ON THE INFLUENCE OF A SUBQUADRATIC CONVECTION TERM IN SINGULAR ELLIPTIC PROBLEMS*

MARIUS GHERGU[a] and VICENŢIU RĂDULESCU[b]

[a] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. BOX 1-764, RO-014700 Bucharest, Romania*
E-mail: marius.ghergu@imar.ro
[b] *Department of Mathematics, University of Craiova, RO-200585 Craiova, Romania*
E-mail: vicentiu.radulescu@math.cnrs.fr

We establish some existence results for the singular elliptic equation $-\Delta u = g(u) + \lambda|\nabla u|^a + \mu f(x,u)$ either in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ or in the whole space. We suppose that λ and μ are positive parameters, $0 < a \leq 2$, f is a nondecreasing function which is sublinear with respect to the second variable, and $g \in C^1(0,\infty)$ is a decreasing function such that $\lim_{s \searrow 0} g(s) = +\infty$. The analysis we develop in this paper emphasizes the central role played by the convection term $|\nabla u|^a$.

Keywords: Singular elliptic equation; Convection term; Maximum principle.

1. Introduction

We are concerned in this paper with singular elliptic equations of the type

$$-\Delta u = g(u) + \lambda|\nabla u|^a + \mu f(x,u), \quad u > 0 \qquad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is either a smooth bounded domain or the whole space, $0 < a \leq 2$ and λ, $\mu \geq 0$. We suppose that $g \in C^1(0,\infty)$ is a positive nonincreasing function such that

$(g1)$ $\lim_{s \searrow 0} g(s) = +\infty$.

We also assume that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function such that $f > 0$ on $\overline{\Omega} \times (0,\infty)$ and is sub-linear with respect to the second variable, that is,

$(f1)$ the mapping $(0,+\infty) \ni s \longmapsto \dfrac{f(x,s)}{s}$ is nonincreasing for all $x \in \overline{\Omega}$;

$(f2)$ $\displaystyle\lim_{s\to\infty}\frac{f(x,s)}{s}=0,$ uniformly for $x\in\overline{\Omega}$.

Problems of this type arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically conducting materials.

Our general setting includes some simple prototype models from boundary-layer theory of viscous fluids (see Ref. 1). If $\lambda = 0$ and $\mu = 0$, problem (1) is called the Lane-Emden-Fowler equation. Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes[2] as a model for long range Van der Waals interactions in thin films spreading on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see Ref. 3), as well as to describe the flow over an impermeable plate (see Refs. 4, 5).

The main feature of this paper is the presence of the convection term $|\nabla u|^a$. As remarked in Refs. 6, 7, the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

In the case where $\lambda = 0$, the problem (1) subject to Dirichlet boundary condition has a unique solution for all $\mu \geq 0$ (see Refs. 8, 9, 10, 11 and the references therein).

If $\lambda > 0$, the following problem was considered in Zhang and Yu[12]

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda|\nabla u|^a + \sigma & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth bounded domain, λ, $\sigma \geq 0$, $\alpha > 0$, and $a \in (0, 2]$. By using the change of variable $v = e^{\lambda u} - 1$ in the case $a = 2$, it is proved in Ref. 12 that problem (2) has classical solutions if $\lambda\sigma < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. This will be used to deduce the existence and nonexistence in the case $0 < a < 2$.

If $f(x, u)$ depends on u, the above change of variable does not preserve the sublinearity condition $(f1)$–$(f2)$ and the monotony of the nonlinear term g in (1). In turn, if $f(x, u)$ does not depend on u and $a = 2$, this method successfully applies to our study and we will be able to give a complete characterization of (1).

Due to the singular term $g(u)$ in (1), we cannot expect to have solutions in $C^2(\overline{\Omega})$ for (1). As it was pointed out in Ref. 12, if $\alpha > 1$ then the solution of (2) is **not** in $C^1(\overline{\Omega})$. We are seeking in this paper classical solutions of (1), that is, solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that verify (1).

2. Singular elliptic equations in bounded domains

We present in this section some existence results for the problem

$$\begin{cases} -\Delta u = g(u) + \lambda |\nabla u|^a + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Theorem 2.1. *Assume that conditions* $(f1)$–$(f2)$, $(g1)$ *are fulfilled and* $0 < a \le 1$. *Then for all* $\lambda, \mu \ge 0$ *the problem* (3) *has at least one solution.*

Proof (Sketch). The proof relies on the sub and super-solution argument. Let us first notice that, by Ref. 13, there exists $v \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta v = g(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Then $\underline{u}_{\lambda\mu} = v$ is a sub-solution of (3). The main point is to find a super-solution $\overline{u}_{\lambda\mu} \in C^2(\Omega) \cap C(\overline{\Omega})$ of (3). This will be done separately for $0 < a < 1$ and $a = 1$. Since g is decreasing, we can easily obtain that $\underline{u}_{\lambda\mu} \le \overline{u}_{\lambda\mu}$ in Ω so (3) has at least one solution.

Case $0 < a < 1$. Let $h \in C^2(0, \eta] \cap C[0, \eta]$ be such that

$$\begin{cases} h''(t) = -g(h(t)), & \text{for all } 0 < t < \eta, \\ h(0) = 0, \\ h > 0 \quad \text{in } (0, \eta]. \end{cases} \tag{5}$$

The existence of h follows by classical arguments of ODE. Since h is concave, there exists $h'(0+) \in (0, +\infty]$. By taking $\eta > 0$ small enough, we can assume that $h' > 0$ in $(0, \eta]$, so h is increasing on $[0, \eta]$. We also have

Lemma 2.1.

(i) $h \in C^1[0, \eta]$ *if and only if* $\int_0^1 g(s)ds < +\infty$;

(ii) *If* $0 < p \le 2$, *then there exist* $c_1, c_2 > 0$ *such that*

$$(h')^p(t) \le c_1 g(h(t)) + c_2, \qquad \textit{for all } \; 0 < t < \eta. \tag{6}$$

Now, we construct a super-solution in the form $\overline{u}_{\lambda\mu} = Mh(c\varphi_1)$ for $M > 1$ large enough and $c > 0$ sufficiently small, where φ_1 represents the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$.

Case $a = 1$. This case was left as an open problem in Ref. 14. Note that the method used in Case $0 < a < 1$ applies here only for small values of λ and μ. Let $R > 0$ be large enough such that $\Omega \subset B_R(0)$, where $B_R(0) = \{x \in \mathbb{R}^N;\ |x| < R\}$. We consider the problem

$$\begin{cases} -\Delta u = g(u) + \lambda|\nabla u| + \mu f(x,u) & |x| < R,\\ u > 0 & |x| < R,\\ u = 0 & |x| = R. \end{cases} \tag{7}$$

In order to provide a super-solution for (7) let us first consider the problem

$$\begin{cases} -\Delta u = g(u) + \lambda|\nabla u| + 1 & |x| < R,\\ u > 0 & |x| < R,\\ u = 0 & |x| = R. \end{cases} \tag{8}$$

We need the following auxiliary result.

Lemma 2.2. *Problem* (8) *has at least one solution.*

Proof. We are looking for radially symmetric solution u of (8), that is, $u = u(r)$, $0 \le r = |x| \le R$ and

$$\begin{cases} -u'' - \dfrac{N-1}{r}u'(r) = g(u(r)) + \lambda|u'(r)| + 1 & 0 \le r < R,\\ u > 0 & 0 \le r < R,\\ u(R) = 0. \end{cases} \tag{9}$$

This implies $-(r^{N-1}u'(r))' \ge 0$ for all $0 \le r < R$, which yields $u'(r) \le 0$ for all $0 \le r < R$. Then (9) gives

$$-\left(u'' + \frac{N-1}{r}u'(r) + \lambda u'(r)\right) = g(u(r)) + 1, \quad 0 \le r < R.$$

We obtain

$$-(e^{\lambda r}r^{N-1}u'(r))' = e^{\lambda r}r^{N-1}(g(u(r)) + 1), \quad 0 \le r < R. \tag{10}$$

From (10) we get

$$u(r)=u(0) - \int_0^r e^{-\lambda t}t^{-N+1}\int_0^t e^{\lambda s}s^{N-1}(g(u(s)) + 1)dsdt,\ 0 \le r < R. \tag{11}$$

Let $w \in C^2(B_R(0)) \cap C(\overline{B}_R(0))$ be the unique radial solution of the problem

$$\begin{cases} -\Delta w = g(w) + 1 & |x| < R, \\ w > 0 & |x| < R, \\ w = 0 & |x| = R. \end{cases} \tag{12}$$

Clearly, w is a sub-solution of (8). As above we get

$$w(r) = w(0) - \int_0^r t^{-N+1} \int_0^t s^{N-1}(g(w(s)) + 1)dsdt, \quad 0 \le r < R. \tag{13}$$

We claim that there exists a solution $v \in C^2[0, R) \cap C[0, R]$ of (11) such that $v > 0$ in $[0, R)$.

Let $A = w(0)$ and define the sequence $(v_k)_{k \ge 1}$ inductively by

$$\begin{cases} v_k(r) = A - \displaystyle\int_0^r e^{-\lambda t} t^{-N+1} \int_0^t e^{\lambda s} s^{N-1}(g(v_{k-1}(s)) + 1)dsdt, \\ \hfill 0 \le r < R,\ k \ge 1, \\ v_0 = w. \end{cases} \tag{14}$$

Note that v_k is decreasing in $[0, R)$ for all $k \ge 0$. From (13) and (14) we easily check that $v_1 \ge v_0$ and by induction we deduce $v_k \ge v_{k-1}$ for all $k \ge 1$. Hence

$$w = v_0 \le v_1 \le ... \le v_k \le ... \le A \quad \text{in } B_R(0).$$

Thus, there exists $v(r) := \lim_{k\to\infty} v_k(r)$, for all $0 \le r < R$ and $v > 0$ in $[0, R)$. We can now pass to the limit in (14) in order to get that v is a solution of (11). By classical regularity arguments we also obtain $v \in C^2[0, R) \cap C[0, R]$. This proves the claim.

We have obtained a super-solution v of (8) such that $v \ge w$ in $B_R(0)$. Hence, the problem (8) has at least one solution and the proof of our Lemma is now complete. □

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the problem (8). For $M > 1$ we have $-\Delta(Mu) \ge g(Mu) + \lambda|\nabla(Mu)| + M$ in Ω. Since f is sublinear, we can choose $M = M(\mu) > 1$ such that $M \ge \mu f(x, M|u|_\infty)$ in $B_R(0)$. Then $\overline{u}_{\lambda\mu} := Mu$ is a super-solution for (1).

This finishes the proof of Theorem 2.1. □

In the case $1 < a \le 2$ we prove the following result.

Theorem 2.2. *Assume $\mu = 1$ and f, g satisfy $(f1)$–$(f2)$ and $(g1)$ respectively. Then there exists $\lambda^* > 0$ such that* (1) *has at least one classical solution for $0 \le \lambda < \lambda^*$ and no solutions exist if $\lambda > \lambda^*$.*

Proof. For small values of $\lambda > 0$ we can construct a super-solution of (3) in the same manner as in the proof of Theorem 2.1.

Set

$$A = \{\ \lambda \ge 0 : \ \text{problem (1) has at least one classical solution}\}.$$

From the above arguments, A is nonempty. Let $\lambda^* = \sup A$. First we claim that if $\lambda \in A$, then $[0, \lambda) \subseteq A$. For this purpose, let $\lambda_1 \in A$ and $0 \le \lambda_2 < \lambda_1$. If u_{λ_1} is a solution of (1) with $\lambda = \lambda_1$, then u_{λ_1} is a super-solution for (1) with $\lambda = \lambda_2$ while v defined in (4) is a sub-solution. Hence, the problem (1) with $\lambda = \lambda_2$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $[0, \lambda^*) \subset A$.

Let us prove that $\lambda^* < +\infty$. For this purpose we use the following result

Lemma 2.3. (see Ref. 15). *If $a > 1$, then there exists a real number $\bar{\sigma} > 0$ such that the problem*

$$\begin{cases} -\Delta u \ge |\nabla u|^a + \sigma & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

has no solutions for $\sigma > \bar{\sigma}$.

Set

$$\tau := \inf_{(x,s) \in \overline{\Omega} \times (0, +\infty)} \Big(g(s) + f(x, s) \Big).$$

Since $\lim_{s \searrow 0} g(s) = +\infty$ and the mapping $(0, +\infty) \ni s \longmapsto \min_{x \in \overline{\Omega}} f(x, s)$ is positive and nondecreasing, we deduce that m is positive. Let $\lambda > 0$ be such that (3) has a solution u_λ. If $w = \lambda^{1/(p-1)} u_\lambda$, then v verifies

$$\begin{cases} -\Delta w \ge |\nabla w|^p + \lambda^{1/(a-1)} \tau & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

By Lemma 2.3 it follows that $\lambda^{1/(a-1)} \tau \le \bar{\sigma}$ which gives $\lambda \le (\bar{\sigma}/\tau)^{a-1}$. This means that λ^* is finite. This completes the proof. □

Theorems 2.1 and 2.2 show the importance of the convection term $\lambda |\nabla u|^a$ in (3). Indeed, according to Ref. 10, Theorem 1.3, for any $\mu > 0$, the boundary value problem

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda|\nabla u|^a + \mu u^\beta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

has a unique solution, provided $\lambda = 0$ and $\alpha, \beta \in (0,1)$. The above theorems show that if λ is not necessarily 0, then the following situations may occur: (i) problem (17) has solutions if $a \in (0,1]$ and for all $\lambda \geq 0$; (ii) if $a \in (1,2)$ then there exists $\lambda^* > 0$ such that problem (17) has a solution for any $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

To better understand the dependence between λ and μ in (3), let us consider the special case $f \equiv 1$ and let

$$m := \lim_{s\to\infty} g(s) \in (0, +\infty).$$

In this case the result concerning (3) is the following.

Theorem 2.3. *Assume that* $a = 2$ *and* $f \equiv 1$. *Then the following properties hold:*

(i) *The problem* (1) *has a solution if and only if* $\lambda(m+\mu) < \lambda_1$;

(ii) *Assume* $\mu > 0$ *is fixed and let* $\lambda^* = \dfrac{\lambda_1}{m+\mu}$. *Then* (1) *has a unique solution* u_λ *for every* $0 < \lambda < \lambda^*$ *and the sequence* $(u_\lambda)_{0<\lambda<\lambda^*}$ *is increasing with respect to* λ. *Moreover, if* $\limsup_{s\searrow 0} s^\alpha g(s) < +\infty$, *for some* $\alpha \in (0,1)$, *then the sequence of solutions* $(u_\lambda)_{0<\lambda<\lambda^*}$ *has the following properties:*

(ii1) $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;

(ii2) $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$ *uniformly on compact subsets of* Ω.

Remark. The assumption $\limsup_{s\searrow 0} s^\alpha g(s) < +\infty$, for some $\alpha \in (0,1)$, has been used in Ref. 8, 10 and it implies the following Keller-Osserman-type growth condition around the origin

$$\int_0^1 \left(\int_0^t g(s)ds\right)^{-1/2} dt < +\infty. \tag{18}$$

As proved by Bénilan, Brezis and Crandall in Ref. 16, condition (18) is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and $-\Delta u = g(u) + h$ a.e. in $\mathbb{R}^N$.

Proof of Theorem 2.3. With the change of variable $v = e^{\lambda u} - 1$, the problem (1) takes the form

$$\begin{cases} -\Delta v = \Psi_{\lambda\mu}(x,u) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where

$$\Psi_{\lambda\mu}(x,s) = \lambda(s+1)g\left(\frac{1}{\lambda}\ln(s+1)\right) + \lambda\mu(s+1)f\left(x, \frac{1}{\lambda}\ln(s+1)\right),$$

for all $(x,s) \in \overline{\Omega} \times (0,\infty)$. The existence and nonexistence results follows now from Ref. 14, Theorem 2.4.

In order to prove the asymptotic behavior of the solution near λ^* we use the following alternative which is due to Hörmander (see Ref. 17, Theorem 4.1.9).

Proposition 2.1. *Let $(u_\lambda)_{0<\lambda<\lambda^*}$ be a sequence of positive super-harmonic functions which are increasing with respect to λ. Then the following alternative holds:*

(i) *either u_λ converges in $L^1_{\mathrm{loc}}(\Omega)$;*
(ii) *or $u_\lambda \to \infty$ uniformly on compact subsets of Ω.*

3. Ground state solutions for singular elliptic problems

We consider in this section the following singular problem

$$\begin{cases} -\Delta u = p(x)(g(u) + f(u) + |\nabla u|^a) & \text{in } \mathbb{R}^N, (N \geq 3), \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{20}$$

where f and g satisfy $(f1)$–$(f2)$ and $(g1)$, $0 < a < 1$, and $p : \mathbb{R}^N \to (0,\infty)$ is a Hölder continuous function of exponent $\gamma \in (0,1)$.

We are concerned here with ground state solutions, that is, positive solutions defined in the whole space and decaying to zero at infinity.

The case $f \equiv 0$ and $a = 0$ was considered in Lair and Shaker[18]. More exactly, it was proved in Ref. 18 that a necessary condition in order to have solution for the problem

$$\begin{cases} -\Delta u = p(x)g(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{21}$$

is

$$\int_1^\infty t\psi(t)dt < \infty, \tag{22}$$

where $\psi(r) = \min_{|x|=r} p(x)$, $r \geq 0$. Note that condition (22) is also necessary for our problem (20), since any solution of (20) is a super-solution of (21). The sufficient condition for existence supplied in Ref. 18 is

$$\int_1^\infty t\phi(t)dt < \infty, \tag{23}$$

where $\phi(r) = \max_{|x|=r} p(x)$, $r \geq 0$. Hence, when p is radially symmetric, the problem (21) has solutions if and only if $\int_1^\infty tp(t)dt < \infty$ (see Ref. 18).

Our result concerning the problem (20) is the following.

Theorem 3.1. *Assume that* $(f1)$–$(f2)$, $(g1)$ *and* (23) *are fulfilled. Then problem* (20) *has at least one solution.*

Proof. The solution of problem (20) is obtained as a limit in $C^{2,\gamma}_{\rm loc}(\mathbb{R}^N)$ of a monotone sequence of solutions associated to (20) in smooth bounded domains. Let $B_n := \{x \in \mathbb{R}^N;\ |x| < n\}$. According to Theorem 2.1, for all $n \geq 1$ there exists $u_n \in C^{2,\gamma}(B_n) \cap C(\overline{B_n})$ such that

$$\begin{cases} -\Delta u_n = p(x)(g(u_n) + f(u_n) + |\nabla u_n|^a) & \text{in } B_n, \\ u_n > 0 & \text{in } B_n, \\ u_n = 0 & \text{on } \partial B_n. \end{cases} \tag{24}$$

We extend u_n by zero outside of B_n. We claim that $u_n \leq u_{n+1}$ in B_n. Assume by contradiction that the inequality $u_n \leq u_{n+1}$ does not hold throughout B_n and let

$$\zeta(x) = \frac{u_n(x)}{u_{n+1}(x)}, \quad x \in B_n.$$

Clearly $\zeta = 0$ on ∂B_n, so that ζ achieves its maximum in a point $x_0 \in B_n$. At this point we have $\nabla\zeta(x_0) = 0$ and $\Delta\zeta(x_0) \leq 0$. This yields

$$-\mathrm{div}(u_{n+1}^2 \nabla\zeta)(x_0) = -\Big(\mathrm{div}(u_{n+1}^2)\nabla\zeta + u_{n+1}^2\Delta\zeta\Big)(x_0) \geq 0.$$

A straightforward computation shows that

$$-\mathrm{div}(u_{n+1}^2\nabla\zeta) = -u_{n+1}\Delta u_n + u_n\Delta u_{n+1}.$$

Hence

$$\Big(-u_{n+1}\Delta u_n + u_n\Delta u_{n+1}\Big)(x_0) \geq 0.$$

The above relation produces

$$\begin{aligned}\left(\frac{g(u_n)+f(u_n)}{u_n}-\frac{g(u_{n+1})+f(u_{n+1})}{u_{n+1}}\right)(x_0)+\\+\left(\frac{|\nabla u_n|^a}{u_n}-\frac{|\nabla u_{n+1}|^a}{u_{n+1}}\right)(x_0)\geq 0.\end{aligned}\tag{25}$$

Since $t\longmapsto \dfrac{g(t)+f(t)}{t}$ is decreasing on $(0,\infty)$ and $u_n(x_0)>u_{n+1}(x_0)$, from (25) we obtain

$$\left(\frac{|\nabla u_n|^a}{u_n}-\frac{|\nabla u_{n+1}|^a}{u_{n+1}}\right)(x_0)>0.\tag{26}$$

On the other hand, $\nabla\zeta(x_0)=0$ implies

$$u_{n+1}(x_0)\nabla u_n(x_0)=u_n(x_0)\nabla u_{n+1}(x_0).$$

Furthermore, relation (26) leads us to $u_n^{a-1}(x_0)-u_{n+1}^{a-1}(x_0)>0$, which is a contradiction since $0<a<1$. Hence $u_n\leq u_{n+1}$ in B_n which means that

$$0\leq u_1\leq\cdots\leq u_n\leq u_{n+1}\leq\ldots\quad \text{in } \mathbb{R}^N.$$

The main point is to find an upper bound for the sequence $(u_n)_{n\geq 1}$. To this aim, set

$$\Phi(r)=r^{1-N}\int_0^r t^{N-1}\phi(t)dt,\qquad \text{for all } r>0.$$

Using the assumption (23) and L'Hôpital's rule, we get $\lim\limits_{r\to\infty}\Phi(r)=\lim\limits_{r\searrow 0}\Phi(r)=0$ and

$$\lim_{r\to\infty}\Phi(r)=\frac{1}{N-2}\int_0^\infty r\phi(r)dr<\infty.$$

Let $k>2$ be such that $k^{1-a}\geq 2\max\limits_{r\geq 0}\Phi^a(r)$ and define

$$\xi(x)=k\int_{|x|}^\infty \Phi(t)dt,\qquad \text{for all } x\in\mathbb{R}^N.$$

Then ξ satisfies

$$\begin{cases}-\Delta\xi=k\phi(|x|) & \text{in } \mathbb{R}^N,\\ \xi>0 & \text{in } \mathbb{R}^N,\\ \xi(x)\to 0 & \text{as } |x|\to\infty.\end{cases}$$

Since the mapping $[0,\infty) \ni t \longmapsto \int_0^t \frac{1}{g(s)+1}ds \in [0,\infty)$ is bijective, we can implicitly define $w : \mathbb{R}^N \to (0,\infty)$ by

$$\int_0^{w(x)} \frac{1}{g(t)+1}dt = \xi(x), \qquad \text{for all } \ x \in \mathbb{R}^N.$$

It is easy to see that $w \in C^2(\mathbb{R}^N)$ and $w(x) \to 0$ as $|x| \to \infty$. Furthermore, we have

$$\begin{cases} -\Delta w \geq p(x)(g(w) + 1 + |\nabla w|^a) & \text{in } \mathbb{R}^N, \\ w > 0 & \text{in } \mathbb{R}^N, \\ w(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{27}$$

Using the assumption $(f1)$, we can find $M > 1$ large enough such that $M > f(Mw)$ in $\mathbb{R}^N$. Multiplying by M in (27) we deduce that $v := Mw$ satisfies

$$\begin{cases} -\Delta v \geq p(x)(g(v) + f(v) + |\nabla v|^a) & \text{in } \mathbb{R}^N, \\ v > 0 & \text{in } \mathbb{R}^N, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

With the same proof as above we deduce that $u_n \leq v$ in B_n, for all $n \geq 1$. This implies $0 \leq u_1 \leq \cdots \leq u_n \leq v$ in $\mathbb{R}^N$. Thus, there exists $u(x) = \lim_{n\to\infty} u_n(x)$, for all $x \in \mathbb{R}^N$ and $u_n \leq u \leq v$ in $\mathbb{R}^N$. Since $v(x) \to 0$ as $|x| \to \infty$, we deduce that $u(x) \to 0$ as $|x| \to \infty$. A standard bootstrap argument (see Gilbarg and Trudinger[19]) implies that $u_n \to u$ in $C^{2,\gamma}_{\text{loc}}(\mathbb{R}^N)$ and that u is a solution of problem (20).

This completes the proof of Theorem 3.1. □

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