u^2_{\alpha_n}\mathrm{d}x = o(1), \tag{4.7}$$

which, together with (4.5) and $I'_{\alpha_n}(u_{\alpha_n})u_{\alpha_n} = 0$, implies that $\|u_{\alpha_n}\| \to 0$ as $\alpha_n \to 0^+$. Hence, it follows from the definition of I_{α_n}, (4.4) and (4.7) that $c_{\alpha_n} \to 0$ as $\alpha_n \to 0^+$. This is a contradiction. $\square$

Our aim in this section is to prove the following result.

Theorem 4.2. *Assume $u_\alpha \in H^1(\mathbb{R}^2)$ is a positive classical solution of (4.1), then there exists $\alpha_1 \in (0, 1]$ such that for $\alpha \in (0, \alpha_1)$, u_α is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center.*

Now we use a variant of the moving plane method (see [15,16,25,37]) to prove this theorem. For simplicity, we write u_α as u. For $\beta \in \mathbb{R}$, we set

$$H_\beta := \{x \in \mathbb{R}^2 : x_1 > \beta\}, \quad \partial H_\beta = \{x \in \mathbb{R}^2 : x_1 = \beta\}.$$

Moreover, for any $x \in \mathbb{R}^2$, denote by x^β the reflection of x with respect to ∂H_β. Set

$$u^\beta(x) = u(x^\beta), \quad w^\beta(x) = w(x^\beta) \text{ for } x \in \mathbb{R}^2,\ \beta \in \mathbb{R},$$

where $w(x) = \frac{1}{\alpha}(\frac{1}{|x|^\alpha} * u^2)$. Set $u_\beta = u^\beta - u$ and $w_\beta = w^\beta - w$, then the following equation holds

$$-\Delta u_\beta + u_\beta = w_\beta u^\beta + (w - \frac{1}{\alpha}\|u\|_2^2 + h_\beta)u_\beta, \quad x \in H_\beta, \tag{4.8}$$

where

$$h_\beta = \begin{cases} \frac{f(u^\beta(x)) - f(u(x))}{u_\beta(x)}, & u_\beta(x) \neq 0, \\ f'(u(x)), & u_\beta(x) = 0. \end{cases}$$

Recalling that $f \in C^1(\mathbb{R}, \mathbb{R})$, there exists $C = C(u) > 0$ such that $\|h_\beta\|_{L^\infty(H_\beta)} \leq C$ for any $\beta \in \mathbb{R}$. From the definition of w, we deduce that

$$w_\beta = \int_{H_\beta} \left(\frac{1}{\alpha|x-y|^\alpha} - \frac{1}{\alpha|x-y^\beta|^\alpha}\right)(u^\beta(y) + u(y))u_\beta(y)\mathrm{d}y$$

for H_β. Since $\frac{|x-y^\beta|}{|x-y|} > 1$ for any $x, y \in H_\beta$, we have $w_\beta \geq 0$ in H_β if $u_\beta \geq 0$ in H_β for every $\beta \in \mathbb{R}$. In what follows, we define the negative part of v by $v^- := \min\{v, 0\}$. Notice that w^- is a non-positive function with the convention. The following lemma contains a key estimate we need in the sequel.

Lemma 4.3. *There exists a constant $\kappa_\alpha > 0$ such that*

$$\|w_\beta^-\|_{L^2(H_\beta)} \leq \kappa_\alpha c_{\alpha,\beta}\|u_\beta^-\|_{L^2(H_\beta)}$$

for every $\beta \in \mathbb{R}$, where

$$c_{\alpha,\beta} = \left(\int_{M_\beta} (y_1 - \beta)^{\frac{\alpha}{1-\alpha}} u^{\frac{1}{1-\alpha}}(y)\mathrm{d}y\right)^{1-\alpha}, \quad M_\beta := \{x \in H_\beta : u_\beta(x) < 0\}.$$

Proof. Observe that for $x, y \in H_\beta$,

$$\begin{aligned} \frac{1}{|x-y|^\alpha} - \frac{1}{|x-y^\beta|^\alpha} &= \frac{|x-y^\beta|^\alpha - |x-y|^\alpha}{|x-y|^\alpha \cdot |x-y^\beta|^\alpha} \\ &\leq \frac{\big(|x-y^\beta| - |x-y|\big)^\alpha}{|x-y|^{2\alpha}} \\ &\leq \frac{|y-y^\beta|^\alpha}{|x-y|^{2\alpha}} = \frac{2^\alpha(y_1-\beta)^\alpha}{|x-y|^{2\alpha}}. \end{aligned}$$

When $u_\beta < 0$, we have $0 \leq u^\beta(y) \leq u(y)$. By the integral representation of w_β, we conclude that for $x \in H_\beta$

$$-w_\beta^-(x) \leq -\int_{M_\beta} \frac{2^\alpha(y_1-\beta)^\alpha}{\alpha|x-y|^{2\alpha}}(u^\beta(y) + u(y))u_\beta^-(y)\mathrm{d}y$$

$$\leq -2^{\alpha+1} \int\limits_{M_\beta} \frac{(y_1-\beta)^\alpha}{\alpha|x-y|^{2\alpha}} u(y) u_\beta^-(y) \mathrm{d}y.$$

It follows from Hardy-Littlewood-Sobolev's inequality and (4.2) that

$$\begin{aligned}
\|w_\beta^-\|_{L^2(H_\beta)}^2 &\leq \kappa_\alpha \Bigg(\int\limits_{M_\beta} \Big((y_1-\beta)^\alpha u(y) u_\beta^-(y) \Big)^{\frac{2}{3-2\alpha}} \mathrm{d}y \Bigg)^{\frac{3-2\alpha}{2}} \|w_\beta^-\|_{L^2(H_\beta)} \\
&\leq \kappa_\alpha \Bigg(\int\limits_{M_\beta} (y_1-\beta)^{\frac{\alpha}{1-\alpha}} u^{\frac{1}{1-\alpha}}(y) \mathrm{d}y \Bigg)^{1-\alpha} \Bigg(\int\limits_{H_\beta} |u_\beta^-|^2 \mathrm{d}y \Bigg)^{\frac{1}{2}} \|w_\beta^-\|_{L^2(H_\beta)} \\
&\leq c_{\alpha,\beta} \kappa_\alpha \|w_\beta^-\|_{L^2(H_\beta)} \|u_\beta^-\|_{L^2(H_\beta)}.
\end{aligned}$$

The conclusion follows immediately and the proof is complete. □

Lemma 4.4. *There exist $\alpha_2, \beta_1 > 0$ such that $u_\beta \geq 0$ in H_β for $\beta \in [\beta_1, +\infty)$ and $\alpha \in (0, \alpha_1)$.*

Proof. By Lemma 2.3 and 4.1, we may choose $\beta_2 > 0$ large enough and $\alpha_1 > 0$ small enough such that

$$w - \frac{1}{\alpha}\|u\|_2^2 + h_\beta \leq 0, \quad \text{for } \beta \geq \beta_2,\ 0 < \alpha < \alpha_1.$$

Here, we used the fact that $h_\beta \in L^\infty(\mathbb{R}^2)$. Multiplying the modified equation (4.8) by u_β^- and then integrating, from Lemma 4.3 we have

$$\begin{aligned}
\|u_\beta^-\|_{L^2(H_\beta)}^2 \leq \|u_\beta^-\|_{H^1(H_\beta)}^2 &= \int\limits_{H_\beta} \Big(w_\beta u^\beta u_\beta^- + (w - \frac{1}{\alpha}\|u\|_2^2 + h_\beta) u_\beta u_\beta^- \Big) \mathrm{d}x \\
&\leq \int\limits_{H_\beta} w_\beta^- u^\beta u_\beta^- \mathrm{d}x \leq \|w_\beta^-\|_{L^2(H_\beta)} \|u^\beta\|_{L^\infty(\mathbb{R}^2)} \|u_\beta^-\|_{L^2(H_\beta)} \\
&\leq c_{\alpha,\beta} \kappa_\alpha \|u^\beta\|_{L^\infty(\mathbb{R}^2)} \|u_\beta^-\|_{L^2(H_\beta)}^2.
\end{aligned}$$

Observe that $c_{\alpha,\beta} \to 0$ as $\beta \to \infty$ for fixed α. Then there exists $\beta_1 > \beta_2$ such that $c_{\alpha,\beta}\kappa_\alpha \|u^\beta\|_{L^\infty(\mathbb{R}^2)} < 1$ for $\beta > \beta_1$, and so $u_\beta^- \equiv 0$ on H_β for $\beta > \beta_1$ and $\alpha \in (0, \alpha_1)$. The proof is complete. □

We have the following auxiliary property, whose proof is based on standard arguments, see [14].

Lemma 4.5. *Either $u_\beta \equiv 0 \equiv w_\beta$ or $u_\beta > 0$, $w_\beta > 0$ on H_β and $\frac{\partial u}{\partial x_1} < 0$, $\frac{\partial w}{\partial x_1} < 0$ on ∂H_β.*

Lemma 4.6. *Let $\beta \in \mathbb{R}$. If $u_\beta > 0$ in H_β, then there exists $\varepsilon > 0$ such that $u_\lambda \geq 0$ in H_λ for $\lambda \in (\beta - \varepsilon, \beta)$.*

4.1. *Proof of Theorem 4.2 completed*

Recalling Lemma 4.4, we set

$$\beta^1 := \inf\{\beta \in \mathbb{R} : u_\beta \geq 0 \text{ in } H_\beta\},$$

and $\beta^1 < \infty$. It follows from (4.2) that $\beta^1 > -\infty$. According to Lemmas 4.5 and 4.6, we get $u_{\beta^1} \equiv 0$ and $w_{\beta^1} \equiv 0$. We use the same arguments to the second coordinate direction x_2, we can find $\beta^2 \in \mathbb{R}$ such that $u_{\beta^2} \equiv 0$ and $w_{\beta^2} \equiv 0$. Consider $\beta = (\beta^1, \beta^2)$, then $\tilde{u}(x) := u(x - \beta)$ and $\tilde{w}(x) := w(x - \beta)$ is a solution of equation (4.8). By the invariance of translation, we can assume that $\tilde{u}(x) = \tilde{u}(-x)$ and $\tilde{w}(x) := w(-x)$ for $x \in \mathbb{R}^2$. So it is not hard to check that each symmetry hyperplane of $\tilde{u}(x)$ and $\tilde{w}(x)$ contains the origin. Thus, repeating the above arguments in an arbitrary direction to replace the x_1-coordinate direction, we get that $\tilde{u}(x)$ and $\tilde{w}(x)$ are symmetric at any hyperplane containing the origin, and so radially symmetric. Moreover, $\tilde{u}(x)$ and $\tilde{w}(x)$ are also strictly decreasing in the distance from the symmetry center. The proof is complete.

5. Proof of Theorem 1.3

By Lemma 3.6, there is a mountain pass type critical point $u_\alpha > 0$ of I_α with $I_\alpha(u_\alpha) = c_\alpha$. In view of Remark 3.4, by (f_2) we have that sequence $\{u_\alpha\}_{\alpha\in(0,\alpha_0]}$ is uniformly bounded in $H^1(\mathbb{R}^2)$ for α. Up to a subsequence, we assume

$$\begin{aligned} &u_\alpha \rightharpoonup u_0 \quad \text{in } H^1(\mathbb{R}^2),\\ &u_\alpha \to u_0 \ a.e. \quad \text{in } \mathbb{R}^2,\\ &u_\alpha \to u_0 \text{ in } L^p(\mathbb{R}^2),\ \ p > 2 \end{aligned} \tag{5.1}$$

as $\alpha \to 0^+$. It follows from Theorem 4.2 that u_α is radially symmetric and strictly decreasing in $r = |x|$. Thus we can obtain the following lemma.

Lemma 5.1. *For fixed $\alpha \in (0, 1)$, there exists $C > 0$ independent of α such that $\|u_\alpha\|_\infty \leq C$.*

Proof. Since u_α is positive, by equation (3.1) and Lemma 3.2 we obtain

$$\begin{aligned} -\Delta u_\alpha + u_\alpha &\leq \int\limits_{|x-y|\leq 1} \frac{|x-y|^{-\alpha} - 1}{\alpha} u_\alpha^2(y)\mathrm{d}y u_\alpha(x) + f(u_\alpha)\\ &\leq C \int\limits_{|x-y|\leq 1} \frac{u_\alpha^2(y)}{|x-y|}\mathrm{d}y u_\alpha(x) + f(u_\alpha). \end{aligned} \tag{5.2}$$

In the spirit of [38], for any $R > 0$, $r \in (0, \frac{R}{2}]$, set $\eta \in C^\infty(\mathbb{R}^2)$, $0 \leq \eta \leq 1$ with $\eta(x) = 0$ if $|x| \leq R - r$ and $\eta(x) = 1$ if $|x| \geq R$ and $|\nabla \eta| \leq \frac{2}{r}$. For $K > 0$, let

$$u_{K,\alpha}(x) = \begin{cases} u_\alpha(x), & u_\alpha(x) \leq K,\\ K, & u_\alpha(x) \geq K, \end{cases}$$

and

$$v_{K,\alpha} = \eta^2 u_{K,\alpha}^{2(\gamma-1)} u_\alpha \quad \text{and } w_{K,\alpha} = \eta u_\alpha u_{K,\alpha}^{\gamma-1}.$$

Here $\gamma > 2$ will be determined later. Multiplying $v_{K,\alpha}$ in two sides of (5.2) and then integrating over the whole space, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla u_\alpha|^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x + \int_{\mathbb{R}^2} u_\alpha^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x \\
\le & C \int_{\mathbb{R}^2} \int_{|x-y|\le 1} \frac{u_\alpha^2(y)}{|x-y|} u_\alpha^2(x) \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^2} f(u_\alpha) u_\alpha \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x \\
& - 2(\gamma-1) \int_{\mathbb{R}^2} u_\alpha u_{K,\alpha}^{2\gamma-3} \eta^2 \nabla u_\alpha \nabla u_{K,\alpha} \mathrm{d}x - 2 \int_{\mathbb{R}^2} \eta u_{K,\alpha}^{2(\gamma-1)} u_\alpha \nabla u_\alpha \nabla \eta \mathrm{d}x.
\end{aligned} \tag{5.3}$$

Using Lemma 2.3(2), taking $R > 0$ (independent of α) large enough, we know that

$$C \int_{\mathbb{R}^2} \int_{|x-y|\le 1} \frac{u_\alpha^2(y)\mathrm{d}y}{|x-y|} u_\alpha^2(x) \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x \le \frac{1}{4} \int_{\mathbb{R}^2} u_\alpha^2(x) \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x. \tag{5.4}$$

It follows from (f_1)-(f_2) that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(u_\alpha)| \le \varepsilon u_\alpha + C_\varepsilon C_\theta u_\alpha (e^{\theta |u_\alpha|^2} - 1), \tag{5.5}$$

where θ is chosen small enough. Thus, using Young's inequality and combining (5.3) with (5.4), we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla u_\alpha|^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x + \int_{\mathbb{R}^2} u_\alpha^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x \\
\le & C \int_{\mathbb{R}^2} u_\alpha^2 |\nabla \eta|^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x + C \int_{\mathbb{R}^2} (e^{\theta |u|^2} - 1) u_\alpha^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)} \mathrm{d}x.
\end{aligned} \tag{5.6}$$

Observe that

$$\nabla w_{K,\alpha} = \nabla \eta u_\alpha u_{K,\alpha}^{\gamma-1} + \eta \nabla u_\alpha u_{K,\alpha}^{\gamma-1} + (\gamma-1) \eta u_\alpha u_{K,\alpha}^{\gamma-2} \nabla u_{K,\alpha},$$

then by (5.3)-(5.6), we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla w_{K,\alpha}|^2 \mathrm{d}x &\le \int_{\mathbb{R}^2} \left(|\nabla \eta| u_\alpha u_{K,\alpha}^{\gamma-1} + \gamma\eta |\nabla u_\alpha| u_{K,\alpha}^{\gamma-1}\right)^2 \mathrm{d}x \\ &\le C\gamma^2\Big(\int_{\mathbb{R}^2} u_\alpha^2 |\nabla\eta|^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u_\alpha|^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x\Big) \\ &\le C\gamma^2\Big(\int_{\mathbb{R}^2} u_\alpha^2 |\nabla\eta|^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x + \int_{\mathbb{R}^2} (e^{\theta|u_\alpha|^2}-1) u_\alpha^2 \eta^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x\Big).\end{aligned} \tag{5.7}$$

Let us fix $t > 2$ and $\theta > 0$ small, then by Lemma 2.2, there exists $C > 0$ (independent of α) such that

$$\|e^{\theta|u_\alpha|^2} - 1\|_{\frac{t}{t-2}} \le C, \tag{5.8}$$

which, together with Hölder's inequality, implies that

$$\int_{\mathbb{R}^2} |\nabla w_{K,\alpha}|^2 \mathrm{d}x \le C\gamma^2\Big(\int_{\mathbb{R}^2} u_\alpha^2|\nabla\eta|^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x + \big(\int_{\mathbb{R}^2} w_{K,\alpha}^t \mathrm{d}x\big)^{2/t}\Big). \tag{5.9}$$

For some fixed $s > t$, by Gagliardo-Nirenberg's inequality and combining with Hölder's inequality, we have

$$\begin{aligned}&\|w_{K,\alpha}\|_s \\ &\le C(\|\nabla w_{K,\alpha}\|_2 + \|w_{K,\alpha}\|_t) \\ &\le C\gamma\Big(\int_{R\ge|x|\ge R-r} u_\alpha^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x + \Big[\int_{|x|\ge R-r} u_\alpha^t u_{K,\alpha}^{(\gamma-1)t}\mathrm{d}x\Big]^{2/t}\Big)^{\frac12} \\ &\le C\gamma\Big(\Big[\int_{R\ge|x|\ge R-r} u_\alpha^t u_{K,\alpha}^{(\gamma-1)t}\mathrm{d}x\Big]^{2/t}\Big[\int_{R\ge|x|\ge R-r} 1\mathrm{d}x\Big]^{\frac{t-2}{t}} + \Big[\int_{|x|\ge R-r} u_\alpha^t u_{K,\alpha}^{(\gamma-1)t}\mathrm{d}x\Big]^{2/t}\Big)^{\frac12} \\ &\le C\gamma\Big(\int_{|x|\ge R-r} u_\alpha^t u_{K,\alpha}^{(\gamma-1)t}\mathrm{d}x\Big)^{1/t}.\end{aligned} \tag{5.10}$$

Based on the above inequality, we check that

$$\begin{aligned}\Big(\int_{|x|\ge R} u_{K,\alpha}^{\gamma s}\mathrm{d}x\Big)^{1/s} \le \Big(\int_{|x|\ge R-r} \eta^s u_{K,\alpha}^{\gamma s}\mathrm{d}x\Big)^{1/s} \le \Big(\int_{|x|\ge R-r} w_{K,\alpha}^s\mathrm{d}x\Big)^{1/s} \\ \le C\gamma\Big(\int_{|x|\ge R-r} u_\alpha^{\gamma t}\mathrm{d}x\Big)^{1/t}.\end{aligned} \tag{5.11}$$

Letting $K \to +\infty$ and applying Fatou's Lemma, we get

$$\left(\int\limits_{|x|\geq R} u_\alpha^{\gamma s}\mathrm{d}x\right)^{\frac{1}{\gamma s}} \leq C^{\frac{1}{\gamma}}\gamma^{\frac{1}{\gamma}}\left(\int\limits_{|x|\geq R-r} u_\alpha^{\gamma t}\mathrm{d}x\right)^{\frac{1}{\gamma t}}. \tag{5.12}$$

Take $\kappa = s/t > 1$, $\gamma = \kappa^n$, so that

$$\left(\int\limits_{|x|\geq R} u_\alpha^{t\kappa^{n+1}}\mathrm{d}x\right)^{\frac{1}{t\kappa^{n+1}}} \leq C^{\frac{1}{\kappa^n}}\kappa^{\frac{n}{\kappa^n}}\left(\int\limits_{|x|\geq R-r} u_\alpha^{t\kappa^n}\mathrm{d}x\right)^{\frac{1}{t\kappa^n}}. \tag{5.13}$$

Finally, we obtain

$$\left(\int\limits_{|x|\geq R} u_\alpha^{t\kappa^{n+1}}\mathrm{d}x\right)^{\frac{1}{t\kappa^{n+1}}} \leq C^{\sum\limits_{i=0}^{n}\frac{1}{\kappa^i}}\kappa^{\sum\limits_{i=0}^{n}\frac{i}{\kappa^i}}\left(\int\limits_{|x|\geq R-r} u_\alpha^{t}\mathrm{d}x\right)^{\frac{1}{t}}, \tag{5.14}$$

from which recalling that u_α is bounded in $H^1(\mathbb{R}^2)$ uniformly for α, we deduce

$$\|u_\alpha\|_{L^\infty(|x|\geq R)} \leq C_1\|u_\alpha\|_{L^t(|x|\geq R/2)} \leq C_2, \tag{5.15}$$

which $C_1, C_2 > 0$ are independent of α. Moreover, by taking $\eta \in C_0^\infty(\mathbb{R}^2, [0,1])$ satisfying $\eta(x) \equiv 1$ if $|x| \leq \bar{R}$ and $\eta(x) = 0$ if $|x| \geq 2\bar{R}$ and $|\nabla\eta| \leq \frac{2}{\bar{R}}$, we can use a similar argument to deduce that there is $C > 0$ independent of α such that

$$\|u_\alpha\|_{L^\infty(|x|\leq\bar{R})} \leq C. \tag{5.16}$$

Indeed, in this case, (5.6) can be rewritten as

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2}|\nabla u_\alpha|^2\eta^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x + \int\limits_{\mathbb{R}^2} u_\alpha^2\eta^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x\\
\leq &C\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2}\frac{u_\alpha^2(y)\mathrm{d}y}{|x-y|}u_\alpha^2(x)\eta^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x + C\int\limits_{\mathbb{R}^2} u_\alpha^2|\nabla\eta|^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x\\
&+C\int\limits_{\mathbb{R}^2}(e^{\theta|u|^2}-1)u_\alpha^2\eta^2 u_{K,\alpha}^{2(\gamma-1)}\mathrm{d}x.
\end{aligned} \tag{5.17}$$

Moreover, by Hardy-Littlewood-Sobolev's inequality, we can easy to get that there exists $C > 0$ (independent of α) such that

$$\int\limits_{|x|\leq 2\bar{R}}\left(\int\limits_{\mathbb{R}^2}\frac{u_\alpha^2(y)\mathrm{d}y}{|x-y|}\right)^{\frac{t}{t-2}}\mathrm{d}x \leq C. \tag{5.18}$$

It follows from (5.8), (5.17) and (5.18) that (5.9) holds also true for this case. Arguing as above, we obtain (5.16). Together with (5.15) and (5.16), we have $\|u_\alpha\|_\infty \leq C$ uniformly for α. The proof is complete. $\square$

Lemma 5.2. *There exist $c, C > 0$ such that*

$$u_\alpha(x) \le C\exp(-c|x|) \quad \textit{for } x \in \mathbb{R}^2$$

for $\alpha \in (0, \alpha_1)$. Here, C, c are independent of α and α_1 has been given in Theorem 4.2.

Proof. We have known that u_α is radially symmetric and strictly decreasing in $r = |x|$ for any fixed $\alpha \in (0, \alpha_1)$. Now recalling Radial Lemma A.IV in [9], there exists $C > 0$ independent of α such that

$$|u_\alpha(x)| \le C|x|^{-1}\|u_\alpha\| \le C|x|^{-1}, \quad x \ne 0$$

which implies that

$$\lim_{|x|\to\infty} |u_\alpha(x)| = 0 \quad \text{uniformly for } \alpha \in (0, \alpha_1).$$

By the uniform boundedness of u_α in $L^\infty(\mathbb{R}^2)$ (see Lemma 5.1), and a comparison principle (see [39]), there exist $C, c > 0$ (independent of α) such that

$$u_\alpha(x) \le C\exp(-c|x|) \quad \text{for } x \in \mathbb{R}^2.$$

The proof is complete. $\square$

The proof of Theorem 1.3. Now we are ready to use the Lebesgue dominated convergence theorem to prove that I'_α is weakly sequence continuous in $H^1(\mathbb{R}^2)$. For any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned} I'_\alpha(u_\alpha)\varphi &= \int_{\mathbb{R}^2} \nabla u_\alpha \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^2} u_\alpha\varphi \mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha}-1}{\alpha} u_\alpha^2(y)\mathrm{d}y u_\alpha(x)\varphi(x)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} f(u_\alpha)\varphi \mathrm{d}x. \end{aligned} \tag{5.19}$$

If $|x-y| \le 1$, then it follows from Lemma 3.2 that

$$\begin{aligned} 0 &\le \left|\frac{|x-y|^{-\alpha}-1}{\alpha} u_\alpha^2(y)u_\alpha(x)\varphi(x)\right| \\ &\le \left|\frac{1}{|x-y|} u_\alpha^2(y)u_\alpha(x)\varphi(x)\right| := g_\alpha(x,y). \end{aligned} \tag{5.20}$$

Using the Hardy-Littlewood-Sobolev inequality, (5.1) and the fact that φ has a compact support, we deduce that sequence $\{g_\alpha(x,y)\}_{\alpha\in(0,1)} \subset L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ has a strongly convergent subsequence in $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$. Thus, by (5.20) and the Lebesgue dominated convergence theorem,

$$\int \int_{|x-y|\le 1} \frac{|x-y|^{-\alpha}-1}{\alpha} u_\alpha^2(y)\mathrm{d}y u_\alpha(x)\varphi(x)\mathrm{d}x \to -\int \int_{|x-y|\le 1} \ln(|x-y|)u_0^2(y)\mathrm{d}y u_0(x)\varphi(x)\mathrm{d}x. \tag{5.21}$$

When $|x-y|\ge 1$, there exists $\tau=\tau(|x-y|)\in(0,1)$ such that

$$0 \ge G_\alpha(x-y) = \frac{|x-y|^{-\alpha}-1}{\alpha} = -|x-y|^{-\tau\alpha}\ln|x-y|, \tag{5.22}$$

where τ depends upon $|x-y|$. Since φ has a compact support, it follows from Lemma 5.2 that,

$$\begin{aligned} &\left|\frac{|x-y|^{-\alpha}-1}{\alpha} u_\alpha^2(y)u_\alpha(x)\varphi(x)\right| \\ &= \left||x-y|^{-\tau\alpha}\ln|x-y|u_\alpha^2(y)u_\alpha(x)\varphi(x)\right| \\ &\le \left|(|x|+|y|)u_\alpha^2(y)u_\alpha(x)\varphi(x)\right|, \qquad \text{for any } x\in \operatorname{spt}(\varphi). \end{aligned} \tag{5.23}$$

Using Lemma 5.2, combining (5.22) with (5.23), similarly to (5.21), by the Lebesgue dominated convergence theorem, one has

$$\begin{aligned} &\int \int_{|x-y|\ge 1} \frac{|x-y|^{-\alpha}-1}{\alpha} u_\alpha^2(y)\mathrm{d}y u_\alpha(x)\varphi(x)\mathrm{d}x \\ &\to -\int \int_{|x-y|\ge 1} \ln|x-y|u_0^2(y)\mathrm{d}y u_0(x)\varphi(x)\mathrm{d}x. \end{aligned} \tag{5.24}$$

Moreover, by Fatou's lemma, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|u_0^2(y)\mathrm{d}y u_0^2(x)\mathrm{d}x\right| \\ &\le \liminf_{\alpha\to 0}\Bigg(\int \int_{|x-y|\le 1} G_\alpha(x-y)u_\alpha^2(y)\mathrm{d}y u_\alpha^2(x)\mathrm{d}x \\ &- \int \int_{|x-y|\ge 1} G_\alpha(x-y)u_\alpha^2(y)\mathrm{d}y u_\alpha^2(x)\mathrm{d}x\Bigg). \end{aligned} \tag{5.25}$$

Similarly to (5.20), we can use Hardy-Littlewood-Sobolev's inequality to get

$$\int \int_{|x-y|\le 1} G_\alpha(x-y)u_\alpha^2(y)\mathrm{d}y u_\alpha^2(x)\mathrm{d}x < +\infty \tag{5.26}$$

uniformly for α. So by Remark 3.4, we further deduce that

$$
\begin{aligned}
&\int\int_{|x-y|\geq 1} G_\alpha(x-y)u_\alpha^2(y)\mathrm{d}yu_\alpha^2(x)\mathrm{d}x \\
&\leq I_\alpha(u_\alpha)+\int\int_{|x-y|\leq 1} G_\alpha(x-y)u_\alpha^2(y)\mathrm{d}yu_\alpha^2(x)\mathrm{d}x+\int_{\mathbb{R}^2} F(x,u_\alpha)\mathrm{d}x-\frac{1}{2}\|u_\alpha\|^2 \\
&<+\infty.
\end{aligned} \tag{5.27}
$$

Together (5.25), (5.26) with (5.27), we have

$$
\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y|u_0^2(y)\mathrm{d}yu_0^2(x)\mathrm{d}x\right|<+\infty. \tag{5.28}
$$

Based on (5.21), (5.24) and (5.28), by taking the limit in (5.19), we have $I'(u_0)=0$ with $I(u_0)<+\infty$, that is, $u_0\in H^1(\mathbb{R}^2)$ solves equation (1.5).
We now claim $u_0\neq 0$. Assume on the contrary that $u_\alpha \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$, and so $u_\alpha\to 0$ in $L^s(\mathbb{R}^2)$ for $s\in(2,+\infty)$. So by (3.9), Lemma 3.2 and Hardy-Littlewood-Sobolev's inequality, we have

$$
\begin{aligned}
I'_\alpha(u_\alpha)u_\alpha &= \|u_\alpha\|^2-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\alpha(x-y)u_\alpha^2(y)u_\alpha^2(x)\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2} f(x,u_\alpha)u_\alpha\mathrm{d}x \\
&\geq \|u_\alpha\|^2-\int\int_{|x-y|\leq 1} G_\alpha(x-y)u_\alpha^2(y)u_\alpha^2(x)\mathrm{d}x\mathrm{d}y+o_\alpha(1) \\
&\geq \|u_\alpha\|^2-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}u_\alpha^2(y)u_\alpha^2(x)\mathrm{d}x\mathrm{d}y+o_\alpha(1) \\
&\geq \|u_\alpha\|^2-C\|u_\alpha\|_{\frac{8}{3}}^4+o_\alpha(1),
\end{aligned}
$$

which means $u_\alpha\to 0$ in $H^1(\mathbb{R}^2)$. Then by Remark 3.4, we have

$$
\begin{aligned}
a &\leq I_\alpha(u_\alpha) \\
&= \frac{1}{2}\|u_\alpha\|^2-\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\alpha(x-y)u_\alpha^2(y)u_\alpha^2(x)\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2} F(x,u_\alpha)\mathrm{d}x \\
&= -\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\alpha(x-y)u_\alpha^2(y)u_\alpha^2(x)\mathrm{d}x\mathrm{d}y+o_\alpha(1) \\
&= o_\alpha(1),
\end{aligned}
$$

which yields a contradiction. Furthermore, similarly to (5.21), (5.24), by Lemma 5.2 and the Lebesgue dominated convergence theorem, we have

$$
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|x-y|^{-\alpha}-1}{\alpha}u_\alpha^2(y)u_\alpha^2(x)\mathrm{d}y\mathrm{d}x\to -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_0^2(y)u_0^2(x)\mathrm{d}x\mathrm{d}y, \tag{5.29}
$$

which implies that $u_\alpha \to u_0$ in $H^1(\mathbb{R}^2)$ as $\alpha \to 0^+$. Thus, u_0 is a positive solution of (1.5). □

6. Proof of Theorem 1.4

In this section, we are ready to restrict the energy functional to the radial space $H_r^1(\mathbb{R}^2)$ in studying the existence of nontrivial solutions to equation (1.5) with nonlinearity f satisfying (f_3)-(f_5). It follows from (f_3) and (f_4) that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon |t|^{p-1}, \quad t \in \mathbb{R},\ p \in (3, +\infty). \tag{6.1}$$

Here, we also study the modified equation (3.1) by using perturbation fashion, since the fact that I is not well defined on $H^1(\mathbb{R}^2)$. Since we do not impose the well-known *4-Ambrosetti-Rabinowitz* condition, the boundedness of the Palais-Smale sequence becomes complicated. In order to overcome this difficulty, we add another perturbation technique developed in [34,36] to equation (3.1). We now give more details to describe such a technique. Set

$$\lambda \in (0, 1], r \in (\max\{p, 4\}, +\infty)$$

and $G_\alpha(x) = \frac{|x|^{-\alpha}-1}{\alpha}$, $\alpha \in (0, 1)$, $x \in \mathbb{R}^2 \setminus \{0\}$. Consider the following modified problem

$$-\Delta u + u - (G_\alpha(x) * u^2)u + \lambda \left(\int_{\mathbb{R}^2} u^2 \mathrm{d}x \right)^{\frac{1}{4}} u = f(u) + \lambda |u|^{r-2}u, u \in H_r^1(\mathbb{R}^2), \tag{6.2}$$

whose associated functional is given by

$$I_{\alpha,\lambda}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^2} (G_\alpha(x) * u^2)u^2 \mathrm{d}x + \frac{2\lambda}{5}\|u\|_2^{\frac{5}{2}} - \int_{\mathbb{R}^2} F(u)\mathrm{d}x - \frac{\lambda}{r}\|u\|_r^r.$$

According to the definition of G_α, using the Hardy-Littlewood-Sobolev inequality, it is not hard for fixed $\alpha > 0$ to show that the perturbation functional $I_{\alpha,\lambda}$ is well-defined on $H^1(\mathbb{R}^2)$ and $I_{\alpha,\lambda} \in C^1(H_r^1(\mathbb{R}^2), \mathbb{R})$ and

$$I'_{\alpha,\lambda}(u)v = I'_\alpha(u)v + \lambda\|u\|_2^{\frac{1}{2}} \int_{\mathbb{R}^2} uv\mathrm{d}x - \lambda \int_{\mathbb{R}^2} |u|^{r-2}uv\mathrm{d}x$$

for $u, v \in H_r^1(\mathbb{R}^2)$. For any critical point $u \in H_r^1(\mathbb{R}^2)$ of $I_{\alpha,\lambda}$, the following Pohozaev identity holds

$$P_{\alpha,\lambda}(u) := \|u\|_2^2 + \frac{1}{\alpha}\|u\|_2^4 - \frac{4-\alpha}{4\alpha}Q(u) + \lambda\|u\|_2^{\frac{5}{2}} - 2\int_{\mathbb{R}^2} (F(u) + \frac{\lambda}{r}|u|^r)\mathrm{d}x = 0. \tag{6.3}$$

The conditions (f_3)-(f_5) imply that the perturbed functional $I_{\alpha,\lambda}$ satisfies the mountain pass geometry. More precisely,

Lemma 6.1. *Suppose* (f_3)-(f_5) *hold, then*
(i) there exist $\rho, \delta_0 > 0$ such that $I_{\alpha,\lambda}|_{S_\rho}(u) \geq \delta_0$ for every $u \in S_\rho = \{u \in H_r^1(\mathbb{R}^2) : \|u\| = \rho\}$;
(ii) there is $e \in H^1(\mathbb{R}^2)$ with $\|e\| > \rho$ such that $I_{\alpha,\lambda}(e) < 0$.

Proof. The proof of conclusion (i) is similar to that of Lemma 3.3, so we omit it. It remains to prove conclusion (ii). Arguing as that of Lemma 3.3, we take $e_0 \in H_r^1(\mathbb{R}^2) \cap C_0^\infty(\mathbb{R}^2)$ such that $e_0(x) \equiv 1$ for $x \in B_{\frac{1}{8}}(0)$, $e_0(x) \equiv 0$ for $x \in \mathbb{R}^2 \setminus B_{\frac{1}{4}}(0)$ and $|\nabla e_0(x)| \leq C$. Then we have

$$\begin{aligned} I_{\alpha,\lambda}(se_0) &\leq \frac{s^2}{2}\|e_0\|^2 - \frac{s^4}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha}-1}{\alpha} e_0^2(y)e_0^2(x)\mathrm{d}y\mathrm{d}x + \frac{2s^{5/2}}{5}\|e_0\|_2^{\frac{5}{2}} - \frac{1}{2}\int_{\mathbb{R}^2} F(se_0)\mathrm{d}x \\ &\leq \frac{s^2}{2}\|e_0\|^2 - \frac{s^4}{4}\int_{|x|\leq\frac{1}{4}}\int_{|y|\leq\frac{1}{4}} \ln\frac{1}{|x-y|} e_0^2(y)e_0^2(x)\mathrm{d}y\mathrm{d}x + \frac{2s^{5/2}}{5}\|e_0\|_2^{\frac{5}{2}} \\ &\leq \frac{s^2}{2}\|e_0\|^2 - \frac{s^4\ln 2}{4}\left(\int_{\mathbb{R}^2} e_0^2(x)\mathrm{d}x\right)^2 + \frac{2s^{5/2}}{5}\|e_0\|_2^{\frac{5}{2}}, \end{aligned} \tag{6.4}$$

which implies that there exists $s_0 > 0$ large enough such that $I_{\alpha,\lambda}(s_0e_0) < 0$. □

Recall the well-known mountain pass theorem (see [46]), then there exists a $(PS)_{c_{\alpha,\lambda}}$ sequence $\{u_n\} \subset H_r^1(\mathbb{R}^2)$, that is,

$$I_{\alpha,\lambda}(u_n) \to c_{\alpha,\lambda} \quad \text{and} \quad I'_{\alpha,\lambda}(u_n) \to 0, \tag{6.5}$$

where $c_{\alpha,\lambda}$ is the mountain pass level characterized by

$$c_{\alpha,\lambda} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\alpha,\lambda}(\gamma(t)) \tag{6.6}$$

with

$$\Gamma := \{\gamma \in C^1([0,1], H_r^1(\mathbb{R}^2)) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\},$$

where e has been given in Lemma 6.1. It is easy to see from Lemma 6.1 that there exist $a, b > 0$ independent of α, λ such that $a < c_{\alpha,\lambda} < b$. We state the following lemma to ensure that functional $I_{\alpha,\lambda}$ has at least a critical point $u \in H_r^1(\mathbb{R}^2)$ at the mountain pass level $c_{\alpha,\lambda}$.

Lemma 6.2. *For fixed* $\lambda \in (0, 1]$, *let* $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ *be a* $(PS)_{c_{\alpha,\lambda}}$ *sequence of* $I_{\alpha,\lambda}$. *Then there exists* $u \in H_r^1(\mathbb{R}^2)$ *such that* $I'_{\alpha,\lambda}(u) = 0$ *and* $I_{\alpha,\lambda}(u) = c_{\alpha,\lambda}$.

Proof. We first show that the sequence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Obviously, there exist $C_1, C_2 > 0$ such that

$$C_1 + C_2\|u_n\| \geq I_{\alpha,\lambda}(u_n) - \frac{1}{4}I'_{\alpha,\lambda}(u_n)u_n$$
$$= \frac{1}{4}\|u_n\|^2 + \frac{3\lambda}{20}\|u_n\|_2^{\frac{5}{2}} + \int_{\mathbb{R}^2} (\frac{1}{4}f(u_n)u_n - F(u_n))\mathrm{d}x + \frac{r-4}{4r}\lambda\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x.$$

By (6.1) it follows that

$$C_1 + C_2\|u_n\| \geq \frac{1}{8}\|u_n\|^2 + \frac{3\lambda}{20}\|u_n\|_2^{\frac{5}{2}} - C_3\int_{\mathbb{R}^2} |u_n|^p \mathrm{d}x + \frac{r-4}{4r}\lambda\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x. \tag{6.7}$$

Observe that for any large $B_1 > 0$, there exists $B_2 > 0$ such that

$$\frac{3}{20}\|u_n\|_2^{\frac{5}{2}} \geq B_1\|u_n\|_2^2 - B_2,$$

which implies by (6.7) that

$$C_1 + \lambda B_2 + C_2\|u_n\| \geq \frac{1}{8}\|u_n\|^2 + \int_{\mathbb{R}^2} \left(\lambda B_1|u_n|^2 - C_3|u_n|^p + \frac{r-4}{4r}\lambda|u_n|^r\right)\mathrm{d}x. \tag{6.8}$$

We remark that $\lambda B_1 t^2 - C_3 t^p + \frac{r-4}{4r}\lambda t^r \geq 0$ for $t \geq 0$, since B_1 can be chosen arbitrary large. Thus, it follows from (6.8) that $\|u_n\| \leq C$ for some C independently of n. It remains to prove the strong convergence of sequence $\{u_n\}$. Up to a subsequence, we assume that there is $u \in H_r^1(\mathbb{R}^2)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } H_r^1(\mathbb{R}^2), \\ &u_n \to u \quad \text{in } L^p(\mathbb{R}^2),\ 3 < p < +\infty, \\ &\|u_n\|_2^2 \to B \ \text{ for some } B \geq 0. \end{aligned} \tag{6.9}$$

Observe that

$$J_{B,\alpha,\lambda}(u_n) \to c_{\alpha,\lambda} + \frac{B^2}{4\alpha} + \frac{\lambda B^{5/4}}{10}$$

and $J'_{B,\alpha,\lambda}(u_n) \to 0$ in H_r^{-1} with $J'_{B,\alpha,\lambda}(u_0) = 0$, where

$$\begin{aligned} J_{B,\alpha,\lambda}(u) :=& \frac{1}{2}\|u\|^2 + (\frac{B}{2\alpha} + \frac{B^{1/4}\lambda}{2})\int_{\mathbb{R}^2} |u|^2 \mathrm{d}x - \frac{1}{4\alpha}\int_{\mathbb{R}^2} (\frac{1}{|x|^\alpha} * u^2)u^2 \mathrm{d}x \\ &- \int_{\mathbb{R}^2} F(u)\mathrm{d}x - \lambda\int_{\mathbb{R}^2} |u|^r \mathrm{d}x. \end{aligned} \tag{6.10}$$

In view of (f_3) and (f_4), one has

$$\int_{\mathbb{R}^2} F(u_n) \to \int_{\mathbb{R}^2} F(u_0), \quad \int_{\mathbb{R}^2} f(u_n)u_n \to \int_{\mathbb{R}^2} f(u_0)u_0. \tag{6.11}$$

Moreover, Hardy-Littlewood-Sobolev's inequality together with (6.9) implies that

$$Q(u_n) \to Q(u_0). \tag{6.12}$$

From (6.9), (6.11) and (6.12), we deduce that $u_n \to u$ in $H_r^1(\mathbb{R}^2)$. Thus, u is a positive critical point of $I_{\alpha,\lambda}$ and $I_{\alpha,\lambda}(u) = c_{\alpha,\lambda}$. $\square$

Based on Lemma 6.2, for fixed $\lambda \in (0, 1]$, there exists $u_\lambda \in H_r^1(\mathbb{R}^2) \setminus \{0\}$ such that $I'_{\alpha,\lambda}(u_\lambda) = 0$. Choosing a sequence $\{\lambda_n\} \subset (0, 1]$ satisfying $\lambda_n \to 0^+$, then we find a sequence of nontrivial critical points sequence $\{u_{\lambda_n}\}$ (still denoted by $\{u_n\}$) of I_{α,λ_n} with $I_{\alpha,\lambda_n}(u_n) = c_{\alpha,\lambda_n}$.

Lemma 6.3. *For fixed $\alpha \in (0, 1)$, there exists $v \in H_r^1(\mathbb{R}^2) \setminus \{0\}$ such that $I'_\alpha(v) = 0$.*

Proof. We now claim sequence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Indeed, by $I'_{\alpha,\lambda_n}(u_n)u_n = 0$, we have

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + u_n^2)\mathrm{d}x + \frac{1}{\alpha}\|u_n\|_2^4 - \frac{1}{\alpha}Q(u_n) - \int_{\mathbb{R}^2} [f(u_n)u_n + \lambda_n |u_n|^r]\mathrm{d}x + \lambda_n \|u_n\|_2^{\frac{5}{2}} = 0. \tag{6.13}$$

Recalling hypothesis (f_5), it follows from (6.13) that for any $a > 0$,

$$\begin{aligned} a\int_{\mathbb{R}^2} F(u_n) \le & \frac{a}{\mu}\int_{\mathbb{R}^2} (|\nabla u_n|^2 + u_n^2)\mathrm{d}x + \frac{a}{\alpha\mu}\|u_n\|_2^4 - \frac{a}{\alpha\mu}Q(u_n) \\ & + \frac{a\lambda_n}{\mu}\|u_n\|_2^{\frac{5}{2}} - \frac{a\lambda_n}{\mu}\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x. \end{aligned} \tag{6.14}$$

Combining (6.14) with (6.3), we have for any $b \in \mathbb{R}$

$$\begin{aligned} (a+b)\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \le & \frac{a}{\mu}\int_{\mathbb{R}^2} |\nabla u_n|^2 \mathrm{d}x + (\frac{a}{\mu} + \frac{b}{2})\int_{\mathbb{R}^2} |u_n|^2 \mathrm{d}x \\ & + \left(\frac{a}{\mu\alpha} + \frac{b}{2\alpha}\right)\|u_n\|_2^4 - \left(\frac{a}{\alpha\mu} + \frac{4-\alpha}{8\alpha}b\right)Q(u_n) \\ & + (\frac{a}{\mu} + \frac{b}{2})\lambda_n \|u_n\|_2^{\frac{5}{2}} - (\frac{a}{\mu} + \frac{b}{2r})\lambda_n \|u_n\|_r^r. \end{aligned} \tag{6.15}$$

Take $a = 1 - b$ in (6.15), then it follows from the definition of I_{α,λ_n} that

$$
\begin{aligned}
c_{\alpha,\lambda_n} =& I_{\alpha,\lambda_n}(u_n) \\
\geq & \left(\frac{1}{2}-\frac{1-b}{\mu}\right)\int\limits_{\mathbb{R}^2}|\nabla u_n|^2\mathrm{d}x+\left(\frac{1}{2}-\frac{b}{2}-\frac{1-b}{\mu}\right)\int\limits_{\mathbb{R}^2}|u_n|^2\mathrm{d}x \\
& +\left(\frac{1}{4\alpha}-\frac{1-b}{\alpha\mu}-\frac{b}{2\alpha}\right)\|u_n\|_2^4+\left(\frac{2}{5}-\frac{1-b}{\mu}-\frac{b}{2}\right)\lambda_n\|u_n\|_2^{\frac{5}{2}} \\
& +\left(\frac{-1}{4\alpha}+\frac{1-b}{\alpha\mu}+\frac{4-\alpha}{8\alpha}b\right)Q(u_n) \\
& +\left(\frac{-1}{r}+\frac{1-b}{\mu}+\frac{b}{2r}\right)\lambda_n\|u_n\|_r^r.
\end{aligned}
\tag{6.16}
$$

By choosing $b=\frac{\mu-4}{2(\mu-2)}$, it is easy to check that all the coefficients are nonnegative, that is,

- $\dfrac{1}{2}-\dfrac{1-b}{\mu}>0$, since $\mu\in(3,4)$;
- $\dfrac{1}{2}-\dfrac{b}{2}-\dfrac{1-b}{\mu}>0$, since $\mu\in(3,4)$;
- $\dfrac{1}{4\alpha}-\dfrac{1-b}{\alpha\mu}-\dfrac{b}{2\alpha}=0$;
- $\dfrac{2}{5}-\dfrac{1-b}{\mu}-\dfrac{b}{2}>0$ since $\mu\in(3,4)$;
- $\dfrac{-1}{4\alpha}+\dfrac{1-b}{\alpha\mu}+\dfrac{4-\alpha}{8\alpha}b>0$;
- $\dfrac{-1}{r}+\dfrac{1-b}{\mu}+\dfrac{b}{2r}>0$ since $\mu\in(3,4)$.

Let $\lim_{n\to+\infty}c_{\alpha,\lambda_n}=c_\alpha\in[a,b]$, then we have $I_{\alpha,\lambda_n}(u_n)=c_\alpha+o(1)$. From (6.16) we have $\|u_n\|\leq C$ for some C independent of n. Moreover, for any $\varphi\in C_0^\infty(\mathbb{R}^2)$, we have

$$
I'_{\alpha,\lambda_n}(u_n)\varphi=I'_\alpha(u_n)\varphi+\lambda_n\|u_n\|_2^{\frac{1}{2}}\int\limits_{\mathbb{R}^2}u_n\varphi\mathrm{d}x+\lambda_n\int\limits_{\mathbb{R}^2}|u_n|^{r-2}u_n\varphi\mathrm{d}x=0.
$$

Thus, $\{u_n\}$ is a bounded Palais-Smale sequence of I_α with level c_α. Observe that $J_{B,\alpha}(u_n)\to c_\alpha+\frac{B^2}{4\alpha}$ and $J'_{B,\alpha}(u_n)\to 0$ in H_r^{-1}, where $J_{B,\alpha}$ has been defined in (3.8). Thus, arguing similarly as in the proof of Lemma 6.2, there exists $v\in H_r^1(\mathbb{R}^2)$ such that $I'_\alpha(v)=0$ and $I_\alpha(v)=c_\alpha$. □

6.1. Proof of Theorem 1.4 completed

In view of Lemma 6.3, we have that for fixed $\alpha\in(0,1)$, there exists $u_\alpha\in H_r^1(\mathbb{R}^2)\setminus\{0\}$ such that $I'_\alpha(u_\alpha)=0$. As can be seen in the proof of Lemma 6.3, we can also prove that $\{u_\alpha\}_{\alpha\in(0,1)}$ is bounded in $H_r^1(\mathbb{R}^2)$ uniformly for $\alpha\in(0,1)$. Now recalling Radial Lemma A.II in [9], there exists $C_1,C_2>0$ independent of α such that

$$
|u_\alpha(x)|\leq C|x|^{\frac{-1}{2}}\|u_\alpha\|\leq C_1|x|^{\frac{-1}{2}}\quad\text{for }|x|\geq C_2,
$$

which implies that

$$\lim_{|x|\to\infty} |u_\alpha(x)| = 0 \quad \text{uniformly for } \alpha \in (0,1).$$

Moreover, the uniform boundedness of u_α in $L^\infty(\mathbb{R}^2)$ can be obtained by a slight modification of the proof of Lemma 5.1. Based on above, by a comparison principle (see [39]), there exist $C, c > 0$ (independent of α) such that

$$u_\alpha(x) \le C\exp(-c|x|) \quad \text{for } x \in \mathbb{R}^2.$$

Furthermore, as arguing in the proof of Theorem 1.3, we prove that (1.5) has at least a positive radial solution $u_0 \in H_r^1(\mathbb{R}^2)$. □

7. Proof of Theorem 1.5

In this section, we turn to prove Theorem 1.5, where gives the existence of positive solutions for a class of concrete nonlinearities that $F(u) = \frac{\varrho}{3}|u|^3$, $\varrho > 0$. In the following, we will use the same procedure as that of Theorem 1.4 to study this case. But a key difference from the previous case is the proof of the boundedness of solution sequence with respect to the perturbation parameters α, λ. More precisely, observe that Lemmas 6.1 and 6.2 still hold in this case. Therefore, the mountain pass theorem (see [46]) implies that for fixed $\lambda \in (0,1]$, there exists $u_{\alpha,\lambda} \in H_r^1(\mathbb{R}^2)$ such that $I'_{\alpha,\lambda}(u_{\alpha,\lambda}) = 0$ with $I_{\alpha,\lambda}(u_{\alpha,\lambda}) = c_{\alpha,\lambda}$, where $c_{\alpha,\lambda}$ is a mountain pass level characterized by

$$c_{\alpha,\lambda} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\alpha,\lambda}(\gamma(t)) \tag{7.1}$$

with

$$\Gamma := \left\{\gamma \in C^1([0,1], H_r^1(\mathbb{R}^2)) : \gamma(0) = 0 \quad \text{and} \quad I_{\alpha,\lambda}(\gamma(1)) < 0\right\}.$$

Let us stress that, there exist $a, b > 0$ independently of α, λ such that $a < c_{\alpha,\lambda} < b$. Now we establish the following lemma to get the boundedness of solution $u_{\alpha,\lambda}$ uniformly for α, λ. Choosing a sequence $\{\lambda_n\} \subset (0,1]$ satisfying $\lambda_n \to 0^+$, denote $\{u_{\alpha,\lambda_n}\}$ by $\{u_n\}$.

Lemma 7.1. *Sequence $\{u_n\}$ is a bounded sequence in $H_r^1(\mathbb{R}^2)$ uniformly for n.*

Proof. Multiplying $I_{\alpha,\lambda_n}(u_n)$, $I'_{\alpha,\lambda_n}(u_n)u_n = 0$ and $P_{\alpha,\lambda_n}(u_n) = 0$ by 1, $-1/2$ and $1/4$ respectively and adding them up, we get

$$\begin{aligned} I_{\alpha,\lambda_n}(u_n) =& \frac{1}{4}\int_{\mathbb{R}^2} u_n^2 \mathrm{d}x + \frac{1}{16}Q(u_n) + \frac{3\lambda_n}{20}\|u_n\|_2^{\frac{5}{2}} \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} f(u_n)u_n - 3F(u_n)\mathrm{d}x + \frac{(r-3)\lambda_n}{2r}\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x, \end{aligned} \tag{7.2}$$

which implies that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^2)$ uniformly for α, λ_n. We now prove that there exists $C > 0$ independent of α, n such that

$$\|\nabla u_n\|_2 \leq C. \tag{7.3}$$

Arguing by contradiction, suppose that, up to subsequence, $\|\nabla u_n\|^2 \to \infty$ as $n \to \infty$. Set $k_n := \|\nabla u_n\|_2^{\frac{-1}{2}}$, then $k_n \to 0$. Let us define $z_n \in H_r^1(\mathbb{R}^2)$ by $z_n(x) := k_n^2 u_n(k_n x)$, and then we have

$$\|\nabla z_n\|_2^2 = 1, \quad \|z_n\|_s^s = k_n^{2s-2}\|u_n\|_s^s \tag{7.4}$$

for all $n \in \mathbb{N}$ and $s \in [1, +\infty)$. In particular, $\|z_n\|_2^2 = k_n^2\|u_n\|_2^2 \to 0$ as $n \to \infty$. On the one hand, multiplying $I_{\alpha,\lambda_n}(u_n)$ by k_n^4, we deduce from (7.2) and (7.4) that there exists $K > 0$ such that

$$\begin{aligned} K k_n^4 \geq & \frac{k_n^2}{4}\int_{\mathbb{R}^2} z_n^2 \mathrm{d}x + \frac{1}{16k_n^\alpha}\int_{\mathbb{R}^4} \frac{z_n^2(x)z_n^2(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ & + \frac{3\lambda_n k_n^{3/2}}{20}\|z_n\|_2^{\frac{5}{2}} + \frac{(r-3)\lambda_n k_n^{6-2r}}{2r}\int_{\mathbb{R}^2} |z_n|^r \mathrm{d}x. \end{aligned} \tag{7.5}$$

Hence,

$$\|z_n\|_2 = o(k_n^\tau), \quad \int_{\mathbb{R}^4} \frac{z_n^2(x)z_n^2(y)}{|x-y|^\alpha}\mathrm{d}x = o(k_n^{3+\tau+\alpha}), \quad \lambda_n\|z_n\|_r^r = o(k_n^{2r-3+\tau}), \ 0 < \tau < 1. \tag{7.6}$$

On the other hand, multiplying the corresponding Pohozaev identity $P_{\alpha,\lambda_n}(u_n) = 0$ by k_n^4, by (7.4) we have

$$\begin{aligned} & k_n^2\|z_n\|_2^2 + \frac{1}{\alpha}\|z_n\|_2^4 - \frac{4-\alpha}{4\alpha}\frac{1}{k_n^\alpha}\int_{\mathbb{R}^4} \frac{z_n^2(x)z_n^2(y)}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ & - 2k_n^4\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x + \lambda_n k_n^{3/2}\|z_n\|_2^{\frac{5}{2}} + \frac{2\lambda_n k_n^{6-2r}}{r}\|z_n\|_r^r = 0. \end{aligned} \tag{7.7}$$

Observe that

$$0 \leq k_n^4\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = k_n^4\int_{\mathbb{R}^2} \frac{\varrho}{3}|u_n|^3\mathrm{d}x = \frac{\varrho}{3}\|z_n\|_3^3. \tag{7.8}$$

Moreover, we use the Gagliardo-Nirenberg inequality and (7.4) to get

$$\|z_n\|_3^3 \leq C\|z_n\|_2^2\|\nabla z_n\|_2 \leq C\|z_n\|_2^2$$

for $n \in \mathbb{N}$. Together (7.6) with (7.7) and (7.8), we infer that

$$k_n^4 \int\limits_{\mathbb{R}^2} F(u_n)\mathrm{d}x = o(1), \quad \left| \int\limits_{\mathbb{R}^4} \left(\frac{|x-y|^{-\alpha}}{\alpha k_n^\alpha} - \frac{1}{\alpha}\right) z_n^2(x) z_n^2(y)\mathrm{d}x\mathrm{d}y \right| = o(1). \tag{7.9}$$

By $I'_{\alpha,\lambda_n}(u_n)u_n = 0$, we also have

$$\begin{aligned} 0 = k_n^4 \Bigg(&\|\nabla u_n\|_2^2 + \|u_n\|_2^2 - \int\limits_{\mathbb{R}^4} \frac{|x-y|^{-\alpha} - 1}{\alpha} u_n^2(x) u_n^2(y)\mathrm{d}x\mathrm{d}y \\ &- \int\limits_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x + \lambda_n \|u_n\|_2^{\frac{5}{2}} - \lambda_n \|u_n\|_r^r \Bigg). \end{aligned} \tag{7.10}$$

It follows from (7.4)-(7.10) that

$$\begin{aligned} 0 &= 1 + k_n^2 \|z_n\|_2^2 + o(1) - \int\limits_{\mathbb{R}^4} \left(\frac{\lambda_n |x-y|^{-\alpha}}{\alpha k_n^\alpha} - \frac{1}{\alpha}\right) z_n^2(x) z_n^2(y)\mathrm{d}x\mathrm{d}y - \lambda_n k_n^4 \int\limits_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x \\ &= 1 + o(1) - \lambda_n k_n^4 \int\limits_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x. \end{aligned} \tag{7.11}$$

Similar to (7.8), we infer that

$$\lambda_n k_n^4 \int\limits_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x = o(1),$$

which contradicts with (7.11). We thus conclude that (7.3) holds. The proof is complete. □

The proof of Theorem 1.5. Based on Lemma 7.1, arguing similarly as in the proof of Lemma 6.2, we obtain that for any fixed $\alpha \in (0,1)$, there exists $u_\alpha \in H_r^1(\mathbb{R}^2)$ such that $I'(u_\alpha) = 0$ and $I(u_\alpha) = c_\alpha$ with $c_\alpha \in [a,b]$. In view of Lemma 7.1, it is easy to check by an easy modification that $\{u_\alpha\}_{\alpha\in(0,1)}$ is bounded in $H_r^1(\mathbb{R}^2)$. The remaining proof is the same as that of Theorem 1.3, we omit it. □

8. Proof of Theorem 1.6

In this section, we are ready to prove the existence of positive solutions to (1.5) under that f satisfies (f_3) and (f_6)-(f_7). Here we still use the same perturbation fashion as that of Theorem 1.4 and Theorem 1.5 to prove the boundedness of (PS) sequences. That is to say, we study the perturbation equation (6.2) in this section under that f satisfies (f_3), (f_6) and (f_7). However, some new tricks will be used to deal with the boundedness of solution sequences with respect to the perturbation parameters α, λ. It is easy to check that Lemma 6.1 and 6.2 is still valid in this case. Therefore, for fixed $\lambda \in (0,1]$, there exists a mountain pass solution $u_{\alpha,\lambda} \in H_r^1(\mathbb{R}^2)$ such that $I'_{\alpha,\lambda}(u_{\alpha,\lambda}) = 0$ with $I_{\alpha,\lambda}(u_{\alpha,\lambda}) = c_{\alpha,\lambda}$. Moreover, there exist $a, b > 0$ independent of α, λ such that $a < c_{\alpha,\lambda} < b$.

Lemma 8.1. *Assume (f_3), (f_6) and (f_7) hold, then for fixed $\alpha \in (0,1)$, there exists $v \in H_r^1(\mathbb{R}^2) \setminus \{0\}$ such that $I'_\alpha(v) = 0$.*

Proof. Choosing a sequence $\{\lambda_n\} \subset (0,1]$ satisfying $\lambda_n \to 0^+$, then we find a sequence of nontrivial critical points sequence $\{u_{\lambda_n}\}$ (still denoted by $\{u_n\}$) of I_{α,λ_n} with $I_{\alpha,\lambda_n}(u_n) = c_{\alpha,\lambda_n}$. Now we claim sequence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^2)$. According to the definition of $I'_{\alpha,\lambda_n}(u_n)u_n = 0$, we have

$$Q(u_n) = \alpha(\|\nabla u_n\|_2^2 + \|u_n\|_2^2) + \|u_n\|_2^4 + \lambda_n\alpha\|u_n\|_2^{\frac{5}{2}} - \alpha\int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x - \lambda_n\alpha\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x. \tag{8.1}$$

It then follows from (7.2) that

$$\begin{aligned} I_{\alpha,\lambda_n}(u_n) \geq &(\frac{1}{4} + \frac{\alpha}{16})\int_{\mathbb{R}^2} u_n^2 \mathrm{d}x + \frac{1}{16}\|u_n\|_2^4 + (\frac{\alpha\lambda_n}{16} + \frac{3\lambda_n}{20})\|u_n\|_2^{\frac{5}{2}} \\ &+ \int_{\mathbb{R}^2} [\frac{7}{16} f(u_n)u_n - \frac{3}{2}F(u_n)]\mathrm{d}x + (\frac{(r-3)\lambda_n}{2r} - \frac{\alpha\lambda_n}{16})\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x, \end{aligned} \tag{8.2}$$

which, together with (f_6), yields that

$$\frac{1}{4}\|u_n\|_2^2 + \frac{1}{16}\|u_n\|_2^4 \leq C(1 + \|u_n\|_p^p), \quad p \in (2,3). \tag{8.3}$$

We first prove that there exists $C > 0$ independent of α, n such that $\|u_n\|_p^p \leq C$. Suppose by contradiction that $\|u_n\|_p^p \to \infty$ as $n \to \infty$. By (8.3) we get that there exists $C > 0$ independent of α, n such that

$$\|u_n\|_p^p \geq C\|u_n\|_2^4,$$

which implies

$$\|u_n\|_2 \leq C\|u_n\|_p^{\frac{p}{4}}. \tag{8.4}$$

Observe that

$$\begin{aligned} I_{\alpha,\lambda_n}(u_n) = &\frac{1}{4}(\|\nabla u_n\|_2^2 + \|u_n\|_2^2) + \frac{3\lambda_n}{20}\|u_n\|_2^{\frac{5}{2}} \\ &+ \int_{\mathbb{R}^2} \frac{1}{4} f(u_n)u_n - F(u_n)\mathrm{d}x + \frac{(r-4)\lambda_n}{4r}\int_{\mathbb{R}^2} |u_n|^r \mathrm{d}x, \end{aligned} \tag{8.5}$$

which, together with (f_3) and (f_6), implies that there exists $C > 0$ independent of α, n such that

$$C\|\nabla u_n\|_2^2 \leq \|u_n\|_p^p. \tag{8.6}$$

By the Gagliardo-Nirenberg inequality and (8.4)-(8.6) we have

$$\|u_n\|_p^p \le C\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} = C\|u_n\|_p^{\frac{p(p-1)}{2}}$$

which is a contradiction. Thus, u_n is bounded in $L^p(\mathbb{R}^2)$ uniformly for n. And by (8.4) and (8.6), we know that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^2)$ uniformly for n. The remaining proof is the same as that of Lemma 6.2, we omit it. $\square$

8.1. *Proof of Theorem 1.6 completed*

Since there exist $a, b > 0$ independent of α, λ such that $a < c_{\alpha,\lambda} < b$, as can be seen in the proof of Lemma 8.1, by an easy modification, we can also prove that $\{u_\alpha\}_{\alpha\in(0,1)}$ is bounded in $H_r^1(\mathbb{R}^2)$ uniformly for $\alpha \in (0, 1)$. As arguing in the proof of Theorem 1.4, we prove that (1.5) has at least a positive radial solution $u_0 \in H_r^1(\mathbb{R}^2)$. $\square$

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