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Planar Schrödinger-Choquard equations with potentials vanishing at infinity: The critical case

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Abstract

We study the following class of stationary Schrödinger equations of Choquard type

$$-\Delta u + V(x)u = \left[|x|^{-\mu} * (Q(x)F(u))\right] Q(x)f(u), \ x \in \mathbb{R}^2,$$

where the potential V and the weight Q decay to zero at infinity like $(1+|x|^{\gamma})^{-1}$ and $(1+|x|^{\beta})^{-1}$ for some (γ, β) in variously different ranges, $*$ denotes the convolution operator with $\mu \in (0, 2)$, and F is the primitive of f that fulfills a critical exponential growth in the Trudinger-Moser sense. By establishing a version of the weighted Trudinger-Moser inequality, we investigate the existence of nontrivial solutions of mountain-pass type for the given problem. Furthermore, we shall establish that the nontrivial solution is a bound state, namely a solution belonging to $H^1(\mathbb{R}^2)$, for some particular (γ, β).

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Keywords: Schrödinger-Choquard equation; Vanishing potentials; Weighted Trudinger-Moser inequality; Mountain-pass; Critical exponential growth; Bound state

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1. Introduction and main results

In the present paper, we are interested in the existence of solutions for the following stationary Schrödinger equation of Choquard type

$$-\Delta u + V(x)u = [|x|^{-\mu} * (Q(x)F(u))]Q(x)f(u),\ x \in \mathbb{R}^2, \tag{1.1}$$

where the potential V and the weight Q decay to zero at infinity like $(1+|x|^\gamma)^{-1}$ and $(1+|x|^\beta)^{-1}$ for some (γ, β) in various different ranges, $*$ denotes the convolution operator with $\mu \in (0,2)$ and F is the primitive of f, which fulfills a critical exponential growth in the Trudinger-Moser sense.

We assume that the potential V and the weight Q satisfy the following hypotheses:

($\mathcal{K}$) $V, Q \in C(\mathbb{R}^2)$ and there exist some positive constants γ, β, a, A and b such that

$$\frac{a}{1+|x|^\gamma} \le V(x) \le A \text{ and } 0 < Q(x) \le \frac{b}{1+|x|^\beta},$$

where $V(x) \sim |x|^{-\gamma}$ and $Q(x) \sim |x|^{-\beta}$ as $|x| \to +\infty$ and (γ, β) satisfies one of the following assumptions:

(i) $0 < \gamma < 2$ and $(4-\mu)\gamma/4 \le \beta < +\infty$, or $0 < \gamma \le 4\beta/(4-\mu) < 2$, where $0 < \mu < 2$;
(ii) $\gamma = 2$ and $(4-\mu)/2 \le \beta < +\infty$;
(iii) $\gamma > 2$ and $(4-\mu)/2 \le \beta < +\infty$.

Inspired by the Trudinger-Moser inequality, we say that a function $f(s)$ possesses *critical exponential growth* if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.2}$$

This definition was introduced by Adimurthi and Yadava [3], see also de Figueiredo, Miyagaki and Ruf [16] for example.

We suppose that the nonlinearity f satisfies (1.2) and the following assumptions

(f_1) $f \in C^1(\mathbb{R})$, $f(s) \equiv 0$ for all $s \le 0$ and $f(s) = o(s^{\frac{2-\mu}{2}})$;
(f_2) there exists a constant $\delta \in [0,1)$ such that

$$\frac{F(s)f'(s)}{f^2(s)} \ge \delta,\ \forall s > 0,\ \text{where } F(s) = \int_0^s f(t)dt;$$

(f_3) there exists some constants $s_0 > 0$, $M_0 > 0$ and $\vartheta \in (0,1]$ such that

$$0 < s^\vartheta F(s) \le M_0 f(s),\ \forall s \ge s_0;$$

(f_4) $\liminf_{s\to+\infty} F(s)/e^{\alpha_0 s^2} \triangleq \beta_0 > 0$.

It is widely known that the term $|x|^{-\mu} * (Q(x)F(u))$ can be regarded as the convolution between the Riesz potential $|x|^{-\mu}$ and $Q(x)F(u)$. Thus, problem (1.1) is closely related to the Choquard equation arising from the study of Bose-Einstein condensation and can be exploited to describe the finite-range many-body interactions between particles. For $N \geq 3$, the Choquard equation under the convolution of the Riesz potential is simply of the form

$$-\Delta u + u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u,\ x \in \mathbb{R}^N. \tag{1.3}$$

In the relevant physical case in which $N = 3$, $\mu = 1$ and $p = 2$, Eq. (1.3) turns into the Choquard-Pekar equation, which was used by Pekar [39] to describe a polaron at rest in the quantum field theory. It was also investigated by Choquard to characterization an electron trapped in its own hole as an approximation to the Hartree-Fock theory for a one component plasma [24]. Subsequently, Lieb [22] and Lions [25] obtained the existence and uniqueness of positive solutions to (1.3) by variational methods. The authors in [28,32] verified the regularity, positivity and radial symmetry of the ground state solutions and established the decay property at infinity. It should be pointed out that Eq. (1.3) was also proposed by Moroz *et al.* in [31] as a model for self-gravitating particles in that context it can be viewed as the classical Schrödinger-Newton equation, see e.g. [13,40,43].

To review the research history of the Choquard equation as (1.3), let us recall the Hardy-Littlewood-Sobolev inequality, which will play a vital role throughout this paper.

Proposition 1.1. *(Hardy-Littlewood-Sobolev inequality [23, Theorem 4.3]). Suppose that $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$, $g \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C = C(s, N, \mu, r) > 0$, independent of g and h, such that*

$$\int_{\mathbb{R}^N} [|x|^{-\mu} * g(x)]h(x)dx \leq C|g|_s|h|_r. \tag{1.4}$$

Suppose that $g(x) = h(x) = |u(x)|^p$ in (1.4) for every $u \in H^1(\mathbb{R}^N)$ with $N \geq 3$. To preserve the variational structure, the Sobolev imbedding theorem, namely $H^1(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ for all $2 \leq t \leq 2^* = 2N/(N-2)$, indicates that the exponent p in Eq. (1.3) should satisfy

$$\frac{2N-\mu}{N} \leq p \leq \frac{2N-\mu}{N-2}.$$

For $(2N-\mu)/N < p < (2N-\mu)/(N-2)$, the authors [32] considered the existence of ground state solutions for Eq. (1.3), where the Pohožaev identity was also established. After it, they investigated the existence and nonexistence of solutions to the equation with nonconstant potential by minimizing arguments for the lower critical exponent case, i.e. $p = (2N-\mu)/N$, in [33]. Gao-Yang studied the upper critical exponent case for $p = (2N-\mu)/(N-2)$ in [17]. In fact, Eq. (1.3) and its variants have received a great number of attentions by many mathematicians because of the appearance of the convolution type nonlinearities over the past several decades. We refer the reader to [1,8,19,20,32,47–49] and the references therein for the consideration of existence, multiplicity, nodal, semiclassic state and concentrating behavior of different type of solutions, even if these references are far to be exhaustive. For the convenience of the interested reader, we shall suggest [34] for a very abundant and meaningful review of the Choquard equations.

However, the case $N = 2$ is very different and special since $2^* = \infty$ in this situation and $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$. As a matter of fact, although the Sobolev imbedding theorem ensures that $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ with $1 \le p < +\infty$ for all bounded domain $\Omega \subset \mathbb{R}^2$, we have that $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Loosely speaking, if one tends to deal with the problem (1.1) variationally, by (1.4), it has to make sure that

$$\int_{\mathbb{R}^2} [|x|^{-\mu}(Q(x)F(u))]Q(x)F(u)dx$$

is well-defined if $Q(x)F(u) \in L^t(\mathbb{R}^2)$ for every $u \in H^1(\mathbb{R}^2)\backslash\{0\}$ and $t > 1$ induced by $2/t + \mu/2 = 2$. Because $f(s)$ satisfies (1.2) and $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$, to overcome the obstacle in this limiting case, the celebrated Trudinger-Moser inequality [30,41,44] may be treated as a suitable substitute of the Sobolev inequality. Firstly, we will introduce the case bounded domain Ω instead of the whole space $\mathbb{R}^2$. The authors in [30,41,44] established the following sharp maximal exponential integrability for functions in $H_0^1(\Omega)$:

$$\sup_{u \in H_0^1(\Omega): \|\nabla u\|_{L^2(\Omega)} \le 1} \int_\Omega e^{\alpha u^2} dx \le C|\Omega| \text{ if } \alpha \le 4\pi, \tag{1.5}$$

where $C > 0$ depends only on α, and $|\Omega|$ denotes the Lebesgue measure of Ω. Subsequently, this inequality was generalized by P. L. Lions in [26]: Let $\{u_n\}$ be a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_n\|_{L^2(\Omega)} = 1$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, then for all $p < \frac{1}{(1-\|\nabla u_0\|_2^2)}$, there holds

$$\limsup_{n\to\infty} \int_\Omega e^{4\pi p u_n^2} dx < +\infty.$$

Unfortunately, the supremum in (1.5) becomes infinite for domains Ω with $|\Omega| = \infty$, and therefore the Trudinger-Moser inequality is not available for the unbounded domains. As to the whole space $\mathbb{R}^2$, the author in [9] established the following version of the Trudinger-Moser inequality (see also [11] for example):

$$e^{\alpha u^2} - 1 \in L^1(\mathbb{R}^2),\ \forall \alpha > 0 \text{ and } u \in H^1(\mathbb{R}^2).$$

Moreover, for all $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{L^2(\mathbb{R}^2)} \le M < +\infty$, there exists a positive constant $C = C(M, \alpha)$ such that

$$\sup_{u \in H^1(\mathbb{R}^2): \|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx \le C \text{ if } \alpha < 4\pi.$$

Concerning some other generalizations, extensions and applications of the Trudinger-Moser inequalities for bounded and unbounded domains, we refer to [16] and its references therein.

It should be noted that the inequality by Cao [11] holds only strictly for $\alpha < 4\pi$, i.e. with subcritical growth. For the sharp case, based on symmetrization and blow-up analysis, Ruf [38], Li and Ruf [21] proved that

$$\sup_{u\in W_0^{1,N}(\mathbb{R}^N),\|u\|_{L^N}^N+\|\nabla u\|_{L^N}^N\le 1}\int_{\mathbb{R}^N}\left(e^{\alpha|u|^{\frac{N}{N-1}}}-\sum_{k=0}^{N-2}\frac{\alpha^k|u|^{kN/(N-1)}}{k!}\right)dx<\infty,\ \text{if}\ \alpha\le\alpha_N,$$

by replacing the L^N norm of ∇u in the supremum with the standard Sobolev norm. This inequality was improved by Souza and do Ó[14] for $N=2$. Let (u_n) be in $W_0^{1,N}(\mathbb{R}^N)$ with $\|u_n\|=1$ and suppose that $u_n\rightharpoonup u_0$ in $W_0^{1,N}(\mathbb{R}^N)$. Then for all $0<p<\frac{4\pi}{1-\|u_0\|^2}$, the authors proved that

$$\sup_n\int_{\mathbb{R}^2}(e^{pu_n^2}-1)dx<\infty.$$

We refer the reader to the references in this paper for more information about the advances on the elliptic equations with critical exponential growth.

Concerning the Choquard equation in $\mathbb{R}^2$ with critical exponential growth, Alves *et al.* [7] studied the existence of ground state solution for Eq. (1.1) with $Q(x)\equiv 1$ if the potential $V(x)$ satisfies

(V_1) $V(x)$ is a 1-periodic continuous function in $\mathbb{R}^2$ and $\inf_{x\in\mathbb{R}^2}V(x)>0$

and the nonlinearity $f(s)$ meets the Ambrosetti-Rabinowitz condition ((AR) in short)

(AR) there exists a constant $K>1$ such that $f(s)s\ge KF(s)$ for all $s>0$.

Moreover, the authors in [7] also considered the semiclassic state solution under the assumption

(V_2) $V\in C(\mathbb{R}^2)$, $V(x)\ge\inf_{x\in\mathbb{R}^2}V(x)\triangleq V_0>0$ and $V_0<V_\infty=\liminf_{|x|\to+\infty}<+\infty$.

Very recently, by still supposing (AR), Qin-Tang [37] generalize the counterpart in [7] to the indefinite case, i.e. the potential $V(x)$ verifies

(V_3) $V(x)$ is a 1-periodic continuous function in $\mathbb{R}^2$ and 0 lies in a gap of the spectrum $-\Delta+V$.

Besides, Albuquerque et al. [5] investigated the existence of nontrivial solutions and ground state solutions for the problem (1.1) by supposing that

(V_4) $V\in C(0,\infty)$, $V(r)>0$ for $r>0$ and there exist $a_0>-2$ and $a>-2$ such that

$$\limsup_{r\to 0^+}V(r)/r^{a_0}<\infty\ \text{and}\ \limsup_{r\to+\infty}V(r)/r^a>0;$$

(Q) $Q\in C(0,\infty)$, $V(r)>0$ for $r>0$ and there exist $b_0>-(4-\mu)/2$ and $b<a(4-\mu)/4$ such that

$$\limsup_{r\to 0^+}Q(r)/r^{b_0}<\infty\ \text{and}\ \limsup_{r\to+\infty}Q(r)/r^b>0.$$

It should be pointed out that (AR) is necessary and indispensable in [5]. For some other interesting works with respect to the Choquard equation in $\mathbb{R}^2$ involving critical exponential growth, please see e.g. [5,7,37] and the references therein. We remark that the semilinear Schrödinger equation involving the decaying potentials and weights are firstly considered in the pioneer work of Ambrosetti-Felli-Malchiodi [6], later studied in [27,42] and the references therein.

Motivated by all of the above mentioned works, particularly by [5,7,37], we shall consider the nonlocal problem (1.1) under $(\mathcal{K})$, (1.2) and $(f_1)-(f_4)$, and aim to study the existence of nontrivial solutions and *bound state* solutions, i.e. a solution belonging to $H^1(\mathbb{R}^2)$. We have to point out that the features of the functions $V(x)$ and $Q(x)$ vanishing at infinity together with the critical exponential growth for the problem bring some new difficulties in our analysis, which can be summarized as follows:

(i). The radial potentials V and weights Q satisfying $(\mathcal{K})$ with $\gamma < 2$ and $(4-\mu)\gamma/4 < \beta$ was studied by Albuquerque et al. in [5], where the growth conditions on (V_4) and (Q) are less restrictive than $(\mathcal{K})$, but a complicated and rigorous interpretation of the function space setting considered was necessary. Moreover, there exist still some questions worth thinking about left over: (1) whether the conclusions in [5] for the non-radial potentials V and weights Q remain true; (2) what would happen whenever γ or β gets the endpoint of (i) ((ii), or (iii)) in $(\mathcal{K})$ such as $\gamma \geq 2$, or $(4-\mu)\gamma/4 = \beta$? (3) whether the nontrivial solution established in [5] is a mountain-pass type, even a *bound state*?

(ii). To obtain the boundedness of the $(C)_c$ sequence of J, different from the cited papers [5,7,37], the (AR) is absent such that we have to propose some new ideas to bridge over this difficulty.

(iii). Restoring the lack of compactness caused by critical exponential growth and the whole space $\mathbb{R}^2$, could we control the minimax level by a suitable threshold which is independent of γ and β?

Before stating the main results briefly, we give several notations and definitions. Let $(X, \|\cdot\|_X)$ be a Banach space with its dual space $(X^{-1}, \|\cdot\|_{X^{-1}})$, and Φ be its functional on X. The Cerami sequence at a level $c \in \mathbb{R}$ ($(C)_c$ sequence in short) corresponding to Φ means that $\Phi(x_n) \to c$ and $(1 + \|\Phi'(x_n)\|_{X^{-1}})\|u_n\|_X \to 0$ as $n \to \infty$, where $\{x_n\} \subset X$. The space $L^p(\mathbb{R}^2)$ stands for the usual Lebesgue space with the norm $|\cdot|_p$ with $1 \leq p \leq +\infty$. Throughout this paper, we shall denote by C and C_i $(i = 1, 2, \cdots)$ for various positive constants whose exact value may change from lines to lines but are never essential to the analysis of the problem. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the function spaces, respectively. Let $|A|$ stand for the Lebesgue measure of the Lebesgue measurable set $A \subset \mathbb{R}^2$. For all $x \in \mathbb{R}^2$ and $\rho > 0$, $B_\rho(x) \triangleq \{y \in \mathbb{R}^2 : |y - x| < \rho\}$ and $B^c_\rho(x) \triangleq \mathbb{R}^2 \backslash B_\rho(x)$.

We introduce the following work space

$$E \triangleq \left\{ u \in D^{1,2}(\mathbb{R}^N) \Big| \int_{\mathbb{R}^2} V(x)u^2 dx < +\infty \right\}$$

endowed with the norm

$$\|u\|^2 \triangleq \int_{\mathbb{R}^2} [|\nabla u|^2 + V(x)u^2] dx$$

is a Banach space. Indeed, as a consequence of the fact that

$$V \in C(\mathbb{R}^2),\ V(x) > 0 \text{ in } \mathbb{R}^2 \text{ and } V(x) \to 0 \text{ as } |x| \to +\infty,$$

one has that the imbedding $E \hookrightarrow H^1_{loc}(\mathbb{R}^2)$ is continuous which together with the definition of Cauchy sequence and Fatou's lemma enables us to verify that $(E, \|\cdot\|)$ is complete. Note that the norm $\|\cdot\|$ can also be induced by the inner product

$$\langle u, v\rangle \triangleq \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)uv]dx,\ \forall u, v \in E.$$

Moreover, we can say that a function $u_0 \in E$ is a weak solution of Eq. (1.1) in the sense that

$$\int_{\mathbb{R}^2} [\nabla u_0 \nabla v + V(x)u_0 v]dx = \int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)vdx,\ \forall v \in E. \tag{1.6}$$

Now, we are ready to state the main results in the present paper. The first result is the following weighted Sobolev imbedding theorem.

Theorem 1.2. *Suppose that* $(\mathcal{K})$ *holds true, for all* $\frac{4-\mu}{2} < p < +\infty$*, the imbedding* $E_r \hookrightarrow L^{p^\mu}_{Q^\mu}(\mathbb{R}^2) \triangleq \{u : \mathbb{R}^2 \to \mathbb{R}$ *is Lebesgue measurable*$|\int_{\mathbb{R}^2} Q^\mu(x)|u|^{p^\mu}dx < +\infty\}$ *is compact with* $E_r \triangleq \{u \in E|u(x) = u(|x|)\}$ *and* $Q^\mu = Q^{4/(4-\mu)}(x)$, $p^\mu = 4p/(4-\mu)$. *If* (i), (ii) *and* (iii) *in* $(\mathcal{K})$ *are replaced by*

- *(i)′* $0 < \gamma < 2$ *and* $(4-\mu)\gamma/4 < \beta < +\infty$, *or* $0 < \gamma < 4\beta/(4-\mu) < 2$, *here* $0 < \mu < 2$;
- *(ii)′* $\gamma = 2$ *and* $(4-\mu)/2 < \beta < +\infty$;
- *(iii)′* $\gamma > 2$ *and* $(4-\mu)/2 < \beta < +\infty$,

respectively, we denote the $(\mathcal{K})$ *by* $(\mathcal{K})'$. *Therefore,* $E \hookrightarrow L^{p^\mu}_{Q^\mu}(\mathbb{R}^2)$ *is compact for every* $\frac{4-\mu}{2} \le p < +\infty$ *if* $(\mathcal{K})'$ *holds.*

As stated before, by Theorem 1.2, it is natural to establish the corresponding weighted Trudinger-Moser inequality of the form as follows

Theorem 1.3. *Suppose that* $(\mathcal{K})$ *holds true, for all* $\alpha > 0$, $0 < \mu < 2$ *and* $u \in E$, *we have*

$$\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2} - 1)dx < +\infty. \tag{1.7}$$

Moreover, if we consider the supremum

$$S_\alpha = S_\alpha(\mu, V, Q) \triangleq \sup_{u\in E, \|u\|\le 1} \int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2} - 1)dx,\ \forall \alpha > 0,$$

then there exists a constant $C = C(\alpha, \mu, V, Q) > 0$ *such that*

$$S_\alpha \le C,\ \forall \alpha < 4\pi \tag{1.8}$$

and

$$S_\alpha = +\infty, \ \forall \alpha > 4\pi. \tag{1.9}$$

Remark 1.4. According to Theorem 1.2, if $\beta = (4-\mu)\gamma/4$ in (i) (resp. (ii), or (iii)), the imbedding $E \hookrightarrow L^{p^\mu}_{Q^\mu}(\mathbb{R}^2)$ for each $\frac{4-\mu}{2} \le p < +\infty$ may not be compact. Consequently, this case is called by the *critical* case, and we shall understand the case in (i)$'$ (resp. (ii)$'$, or (iii)$'$) should correspond to the *subcritical* case, respectively. Similarly, we can also regard as the inequality obtained in Theorem 1.3 is *subcritical* in the sense that the supremum holds only for the open interval $(0, 4\pi)$. Even if our result does not cover the *critical* case $\alpha = 4\pi$, the *subcritical* inequality expressed in Theorem 1.3 is enough to study the existence of nontrivial solutions for Eq. (1.1).

Combining Theorems 1.2 and 1.3, we establish the following existence result.

Theorem 1.5. *Suppose that* $(\mathcal{K})$ *with* V *and* Q *being radially symmetric, or* $(\mathcal{K})'$*, holds true. If* f *satisfies* (1.2) *and* $(f_1)-(f_4)$*, then Eq.* (1.1) *admits at least a nontrivial solution* $u_0 \in E$*. Moreover, if* $\delta = 0$ *in* (f_2)*, we deduce that* $J(u_0) = \inf_{u\in\mathcal{N}} J(u)$*, where the functional* $J : E \to \mathbb{R}$ *defined by*

$$J(u) \triangleq \frac{1}{2}\int\limits_{\mathbb{R}^2} [|\nabla u|^2 + V(x)u^2]dx - \frac{1}{2}\int\limits_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u))]Q(x)F(u)dx, \ \forall u \in E \tag{1.10}$$

and $\mathcal{N} \triangleq \{u \in E\backslash\{0\} | \langle J'(u), u\rangle = 0\}$.

Remark 1.6. Proceeding as [32,45], one can conclude that the variational functional J defined by (1.10) is of class C^1 and then the Nehari manifold $\mathcal{N}$ is well-defined. We mention here that the condition (f_2) is motivated by [12], where the case $\delta \in (0, 1)$ was investigated. However, $\delta = 0$ may occur in Theorem 1.5, therefore some new analytic techniques have to put forward to overcome this obstacle. It's worthy pointing out that the case $\delta = 0$ is a generalization and supplement to the (AR) exploited in [5,7,37]. Indeed, by (f_2), one derives $f'(s) \ge 0$ for all $s > 0$ and then f is nondecreasing on $s \in (0, +\infty)$. As a consequence, there holds

$$0 < F(s) = \int\limits_0^s f(t)dt \le f(s)s, \ \forall s > 0 \tag{1.11}$$

which is the (AR) with the constant $K \equiv 1$. Besides, we can use (f_2) and (1.11) to show that

$$0 < F(s) \le (1-\delta)f(s)s, \ \forall s > 0. \tag{1.12}$$

Since (f_2) indicates that $(F(s)/f(s))' \le 1-\delta$ for any $s > 0$, then for all $\varepsilon \in (0, s)$, one has

$$\frac{F(s)}{f(s)} - \frac{F(\varepsilon)}{f(\varepsilon)} = \int\limits_\varepsilon^s \frac{d}{dt}\left(\frac{F(t)}{f(t)}\right)dt \le (1-\delta)\int\limits_\varepsilon^s dt = (1-\delta)(s-\varepsilon)$$

which together with $\lim_{\varepsilon\to 0^+} F(\varepsilon)/f(\varepsilon) = 0$ by (1.11) yields that (1.12). Moreover, we conclude that

$$f(s)s - F(s) \text{ is nondecreasing on } s \in (0, +\infty). \tag{1.13}$$

In fact, for all $0 < s_1 < s_2$, since f is nondecreasing on $s \in (0, +\infty)$, we obtain

$$F(s_2) - F(s_1) = \int_{s_1}^{s_2} f(t)dt \le f(s_2)(s_2 - s_1) \le f(s_2)s_2 - f(s_1)s_1$$

showing the desired result.

Because we have explained in Remark 1.4, the cases in $(\mathcal{K})'$ are *subcritical* which allow us to verify that the energy of u_0 established by Theorem 1.5 equals to the mountain-pass energy of J, i.e.

$$J(u_0) = c \triangleq \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)), \tag{1.14}$$

where $\Gamma \triangleq \{\gamma \in C([0,1], E) | \gamma(0) = 0, J(\gamma(1)) < 0\}$. However, concerning the cases in $(\mathcal{K})$ which are *critical*, we cannot derive that (1.14) holds immediately. To solve this difficulty, motivated by [14,26], we have to establish a version of concentration-compactness principle in our analysis settings. More precisely, we prove the following theorem.

Theorem 1.7. *Suppose that* $(\mathcal{K})$ *holds true and* $\{u_n\} \subset E$ *to be a sequence satisfying* $\|u_n\| \equiv 1$ *and* $u_n \rightharpoonup u \neq 0$ *in* E*, then*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p|u_n|^2} - 1)dx < +\infty,\ \forall 0 < p < P_{\alpha_0}(u), \tag{1.15}$$

where the sharp constant P_{α_0} *is defined by*

$$P_{\alpha_0}(u) = \begin{cases} \frac{1}{1-\|u\|^2}, & \text{if } \|u\| < 1, \\ +\infty, & \text{if } \|u\| = 1. \end{cases}$$

As a by-product of Theorem 1.7, we obtain the following result.

Corollary 1.8. *Suppose that* $(\mathcal{K})$ *with* V *and* Q *being radially symmetric, or* $(\mathcal{K})'$*, holds true. If* f *satisfies* (1.2) *and* $(f_1) - (f_4)$*, then Eq.* (1.1) *possesses a nontrivial solution* $u_0 \in E$ *with* $J(u_0) = c$.

For our next existence result, we replace the conditions (f_4) by the following condition

(f_5) there are constants $q > (4-\mu)/2$ and $C_q > 0$ such that $f(s) \geq C_q s^{q-1}$ for all $s > 0$, where

$$C_q > \sqrt{q\mathbb{S}_{\mu,q}}\left(\frac{(q-1)\mathbb{S}_{\mu,q}\alpha_0}{(4-\mu)\pi q}\right)^{\frac{q-1}{2}}$$

and

$$\mathbb{S}_{\mu,q} \triangleq \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^2}[|\nabla u|^2 + V(x)u^2]dx}{(\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)|u|^q)]Q(x)|u|^q dx)^{1/q}} > 0. \tag{1.16}$$

Theorem 1.9. *Suppose that* $(\mathcal{K})$ *with* V *and* Q *being radially symmetric, or* $(\mathcal{K})'$*, holds true. If* f *fulfills* (1.2) *and* $(f_1) - (f_3)$ *with* (f_5)*, the conclusions in Theorem 1.5, or Corollary 1.8, remain true.*

Remark 1.10. It follows from $(\mathcal{K})$ or $(\mathcal{K})'$ that $0 < Q(x) \leq b$ for all $x \in \mathbb{R}^2$. Then, by $4q/(4-\mu) > 2$, we can apply (1.4) and Theorem 1.2 to conclude that

$$\begin{aligned}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)|u|^q)]Q(x)|u|^q dx &\leq C\left(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u|^{\frac{4q}{4-\mu}} dx\right)^{\frac{4-\mu}{2}} \\ &\leq \widetilde{C}\left(\int_{\mathbb{R}^2}[|\nabla u|^2 + V(x)u^2]dx\right)^q\end{aligned}$$

indicating that $\mathbb{S}_{\mu,q} > 0$ presented by (1.16) is well-defined. Moreover, arguing as [32] together with Theorem 1.2, we derive the constant $\mathbb{S}_{\mu,q}$ can be attained by a nontrivial function belonging to E.

Finally, we are concerned with the existence of *bound state* solutions of Eq. (1.1).

Theorem 1.11. *Let* $(\mathcal{K}')$ *with* (i)$'$ *hold true and* $u_0 \in E$ *be the nontrivial solution established by Theorem 1.5 of Eq.* (1.1)*,* i.e. u_0 *satisfies* (1.6)*, then* $u_0 \in L^2(\mathbb{R}^2)$ *and hence* $u_0 \in H^1(\mathbb{R}^2)$.

Let's recall the celebrated paper by A. Ambrosetti et al. in [6], the so-called *bound state* solutions have important physics meaning if they exist. For instance, from the physical point of view, according to the well-known probabilistic interpretation of quantum mechanics, the standing wave solutions u_0 which possesses a finite L^2-norm of nonlinear Choquard equations (1.1) are the most relevant because they correspond to localized elementary particles in space by proving that $\lim_{|x|\to\infty} u_0(x) = 0$.

In the sequel, we shall say that $(\gamma, \beta) \in$ (i)$'$ (resp. $(\gamma, \beta) \in$ (ii)$'$, or $(\gamma, \beta) \in$ (iii)$'$) if γ, β satisfy (i)$'$ (resp. (ii)$'$, or (iii)$'$) in this paper, for simplicity.

The paper is organized as follows. In Section 2, we introduce some useful preliminaries and present the proofs of Theorems 1.2 and 1.3. Section 3 is devoted to the proofs of existence results in Theorem 1.5, Corollary 1.8 and Theorem 1.9 and the concentration-compactness principle in Theorem 1.7. In Section 4, we show that some particular nontrivial solutions of Eq. (1.1) are *bound state* solutions.

2. Preliminaries and the weighted Trudinger-Moser inequality

In this section, we mainly focus on the proofs of the imbedding $E \hookrightarrow L^{p^\mu}_{Q^\mu}(\mathbb{R}^2)$ with $2 \le p < +\infty$ and the weighted Trudinger-Moser inequality expressed in Theorems 1.2 and 1.3, respectively. As far as we are concerned, the results in Theorems 1.2 and 1.3 are new and we do believe that they would be utilized in some other fields of the nonlinear Choquard equation like Eq. (1.1) involving vanishing potentials at infinity.

Since the proof of Theorem 1.2 can be learnt from the proof of Theorem 1.3, we shall firstly prove Theorem 1.3 in detail and then take some necessary modifications to show Theorem 1.2. To obtain the proof of (1.8) in Theorem 1.3, we could combine the ideas introduced by Kufner and Opic in [35] together with the procedures concerning a local version of the classic Trudinger-Moser inequality proposed by Y. Yang and X. Zhu in [46].

Proposition 2.1. *(see [46]) There exists a constant $C > 0$ such that for every $y \in \mathbb{R}^2$, $R > 0$ and any $u \in H^1_0(B_R(y))$ with $|\nabla u|_2 \le 1$, we have*

$$\int\limits_{B_R(y)} (e^{4\pi u^2} - 1)dx \le CR^2 \int\limits_{B_R(y)} |\nabla u|^2 dx.$$

Moreover, we also need the following well-known global result.

Proposition 2.2. *(see e.g. [2,9,36]) There exists a constant $C_\alpha > 0$ such that*

$$\sup_{u\in H^1(\mathbb{R}^2), |\nabla u|_2 \le 1} \frac{1}{|u|_2^2} \int\limits_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx \begin{cases} \le C_\alpha, & \text{if } 0 < \alpha < 4\pi, \\ = +\infty, & \text{if } \alpha \ge 4\pi. \end{cases}$$

As to the proof of (1.9) in Theorem 1.3, we depend on the sharpness of the following Trudinger-Moser inequality due to B. Ruf [38].

Proposition 2.3. *(see [38]) Let $\Omega \subset \mathbb{R}^2$ be a domain (possibly unbounded) and let $\tau > 0$. For every $\alpha \in [0, 4\pi]$, there exists a constant $C_\tau > 0$ such that*

$$R_\alpha(\tau, \Omega) \triangleq \sup_{u\in H^1_0(\Omega), |\nabla u|_2^2 + \tau |u|_2^2 \le 1} \int\limits_{\Omega} (e^{\alpha u^2} - 1)dx \le C_\tau$$

and the above inequality is shape, i.e.

$$R_\alpha(\tau, \Omega) = +\infty, \ \forall \alpha > 4\pi.$$

Now, we begin to give the proof of Theorem 1.3. Firstly, we verify (1.9) in Theorem 1.3:

Proof of (1.9) in Theorem 1.3. Since $V, Q \in C(\mathbb{R}^2)$ are positive and continuous by $(\mathcal{K})$, we can define

$$V_1 \triangleq \max_{x\in B_1(0)} V(x) \in (0, +\infty) \text{ and } Q_1 \triangleq \min_{x\in B_1(0)} Q(x) \in (0, +\infty)$$

indicating that

$$S_\alpha \ge Q_1^{\frac{4}{4-\mu}} \sup_{u\in H_0^1(B_1(0)),\|u\|\le 1} \int_{B_1(0)} (e^{\alpha u^2}-1)dx,\ \forall \alpha > 4\pi.$$

Recalling that the continuous imbedding $H_0^1(B_1(0)) \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow E$, one has

$$\|u\|^2 \le |u|_2^2 + V_1|u|_2^2,\ \forall u \in H_0^1(B_1(0)).$$

Therefore, we can conclude that

$$\begin{aligned} S_\alpha \ge & Q_1^{\frac{4}{4-\mu}} \sup_{u\in H_0^1(B_1(0)),|u|_2^2+V_1|u|_2^2\le 1} \int_{B_1(0)} (e^{\alpha u^2}-1)dx \\ &= Q_1^{\frac{4}{4-\mu}} R_\alpha(V_1, B_1(0)) = +\infty,\ \forall \alpha > 4\pi, \end{aligned}$$

where we have applied Proposition 2.3 with $\tau = V_1$ and $\Omega = B_1(0)$. The proof is complete. □

Next, we concentrate on the proof of (1.8) in Theorem 1.3. To end it, we split it into three cases.

2.1. *Case 1:* $0 < \gamma < 2$ *and* $(4-\mu)\gamma/4 \le \beta < +\infty$*, or* $0 < \gamma \le 4\beta/(4-\mu) < 2$

In this case, the assumption $0 < \gamma < 2$ plays a significant role so that we only consider (1.8) when $0 < \gamma < 2$ and $(4-\mu)\gamma/4 \le \beta < +\infty$. Throughout this case, for every $\alpha \in (0, 4\pi)$, we take the fixed small constants $\epsilon \in (0,1)$ and $\delta \in (0,\epsilon)$ such that

$$\alpha = 4\pi(1-\epsilon) \text{ and } \alpha < 4\pi(1-\delta). \tag{2.1}$$

Let $u \in E$ with $\|u\| \le 1$ and choose a sufficiently large constant $R > 0$ determined later, then

$$\begin{aligned} &\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \\ &= \int_{B_R(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx + \int_{B_R^c(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \\ &\le b^{\frac{4}{4-\mu}} \int_{B_R(0)} (e^{\alpha u^2}-1)dx + \int_{B_R^c(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx, \end{aligned} \tag{2.2}$$

where we have exploited the fact $0 < Q(x) \le b$ for all $x \in \mathbb{R}^2$. To estimate the first integral in (2.2), proceeding as [46], we assume $\psi \in C_0^\infty(B_{2R}(0))$ to be a cutoff function satisfying

$$0 \le \psi \le 1 \text{ in } B_{2R}(0),\ \psi \equiv 1 \text{ in } B_R(0) \text{ and } |\nabla\psi| \le \frac{C}{R} \text{ in } B_{2R}(0) \tag{2.3}$$

for some universal constant $C > 0$. Clearly, $\psi u \in H_0^1(B_{2R}(0))$. As $V(x)(1+(2R)^\gamma) \geq V(x)(1+|x|^\gamma) \geq a > 0$ in $B_{2R}(0)$ by $(\mathcal{K})$, it follows from the Young's inequality that

$$\begin{aligned}
&\int_{B_{2R}(0)} |\nabla(\psi u)|^2 dx \\
&\leq (1+\epsilon) \int_{B_{2R}(0)} \psi^2 |\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right) \int_{B_{2R}(0)} u^2 |\nabla \psi|^2 dx \\
&\leq (1+\epsilon) \int_{B_{2R}(0)} |\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right) \frac{C^2}{R^2} \int_{B_{2R}(0)} u^2 dx \\
&\leq (1+\epsilon) \int_{B_{2R}(0)} |\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right) \frac{C^2(1+(2R)^\gamma)}{aR^2} \int_{B_{2R}(0)} V(x) u^2 dx.
\end{aligned}$$

Since $\gamma < 2$, there is a constant $\overline{R} = \overline{R}(\epsilon, a, \gamma, C) > 0$ independent of $u \in E$ such that

$$\left(1+\frac{1}{\epsilon}\right) \frac{C^2(1+(2\overline{R})^\gamma)}{a\overline{R}^2} \leq 1+\epsilon$$

which implies that

$$\int_{B_{2R}(0)} |\nabla(\psi u)|^2 dx \leq (1+\epsilon) \int_{B_{2R}(0)} (|\nabla u|^2 + V(x) u^2) dx \leq 1+\epsilon, \ \forall R \geq \overline{R}.$$

So, set $v \triangleq \sqrt{1-\epsilon}\,\psi u \in H_0^1(B_{2R}(0))$ and then $|\nabla v|_2^2 \leq 1-\epsilon^2 < 1$. By Proposition 2.1, for $R \geq \overline{R}$,

$$\int_{B_R(0)} (e^{4\pi(1-\epsilon)u^2} - 1) dx = \int_{B_R(0)} (e^{4\pi(1-\epsilon)(\psi u)^2} - 1) dx \leq \int_{B_{2R}(0)} (e^{4\pi v^2} - 1) dx \leq C_1 R^2. \tag{2.4}$$

Then, we estimate the second integral in (2.2). To achieve this purpose, motivated by [6], we shall apply the Besicovitch covering lemma (see e.g. [18]). More precisely, let $\overline{n} \in \mathbb{N}^+$ be a sufficiently large constant chosen later. For any fixed $n \geq \overline{n}$, we introduce the covering of $B_n^c(0)$ of all annuli A_n^σ with $\sigma > n$ defined by

$$A_n^\sigma \triangleq \{x \in \mathbb{R}^2 | n < |x| < \sigma\} \subset B_n^c(0).$$

For any $\sigma > \overline{n}$, by means of the Besicovitch covering lemma, there is a sequence of points $\{x_k\} \subset A_{\overline{n}}^\sigma$ and a universal constant $\vartheta > 0$ such that

- $A_{\overline{n}}^\sigma \subset \cup_k U_k^{1/2}$, where $U_k^{1/2} \triangleq B_{|x_k|/6}(x_k)$;
- $\sum_k \chi_{U_k}(x) \leq \vartheta$ for any $x \in \mathbb{R}^2$, where χ_{U_k} is the characteristic function of $U_k \triangleq B_{|x_k|/3}(x_k)$.

Arguing as [35], we introduce the following set of indices

$$K_{n,\sigma} \triangleq \{k \in \mathbb{N}^+ | U_k^{1/2} \cap B_{3n}^c \neq \emptyset\}.$$

For all $\sigma > 3n$, it's simple to see that

$$A_{3n}^\sigma \subset A_{\overline{n}}^\sigma \subset \bigcup_{k\in\mathbb{N}^+} U_k^{1/2} = \bigcup_{k\in K_{n,\sigma}} U_k^{1/2} \subset \bigcup_{k\in K_{n,\sigma}} U_k \subset B_n^c(0) \subset B_{\overline{n}}^c(0) \tag{2.5}$$

indicating that

$$\int_{A_{3n}^\sigma} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le \sum_{k\in K_{n,\sigma}} \int_{U_k^{1/2}} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx. \tag{2.6}$$

Let's observe that $2|x_k|/3 \le |y| \le 4|x_k|/3$ for all $y \in U_k$, then by $(\mathcal{K})$, one has

$$V(y) \ge \frac{a}{1+C_\gamma |x_k|^\gamma} \text{ and } Q(y) \le \frac{b}{1+C_\beta |x_k|^\beta}, \ \forall y \in U_k, \tag{2.7}$$

where $C_\gamma = (4/3)^\gamma$ and $C_\beta = (2/3)^\beta$. Consequently, by (2.7), we obtain

$$\int_{U_k^{1/2}} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le \frac{b^{\frac{4}{4-\mu}}}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{U_k^{1/2}} (e^{\alpha u^2}-1)dx. \tag{2.8}$$

Proceeding as before, to estimate the integral $e^{\alpha u^2}-1$ over $U_k^{1/2}$, assume $\psi_k \in C_0^\infty(U_k)$ to be a cutoff function satisfying

$$0 \le \psi_k \le 1 \text{ in } U_k, \ \ \psi_k \equiv 1 \text{ in } U_k^{1/2} \text{ and } |\nabla \psi_k| \le \frac{C}{|x_k|} \text{ in } U_k$$

for a universal constant $C > 0$. Clearly, $\psi_k u \in H_0^1(U_k)$. In view of the constant δ in (2.1), by (2.7)

$$\begin{aligned}
\int_{U_k} |\nabla(\psi_k u)|^2 dx &\le (1+\delta)\int_{U_k} \psi_k^2 |\nabla u|^2 dx + \left(1+\frac{1}{\delta}\right)\int_{U_k} u^2 |\nabla \psi_k|^2 dx \\
&\le (1+\delta)\int_{U_k} |\nabla u|^2 dx + \left(1+\frac{1}{\delta}\right)\frac{C^2}{|x_k|^2}\int_{U_k} u^2 dx \\
&\le (1+\delta)\int_{U_k} |\nabla u|^2 dx + \left(1+\frac{1}{\delta}\right)\frac{C^2(1+C_\gamma|x_k|^\gamma)}{a|x_k|^2}\int_{U_k} V(x)u^2 dx.
\end{aligned}$$

Given a $k \in K_{n,\sigma}$, then $x_k \in B_{\overline{n}}^c$. Since $\gamma < 2$, there exists a sufficiently large $\overline{n} = \overline{n}(\delta, a, \gamma, C) \in \mathbb{N}^+$ with $\{x_k\} \subset A_{\overline{n}}^\sigma$ such that

$$\int_{U_k} |\nabla(\psi_k u)|^2 dx \le (1+\delta)\int_{U_k} (|\nabla u|^2 + V(x)u^2)dx \le 1+\delta,\ \forall k \in K_{n,\sigma} \text{ and } n \ge \overline{n}.$$

Therefore, let's define $v_k \triangleq \sqrt{1-\delta}\psi_k u \in H_0^1(U_k) \subset H^1(\mathbb{R}^2)$ and then $|\nabla v_k|_2^2 \le 1-\delta^2 < 1$. Recalling that $\alpha/(1-\delta) < 4\pi$ by (2.1), then it follows from Proposition 2.2 that

$$\int_{U_k^{1/2}} (e^{\alpha u^2}-1)dx = \int_{U_k^{1/2}} (e^{\alpha(\psi_k u)^2}-1)dx \le \int_{U_k} (e^{[\alpha/(1-\delta)]v_k^2}-1)dx \le C_1 \int_{\mathbb{R}^2} |v_k|^2 dx$$

for some positive constant C_1 independent of $u \in E$. So, by means of (2.7) again, we derive

$$\int_{U_k^{1/2}} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le \frac{b^{\frac{4}{4-\mu}}}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{U_k^{1/2}} (e^{\alpha u^2}-1)dx$$
$$\le \frac{C_1(1-\delta)b^{\frac{4}{4-\mu}}}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{U_k} u^2 dx \le \frac{C(1-\delta)b^{\frac{4}{4-\mu}}(1+C_\gamma|x_k|^\gamma)}{a(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{U_k} V(x)u^2 dx$$

which together with (2.6) implies that

$$\int_{A_{3n}^\sigma} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le \frac{C(1-\delta)b^{\frac{4}{4-\mu}}}{a} \sum_{k\in K_{n,\sigma}} \frac{1+C_\gamma|x_k|^\gamma}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{U_k} V(x)u^2 dx$$
$$\le \frac{C(1-\delta)b^{\frac{4}{4-\mu}}}{a} \sum_{k\in K_{n,\sigma}} \frac{1+C_\gamma|x_k|^\gamma}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{B_n^c} V(x)u^2 \chi_{U_k}(x) dx.$$

In view of (2.5), we have

$$\frac{1+C_\gamma|x_k|^\gamma}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \le \mathcal{B}_n \triangleq \sup_{x\in B_n^c} \frac{1+C_\gamma|x|^\gamma}{(1+C_\beta|x|^\beta)^{\frac{4}{4-\mu}}},\ \forall k \in K_{n,\sigma}.$$

As a consequence, in view of the constant ϑ in the Besicovitch covering lemma, we have

$$\int_{A_{3n}^\sigma} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le \frac{C(1-\delta)b^{\frac{4}{4-\mu}}\mathcal{B}_n\vartheta}{a} \int_{B_n^c} V(x)u^2 dx.$$

Letting $\sigma \to +\infty$, there holds

$$\int_{B_{3n}^c} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le \frac{C(1-\delta)b^{\frac{4}{4-\mu}}\mathcal{B}_n\vartheta}{a} \int_{B_n^c} V(x)u^2 dx. \tag{2.9}$$

Let's recall that $(4-\mu)\gamma/4 \le \beta$, i.e. $\gamma \le 4\beta/(4-\mu)$, one has

$$\lim_{n\to+\infty}\mathcal{B}_n = \lim_{n\to+\infty}\frac{1+C_\gamma n^\gamma}{(1+C_\beta n^\beta)^{\frac{4}{4-\mu}}} = \begin{cases} 0, & \text{if } \gamma < 4\beta/(4-\mu) \\ C_\gamma/C_\beta^{4/(4-\mu)}, & \text{if } \gamma = 4\beta/(4-\mu). \end{cases} \tag{2.10}$$

Combining (2.2), (2.4) and (2.9)-(2.10), we can finish the proof in this case.

2.2. Case 2: $\gamma = 2$ and $(4-\mu)/2 \le \beta < +\infty$

Let $u \in E\backslash\{0\}$ satisfy $\|u\| \le 1$, denoting $\Sigma_j \triangleq \{x \in \mathbb{R}^2 : 2^j \le |x| < 2^{j+1}\}$ for every $j \in \mathbb{N}^+ \cup \{0\}$,

$$\int\limits_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx = \int\limits_{B_1(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx + \sum_{j=0}^{\infty}\int\limits_{\Sigma_j} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx. \tag{2.11}$$

Since $Q \in C(\mathbb{R}^2)$ is positive, then $\max_{x\in B_1(0)} Q(x) = Q^1 \in (0,+\infty)$ and

$$\int\limits_{B_1(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le (Q^1)^{\frac{4}{4-\mu}} \int\limits_{B_1(0)} (e^{\alpha u^2}-1)dx. \tag{2.12}$$

To estimate the right term of the above formula, arguing as before, for a constant $R > 1$ defined later, we chose $\psi \in C_0^\infty(B_{2R}(0))$ to be a cutoff function satisfying (2.3). So, $\psi u \in H_0^1(B_{2R}(0)) \subset H_0^1(B_2(0))$. By (V), it simply concludes that $5V(x) \ge (1+|x|^2)V(x) \ge a$ on $B_2(0)$, then

$$\begin{aligned}\int\limits_{B_2(0)} |\nabla(\psi u)|^2 dx &\le (1+\epsilon)\int\limits_{B_2(0)} \psi^2|\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right)\int\limits_{B_2(0)} u^2|\nabla\psi|^2 dx \\ &\le (1+\epsilon)\int\limits_{B_{2R}(0)} |\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right)\frac{5}{a}\int\limits_{B_2(0)} V(x)u^2|\nabla\psi|^2 dx \\ &\le (1+\epsilon)\int\limits_{B_{2R}(0)} |\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right)\frac{5C^2}{aR^2}\int\limits_{B_{2R}(0)} V(x)u^2 dx \\ &\le (1+\epsilon)\int\limits_{B_{2R}(0)} (|\nabla u|^2 + V(x)u^2)dx \le 1+\epsilon\end{aligned}$$

provided that we pick the constant $\underline{R} = \underline{R}(\epsilon, a, C) > 1$ independent of $u \in E$ large enough such that

$$\left(1+\frac{1}{\epsilon}\right)\frac{5C^2}{a\underline{R}^2} \le 1+\epsilon,$$

where ϵ is given by (2.1). Thereby, we would define $v \triangleq \sqrt{1-\epsilon}\psi u \in H_0^1(B_2(0))$ and $|\nabla v|_2^2 \leq 1-\epsilon^2 < 1$ immediately. Recalling that $R > \underline{R}$, then $\psi \equiv 1$ in $B_1(0)$. By (2.1) and Proposition 2.1, we obtain

$$\int\limits_{B_1(0)} (e^{\alpha u^2} - 1)dx = \int\limits_{B_1(0)} (e^{4\pi(1-\epsilon)(\psi u)^2} - 1)dx \leq \int\limits_{B_2(0)} (e^{4\pi v^2} - 1)dx \leq C < +\infty,$$

where $C > 0$ is independent of $u \in E$ with $\|u\| = 1$, which together with (2.12) yields that

$$\int\limits_{B_1(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2} - 1)dx \leq C + \infty. \tag{2.13}$$

Next, let's take the estimate for the third term in (2.11). By $(\mathcal{K})$, one has

$$0 < Q(x) \leq \frac{b}{1+|x|^\beta} \leq \frac{b}{2^{\beta j}}, \ \forall x \in \Sigma_j = \{x \in \mathbb{R}^2 : 2^j \leq |x| < 2^{j+1}\}$$

showing that

$$\int\limits_{\Sigma_j} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2} - 1)dx$$

$$\leq \frac{b^{\frac{4}{4-\mu}}}{2^{\frac{2(2\beta-(4-\mu))j}{4-\mu}}} \int\limits_{\Sigma_0} (e^{\alpha u_j^2} - 1)dy = b^{\frac{4}{4-\mu}} \mathcal{C}_j \int\limits_{\Sigma_0} (e^{\alpha u_j^2} - 1)dy, \tag{2.14}$$

where a change of variables $y = 2^{-j}x$ is performed with $u_j(y) = u(2^j y)$ and

$$0 < \mathcal{C}_j \leq 1, \ \forall j \in \mathbb{N}^+ \cup \{0\} \text{ and } \lim_{j\to+\infty} \mathcal{C}_j = \lim_{j\to+\infty} \frac{1}{2^{\frac{2(2\beta-(4-\mu))j}{4-\mu}}} = 0, \text{ if } \beta > (4-\mu)/2. \tag{2.15}$$

In order to estimate for the integral term in (2.11), we shall firstly observe that $\overline{\Sigma_0} \subset \cup_{y\in\Sigma_0} B_{R_y/2}(y)$, where $R_y \triangleq \text{dist}(y, \partial\Sigma_0)$. Clearly, $B_{R_y}(y) \subset \Sigma_0$ and there are finitely many, say $k \in \mathbb{N}^+$, such balls to cover Σ_0 form the compactness of $\overline{\Sigma_0}$ in $\mathbb{R}^2$, that is, $\Sigma_0 \subset \cup_{i=1}^k B_{R_{y_i}/2}(y_i)$. Given an $i \in \{1, 2, \cdots, k\}$, we can pick the constant $R_i > R_{y_i}$ determined later and chose the corresponding cutoff function $\psi_i \in C_0^\infty(B_{R_i}(y_i))$ such that

$$0 \leq \psi_i \leq 1 \text{ in } B_{R_i}(y_i), \ \psi_i \equiv 1 \text{ in } B_{R_i/2}(y_i) \text{ and } |\nabla\psi_i| \leq \frac{C}{R_i} \text{ in } B_{R_i}(y_i)$$

for some universal constant $C > 0$. So, one concludes that $\psi_i u_j \in H_0^1(B_{R_i}(y_i)) \subset H_0^1(B_{R_{y_i}}(y_i))$. In view of $(\mathcal{K})$, it simply gets that $2^{2(j+2)}V(x) \geq (1+|x|^2)V(x) \geq a$ on Σ_j for all $j \in \mathbb{N} \cup \{0\}$. Then in view of the constant $\epsilon > 0$ given by (2.1), as a consequence of Young's inequality one has

$$\begin{aligned}\int_{B_{R_{y_i}}(y_i)} |\nabla(\psi_i u_j)|^2 dy &\le (1+\epsilon)\int_{B_{R_{y_i}}(y_i)} \psi_i^2|\nabla u_j|^2 dy + \left(1+\frac{1}{\epsilon}\right)\int_{B_{R_{y_i}}(y_i)} u_j^2|\nabla\psi_i|^2 dy\\
&\le (1+\epsilon)\int_{\Sigma_0}|\nabla u_j|^2 dy + \left(1+\frac{1}{\epsilon}\right)\frac{C^2}{R_i^2}\int_{\Sigma_0} u_j^2 dy\\
&= (1+\epsilon)\int_{\Sigma_j}|\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right)\frac{C^2}{2^{2j}R_i^2}\int_{\Sigma_j} u^2 dx\\
&\le (1+\epsilon)\int_{\Sigma_j}|\nabla u|^2 dx + \left(1+\frac{1}{\epsilon}\right)\frac{16C^2}{aR_i^2}\int_{\Sigma_j} V(x)u^2 dx\\
&\le (1+\epsilon)\int_{\Sigma_j}(|\nabla u|^2 + V(x)u^2)dx \le 1+\epsilon\end{aligned}$$

provided that we chose a constant $R_i = R_i(\epsilon, a, C) > 1$ independent of $u \in E$ large enough such that

$$\left(1+\frac{1}{\epsilon}\right)\frac{16C^2}{aR_i^2} \le 1+\epsilon.$$

So, we also define $v_{ij} \triangleq \sqrt{1-\epsilon}\,\psi_i u_j \in H_0^1(B_{R_{y_i}}(y_i))$ and $|\nabla v_{ij}|_2^2 \le 1-\epsilon^2 < 1$ immediately. Recalling that $R_i > R_{y_i}$, then $\psi_i \equiv 1$ in $B_{R_{y_i}/2}(y_i)$ and it follows from Proposition 2.1 that

$$\begin{aligned}\int_{B_{R_{y_i}/2}(y_i)} (e^{\alpha u_j^2}-1)dy &= \int_{B_{R_{y_i}/2}(y_i)} (e^{4\pi(1-\epsilon)(\psi_i u_j)^2}-1)dy \le \int_{B_{R_{y_i}}(y_i)} (e^{4\pi v_{ij}^2}-1)dy\\
&\le CR_{y_i}^2\int_{\Sigma_j}(|\nabla u|^2+V(x)u^2)dx \le \overline{C}\int_{\Sigma_j}(|\nabla u|^2+V(x)u^2)dx\end{aligned}$$

where $\overline{C} > 0$ is independent of $u \in E$ with $\|u\| = 1$. Since $\Sigma_0 \subset \cup_{i=1}^k B_{R_{y_i}/2}(y_i)$, we obtain

$$\int_{\Sigma_0}(e^{\alpha u_j^2}-1)dy \le \sum_{i=1}^k \int_{B_{R_{y_i}/2}(y_i)} (e^{\alpha u_j^2}-1)dy \le C\int_{\Sigma_j}(|\nabla u|^2+V(x)u^2)dx$$

which together with (2.14)-(2.15) indicates that

$$\sum_{j=0}^{\infty}\int_{\Sigma_j} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx \le Cb^{\frac{4}{4-\mu}}\sum_{j=0}^{\infty}\int_{\Sigma_j} \mathcal{C}_j(|\nabla u|^2+V(x)u^2)dx \le C < +\infty. \tag{2.16}$$

Combining (2.11), (2.13) and (2.16), we can finish the proof in this case.

2.3. Case 3: $\gamma > 2$ and $(4-\mu)/2 \le \beta < +\infty$

Let $u \in E\backslash\{0\}$ satisfy $\|u\| \le 1$, denoting $\Sigma_j \triangleq \{x \in \mathbb{R}^2 : 2^j \le |x| < 2^{j+1}\}$ for every $j \in \mathbb{N}^+ \cup \{0\}$. Assume (2.11) and (2.12) still to be satisfied in this case, for a constant $R > 1$ chosen later, we let ψ be a cutoff function given as in (2.3). Suppose $\lambda_1 > 0$ to be the principle eigenvalue of the Dirichlet problem $(-\Delta, H_0^1(B_2(0)))$, for $\epsilon > 0$ in (2.1), we apply the Young's inequality to derive

$$\begin{aligned}\int_{B_2(0)} |\nabla(\psi u)|^2 dx &\le \left(1+\frac{\epsilon}{2}\right)\int_{B_2(0)} \psi^2|\nabla u|^2 dx + \left(1+\frac{2}{\epsilon}\right)\int_{B_2(0)} u^2|\nabla\psi|^2 dx\\ &\le \left(1+\frac{\epsilon}{2}\right)\int_{B_2(0)} |\nabla u|^2 dx + \left(1+\frac{2}{\epsilon}\right)\frac{C^2}{R^2}\int_{B_2(0)} u^2 dx\\ &\le 1+\frac{\epsilon}{2}+\left(1+\frac{2}{\epsilon}\right)\frac{C^2}{\lambda_1 R^2}\int_{B_2(0)} |\nabla u|^2 dx\\ &\le 1+\frac{\epsilon}{2}+\left(1+\frac{2}{\epsilon}\right)\frac{C^2}{\lambda_1 R^2} \le 1+\epsilon\end{aligned}$$

if we chose the constant $R' = R'(\epsilon, \lambda_1, C) > 1$ independent of $u \in E$ large enough such that

$$\left(1+\frac{1}{\epsilon}\right)\frac{5C^2}{\lambda_1 (R')^2} \le \frac{\epsilon}{2}.$$

The remaining part is totally similar to the proof of Case 2, we omit it here. Consequently, we would finish the proof in this case.

Proof of (1.8) in Theorem 1.3. In summary, we have verified the validity of (1.8) in the Cases 1, 2 and 3, respectively. So, the proof is complete. □

Now, we end the proof of Theorem 1.3 by showing (1.9). For this purpose, with the help of the density of $C_0^\infty(\mathbb{R}^2)$ into E (see the Appendix A), we shall combine (1.8) and the Young's inequality to receive this goal. Consequently, we derive the following

Proof of (1.9) in Theorem 1.3. For every $\alpha > 0$ and $u \in E$, there exists a function $u_0 \in C_0^\infty(\mathbb{R}^2)$ such that $\|u - u_0\| \le \alpha^{-1/2}$. Moreover, $u^2 \le 2(u-u_0)^2 + 2u_0^2$. By choosing $R > 0$ to be such that $\operatorname{supp} u_0 \subset B_R(0)$, then

$$\begin{aligned}\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha u^2}-1)dx &\le \int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{2\alpha(u-u_0)^2}e^{2\alpha u_0^2}-1)dx\\ &\le \frac{1}{2}\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{4\alpha\|u-u_0\|^2(|u-u_0|^2/\|u-u_0\|^2)}-1)dx + \frac{1}{2}\int_{B_R(0)} Q^{\frac{4}{4-\mu}}(x)(e^{4\alpha u_0^2}-1)dx\end{aligned}$$

$$\leq \frac{1}{2}S_4 + \frac{b^{\frac{4}{4-\mu}}}{2}|B_R(0)|e^{4\alpha|u_0|_\infty^2} < +\infty,$$

where we depend on the fact that $4\alpha\|u-u_0\|^2 \leq 4 < 4\pi$ in (1.8). The proof is complete. □

After showing the proof of Theorem 1.3 successfully, we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. If $(\mathcal{K})$ holds, proceeding as [5], we would omit the proof of $E_r \hookrightarrow L^{p\mu}_{Q^\mu}(\mathbb{R}^2)$ with $(4-\mu)/2 < p < +\infty$. Now, we would suppose that $(\mathcal{K}')$ holds. For this goal, if $u_m \rightharpoonup u$ in E, we aim to conclude that $u_m \to u$ in $L^{4p/(4-\mu)}(\mathbb{R}^2)$ after passing to a subsequence if necessary. As we know, $u_m \to u$ in $L^{4p/(4-\mu)}(\Omega)$ for every bounded domain $\Omega \subset \mathbb{R}^2$. We just need to obtain that $u_m \to u$ in $L^{4p/(4-\mu)}(B_R^c(0))$ for some sufficiently large $R > 0$. Without loss of generality, we assume that $u \equiv 0$ for simplicity. To end it, we split it into three cases.

(i)′ Proceeding as the Case 1 in Section 2.1, we only consider $0 < \gamma < 2$ and $(4-\mu)\gamma/4 < \beta < +\infty$. We recall the process in Section 2.1 to derive

$$\int_{A_{3n}^\sigma} Q^{\frac{4}{4-\mu}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \leq b^{\frac{4}{4-\mu}} \sum_{k\in K_{n,\sigma}} \frac{1+C_\gamma|x_k|^\gamma}{(1+C_\beta|x_k|^\beta)^{\frac{4}{4-\mu}}} \int_{B_n^c} V(x)u_m^2\chi_{U_k}(x)dx.$$

Letting $\sigma \to +\infty$ and $\|u_m\|^2 \leq C < +\infty$ uniformly in $m \in \mathbb{N}^+$, there holds

$$\int_{B_{3n}^c} Q^{\frac{4}{4-\mu}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \leq b^{\frac{4}{4-\mu}}\mathcal{B}_n\vartheta \int_{B_n^c} V(x)u_m^2 dx \leq b^{\frac{4}{4-\mu}}\mathcal{B}_n\vartheta\|u_m\|^2 \leq b^{\frac{4}{4-\mu}}C\mathcal{B}_n\vartheta.$$

In view of (2.10), for all $\varepsilon > 0$, there exists an integer $m_0 > 0$ such that

$$\int_{B_{3n}^c} Q^{\frac{4}{4-\mu}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \leq \varepsilon,\ \forall m \geq m_0$$

yielding the desired result since $\varepsilon > 0$ is arbitrary.

(ii)′ We still exploit the ideas in Section 2.2, then

$$\int_{\Sigma_j} Q^{\frac{4}{4-\mu}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \leq \frac{b^{\frac{4}{4-\mu}}}{2^{\frac{2(2\beta-(4-\mu))j}{4-\mu}}} \int_{\Sigma_0} |(u_m)_j|^{\frac{4p}{4-\mu}}dy = b^{\frac{4}{4-\mu}}\mathcal{C}_j \int_{\Sigma_0} |(u_m)_j|^{\frac{4p}{4-\mu}}dy$$

where $(u_m)_j = u_m(2^j y)$ and $\mathcal{C}_j$ satisfies (2.15). Let's observe that $H^1(\Sigma_0) \hookrightarrow L^{4p/(4-\mu)}(\Sigma_0)$ because $4p/(4-\mu) \geq 2$, there exists a constant $C > 0$ such that

$$\int_{\Sigma_j} Q^{\frac{4-\mu}{4}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \leq Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\left(\int_{\Sigma_0}[|\nabla(u_m)_j|^2 + |(u_m)_j|^2]dx\right)^{\frac{2p}{4-\mu}}$$

$$= Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\Big(\int\limits_{\Sigma_j}[|\nabla u_m|^2 + 2^{-2j}|u_m|^2]dx\Big)^{\frac{2p}{4-\mu}}$$

$$\le Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\Big(\int\limits_{\Sigma_j}[|\nabla u_m|^2 + 16a^{-1}V(x)|u_m|^2]dx\Big)^{\frac{2p}{4-\mu}}$$

$$\le Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\Big(\int\limits_{\Sigma_j}[|\nabla u_m|^2 + V(x)|u_m|^2]dx\Big)^{\frac{2p}{4-\mu}}$$

which together with the fact that the map $s \mapsto s^{2p/(4-\mu)}$ is super additive indicates that

$$\sum_{j=0}^{\infty}\int\limits_{\Sigma_j} Q^{\frac{4-\mu}{4}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \le Cb^{\frac{4}{4-\mu}}\sum_{j=0}^{\infty}\mathcal{C}_j\Big(\int\limits_{\Sigma_j}[|\nabla u_m|^2 + V(x)|u_m|^2]\Big)^{\frac{2p}{4-\mu}}.$$

Recalling that (2.15), for all $\varepsilon > 0$, there exists a $j_0 \in \mathbb{N}^+$ such that

$$\sum_{j=j_0}^{\infty}\int\limits_{\Sigma_j} Q^{\frac{4-\mu}{4}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \le Cb^{\frac{4}{4-\mu}}\varepsilon\sum_{j=j_0}^{\infty}\Big(\int\limits_{\Sigma_j}[|\nabla u_m|^2 + V(x)|u_m|^2]\Big)^{\frac{2p}{4-\mu}}$$

$$\le Cb^{\frac{4}{4-\mu}}\|u_m\|^{\frac{4p}{4-\mu}}\varepsilon$$

yielding the desired result since $\varepsilon > 0$ is arbitrary.

(iii)′ Since Σ_0 is bounded, we can exploit $H^1(\Sigma_0)$ under the norm $|\nabla\cdot|_2$ to obtain

$$\int\limits_{\Sigma_j} Q^{\frac{4-\mu}{4}}(x)|u_m|^{\frac{4p}{4-\mu}}dx \le b^{\frac{4}{4-\mu}}\mathcal{C}_j\int\limits_{\Sigma_0}|(u_m)_j|^{\frac{4p}{4-\mu}}dx \le Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\Big(\int\limits_{\Sigma_0}|\nabla(u_m)_j|^2dx\Big)^{\frac{2p}{4-\mu}}$$

$$= Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\Big(\int\limits_{\Sigma_j}|\nabla u_m|^2dx\Big)^{\frac{2p}{4-\mu}} \le Cb^{\frac{4}{4-\mu}}\mathcal{C}_j\Big(\int\limits_{\Sigma_j}[|\nabla u_m|^2 + V(x)|u_m|^2]dx\Big)^{\frac{2p}{4-\mu}}.$$

The remainder is totally similar to (ii)′, we omit it here. So, the proof of Theorem 1.2 is finished. □

3. The existence result

In this section, we try to investigate the existence of nontrivial solutions of Eq. (1.1). To this aim, the critical point theorem introduced in [10,29] will be used to search for the existence of solutions.

Proposition 3.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and $\Psi \in C^1(X, \mathbb{R})$ satisfy the condition*

$$\inf_{\{u\in X \mid \|u\|_X=\rho\}} \Psi(u) \geq \varrho > \eta = \max\{\Psi(0), \Psi(\gamma(1))\}$$

for some constants ϱ, η and $\rho > 0$ with $\|\gamma(1)\|_X > \rho$ and γ given by (1.14). Let $c \geq \varrho$ be characterized by (1.14), then there exists a sequence $(C)_c$ sequence $\{u_n\} \subset X$ of Ψ.

Firstly, we formulate the functional setting for a variational approach to Eq. (1.1). Since the nonlinearity f satisfies (1.2), $f(0) = 0$ and (f_1), for fixed $\alpha > \alpha_0$, $q \geq 1$ and for all $\varepsilon > 0$ we have

$$|f(s)| \leq \varepsilon |s|^{\frac{2-\mu}{2}} + C(\alpha, q, \varepsilon)|s|^{q-1}(e^{\alpha s^2} - 1), \ \forall s \in \mathbb{R}. \tag{3.1}$$

As a consequence of (1.11), there holds

$$|F(s)| \leq \varepsilon |s|^{\frac{4-\mu}{2}} + C(\alpha, q, \varepsilon)|s|^{q}(e^{\alpha s^2} - 1), \ \forall s \in \mathbb{R}. \tag{3.2}$$

Given a function $u \in E$, by (1.4), we utilized (3.2) with $\alpha > \alpha_0$ and $q \geq 2$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u))]Q(x)F(u)dx &\leq C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)F^{\frac{4}{4-\mu}}(u)dx\Big)^{\frac{4-\mu}{2}} \\
&\leq C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u|^{\frac{4q}{4-\mu}}(e^{\frac{4\alpha}{4-\mu}u^2} - 1)dx\Big)^{\frac{4-\mu}{2}} + C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u|^{\frac{4}{4-\mu}\cdot\frac{4-\mu}{2}}dx\Big)^{\frac{4-\mu}{2}} \\
&\leq C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u|^{\frac{4q\nu'}{4-\mu}}\Big)^{\frac{4-\mu}{2\nu'}}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha\nu}{4-\mu}u^2} - 1)dx\Big)^{\frac{4-\mu}{2\nu}} + C\|u\|^{4-\mu} \\
&\leq C\|u\|^{2q}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha\nu}{4-\mu}u^2} - 1)dx\Big)^{\frac{4-\mu}{2\nu}} + C\|u\|^{4-\mu} < +\infty,
\end{aligned} \tag{3.3}$$

where we have exploited Theorem 1.2 and (1.7) together with $\nu > 1$ and $1/\nu + 1/\nu' = 1$. Consequently, the functional J defined by (1.10) is well-defined and J is of class $\mathcal{C}^1$.

From now on, for simplicity, we rewrite the assumptions in Theorem 1.5 in this paper as follows

- $(\mathbb{H}_1)$ Let $(\mathcal{K})$ with V and Q being radially symmetric, or $(\mathcal{K})'$, hold true;
- $(\mathbb{H}_2)$ f satisfies (1.2) and $(f_1) - (f_4)$ with $\delta \in (0, 1)$ in (f_2);
- $(\mathbb{H}_3)$ f satisfies (1.2) and $(f_1) - (f_4)$ with $\delta = 0$ in (f_2);
- $(\mathbb{H}_4)$ f satisfies (1.2) and $(f_1) - (f_3)$ with (f_5).

Now, we show that the functional J defined by (1.10) possesses the mountain-pass geometry.

Lemma 3.2. *Suppose that $(\mathbb{H}_1)$ and* **one** *of $(\mathbb{H}_2)$, $(\mathbb{H}_3)$ and $(\mathbb{H}_4)$ hold true, then*

(i) there exist $\varrho > 0$ and $\rho > 0$ such that $J(u) \geq \varrho$ for all $u \in E$ with $\|u\| = \rho$;
(ii) there exists a function $e \in E$ with $\|e\| > \rho$ such that $J(e) < 0$.

Proof. (i). In view of (3.3), we can proceed as [7, Lemma 3.1] to get the desired result.

(ii). Obviously, we would deduce that $\lim_{|s|\to\infty} F(s)/s = +\infty$ either from (f_4) or (f_5). Let's choose a positive function ψ with support in $B_1(0)$ belonging to $C_c^\infty(\mathbb{R}^2)$. Since $Q_1 = \min_{x\in B_1(0)} Q(x) \in (0, +\infty)$, by means of the Fatou's lemma, we obtain

$$\frac{1}{t^2}\int\limits_{B_1(0)}\bigg(\int\limits_{B_1(0)}\frac{Q(y)F(t\psi(y))}{|x-y|^\mu}dy\bigg)Q(x)F(t\psi(x))dx \geq \frac{Q_1^2}{2}\bigg(\int\limits_{B_1(0)}\frac{F(t\psi)}{t\psi}\psi dx\bigg)^2 \to -\infty$$

yielding that $\lim_{t\to+\infty} J(tu) = -\infty$. We'll finish the proof by letting $e = t_0\psi$ with a large $t_0 > 0$. $\square$

As a consequence of Proposition 3.1 and Lemma 3.2, we obtain the existence of (C) sequence of J at the level c defined by (1.14), that is, $J(u_n) \to c$ and $(1 + \|u_n\|)\|J'(u_n)\|_{E^{-1}} \to 0$. To restore the compactness of $\{u_n\}$, we firstly derive the upper estimate for the mountain-pass level c. With this aim in mind, we shall deal with it by $(f_3) - (f_4)$ and (f_5), respectively. Inspired by [4,7,9,15,16], we consider the Moser sequence defined by

$$\overline{w}_n(x) \triangleq \frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & \text{if } 0 \leq |x| \leq \frac{1}{n},\\ \frac{\log(\frac{1}{|x|})}{\sqrt{\log n}}, & \text{if } \frac{1}{n} < |x| \leq 1,\\ 0, & \text{if } |x| > 1.\end{cases}$$

Lemma 3.3. *Suppose that* $(\mathbb{H}_1)$ *and* **one** *of* $(\mathbb{H}_2)$*,* $(\mathbb{H}_3)$ *and* $(\mathbb{H}_4)$ *hold true, then* $0 < c < c^* \triangleq \frac{(4-\mu)\pi}{2\alpha_0}$.

Proof. It follows Lemma 3.2-(i) that $c > 0$. Thanks to Lemma 3.2-(ii), we shall easily conclude that $c = \inf_{\gamma\in\Gamma}\max_{t\in(0,1]} J(\gamma(t)) \leq \inf_{u\in E\setminus\{0\}}\max_{t>0} J(tu)$. As a consequence, it suffices to derive that there exists a function $w \in E\setminus\{0\}$ such that $\max_{t>0} J(tw) < c^*$. Proceeding as [7], we obtain

$$\begin{aligned}\|\overline{w}_n\|^2 &= \int\limits_{B_1(0)}|\nabla\overline{w}_n|^2dx + \int\limits_{B_1(0)}V(x)|\overline{w}_n|^2dx = 1 + \int\limits_{B_1(0)}V(x)|\overline{w}_n|^2dx\\ &\leq 1 + A\int_0^{1/n} r\log n dr + A\int_{1/n}^1\frac{\log^2(1/r)}{\log n}rdr = 1 + \delta_n,\end{aligned}$$

where $A > 0$ is a constant given by $(\mathcal{K})$ or $(\mathcal{K})'$ and

$$\delta_n \triangleq A\bigg(\frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2}\bigg) > 0. \tag{3.4}$$

Then, we set $w_n \triangleq \overline{w}_n/\sqrt{1+\delta_n} \in E\backslash\{0\}$ with $\|w_n\| \le 1$. We claim that there is a $n \in \mathbb{N}^+$ such that

$$\max_{t>0} J(tw_n) < c^*. \tag{3.5}$$

Otherwise, for all $n \in \mathbb{N}^+$, there exists a $t_n > 0$ corresponding to the maximum point of $\max_{t>0} J(tw_n)$

$$\langle J'(t_n w_n), t_n w_n\rangle = 0 \text{ and } J(t_n w_n) = \max_{t>0} J(tw_n) \ge c^* \tag{3.6}$$

in which of the first formula together with $\|w_n\| \le 1$ implies that

$$t_n^2 \ge \int_{\mathbb{R}^2} (|x|^{-\mu} * (Q(x)F(t_n w_n)))Q(x) f(t_n w_n) t_n w_n dx. \tag{3.7}$$

Since $F(s) \ge 0$ for all $s \in \mathbb{R}$, we infer from the second formula in (3.6) and $\|w_n\| \le 1$ that

$$t_n^2 \ge 2c^* = (4-\mu)\pi/\alpha_0, \ \forall n \in \mathbb{N}^+. \tag{3.8}$$

If (f_3) and (f_4) hold true in $(\mathbb{H}_2)$ or $(\mathbb{H}_3)$, for all $\varepsilon \in (0, \beta_0)$, there is a constant $R_\varepsilon > 0$ such that

$$F(s)f(s)s \ge M_0^{-1}(\beta_0 - \varepsilon)s^{\vartheta+1}e^{2\alpha_0 s^2}, \ \forall s \ge R_\varepsilon. \tag{3.9}$$

Notice that $\min_{x\in B_{1/n}(0)} Q(x) \ge \min_{x \in B_1(0)} Q(x) = Q_1 > 0$, by (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned}
t_n^2 &\ge Q_1^2 \int_{B_{1/n}(0)} \bigg(\int_{B_{1/n}(0)} \frac{F(t_n w_n(y))}{|x-y|^\mu} dy\bigg) f(t_n w_n(x)) t_n w_n(x) dx \\
&= F\bigg(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}\sqrt{1+\delta_n}}\bigg) f\bigg(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}\sqrt{1+\delta_n}}\bigg) t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}\sqrt{1+\delta_n}} \int_{B_{1/n}(0)} \int_{B_{1/n}(0)} \frac{1}{|x-y|^\mu} dy dx \\
&\ge M_0^{-1}(\beta_0-\varepsilon) t_n^{\vartheta+1} \bigg(\frac{\log n}{2\pi(1+\delta_n)}\bigg)^{\frac{\vartheta+1}{2}} \bigg(e^{\alpha_0 t_n^2 \pi^{-1}(1+\delta_n)^{-1}\log n}\bigg) \frac{n^\mu}{2} |B_{1/n}(0)|^2 \\
&= \frac{\pi^2}{2M_0}(\beta_0-\varepsilon) t_n^{\vartheta+1} \bigg(\frac{1}{2\pi(1+\delta_n)}\bigg)^{\frac{\vartheta+1}{2}} e^{[\alpha_0 t_n^2 \pi^{-1}(1+\delta_n)^{-1} - (4-\mu)]\log n + \frac{\vartheta+1}{2}\log(\log n)}.
\end{aligned} \tag{3.10}$$

By (3.8) and $(\vartheta+1)\log(\log n)/2 > 0$, we can deduce that

$$\begin{aligned}
(1-\vartheta)\log t_n \ge{}& \log\bigg[\frac{\pi^2}{2M_0}(\beta_0-\varepsilon)\bigg(\frac{1}{2\pi(1+\delta_n)}\bigg)^{\frac{\vartheta+1}{2}}\bigg] \\
&+ [\alpha_0 t_n^2 \pi^{-1}(1+\delta_n)^{-1} - (4-\mu)]\log n.
\end{aligned} \tag{3.11}$$

If $\{t_n\}$ is unbounded, up to a subsequence if necessary, we can assume that $t_n \to +\infty$ and then

$$\frac{(1-\vartheta)\log t_n}{t_n^2} \geq t_n^{-2}\log\left[\frac{\pi^2}{2M_0}(\beta_0-\varepsilon)\left(\frac{1}{2\pi(1+\delta_n)}\right)^{\frac{\vartheta+1}{2}}\right] + [\alpha_0\pi^{-1}(1+\delta_n)^{-1} - t_n^{-2}(4-\mu)]\log n$$

which together with $\delta_n \to 0$ in (3.4) yields a contradiction if we tend $n \to \infty$. Thereby, passing to a subsequence if necessary, there exists a positive constant t_0 such that

$$\lim_{n\to\infty} t_n^2 = t_0^2 \geq (4-\mu)\pi/\alpha_0,$$

where (3.8) gives the inequality. Moreover, we conclude that $t_0^2 = (4-\mu)\pi/\alpha_0$. Otherwise, we obtain a contradiction by letting $n \to \infty$ in (3.11). Let's tend $n \to \infty$ in (3.10), there holds

$$(4-\mu)\pi/\alpha_0 = t_0^2 \geq \frac{\pi^2}{2M_0}(\beta_0-\varepsilon)t_0^{\vartheta+1}\left(\frac{1}{2\pi}\right)^{\frac{\vartheta+1}{2}} \lim_{n\to\infty} e^{\frac{\vartheta+1}{2}\log(\log n)} = +\infty,$$

a contradiction. So, (3.5) holds true.

Next, we suppose $(\mathbb{H}_3)$ to verify that $c < c^*$ still holds true. In view of Remark 1.10, the constant $\mathbb{S}_{\mu,q}$ defined by (1.16) can be attained by a nontrivial $\psi \in E$, that is,

$$\|\psi\|^2 = \mathbb{S}_{\mu,q} \text{ and } \int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)|\psi|^q)]Q(x)|\psi|^q dx = 1. \tag{3.12}$$

Since $2q > 4-\mu > 2$, it's easy to prove that $\lim_{t\to+\infty} J(t\psi) = -\infty$ which indicates the existence of a sufficient large $t_0 > 0$ such that $J(t_0\psi) < 0$. Thus, $\gamma_0(t) = tt_0\psi \in \Gamma$. It follows from the definition of c given by (1.14) that

$$c \leq \max_{t\in(0,1]} J(tt_0\psi) \leq \max_{t\in(0,+\infty)} J(t\psi),$$

which together with (f_5) and (3.12) gives that

$$c \leq \max_{t\in(0,+\infty)}\left(\frac{\mathbb{S}_{\mu,q}}{2}t^2 - \frac{C_q^2}{2q^2}t^{2q}\right) = \frac{(q-1)\mathbb{S}_{\mu,q}}{2q}\left(\frac{q\mathbb{S}_{\mu,q}}{C_q^2}\right)^{1/(q-1)} < \frac{(4-\mu)\pi}{2\alpha_0} = c^*,$$

where the constant $C_q > 0$ comes from (f_5). The proof is complete. □

Before investigating the boundness of the $(C)_c$ sequence $\{u_n\} \subset E$, we shall establish the following lemma which is very significant in the present paper. We emphasize here that this lemma does not require the assumption that $\{u_n\}$ is a $(C)_c$ sequence. Thereby, it can be regarded as an generalization to [7, Lemma 2.4].

Lemma 3.4. *Suppose that* $(\mathbb{H}_1)$ *and* **one** *of* $(\mathbb{H}_2)$, $(\mathbb{H}_3)$ *and* $(\mathbb{H}_4)$ *hold true. If* $\{u_n\} \subset E$ *satisfies* $u_n \rightharpoonup u_0$ *in* E *as* $n \to \infty$ *and there is a constant* $K_0 > 0$ *such that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)u_n dx \leq K_0. \tag{3.13}$$

Then, going to a subsequence if necessary, there holds

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[|x|^{-\mu}*(Q(x)F(u_n))]Q(x)F(u_n)dx = \int_{\mathbb{R}^2}[|x|^{-\mu}*(Q(x)F(u_0))]Q(x)F(u_0)dx. \tag{3.14}$$

Moreover, for all $\psi\in C_0^\infty(\mathbb{R}^2)$, going to a subsequence if necessary, we can conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[|x|^{-\mu}*(Q(x)F(u_n))]Q(x)f(u_n)\psi dx = \int_{\mathbb{R}^2}[|x|^{-\mu}*(Q(x)F(u_0))]Q(x)f(u_0)\psi dx. \tag{3.15}$$

Proof. Combining (3.13) and the Fatou's lemma, one has

$$\int_{\mathbb{R}^2}[|x|^{-\mu}*(Q(x)F(u_0))]Q(x)f(u_0)u_0dx \le K_0. \tag{3.16}$$

In view of (f_3) in **one** of $(\mathbb{H}_2)$, $(\mathbb{H}_3)$ and $(\mathbb{H}_4)$, it follows that

$$0\le \lim_{s\to+\infty}\frac{F(s)}{f(s)s}\le \lim_{s\to+\infty}\frac{M_0}{s^{\vartheta+1}}=0$$

and for all $\varepsilon>0$, there exists a constant $\overline{s}=\overline{s}(\varepsilon)>1$ such that

$$F(s)\le \varepsilon f(s)s,\ \forall s\ge \overline{s}$$

which together with (3.13) and (3.16) gives that

$$\sup_{n\in\mathbb{N}}\int_{|u_n|\ge\overline{s}}[|x|^{-\mu}*(Q(x)F(u_n))]Q(x)F(u_n)dx\le K_0\varepsilon \tag{3.17}$$

and

$$\int_{|u_0|\ge\overline{s}}[|x|^{-\mu}*(Q(x)F(u_0))]Q(x)F(u_0)dx\le K_0\varepsilon. \tag{3.18}$$

Let's define

$$\Theta_n \triangleq \int\limits_{|u_n|<\overline{s}} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx$$
$$= \int\limits_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)\chi_{\{|u_n|<\overline{s}\}}dx$$
$$= \int\limits_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n)\chi_{\{|u_n|<\overline{s}\}})]Q(x)F(u_n)dx = \int\limits_{\mathbb{R}^2} Q(x)\xi_n F(u_n)dx$$

and similarly

$$\Theta_0 \triangleq \int\limits_{|u_0|<\overline{s}} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)F(u_0)dx = \int\limits_{\mathbb{R}^2} Q(x)\xi_0 F(u_0)dx,$$

where

$$\xi_n(x) \triangleq |x|^{-\mu} * (Q(x)F(u_n)\chi_{\{|u_n|<\overline{s}\}}) = \int\limits_{\mathbb{R}^2} \frac{Q(y)F(u_n)\chi_{\{|u_n|<\overline{s}\}}}{|x-y|^{\mu}}dy,\ x \in \mathbb{R}^2$$

and

$$\xi_0(x) \triangleq |x|^{-\mu} * (Q(x)F(u_0)\chi_{\{|u_0|<\overline{s}\}}) = \int\limits_{\mathbb{R}^2} \frac{Q(y)F(u_0)\chi_{\{|u_0|<\overline{s}\}}}{|x-y|^{\mu}}dy,\ x \in \mathbb{R}^2.$$

Claim 1. $\{\xi_n\}$ is uniformly bounded in $n \in \mathbb{N}$ and $\xi_n \to \xi_0$ a.e. in $\mathbb{R}^2$.
Verification: Let $\varepsilon = 1$ and $q = 1$ in (3.2), then there exists a constant $C(\overline{s}) > 0$ such that

$$|F(s)| \le C(\overline{s})|s|^{(4-\mu)/2},\ \forall |s| \le \overline{s}. \tag{3.19}$$

Combining the Hölder's inequality and (3.19), we obtain

$$|\xi_n| \le \left(\int\limits_{|x-y|\le 1} \left|Q(y)F(u_n)\chi_{\{|u_n|<\overline{s}\}}\right|^{\frac{2+\mu}{2-\mu}}dy\right)^{\frac{2-\mu}{2+\mu}} \left(\int\limits_{|x-y|\le 1} \frac{1}{|x-y|^{\frac{2+\mu}{2}}}dy\right)^{\frac{2\mu}{2+\mu}}$$
$$+ \left(\int\limits_{|x-y|>1} \left|Q(y)F(u_n)\chi_{\{|u_n|<\overline{s}\}}\right|^{\frac{4-\mu}{2(2-\mu)}}dy\right)^{\frac{2(2-\mu)}{4-\mu}} \left(\int\limits_{|x-y|>1} \frac{1}{|x-y|^{4-\mu}}dy\right)^{\frac{\mu}{4-\mu}}$$
$$\le \left(\frac{4}{2-\mu}\right)^{2\mu/(2+\mu)} b\pi \max_{|s|\le\overline{s}} F(s)$$
$$+ C(\overline{s})b^{\frac{\mu^2}{(4-\mu)^2}}\left(\frac{2\pi}{2-\mu}\right)^{\frac{\mu}{4-\mu}}\left(\int\limits_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4}{4-\mu}\cdot\frac{(4-\mu)^3}{16(2-\mu)}}dx\right)^{\frac{2(2-\mu)}{4-\mu}}$$

$$\leq \left(\frac{4}{2-\mu}\right)^{2\mu/(2+\mu)} b\pi \max_{|s|\leq\overline{s}} F(s)$$
$$+ CC(\overline{s})b^{\frac{\mu^2}{(4-\mu)^2}}\left(\frac{2\pi}{2-\mu}\right)^{\frac{\mu}{4-\mu}}\|u_n\|^{\frac{4-\mu}{2}} \leq C < +\infty,$$

where we have exploited Theorem 1.2 because $(4-\mu)^3/[16(2-\mu)] > (4-\mu)/2$. Then, we will deduce that $\xi_n \to \xi_0$ a.e. in $\mathbb{R}^2$. Recalling that $\{\|u_n\|\}$ is bounded, there is a constant $C_\mu \in (0,+\infty)$ such that $\|u_n\|^{(4-\mu)/2} + \|u_0\|^{(4-\mu)/2} \leq C_\mu$. As a consequence, for every $R > 0$, we shall apply the Hölder's inequality and (3.19) again to derive

$$|\xi_n - \xi_0| \leq \left(\int_{|x-y|\leq R} \left|Q(y)[F(u_n)\chi_{\{|u_n|<\overline{s}\}} - F(u_0)\chi_{\{|u_0|<\overline{s}\}}]\right|^{\frac{2+\mu}{2-\mu}} dy\right)^{\frac{2-\mu}{2+\mu}}$$
$$\times \left(\int_{|x-y|\leq R} \frac{1}{|x-y|^{\frac{2+\mu}{2}}} dy\right)^{\frac{2\mu}{2+\mu}}$$
$$+ \left(\int_{|x-y|>R} \left|Q(y)[F(u_n)\chi_{\{|u_n|<\overline{s}\}} - F(u_0)\chi_{\{|u_0|<\overline{s}\}}]\right|^{\frac{4-\mu}{2(2-\mu)}} dy\right)^{\frac{2(2-\mu)}{4-\mu}}$$
$$\times \left(\int_{|x-y|>R} \frac{1}{|x-y|^{4-\mu}} dy\right)^{\frac{\mu}{4-\mu}}$$
$$\leq \left(\frac{4\pi R^{(2-\mu)/2}}{2-\mu}\right)^{2\mu/(2+\mu)} b^{\frac{\mu(6-\mu)}{(4-\mu)(2+\mu)}} C(\overline{s})$$
$$\times \left(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n - u_0|^{\frac{4}{4-\mu}\cdot\frac{(4-\mu)^2(2+\mu)}{8(2-\mu)}} dx\right)^{\frac{2-\mu}{2+\mu}}$$
$$+ CC(\overline{s})b^{\frac{\mu^2}{(4-\mu)^2}}\left(\frac{2\pi}{(2-\mu)R^{2-\mu}}\right)^{\frac{\mu}{4-\mu}}\left(\|u_n\|^{\frac{4-\mu}{2}} + \|u_0\|^{\frac{4-\mu}{2}}\right).$$

Because $(4-\mu)^2(2+\mu)/[8(2-\mu)] > (4-\mu)/2$, we can let $n \to \infty$ in the above formula by Theorem 1.2, and then the claim would be true by tending $R \to \infty$.

Claim 2. $\Theta_n \to \Theta_0$ as $n \to \infty$.

Verification: Because $u_n \rightharpoonup u_0$ in E together with $(4-\mu)^2(2+\mu)/[8(2-\mu)] > (4-\mu)/2$ and $(4-\mu)^3/[16(2-\mu)] > (4-\mu)/2$, up to a subsequence if necessary, combining the Lebesgue theorem and Theorem 1.2, there exists functions $g, h \in L^1(\mathbb{R}^2)$ such that

$$Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4}{4-\mu}\cdot\frac{(4-\mu)^2(2+\mu)}{8(2-\mu)}} \leq |g| \text{ and } Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4}{4-\mu}\cdot\frac{(4-\mu)^3}{16(2-\mu)}} \leq |h| \text{ a.e. in } \mathbb{R}^2. \tag{3.20}$$

Arguing as in the proof of Claim 1, we have that

$$
\begin{aligned}
|\xi_n| \le & \left(\int\limits_{|x-y|\le 1} \left| Q(y)F(u_n)\chi_{\{|u_n|<\overline{s}\}}\right|^{\frac{2+\mu}{2-\mu}} dy \right)^{\frac{2-\mu}{2+\mu}} \left(\int\limits_{|x-y|\le 1} \frac{1}{|x-y|^{\frac{2+\mu}{2}}} dy \right)^{\frac{2\mu}{2+\mu}} \\
& + \left(\int\limits_{|x-y|>1} \left| Q(y)F(u_n)\chi_{\{|u_n|<\overline{s}\}}\right|^{\frac{4-\mu}{2(2-\mu)}} dy \right)^{\frac{2(2-\mu)}{4-\mu}} \left(\int\limits_{|x-y|>1} \frac{1}{|x-y|^{4-\mu}} dy \right)^{\frac{\mu}{4-\mu}} \\
\le & \left(\frac{4\pi}{2-\mu} \right)^{2\mu/(2+\mu)} b^{\frac{\mu(6-\mu)}{(4-\mu)(2+\mu)}} C(\overline{s}) \left(\int\limits_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4}{4-\mu}\cdot\frac{(4-\mu)^2(2+\mu)}{8(2-\mu)}} dx \right)^{\frac{2-\mu}{2+\mu}} \\
& + \left(\frac{2\pi}{2-\mu} \right)^{\frac{\mu}{4-\mu}} b^{\frac{\mu^2}{(4-\mu)^2}} C(\overline{s}) \left(\int\limits_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4}{4-\mu}\cdot\frac{(4-\mu)^3}{16(2-\mu)}} dx \right)^{\frac{2(2-\mu)}{4-\mu}} . \qquad (3.21)
\end{aligned}
$$

It's simple to observe that

$$
|Q(x)\xi_n F(u_n)\chi_{\{|u_n|<\overline{s}\}}| \le b \max_{|s|<\overline{s}} F(s)|\xi_n|
$$

which together with Claim 1, (3.20)-(3.21) and the Dominated Convergence theorem yields that

$$
\lim_{n\to\infty} \int\limits_{\mathbb{R}^2} Q(x)\xi_n F(u_n)\chi_{\{|u_n|<\overline{s}\}} dx = \int\limits_{\mathbb{R}^2} Q(x)\xi_0 F(u_0)\chi_{\{|u_0|<\overline{s}\}} dx. \qquad (3.22)
$$

In view of (3.17)-(3.18), we apply (3.22) to deduce that

$$
\begin{aligned}
& \lim_{n\to\infty} |\Theta_n - \Theta_0| \\
& = \lim_{n\to\infty} \left| \int\limits_{|u_n|\ge\overline{s}} Q(x)\xi_n F(u_n) dx + \int\limits_{|u_n|<\overline{s}} Q(x)\xi_n F(u_n) dx \right. \\
& \qquad \left. - \int\limits_{|u_0|\ge\overline{s}} Q(x)\xi_0 F(u_0) dx - \int\limits_{|u_0|<\overline{s}} Q(x)\xi_0 F(u_0) dx \right| \\
& \le 2K_0\varepsilon + \lim_{n\to\infty} \left| \int\limits_{|u_n|<\overline{s}} Q(x)\xi_n F(u_n) dx - \int\limits_{|u_0|<\overline{s}} Q(x)\xi_0 F(u_0) dx \right| \\
& = 2K_0\varepsilon + \lim_{n\to\infty} \left| \int\limits_{\mathbb{R}^2} Q(x)\xi_n F(u_n)\chi_{\{|u_n|<\overline{s}\}} dx - \int\limits_{\mathbb{R}^2} Q(x)\xi_0 F(u_0)\chi_{\{|u_0|<\overline{s}\}} dx \right| \\
& = 2K_0\varepsilon
\end{aligned}
$$

showing the Claim 2 since $\varepsilon > 0$ is arbitrary. In consideration of the given $\varepsilon > 0$, there is a sufficiently large $n_0 \in \mathbb{N}^+$ such that $|\Theta_n - \Theta_0| \le \varepsilon$ for all $n \ge n_0$. Now, as a consequence of (3.17)-(3.18),

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx \\
&\quad - \int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_0))]Q(x)F(u_0)dx\Big| \\
&\leq \int_{|u_n|\geq \overline{s}} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx \\
&\quad + \int_{|u_0|\geq \overline{s}} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)F(u_0)dx \\
&\quad + \Big| \int_{|u_n|< \overline{s}} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx \\
&\quad - \int_{|u_0|< \overline{s}} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)F(u_0)dx\Big| \\
&\leq (2K_0+1)\varepsilon, \ \forall n \geq n_0.
\end{aligned}
$$

So, (3.14) holds true.

For all $\varepsilon > 0$, denoting $s_\varepsilon \triangleq \varepsilon^{-1}(K_0+1)|\psi|_\infty$, by means of (3.13), one easily observes that

$$
\begin{aligned}
&\int_{|u_n|\geq s_\varepsilon} \big|[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)\psi\big|dx \\
&\leq \frac{\varepsilon}{K_0+1} \int_{|u_n|\geq s_\varepsilon} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)u_n dx < \varepsilon.
\end{aligned}
\tag{3.23}
$$

Similarly, in view of (3.16), there holds

$$
\int_{|u_n|\geq s_\varepsilon} \big|[|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)\psi\big|dx < \varepsilon. \tag{3.24}
$$

Define $\Omega \triangleq \operatorname{supp}\psi$, then $|\Omega| < +\infty$. Combining (3.23) and (3.24), we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)\psi dx - \int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)\psi dx\Big| \\
&= \Big|\int_{\Omega}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)\psi dx - \int_{\Omega}[|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)\psi dx\Big|
\end{aligned}
$$

$$\le 2\varepsilon + \Bigg| \int\limits_{\Omega\cap\{|u_n|<s_\varepsilon\}} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)\psi dx$$

$$- \int\limits_{\Omega\cap\{|u_0|<s_\varepsilon\}} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)\psi dx \Bigg|$$

$$= 2\varepsilon + \Bigg| \int\limits_\Omega Q(x)\overline{\xi}_n F(u_n)dx - \int\limits_\Omega Q(x)\overline{\xi}_0 F(u_0)dx \Bigg|,$$

where

$$\overline{\xi}_n \triangleq |x|^{-\mu} * (Q(x)f(u_n)\psi\chi_{\{|u_n|<s_\varepsilon\}} \text{ and } \overline{\xi}_0 \triangleq |x|^{-\mu} * (Q(x)f(u_0)\psi\chi_{\{|u_0|<s_\varepsilon\}},\ x \in \Omega.$$

To arrive at (3.15), it suffices to show that $Q(x)\overline{\xi}_n F(u_n) \to Q(x)\overline{\xi}_0 F(u_0)$ in $L^1(\Omega)$ for every fixed $\varepsilon > 0$. Since $|\Omega| < \infty$, going to a subsequence if necessary, $u_n \to u_0$ in $L^2(\Omega)$, there exists a function $\varphi \in L^1(\Omega)$ such that $|u_n|^2 \le \varphi$ a.e. in Ω. Similar to (3.19), there exists a constant $s_\varepsilon > 0$ such that

$$|f(s)| \le C(s_\varepsilon)|s|^{(2-\mu)/2},\ \forall |s| \le s_\varepsilon. \tag{3.25}$$

Combining (1.11) and (3.13), we apply the Cauchy-Schwarz inequality in [23] and (3.25) to obtain

$$\Bigg| \int\limits_\Omega Q(x)\overline{\xi}_n F(u_n)dx \Bigg|$$

$$\le K_0^{\frac{1}{2}} \Bigg(\int\limits_\Omega [|x|^{-\mu} * (Q(x)f(u_n)\psi\chi_{\{|u_n|<s_\varepsilon\}})]Q(x)f(u_n)\psi\chi_{\{|u_n|<s_\varepsilon\}}dx \Bigg)^{\frac{1}{2}}$$

$$\le K_0^{\frac{1}{2}} bC(s_\varepsilon)\big||u_n|^{(2-\mu)/2}\psi\big|_{4/(4-\mu)} \le K_0^{\frac{1}{2}} bC(s_\varepsilon)\Bigg(\int\limits_\Omega |u_n|^2 dx \Bigg)^{\frac{2-\mu}{4}} \Bigg(\int\limits_\Omega |\psi|^2 dx \Bigg)^{\frac{1}{2}}$$

$$\le K_0^{\frac{1}{2}} bC(s_\varepsilon)\Bigg(\int\limits_\Omega \varphi dx \Bigg)^{\frac{2-\mu}{4}} \Bigg(\int\limits_\Omega |\psi|^2 dx \Bigg)^{\frac{1}{2}}$$

indicating that $\{Q(x)\overline{\xi}_n F(u_n)\}$ is uniformly integrable on Ω. Proceeding as the Claim 2, one derives $\overline{\xi}_n \to \overline{\xi}_0$ a.e. in Ω. Consequently, we can exploit the Vitali's Convergence theorem to conclude that $Q(x)\overline{\xi}_n F(u_n) \to Q(x)\overline{\xi}_0 F(u_0)$ in $L^1(\Omega)$ for every fixed $\varepsilon > 0$. The proof is complete. □

Now, we begin to verify that any $(C)_c$ sequence $\{u_n\} \subset E$ of J is bounded.

Lemma 3.5. *Suppose that* $(\mathbb{H}_1)$ *and* $(\mathbb{H}_2)$ *hold, then any* $(C)_c$ *sequence* $\{u_n\} \subset E$ *of* J *is bounded.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence of J, that is, $J(u_n) \to c$ and $(1 + \|u_n\|)\|J'(u_n)\|_{E^{-1}} \to 0$

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx = c + o_n(1) \tag{3.26}$$

and for all $\{\psi_n\} \subset E$, there holds

$$\left|\langle u_n, \psi_n\rangle - \int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)\psi_n dx\right| \le o_n(1)\|\psi_n\|, \tag{3.27}$$

where $o_n(1) \to 0$ as $n \to \infty$.

Without loss of generality, we can suppose that $u_n \neq 0$. Inspired by [12], we let $\psi_n \triangleq F(u_n)/f(u_n)$. Since $\{u_n\} \subset E$, by using (1.11), one has

$$\int_{\mathbb{R}^2} V(x)\psi_n^2 dx \le \int_{\mathbb{R}^2} V(x)u_n^2 dx < +\infty$$

and the computation $\nabla v_n = [f^2(u_n) - F(u_n)f'(u_n)]\nabla u_n/f^2(u_n)$ with (f_2) gives that

$$\int_{\mathbb{R}^2} |\nabla\psi_n|^2 dx \le (1-\delta)^2 \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \le \int_{\mathbb{R}^2} |\nabla u_n|^2 dx < +\infty.$$

So, $\{\psi_n\} \subset E$ and can be applied in (3.27). Moreover, by (f_2) and (1.12) it's easy to calculate that

$$\langle u_n, \psi_n\rangle = \int_{\mathbb{R}^2} \left[\frac{f^2(u_n) - F(u_n)f'(u_n)}{f^2(u_n)}|\nabla u_n|^2 + V(x)\frac{F(u_n)u_n}{f(u_n)}\right]dx \le (1-\delta)\|u_n\|^2$$

which together with (3.26) and (3.27) indicates that

$$\|u_n\|^2 \le 2c + o_n(1) + \langle u_n, \psi_n\rangle + o_n(1)\|\psi_n\| \le 2c + o_n(1) + (1-\delta)\|u_n\|^2 + o_n(1)\|u_n\|,$$

where we have used the fact that $\|\psi_n\| \le \|u_n\|$. Because $\delta \in (0, 1)$ in $(\mathbb{H}_2)$, we derive that $\{\|u_n\|\}$ is bounded. □

Lemma 3.6. *Suppose that* $(\mathbb{H}_1)$ *and* $(\mathbb{H}_3)$ *hold, then any* $(C)_c$ *sequence* $\{u_n\} \subset E$ *of* J *is bounded.*

Proof. We argue it by the contradiction and assume, up to a subsequence if necessary, that $\|u_n\| \to \infty$. Define $v_n = \sigma u_n/\|u_n\|$ with $\sigma = \sqrt{c + c^*}$, according to Lemma 3.3, then we have that

$$2c < \|v_n\|^2 = \sigma^2 = c + c^* < 2c^* = (4-\mu)\pi/\alpha_0. \tag{3.28}$$

Thereby, we shall chose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $1/\nu + 1/\nu' = 1$ and

$$\frac{4\alpha\nu\|v_n\|^2}{4-\mu} < 4\pi(1-\epsilon) \text{ for some suitable } \epsilon \in (0,1). \tag{3.29}$$

With this choice of $\alpha > \alpha_0$ and $\nu > 1$, combining (1.4) and (1.11), we apply (3.1) and (3.29) to derive

$$\begin{aligned}\int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(v_n))]Q(x)f(v_n)v_n dx &\le C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(f(v_n)v_n)^{\frac{4}{4-\mu}}dx\Big)^{\frac{4-\mu}{2}} \\ &\le C\|v_n\|^{2q}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha\nu\|v_n\|^2}{4-\mu}(v_n^2/\|v_n\|^2)}-1)dx\Big)^{\frac{4-\mu}{2\nu}} + C\|v_n\|^{4-\mu} \\ &\le C\sigma^{2q}S_{4\pi/(1-\epsilon)}^{\frac{4-\mu}{2\nu}} + C\sigma^{\frac{4-\mu}{2}} < +\infty.\end{aligned} \tag{3.30}$$

Going to a subsequence if necessary, there exists a function $v \in E$ such that $v_n \rightharpoonup v$ in E. We claim that $v \not\equiv 0$ in $\mathbb{R}^2$, otherwise, we should suppose that $v \equiv 0$ a.e. in $\mathbb{R}^2$. By means of (3.14) and (3.30),

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(v_n))]Q(x)F(v_n)dx = 0$$

which indicates that

$$\lim_{n\to\infty} J(v_n) = \frac{\sigma^2}{2} - \lim_{n\to\infty}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(v_n))]Q(x)F(v_n)dx = \frac{\sigma^2}{2}. \tag{3.31}$$

Since $\|u_n\| \to +\infty$ as $n \to \infty$, $\sigma/\|u_n\| \in (0,1)$ for some sufficiently large $n \in \mathbb{N}$. It's clear to compute that $\max_{t\in(0,1]} J(tu_n)$ can be achieved at some $t_n \in (0,1]$ and then $\langle J'(t_nu_n), t_nu_n\rangle = 0$. Thereby, it follows from (1.13) that

$$\begin{aligned} J(v_n) = J(\sigma\|u_n\|^{-1}u_n) &\le \max_{t\in(0,1]} J(tu_n) = J(t_nu_n) = J(t_nu_n) - \frac{1}{2}\langle J'(t_nu_n), t_nu_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(t_nu_n))]Q(x)[f(t_nu_n)t_nu_n - F(t_nu_n)]dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)[f(u_n)u_n - F(u_n)]dx \\ &= J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle. \end{aligned} \tag{3.32}$$

Let's recall that $\{u_n\}$ is a $(C)_c$ sequence of J, taking the limit $n \to \infty$ in (3.32), we would conclude that $\sigma^2 \le 2c$ by (3.31), a contradiction to (3.28). Consequently, $v \ne 0$ in $\mathbb{R}^2$ and then there exists a constant $R > 0$ such that $B_R(0) \cap \Upsilon$ admits positive Lebesgue measure, where $\Upsilon \triangleq \{x \in \mathbb{R}^2 | v(x) \ne 0\}$. Since $\|u_n\| \to \infty$, one knows $|u_n| \to \infty$ on $B_R(0) \cap \Upsilon$. It infers from (f_4) that $F(u_n)/|u_n| \to +\infty$ as $|u_n| \to \infty$. Let's denote $Q_{\min,\mathrm{R}} \triangleq \min_{x \in B_R(0)} > 0$ by $(\mathbb{H}_1)$, then via the Fatou's lemma, we have

$$\begin{aligned}
&\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx\\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{Q(y)F(u_n(y))}{|x-y|^{\mu}\|u_n\|}dy\right)\frac{Q(x)F(u_n(x))}{\|u_n\|}dx\\
&\ge \frac{Q^2_{\min,\mathrm{R}}}{2R^{\mu}}\left(\int_{B_R(0)\cap\Upsilon}\frac{F(u_n)}{|u_n|}|v_n|dx\right)^2 \to +\infty \text{ as } n \to \infty.
\end{aligned}$$

Recalling that $\{u_n\}$ is a $(C)_c$ sequence, then

$$0 = \liminf_{n\to\infty}\frac{J(u_n)}{\|u_n\|^2} \le \frac{1}{2} - \limsup_{n\to\infty}\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx = -\infty,$$

a contradiction. The proof of this lemma is now complete. □

Proof of Theorem 1.5. By the discussions above, there exist a bounded $(C)_c$ sequence $\{u_n\}$ of J and a function $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in E in the sense of a subsequence. Moreover, let us chose $\{\psi_n\}$ to be $\{u_n\}$, there exists a constant $K_0 > 0$ such that (3.13) holds true. Since $C_0^\infty(\mathbb{R}^2)$ is dense in E, one knows that $J'(u_0) = 0$ by (3.15). Now, we would affirm that $u_0 \ne 0$. Otherwise, thanks to (3.14) and (3.26), we apply Lemma 3.3 to obtain $\limsup_{n\to\infty}\|u_n\|^2 = 2c < 2c^*$. Proceeding as (3.28), (3.29) and (3.30), we can obtain a constant $\overline{K}_0 > 0$ such that

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)f(u_n)u_n)]Q(x)f(u_n)u_n dx \le \overline{K}_0. \tag{3.33}$$

By using (3.14) again, we use (3.33) and the Cauchy-Schwarz inequality in [23] to have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)u_n dx\right|\\
&\le \left(\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)F(u_n))]Q(x)F(u_n)dx\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{\mathbb{R}^2}[|x|^{-\mu} * (Q(x)f(u_n)u_n)]Q(x)f(u_n)u_n dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \overline{K}_0\bigg(\int_{\mathbb{R}^2}[|x|^{-\mu}*(Q(x)F(u_n))]Q(x)F(u_n)dx\bigg)^{\frac{1}{2}}\to 0$$

which together with $\langle J'(u_n),u_n\rangle\to 0$ yields that $\|u_n\|\to 0$. Thus, $c=\lim_{n\to\infty}J(u_n)=0$, violating to Lemma 3.3. So, $u_0\neq 0$ and it is a nontrivial solution of Eq. (1.1). If $\delta=0$ in (f_2), by (1.11)

$$\begin{aligned}
c&=\liminf_{n\to\infty}J(u_n)=\liminf_{n\to\infty}[J(u_n)-\frac{1}{2}\langle J'(u_n),u_n\rangle]\\
&=\frac{1}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^2}(|x|^{-\mu}*(Q(x)F(u_n)))Q(x)[f(u_n)u_n-F(u_n)]dx\\
&\geq\frac{1}{2}\int_{\mathbb{R}^2}(|x|^{-\mu}*(Q(x)F(u_0)))Q(x)[f(u_0)u_0-F(u_0)]dx=J(u_0)-\frac{1}{2}\langle J'(u_0),u_0\rangle\\
&=J(u_0)
\end{aligned}$$

which completes the last part of the theorem. □

Next, we are concerned with the proof of Theorem 1.7.

Proof of Theorem 1.7. Since $\|u\|\leq\liminf_{n\to\infty}\|u_n\|\equiv 1$, we can split the proof into two cases.

Case 1: $\|u\|<1$. Arguing it by contradiction that for some $0<p_1<P_{\alpha_0}(u)$, where $P_{\alpha_0}(u)$ is given by (1.15), there holds

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^2}Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2}-1)dx=+\infty. \tag{3.34}$$

In light of a constant $L\in(0,+\infty)$ which is determined later and $v\in E$, set

$$G_L(v)=\begin{cases}L, & \text{if } v>L,\\ -L, & \text{if } v<-L,\\ v, & \text{if } |v|\leq L,\end{cases}\ \text{and}\ T_L(v)=v-G_L(v).$$

Plainly, there exists a constant $\varepsilon\in(0,1)$ such that

$$p_1(1+\varepsilon)^2<\frac{1}{1-\|u\|^2}.$$

Obviously, $\|G_L(u)\|\to\|u\|$ as $L\to+\infty$, then one can choose a sufficiently large $L>0$ such that

$$p_1(1+\varepsilon)^2<\frac{1}{1-\|G_L(u)\|^2}. \tag{3.35}$$

We claim that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2}[|\nabla T_L(u_n)|^2+V(x)|T_L(u_n)|^2]dx<\frac{1}{p_1(1+\varepsilon)}. \tag{3.36}$$

Suppose that (3.36) doesn't hold, then going to a subsequence of $\{T_L(u_n)\}$ if necessary, we have

$$\|T_L(u_n)\|^2=\int_{\mathbb{R}^2}[|\nabla T_L(u_n)|^2+V(x)|T_L(u_n)|^2]dx\geq\frac{1}{p_1(1+\varepsilon)^2},\ \forall n\in\mathbb{N}^+$$

which together with the facts $T_L(u_n)G_L(u_n)\geq 0$ and $\nabla T_L(u_n)\nabla G_L(u_n)\equiv 0$ yields that

$$1=\|u_n\|^2\geq\|T_L(u_n)\|^2+\|G_L(u_n)\|^2\geq\frac{1}{p_1(1+\varepsilon)^2}+\|G_L(u_n)\|^2.$$

Since $\{G_L(u_n)\}$ is bounded in E and $G_L(u_n)\rightharpoonup G_L(u)$ in E, by using the above formula, we derive

$$p_1(1+\varepsilon)^2\geq\frac{1}{1-\|G_L(u_n)\|^2}$$

which is in contradicts with (3.36). So, (3.36) holds true. Up to a subsequence if necessary, we can suppose that $4\pi p_1(1+\varepsilon)^2\|T_L(u_n)\|^2<4\pi$ for all $n\in\mathbb{N}$. In view of (1.8), we obtain

$$\begin{aligned}&\sup_{n\in\mathbb{N}}\int_{\Omega_{n,L}}Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1(1+\varepsilon)^2|u_n-L|^2}-1)dx\\&\leq\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^2}Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1(1+\varepsilon)^2\|T_L(u_n)\|^2(|T_L(u_n)|^2/\|T_L(u_n)\|^2)}-1)dx<+\infty,\end{aligned}\tag{3.37}$$

where $\Omega_{n,L}\triangleq\{x\in\mathbb{R}^2:|u_n(x)|\geq L\}$. By means of Theorem 1.2, we derive

$$|\Omega_{n,L}|_{Q^\mu}\triangleq\int_{\Omega_{n,L}}Q^{\frac{4}{4-\mu}}(x)dx\leq\frac{1}{L^2}\int_{\Omega_{n,L}}Q^{\frac{4}{4-\mu}}(x)|u_n|^2dx\leq\frac{C\|u_n\|^2}{L^2}=\frac{C}{L^2}<+\infty, \tag{3.38}$$

where $C>0$ is a constant dependent of n by the imbedding of $E\hookrightarrow L^2_{Q^\mu}(\mathbb{R}^2)$. To get a contradiction, let's write

$$\begin{aligned}\int_{\mathbb{R}^2}Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2}-1)dx=&\int_{\Omega_{n,L}}Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2}-1)dx\\&+\int_{\Omega_{n,L}^c}Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2}-1)dx.\end{aligned}$$

Combining (3.37) and (3.38), we apply the following two type Young's inequalities

$$|u_n|^2 = |u_n - L + L|^2 \le (1+\varepsilon)|u_n - L|^2 + (1+\varepsilon^{-1})|L|^2$$

and

$$ab - 1 \le \frac{1}{1+\varepsilon}(a^{1+\varepsilon} - 1) + \frac{\varepsilon}{1+\varepsilon}(b^{\frac{1+\varepsilon}{\varepsilon}} - 1),\ \forall a, b > 0$$

to conclude that for all $n \in \mathbb{N}^+$

$$\begin{aligned}
\int_{\Omega_{n,L}} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2} - 1)dx \le & \int_{\Omega_{n,L}} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1[(1+\varepsilon)|u_n - L|^2 + (1+\varepsilon^{-1})|L|^2]} - 1)dx \\
\le & \frac{1}{1+\varepsilon}\int_{\Omega_{n,L}} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1(1+\varepsilon)^2|u_n - L|^2} - 1)dx \\
& + \frac{\varepsilon}{1+\varepsilon}\int_{\Omega_{n,L}} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1(1+\varepsilon)^2\varepsilon^{-2}|L|^2} - 1)dx \\
\le & \frac{1}{1+\varepsilon}\int_{\Omega_{n,L}} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1(1+\varepsilon)^2|u_n - L|^2} - 1)dx \\
& + \frac{\varepsilon}{1+\varepsilon}(e^{4\pi p_1(1+\varepsilon)^2\varepsilon^{-2}|L|^2} - 1)|\Omega_{n,L}|_{Q^\mu} \le C < +\infty.
\end{aligned}$$

On the other hand, in view of Theorem 1.2, we have that for all $n \in \mathbb{N}^+$

$$\begin{aligned}
& \int_{\Omega_{n,L}^c} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2} - 1)dx = \int_{\{|u_n(x)|<L\}} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_1|u_n|^2} - 1)dx \\
& = \int_{\{|u_n(x)|<L\}} Q^{\frac{4}{4-\mu}}(x)\sum_{j=1}^{\infty}\frac{(4\pi p_1 L^2)^j}{j!}\left|\frac{u_n}{L}\right|^{2j}dx \\
& \le \sum_{j=1}^{\infty}\frac{(4\pi p_1 L^2)^j}{j!}\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)\left|\frac{u_n}{L}\right|^{\frac{4}{4-\mu}\cdot\frac{4-\mu}{2}}dx \\
& \le C\left\|\frac{u_n}{L}\right\|^2\sum_{j=1}^{\infty}\frac{(4\pi p_1 L^2)^j}{j!} = CL^{-2}\sum_{j=1}^{\infty}\frac{(4\pi p_1 L^2)^j}{j!} = CL^{-2}(e^{4\pi p_1 L^2} - 1) \le C < +\infty.
\end{aligned}$$

The above two formulas reveal a contradiction to (3.34). So, the theorem in this case holds true.

Case 2: $\|u\| = 1$. Since $u_n \rightharpoonup u$ in E, one derives $\lim_{n\to\infty}\|u_n - u\|^2 = \lim_{n\to\infty}\|u_n\|^2 - \|u\|^2 = 0$ which shows that $u_n \to u$ in E. Recalling that the Lebesgue theorem, there is a function $v \in E$ such that $|u_n| \le v$ a.e. in $\mathbb{R}^N$ which together with (1.7) yields (1.15).

Next, we turn to focus on the sharpness of $P_{\alpha_0}(u)$, that is, there is a sequence $\{u_n\} \subset E$ satisfying $\|u_n\| \equiv 1$ and $u_n \rightharpoonup u \ne 0$ in E such that the supremum given by (1.15) is infinite for each $p \ge P_{\alpha_0}(u)$. To this aim, for some constants $r > 0$ and $R = 3r$, we define $\underline{w}_n(x)$ as

$$\underline{w}_n(x) \triangleq \frac{1}{\sqrt{2\pi}} \begin{cases} 2^{-\frac{1}{2}} n^{\frac{1}{2}}, & \text{if } 0 \le |x| \le re^{-\frac{n}{2}}, \\ 2^{\frac{1}{2}} \log(r/|x|) n^{-\frac{1}{2}}, & \text{if } re^{-\frac{n}{2}} < |x| \le r, \\ 0, & \text{if } |x| > r, \end{cases}$$

and $u \in E$ as

$$u \triangleq \begin{cases} \underline{A}, & \text{if } 0 \le |x| \le \frac{2}{3}R, \\ 3\underline{A}(1 - \frac{|x|}{R}), & \text{if } \frac{2}{3}R < |x| \le R, \\ 0, & \text{if } |x| > R, \end{cases}$$

respectively. Here, the constant $\underline{A} > 0$ is chosen in such a way that $\|u\| = \sigma < 1$. We set

$$\underline{u}_n = (1 - \sigma^2)^{1/2} \underline{w}_n + u.$$

Let's recall the constant $A > 0$ appearing in $(\mathcal{K})$, it's simple to compute that

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla \underline{w}_n|^2 dx = \frac{1}{\pi n} \int_{re^{-\frac{n}{2}} < |x| \le r} \frac{1}{|x|^2} dx = \frac{2}{n} \int_{re^{-n/2}}^{r} \frac{1}{\rho} d\rho = 1, \\ &0 \le \int_{\mathbb{R}^2} V(x) |\underline{w}_n|^2 dx \le \frac{Anr^2}{4e^n} + \frac{Ar^2}{4n}\left(2 - \frac{n^2 + 2n + 2}{e^n}\right) \triangleq \underline{\delta}_n \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.39}$$

Since $B_r(0) \cap B^c_{2R/3}(0) = \emptyset$ by $R = 3r$, one has $\nabla \underline{w}_n \nabla u \equiv 0$ for all $x \in \mathbb{R}^2$ and then

$$\int_{\mathbb{R}^2} |\nabla \underline{u}_n|^2 dx = (1 - \sigma^2) \int_{\mathbb{R}^2} |\nabla \underline{w}_n|^2 dx + \int_{\mathbb{R}^2} |\nabla u|^2 dx = (1 - \sigma^2) + \int_{\mathbb{R}^2} |\nabla u|^2 dx \tag{3.40}$$

It can be inferred from (3.39) and the Holder's inequality that

$$\begin{aligned} \int_{\mathbb{R}^2} V(x) |\underline{u}_n|^2 dx &= (1 - \sigma^2) \int_{\mathbb{R}^2} V(x) |\underline{w}_n|^2 dx \\ &\quad + \int_{\mathbb{R}^2} V(x) |u|^2 dx + 2(1 - \sigma^2)^{1/2} \int_{\mathbb{R}^2} V(x) \underline{w}_n u dx \\ &\le \int_{\mathbb{R}^2} V(x) |u|^2 dx + [(1 - \sigma^2) + 2(1 - \sigma^2)^{1/2} \sigma^{1/2}] \underline{\delta}_n^{1/2}. \end{aligned} \tag{3.41}$$

So, we can derive that $\|\underline{u}_n\|^2 \le 1 + [(1 - \sigma^2) + 2(1 - \sigma^2)^{1/2} \sigma^{1/2}] \underline{\delta}_n^{1/2} \triangleq 1 + \underline{\tau}_n$ with $\underline{\tau}_n \to 0$ by (3.39). Actually, we'll also verify that $\|\underline{u}_n\| \to 1$ via (3.40) and (3.41). Therefore, without loss of generality, we may suppose that $\|\underline{u}_n\| = 1 + \underline{\tau}_n$. Now, we shall define $u_n \triangleq \underline{u}_n/(1+\underline{\tau}_n)^{1/2}$. Clearly,

$$\|u_n\| \equiv 1 \text{ and } u_n \rightharpoonup u \neq 0 \text{ as } n \to \infty.$$

As a consequence, for all $\varepsilon_0 \geq 0$ and $p_{\varepsilon_0} = (1+\varepsilon_0)P_{\alpha_0}(u) = (1+\varepsilon_0)/(1-\sigma^2) \geq P_{\alpha_0}(u)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_{\varepsilon_0}|u_n|^2}-1)dx &= \int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi(1+\varepsilon_0)(1-\sigma^2)^{-1}|u_n|^2}-1)dx \\
&\geq Q_1^{\frac{4}{4-\mu}} \int_{B_{re^{-n/2}}(0)} (e^{4\pi(1+\varepsilon_0)(1-\sigma^2)^{-1}(\underline{A}+(1-\sigma^2)^{1/2}\underline{w}_n)^2}-1)dx \\
&\geq Q_1^{\frac{4}{4-\mu}} \int_{B_{re^{-n/2}}(0)} (e^{4\pi C_{\sigma,\underline{A}}(1+\varepsilon_0)(1-\sigma^2)^{-1}(1+\underline{w}_n)^2}-1)dx \\
&\geq Q_1^{\frac{4}{4-\mu}} (e^{4\pi C_{\sigma,\underline{A}}(1+\varepsilon_0)(1-\sigma^2)^{-1}(1+\sqrt{n})^2}-1)|B_{re^{-n/2}}(0)| \\
&= \pi r^2 Q_1^{\frac{4}{4-\mu}} (e^{4\pi C_{\sigma,\underline{A}}(1+\varepsilon_0)(1-\sigma^2)^{-1}(1+\sqrt{n})^2}-1)e^{-n} \to +\infty \text{ as } n \to \infty,
\end{aligned}$$

where $Q_1 = \min_{x\in B_1(0)} Q(x) > 0$ and $C_{\sigma,\underline{A}} = \min\{1-\sigma^2, \underline{A}^2\} > 0$. The proof is complete. □

As a by-product of Theorem 1.7, we will certify that the functional J satisfies the so-called $(C)_c$ condition, i.e. every $(C)_c$ sequence $\{u_n\} \subset E$ of the functional J contains a strongly convergent subsequence.

Lemma 3.7. *Let* $(\mathbb{H}_1)$ *and* **one** *of* $(\mathbb{H}_2)$, $(\mathbb{H}_3)$ *and* $(\mathbb{H}_4)$ *hold true, then* J *satisfies the* $(C)_c$ *condition.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence of J, then $\{u_n\}$ is bounded by Lemmas 3.5 and 3.6. Chosen ψ_n to be u_n in (3.27), we obtain (3.13). Passing to a subsequence if necessary, there exists a function $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in E. Thereby, according to (3.13), we derive (3.14) and (3.15) by Lemma 3.4. As a consequence, we have that $J'(u_0) = 0$ by (3.15) which together with (1.11) indicates that

$$\begin{aligned}
J(u_0) &= J(u_0) - \frac{1}{2}\langle J'(u_0), u_0\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)[f(u_0)u_0 - F(u_0)]dx \geq 0. \qquad (3.42)
\end{aligned}$$

Since $c > 0$ by Lemma 3.3, we only have to the following two cases.

Case 1: $u_0 \equiv 0$**.** We apply (3.14) with $u_0 = 0$ and (3.26) together with Lemma 3.3 to obtain

$$\limsup_{n\to\infty} \|u_n\|^2 = 2c < 2c^*.$$

Proceeding as in the proof of Theorem 1.5, one has $\|u_n\| \to 0$ which is a contradict with the fact that $c > 0$.

Case 2: $u_0 \neq 0$. By the Fatou's lemma, there holds

$$0 < \|u_0\| \leq \liminf_{n\to\infty} \|u_n\|. \tag{3.43}$$

Up to a subsequence if necessary, we define

$$v_n \triangleq \frac{u_n}{\|u_n\|} \text{ and } v = \frac{u_0}{\lim_{n\to\infty} \|u_n\|}.$$

Obviously, $0 < \|v\| \leq 1$ by (3.43). If $\|v\| = 1$, we have $\|u_n\| \to \|u_0\|$ which together with $u_n \rightharpoonup u_0$ in E yields that $u_n \to u_0$ in E. Thereby, the proof is finished. Let's suppose that $0 < \|v\| < 1$. In this situation, combining Lemma 3.3, (3.42), (3.26), (3.14) and the Fatou's lemma, we conclude that

$$\begin{aligned} 2c^* > 2c \geq 2[c - J(u_0)] &= \limsup_{n\to\infty}(\|u_n\|^2 - \|u_0\|^2) = \limsup_{n\to\infty} \|u_n\|^2\left(1 - \left\|\frac{u_0}{\|u_n\|}\right\|^2\right) \\ &\geq (1 - \|v\|^2) \limsup_{n\to\infty} \|u_n\|^2 \end{aligned}$$

which gives that

$$\limsup_{n\to\infty} \|u_n\|^2 < \frac{(4-\mu)\pi}{\alpha_0(1 - \|v\|^2)}.$$

Then, we would choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu > 1$ sufficiently close to 1 in such a way that $1/v + 1/v' = 1$ and

$$\frac{4\alpha\nu\|u_n\|^2}{4-\mu} < \frac{4\pi(1-\epsilon)}{1 - \|v\|^2} \triangleq 4\pi p_\epsilon, \text{ for some suitable } \epsilon \in (0, 1),$$

where $0 < p_\varepsilon = (1 - \varepsilon)/(1 - \|v\|^2) < P_{\alpha_0}(v)$. So, by (1.15) and $|u_n|^2 = \|u_n\|^2|v_n|^2$, we have that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha\nu}{4-\mu}|u_n|^2} - 1)dx \leq \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{4\pi p_\epsilon |v_n|^2} - 1)dx < +\infty. \tag{3.44}$$

To finish the proof, we claim that

$$\int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)(u_n - u_0)dx \to 0. \tag{3.45}$$

Indeed, since $\langle J'(u_n), u_0 - u_n\rangle \to 0$, we apply the convexity of the functional $I(u) \triangleq \|u\|^2/2$ to get

$$\begin{aligned}\frac{1}{2}\|u_0\|^2 &= I(u_0) \ge I(u_n) + \langle I'(u_n), u_0 - u_n\rangle \\ &= \frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^2} [\nabla u_n \nabla(u_0 - u_n) + V(x)u_n(u_0 - u_n)]dx \\ &= \frac{1}{2}\|u_n\|^2 + \langle J'(u_n), u_0 - u_n\rangle - \int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)(u_n - u_0)dx\end{aligned}$$

which indicates that $\limsup_{n\to\infty}\|u_n\|^2 \le \|u_0\|^2$. So, we derive $u_n \to u_0$ in E by Fatou's lemma.

The remainder is to verify the validity of (3.45). In view of (1.4), there holds

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} [|x|^{-\mu} * (Q(x)F(u_n))]Q(x)f(u_n)(u_n - u_0)dx\right| \\ &\le C|Q(x)F(u_n)|_{\frac{4}{4-\mu}}|Q(x)f(u_n)(u_n - u_0)|_{\frac{4}{4-\mu}} \triangleq CJ_n^1J_n^2.\end{aligned}$$

Combining (3.2) and (3.44) together with Theorem 1.2, one sees that

$$\begin{aligned}(J_n^1)^{\frac{4}{4-\mu}} &\le C\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4q}{4-\mu}}(e^{\frac{4\alpha}{4-\mu}|u_n|^2} - 1)dx + C\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4}{4-\mu}\cdot\frac{4-\mu}{2}}dx \\ &\le C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4q\nu'}{4-\mu}}\Big)^{\frac{1}{\nu'}}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha\nu}{4-\mu}|u_n|^2} - 1)dx\Big)^{\frac{1}{\nu}} + C\|u_n\|^2 \\ &\le C\|u_n\|^{\frac{4q}{4-\mu}}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha\nu}{4-\mu}|u_n|^2} - 1)dx\Big)^{\frac{1}{\nu}} + C\|u_n\|^2 \le \overline{C} < +\infty.\end{aligned}$$

On the other hand, for all $\varepsilon > 0$ and $q = \nu' > (4-\mu)/2$ ($\Leftrightarrow \nu = q/(q-1)$) in (3.1), by (3.44),

$$\begin{aligned}(J_n^2)^{\frac{4}{4-\mu}} &\le C_\varepsilon\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n - u_0|^{\frac{4}{4-\mu}}|u_n|^{\frac{4(q-1)}{4-\mu}}(e^{\frac{4\alpha}{4-\mu}u_n^2} - 1)dx \\ &\quad + \varepsilon\|u_n\|^{\frac{2(2-\mu)}{4-\mu}}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n - u_0|^2dx\Big)^{\frac{2}{4-\mu}} \\ &\le C\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n - u_0|^{\frac{4q}{4-\mu}}\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n|^{\frac{4q}{4-\mu}}(e^{\frac{4\nu\alpha}{4-\mu}u_n^2} - 1)dx\Big)^{\frac{q-1}{q}} \\ &\quad + \varepsilon\|u_n\|^{\frac{2(2-\mu)}{4-\mu}}\Big(\int_{\mathbb{R}^2} Q^{\frac{4}{4-\mu}}(x)|u_n - u_0|^2dx\Big)^{\frac{2}{4-\mu}}.\end{aligned}$$

Letting $n \to \infty$ by Theorem 1.2 and then $\varepsilon \to 0^+$, hence $J_n^2 \to 0$. Combining the above three formulas, we can get (3.45). The proof is complete. □

With Lemma 3.7 in hand, arguing as the proof of Theorem 1.5, it's simple to prove Corollary 1.8 and Theorem 1.9, respectively. So, we omit the details.

4. Bound state solution

The aim of this section is to verify that the nontrivial solution $u_0 \in E$ obtained in Theorem 1.5, or Corollary 1.8 and Theorem 1.9, is a *bound state*, that is, $u_0 \in H^1(\mathbb{R}^2)$. For this purpose, we need to establish some integration estimates on u_0, which are essentially motivated by [6]. However, we should mention that in the proof of the following Lemmas 4.1 and 4.2, it is crucial to estimate the integration on the nonlinearity properly. It seems some difficulties to have the desired because of the appearance of the nonlocal term with critical exponential growth.

Lemma 4.1. *Let* $(\mathcal{K})'$ *with* $(\gamma, \beta) \in$ (i)$'$ *and* $\alpha \in (0, 4\pi)$, *then for each* $v \in E\backslash\{0\}$ *with* $\|v\| \leq 1$ *and any* $\varepsilon > 0$, *there exists a constant* $\overline{n} = \overline{n}(\alpha, a, \gamma) > 1$ *independent of* v *such that for every* $n \geq \overline{n}$

$$\int_{B_{3n}^c(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha|v|^2} - 1)dx \leq \varepsilon. \tag{4.1}$$

Proof. In view of (2.9)-(2.10), the proof can be obtained immediately. □

Combining Lemma 4.1 and [6, Proposition 11], we establish the following result.

Lemma 4.2. *Suppose that* $(\mathcal{K})'$ *with* $(\gamma, \beta) \in$ (i)$'$ *hold true and let* $\alpha > 0$ *and* $u \in E\backslash\{0\}$ *be fixed, then for all* $\varepsilon > 0$, *there exists a constant* $\overline{R} = \overline{R}(u, \alpha, a, \gamma)$ *such that*

$$\int_{B_R^c(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha|u|^2} - 1)dx \leq \varepsilon, \ \forall R \geq \overline{R}. \tag{4.2}$$

Proof. Given a constant $R > 1$, let $\overline{\psi}_R : \mathbb{R}^+ \to [0, 1]$ be a smooth nondecreasing function such that

$$\overline{\psi}_R(r) \triangleq \begin{cases} 0, & \text{if } 0 \leq r \leq R - R^{\gamma/2}, \\ 1, & \text{if } r \geq R, \end{cases} \text{ and } |\overline{\psi}_R'(r)| \leq \frac{2}{R^{\gamma/2}}, \ \forall r > 0.$$

Define, in polar coordinates $(r, \theta) \in [0, +\infty) \times \mathbb{S}^1$,

$$\overline{u}_R(r, \theta) \triangleq \begin{cases} 0, & \text{if } 0 \leq r \leq R - R^{\gamma/2}, \\ \overline{\psi}_R(r)u(2R - r, \theta), & \text{if } R - R^{\gamma/2} \leq r \leq R, \\ u(r, \theta), & \text{if } r \geq R. \end{cases}$$

Let's recall [6, Proposition 11], one has the following result

$$\int_{A_R} [|\nabla \overline{u}_R|^2 + V(x)|\overline{u}_R|^2]dx \le C \int_{B_R^c(0)} [|\nabla \overline{u}_R|^2 + V(x)|\overline{u}_R|^2]dx,$$

where $A_R \triangleq \{x \in \mathbb{R}^2 | R - R^{\gamma/2} \le |x| \le R\}$. Consequently, since $A_R \cap B_R^c(0) = \emptyset$, we obtain

$$\begin{aligned}\|\overline{u}_R\|^2 &= \int_{B^c_{R-R^{\gamma/2}}(0)} [|\nabla \overline{u}_R|^2 + V(x)|\overline{u}_R|^2]dx = \left(\int_{A_R} + \int_{B_R^c(0)}\right)[|\nabla \overline{u}_R|^2 + V(x)|\overline{u}_R|^2]dx \\ &\le (1+C) \int_{B_R^c(0)} [|\nabla \overline{u}_R|^2 + V(x)|\overline{u}_R|^2]dx. \end{aligned} \tag{4.3}$$

According to $u \in E$, there exists a sufficiently large constant $\overline{R} = \overline{R}(u, \alpha) > 1$ such that

$$\int_{B^c_{\overline{R}}(0)} [|\nabla \overline{u}_R|^2 + V(x)|\overline{u}_R|^2]dx = \int_{B^c_{\overline{R}}(0)} [|\nabla u|^2 + V(x)|u|^2]dx < \frac{4\pi}{(1+C)\alpha}$$

which together with (4.3) indicates that $\alpha\|\overline{u}_R\|^2 < 4\pi$ for every $R \ge \overline{R}$. Therefore, to apply Lemma 4.1, we can choose an $\overline{R} = \overline{R}(u, \alpha, a, \gamma) > 0$ large enough such that $\overline{R} - \overline{R}^{\gamma/2} \ge 3\overline{n}$ and $v = u_R/\|u_R\|$ in (4.1), we have

$$\begin{aligned}\int_{B_R^c(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha|u|^2} - 1)dx &= \int_{B_R^c(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha\|\overline{u}_R\|^2(|\overline{u}_R|^2/\|\overline{u}_R\|^2)} - 1)dx \\ &\le \int_{B^c_{R-R^{\gamma/2}}(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\alpha\|\overline{u}_R\|^2(|\overline{u}_R|^2/\|\overline{u}_R\|^2)} - 1)dx \le \varepsilon,\ \forall R \ge \overline{R}.\end{aligned}$$

The proof is complete. □

Let's denote $u_0 \in E\backslash\{0\}$ by the nontrivial solution of Eq. (1.1) throughout this section. Now, we can prove the following lemma.

Lemma 4.3. *There exists a constant $\widetilde{R} > 0$ such that for any $n \in \mathbb{N}^+$ satisfying $R_n \triangleq n^{2/(2-\gamma)} \ge \widetilde{R}$, there holds*

$$\int_{B^c_{R_{n+1}}(0)} [|\nabla u_0|^2 + V(x)u_0^2]dx \le \frac{3}{4} \int_{B^c_{R_n}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2]dx$$

Proof. Arguing as [6, Proposition 17], let χ_n be a piecewise affine function such that

$$\chi_n(x) \triangleq \begin{cases} 0, & \text{if } |x| \le R_n, \\ 1, & \text{if } |x| \le R_{n+1}. \end{cases}$$

Moreover, one can prove that

$$|\nabla \chi_n(x)|^2 \le V(x)$$

and

$$\int_{B^c_{R_{n+1}}(0)} [|\nabla u_0|^2 + V(x)u_0^2]dx \le \int_{B^c_{R_n}(0)} [|\nabla u_0|^2 + V(x)u_0^2]\chi_n dx.$$

Taking $v = \chi_n u_0 \in E$ in (1.6), exploiting it with the above two formulas, we may apply the Hölder's equality to obtain

$$\begin{aligned}
&\int_{B^c_{R_{n+1}}(0)} [|\nabla u_0|^2 + V(x)u_0^2]dx \le \int_{B^c_{R_n}(0)} [|\nabla u_0|^2 + V(x)u_0^2]\chi_n dx \\
&= \int_{B^c_{R_n}(0)} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)u_0\chi_n dx - \int_{B^c_{R_n}(0)} u_0\nabla u_0 \nabla \chi_n dx \\
&\le \int_{B^c_{R_n}(0)} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)u_0 dx + \frac{1}{2}\Big(\int_{B^c_{R_n}(0)} (|\nabla u_0|^2 + |\nabla \chi_n|^2|u_0|^2)dx\Big) \\
&\le \int_{B^c_{R_n}(0)} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)u_0 dx + \frac{1}{2}\Big(\int_{B^c_{R_n}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2]dx\Big).
\end{aligned} \tag{4.4}$$

Next, we are concerned with the estimate for $\int_{B^c_{R_n}(0)}[|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)u_0 dx$. By (3.2),

$$|f(u_0)u_0| \le C(\alpha)|s|(e^{\alpha s^2} - 1),\ \forall |s| \ge \widetilde{R} \tag{4.5}$$

for some $\widetilde{R} > 1$. Combining (1.11) and (1.4) with (4.5), we have that

$$\begin{aligned}
&\int_{B^c_{R_n}(0)} [|x|^{-\mu} * (Q(x)F(u_0))]Q(x)f(u_0)u_0 dx \le C\Big(\int_{B^c_{R_n}(0)} |Q(x)f(u_0)u_0|^{\frac{4}{4-\mu}} dx\Big)^{\frac{4-\mu}{2}} \\
&\le CC(\alpha)^2\Big(\int_{B^c_{R_n}(0)} Q^{\frac{4}{4-\mu}}(x)|u_0|^{\frac{4}{4-\mu}}(e^{\frac{4\alpha}{4-\mu}|u_0|^2} - 1)dx\Big)^{\frac{4-\mu}{2}} \\
&\le C\Big(\int_{B^c_{R_n}(0)} Q^{\frac{4}{4-\mu}}(x)|u_0|^2 dx\Big)\Big(\int_{B^c_{R_n}(0)} Q^{\frac{4}{4-\mu}}(x)(e^{\frac{4\alpha}{2-\mu}|u_0|^2} - 1)dx\Big)^{\frac{2-\mu}{2}}
\end{aligned}$$

$$\leq \int_{B^c_{R_n}(0)} Q^{\frac{4}{4-\mu}}(x)|u_0|^2 dx \leq \mathcal{B}(\widetilde{R}) \int_{B^c_{R_n}(0)} V(x)|u_0|^2 dx$$

$$\leq \mathcal{B}(\widetilde{R}) \int_{B^c_{R_n}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2] dx, \tag{4.6}$$

where we have used (4.2) with $\varepsilon \in (0, C^{2/(\mu-2)})$ in the fourth inequality and

$$\mathcal{B}(\widetilde{R}) \triangleq \frac{b^{\frac{4}{4-\mu}}(1+\widetilde{R}^{\gamma})}{a(1+\widetilde{R}^{\beta})^{4/(4-\mu)}} \geq \sup_{x\in B^c_{\widetilde{R}}(0)} \frac{b^{\frac{4}{4-\mu}}(1+|x|^{\gamma})}{a(1+|x|^{\beta})^{4/(4-\mu)}}$$

$$\geq \sup_{x\in B^c_{\widetilde{R}}(0)} \frac{Q^{\frac{4}{4-\mu}}(x)}{V(x)} \geq \sup_{x\in B^c_{R_n}(0)} \frac{Q^{\frac{4}{4-\mu}}(x)}{V(x)}.$$

Since $\gamma < 4\beta/(4-\mu)$, one sees that $\lim_{\widetilde{R}\to+\infty} \mathcal{B}(\widetilde{R}) = 0$ which indicates that $\mathcal{B}(\widetilde{R}) \leq 1/4$ for some sufficiently large $\widetilde{R} > 0$. As a consequence of (4.4) and (4.6), we accomplish the proof. □

Lemma 4.4. *There exist constants $\widetilde{R} > 0$ and $C > 0$ such that for all $\zeta > 2\widetilde{R}$, there holds*

$$\int_{B^c_{\zeta}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2] dx \leq Ce^{(\log\frac{3}{4})\zeta^{(2-\gamma)/2}}.$$

Proof. With Lemma 4.3 in hand, we can obtain the desired result immediately followed by [6, Lemma 18]. For the reader's convenience, we present it here in detail. Let $\widetilde{R}$ and $\{R_n\}$ be as in Lemma 4.3 and $\zeta > 2\widetilde{R}$, there exist two positive integers $\overline{n} > \widetilde{n}$ such that

$$R_{\widetilde{n}} \leq \widetilde{R} \leq R_{\widetilde{n}+1} \text{ and } R_{\overline{n}-1} \leq \zeta \leq R_{\overline{n}}$$

and then

$$\overline{n} - \widetilde{n} = R_{\overline{n}}^{(2-\gamma)/2} - R_{\widetilde{n}}^{(2-\gamma)/2} \geq \zeta^{(2-\gamma)/2} - \widetilde{R}^{(2-\gamma)/2} > \widetilde{R}^{(2-\gamma)/2}(2^{(2-\gamma)/2} - 1) > 2$$

provided $\widetilde{R} > 0$ is sufficiently large. So, $\overline{n} - \widetilde{n} \geq 3$ and $R_{\overline{n}-2} \geq R_{\widetilde{n}+1} \geq \widetilde{R}$. By Lemma 4.3,

$$\int_{B^c_{\zeta}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2] dx \leq \int_{B^c_{R_{\overline{n}-1}}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2] dx$$

$$\leq \frac{3}{4} \int_{B^c_{R_{\overline{n}-2}}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2] dx \leq \cdots$$

$$\leq \left(\frac{3}{4}\right)^{\overline{n}-\widetilde{n}-2} \int_{B^c_{R_{\widetilde{n}+1}}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2] dx$$

$$\leq \left(\frac{3}{4}\right)^{\overline{n}-\widetilde{n}-2} \int\limits_{B^c_{\widetilde{R}}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2]dx$$
$$= C(\widetilde{R})e^{(\log \frac{3}{4})\zeta^{(2-\gamma)/2}} \int\limits_{B^c_{\widetilde{R}}(0)} [|\nabla u_0|^2 + V(x)|u_0|^2]dx$$

showing the desired result. The proof is complete. □

Now, we are in a position to give the proof of Theorem 1.11.

Proof of Theorem 1.11. Since $V(x)(1+2^\gamma) \geq V(x)(1+|x|^\gamma) \geq a$ for all $x \in B_2(0)$, one has

$$\int\limits_{B_2(0)} u_0^2 dx \leq \frac{1+2^\gamma}{a} \int\limits_{B_2(0)} V(x)u_0^2 dx < +\infty.$$

Hence, to derive $u_0 \in L^2(\mathbb{R}^2)$, it suffices to verify $\int_{B_2^c(0)} u_0^2 dx < +\infty$. Let's denote $\Sigma_j = \{x \in \mathbb{R}^2 | 2^j \leq |x| < 2^{j+1}\}$ with $j \in \mathbb{N}^+$ as before. Since $2^{(j+2)\gamma}V(x) \geq (1+|x|^\gamma)V(x) \geq a$ on Σ_j, we obtain

$$\int\limits_{\Sigma_j} u_0^2 dx \leq \frac{2^{(j+2)\gamma}}{a} \int\limits_{\Sigma_j} V(x)u_0^2 dx \leq \frac{2^{(j+2)\gamma}}{a} \int\limits_{\Sigma_j} [|\nabla u_0|^2 + V(x)u_0^2]dx$$
$$\leq \frac{2^{(j+2)\gamma}}{a} \int\limits_{B^c_{2^j}(0)} [|\nabla u_0|^2 + V(x)u_0^2]dx \leq \frac{2^{(j+2)\gamma}}{a} Ce^{(\log \frac{3}{4})2^{(2-\gamma)j/2}} \tag{4.7}$$

for each $\zeta \triangleq 2^j \geq 2\widetilde{R}$ in Lemma 4.4. Therefore, there exists an integral $j_0 > 0$ such that (4.7) holds true for every $j \geq j_0 + 1$. As a consequence, with the help of $\gamma < 2$ and $\log(3/4) < 0$, we have

$$\int\limits_{B_2^c(0)} u_0^2 dx = \sum_{j=1}^{\infty} \int\limits_{\Sigma_j} u_0^2 dx = \sum_{j=1}^{j_0} \int\limits_{\Sigma_j} u_0^2 dx + \sum_{j=j_0+1}^{\infty} \int\limits_{\Sigma_j} u_0^2 dx$$
$$\leq \sum_{j=1}^{j_0} \int\limits_{\Sigma_j} u_0^2 dx + \frac{C}{a} \sum_{j=j_0+1}^{\infty} 2^{(j+2)\gamma} e^{(\log \frac{3}{4})2^{(2-\gamma)j/2}} < +\infty.$$

The proof is now complete. □

Remark 4.5. It is worthy mentioning here that our approach simplifies the proof that the nontrivial solution $u_0 \in E$ is indeed a bound state whence $(\gamma, \beta) \in (\mathrm{i})'$ since the references, [6, Proof of Theorem 16] and [27, Proof of Theorem 1.1], strongly relied on the Borel finite covering lemma with respect to the domain $B_5(0) \backslash B_2(0)$. In addition, as explained by Su et al. in [42, Remark 2], one could show the exponential decay of u_0 for each $(\gamma, \beta) \in (\mathrm{i})'$ which indicates that $\lim_{|x|\to\infty} u_0(x) = 0$, however, u_0 cannot have this exponential decay if $(\gamma, \beta) \in$ (ii)$'$, or $(\gamma, \beta) \in$ (iii)$'$, by a comparison argument concerning an explicit solution to $-\Delta u + c|x|^{-2}u = 0$.

Appendix A. Suppose that hypothesis $(\mathcal{K})$ holds. Then $C_0^\infty(\mathbb{R}^2)$ is dense in E under the norm $\|\cdot\|$.

Proof. Motivated by [27, Lemma 2.3], we know that the space $E_0 \triangleq \{u \in E | u$ has a compact support$\}$ is dense in E under the norm $\|\cdot\|$. Let us point out that $C_0^\infty(\mathbb{R}^2)$ is dense in $(E_0, \|\cdot\|)$. So, to finish the proof of this lemma, it suffices to obtain that E_0 is dense in E.

For every $R > 1$, we can choose a function $\psi_R \in C_0^\infty(\mathbb{R}^2, [0,1])$ to satisfy $\psi_R(x) \equiv 1$ for all $|x| \le R$, $\psi_R(x) \equiv 1$ for each $|x| \ge 2R$, and $|\nabla \psi_R| \le 2/R$ for every $x \in \mathbb{R}^2$. Considered a function $u \in E$, it's simple to find that $\psi_R u \in E_0$ for each fixed $R > 1$. Next, we shall conclude that $\|\psi_R u - u\| \to 0$ as $R \to \infty$.

Obviously, $V^{1/2}(x)(\psi_R - 1)u \to 0$ a.e. in $\mathbb{R}^2$ as $R \to \infty$ and $V^{1/2}(x)|\psi_R u - u| \le 2V^{1/2}(x)u \in L^2(\mathbb{R}^2)$, then by means of the Lebesgue Dominated Convergence theorem, one has

$$\int_{\mathbb{R}^2} V(x)|\psi_R u - u|^2 dx \to 0 \text{ as } R \to \infty. \tag{4.8}$$

Similarly, by the fact that $|\nabla u|^2 \in L^1(\mathbb{R}^2)$ since $u \in E$, we can derive that

$$\int_{\mathbb{R}^2} |(\psi_R - 1)\nabla u|^2 dx \to 0 \text{ as } R \to \infty. \tag{4.9}$$

Now, we claim that

$$\int_{\mathbb{R}^2} |u \nabla \psi_R|^2 dx \to 0 \text{ as } R \to \infty \tag{4.10}$$

whose proof is postponed. If (4.10) holds true, we deduce that $|\nabla(\psi_R u - u)| \to 0$ in $L^2(\mathbb{R}^2)$ by (4.9) and the Young's inequality. In view of (4.8), one sees immediately that $\|\psi_R u - u\| \to 0$ as $R \to \infty$.

Let's focus on showing (4.10). Define

$$\lambda_{12} \triangleq \inf\left\{ \int_{B_2(0)\backslash B_1(0)} |\nabla v|^2 dx \,\middle|\, v \in H_0^1(B_2(0)\backslash B_1(0)) \text{ and } \int_{B_2(0)\backslash B_1(0)} |v|^2 dx = 1 \right\}.$$

As a consequence, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |u\nabla\psi_R|^2 dx &= \int_{B_{2R}(0)\backslash B_R(0)} |u\nabla\psi_R|^2 dx \le \frac{4}{R^2} \int_{B_{2R}(0)\backslash B_R(0)} |u|^2 dx \\ &= 4 \int_{B_2(0)\backslash B_1(0)} |u_R(y)|^2 dy \le \frac{4}{\lambda_{12}} \int_{B_2(0)\backslash B_1(0)} |\nabla u_R|^2 dy \end{aligned}$$

$$= \frac{4}{\lambda_{12}} \int_{B_{2R}(0)\setminus B_R(0)} |\nabla u|^2 dx \to 0 \text{ as } R \to \infty$$

yielding (4.10), where $u_R(y) = u(Ry)$. The proof is complete. □

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