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A SHARP EIGENVALUE THEOREM FOR FRACTIONAL ELLIPTIC EQUATIONS

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ABSTRACT

By using variational methods, in this paper we study a nonlinear elliptic problem defined in a bounded domain $\Omega \subset \mathbb{R}^N$, with smooth boundary $\partial\Omega$, involving fractional powers of the Laplacian operator together with a suitable nonlinear term f. More precisely, we prove a characterization theorem on the existence of one weak solution for the elliptic problem

$$\begin{cases} (-\Delta)^{\alpha/2}u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in (0,2)$, $N > \alpha$, $\lambda > 0$ and $(-\Delta)^{\alpha/2}$ denotes the nonlocal fractional Laplacian operator. Our result extends to the nonlocal setting recent theorems for ordinary and classical elliptic equations, as well as a characterization for elliptic problems on certain non-smooth domains. To make the nonlinear methods work, some careful analysis of the fractional spaces involved is necessary.

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Contents

1. Introduction

Recently, a lot of interest has been devoted to elliptic equations involving the fractional Laplace operator proving several interesting mathematical results (see, among others, the works of Caffarelli and Silvestre [12, 13, 14]). Furthermore, this operator is of nonlocal diffusion type and arises in several physical phenomena like frames propagation, American options in finance, population dynamics and Lévy processes (see, for instance, [2, 6, 17]).

Motivated by this large interest in the current literature, the aim of this paper is to prove a characterization result on the existence of one positive solution for fractional nonlocal equations. Our result reads as follows:

Theorem 1: *Let $f : [0,+\infty[\to [0,+\infty[$ be a continuous function with $f(0) = 0$ and such that, for some $a > 0$, the map $h :]0,+\infty[\to [0,+\infty[$ defined by*

$$h(\xi) := \frac{F(\xi)}{\xi^2}$$

is nonincreasing in the real interval $]0, a]$, where

$$F(\xi) := \int_0^\xi f(t)dt,$$

for each $\xi \in [0,+\infty[$. Then, the following assertions are equivalent:

(h_1) *h is not constant in $]0,\zeta]$ for each $\zeta > 0$;*

($h_2^\star$) *f is subcritical with $\lim_{\xi\to 0^+} h(\xi) > 0$ and for each $r > 0$ there exists an open interval $J_r \subseteq]0,+\infty[$ such that, for every $\lambda \in J_r$, the nonlocal problem given by*

$$(S_\lambda) \qquad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution in $H_0^{\alpha/2}(\Omega)$, whose norm is less than r.

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $(-\Delta)^{\alpha/2}u$ denotes the fractional Laplacian acting on u, λ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is a suitable continuous function.

For a smooth function $u : \mathbb{R}^N \to \mathbb{R}$ the fractional Laplacian can be defined either by using the Riesz potential

$$(-\Delta)^{\alpha/2}u(x) := C_{N,\alpha} \lim_{\varrho\to 0^+} \int_{\mathbb{R}^N\setminus B_\varrho} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\, dy,$$

where $C_{N,\alpha}$ is a suitable normalization constant depending on N and α, or by

$$(-\Delta)^{\alpha/2}u(x) := \mathcal{F}^{-1}[|y|^\alpha \mathcal{F}[u](y)](x),$$

where $\mathcal{F}[\cdot]$ denote respectively the classical Fourier transform and and $\mathcal{F}^{-1}[\cdot]$ its inverse.

Alternatively, following the work of Caffarelli and Silvestre [12], the fractional Laplacian operator in $\mathbb{R}^N$ can be defined as a Dirichlet to a Neumann map:

$$(-\Delta)^{\alpha/2}u(x) := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x,y),$$

where κ_α is a suitable constant and w is the α-harmonic extension of u. In other words, w is the function defined on the upper half-space $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times]0,+\infty[$ which is solution to the local elliptic problem

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w(x,0) = u(x) & \text{in } \mathbb{R}^N. \end{cases}$$

In order to define the fractional Laplacian operator in bounded domains, the above procedure has been adapted in [7] and [11] (see Section 2 for details). Successively, several authors have considered this definition for the operator $(-\Delta)^{\alpha/2}$ in a bounded domain with zero Dirichlet boundary data (see [10, 11, 16]).

For completeness, we point out that two notions of fractional operators on bounded domains were considered in the literature, namely the previous one (called also the **spectral** Laplacian operator) and the **integral** one (in this setting, see, among others, the papers [32, 33, 36, 37, 38, 41] and [22, 23, 24, 25, 28, 29, 30]). Servadei and Valdinoci in [39, Theorem 1] compare these two operators, studying their spectral properties and obtaining, as a consequence of this careful analysis, that these two operators are different. See also the recent paper [31] of Musina and Nazarov for an exhaustive study of this comparison.

In the sequel we use the spectral definition, having some technical advantages to overcome certain mathematical difficulties in proving our result. For instance, one of the main tools is the validity of the Maximum Principle for the augmented nonlocal problem proved by Cabré and Sire (see [10]) as well as a regularity result and a priori estimates for solutions of nonlocal equations in terms of the data due to Di Blasio and Volzone (see [18]), which extend the well known ones for the standard Laplacian case. For completeness, see also the recent papers of Barrios, Colorado, Servadei and Soria [4] and Kuusi, Mingione and Sire [20, 21] where interesting regularity results for nonlocal problems have been studied.

We also point out that elliptic equations in $\mathbb{R}^N$, driven by a nonlocal integro-differential operator, whose standard prototype is the fractional Laplace operator, have been studied very recently by Autuori and Pucci in [3].

Theorem 1 can be regarded as an elliptic version, for nonlocal fractional equations, of a very recent result obtained by Ricceri in [35, Theorem 1] for a two-point boundary value problem. In this paper the author first proves an original critical point theorem on Hilbert spaces and successively uses this abstract tool in order to obtain the cited characterization result (see also [34] for related topics on the abstract variational setting).

In the mentioned result the author uses the compact embedding

$$W_0^{1,2}(]0,1[) \hookrightarrow C^0([0,1]),$$

as well as the estimation

$$\sup_{u \in W_0^{1,2}(]0,1[)\setminus\{0\}} \frac{\max_{x\in[0,1]}|u(x)|}{(\int_0^1 |u'(t)|^2\,dt)^{1/2}} < \frac{1}{2},$$

in a crucial way. In [26], a similar technical approach was adopted studying elliptic equations defined on the Sierpiński gasket or, more generally, on self-similar fractal domains whose spectral dimension $\nu \in]0,2[$.

Contrary to the above cases, in the standard higher-dimensional setting the same strategy cannot be directly used treating elliptic equations involving the classical Laplacian operator. In such a case a different proof, again based on variational methods, was developed by Anello in [1].

The extension of the cited results to (S_λ) is not trivial and requires overcoming some technical difficulties which arise in this new analytic nonlocal setting. In particular, for our goal, it is necessary to exploit some basic properties of the space $H_0^{\alpha/2}(\Omega)$ and to use the distribution of the spectrum of the corresponding linear fractional problem.

Further, the regularity result obtained in [18, Theorem 4.1] is an essential argument proving that assumption (h_1) implies condition (h_2) defined at the end of Section 2.

At the end of the following preliminary section, we point out that this paper is inspired by the pioneering work on sublinear elliptic equations by Brezis and Oswald [9]. Indeed, our hypothesis that h is nonincreasing in some right-neighborhood of the origin implies a sublinear decay of the nonlinear term f. For instance, in the case of power-type nonlinearities $f(u) = u^p$, this assumption is fulfilled if and only if $p \in (0, 1]$. Next, property (h_1) excludes the linear case that corresponds to $p = 1$. In [9], the semilinear case described by the Laplace operator under the basic assumption that $f(u)/u$ is decreasing on $(0, \infty)$ is studied. The differences between our main result and Theorem 1 in [9] are the following:

(i) In [9], the global assumption that $f(u)/u$ is decreasing on the whole positive semi-axis is used, while in our case a local monotonicity hypothesis is used, namely $F(\xi)/\xi^2$ is nondecreasing in some interval $]0, a[$. The global assumption on $f(u)/u$ is used in [9] to show that the problem has at most one solution, hence to establish a uniqueness property.

(ii) The existence of solutions is deduced in [9] in accordance with the sign of the principal eigenvalues of some linear operators that depend on the growth rate of $f(u)/u$ near the origin and infinity. In our case, we do not necessarily have a unique solution and the existence of solutions depends on the location of the parameter λ in a certain interval. Such a parameter does not exist in the problem studied by Brezis and Oswald [9]. Moreover, in the present paper, we provide an estimate of the norm of solutions in a suitable Sobolev space and in a relationship with a prescribed positive real number. Finally, it is striking to point out that the main result in this paper establishes an existence property by assuming a local behavior of the nonlinear term (as described by the mapping h) and without assuming any growth for large values of the argument. The key role here is played by the parameter λ. We argue that this type of argument can be extended in order to obtain existence results for large classes of elliptic equations under local information on the nonlinear term.

The paper is organized as follows. In the next section we collect some properties of the fractional Laplacian operator in a bounded domain defined by using the notion of α-harmonic extension, as well as providing some basic notions on the Sobolev spaces $X_0^\alpha(\mathcal{C}_\Omega)$ and $H_0^{\alpha/2}(\Omega)$. Finally, Section 3 is devoted to the proof of Theorem 1.

We refer to the recent book [27] for the basic variational methods used in the present paper.

2. Some preliminaries

2.1. The Sobolev space $H_0^{\alpha/2}(\Omega)$. The powers $(-\Delta)^{\alpha/2}$ of the Laplace operator $-\Delta$ in a bounded domain Ω with zero boundary conditions are defined through the spectral decomposition using the powers of the eigenvalues of the original operator.

Hence, according to classical results on positive operators in Ω, if $\{\varphi_j, \lambda_j\}_{j\in\mathbb{N}}$ are the eigenfunctions and eigenvalues of the usual linear Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

then $\{\varphi_j, \lambda_j^{\alpha/2}\}_{j\in\mathbb{N}}$ are the eigenfunctions and eigenvalues of the corresponding fractional one:

$$\begin{cases} (-\Delta)^{\alpha/2} u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Indeed, the operator $(-\Delta)^{\alpha/2}$ is well defined on the Sobolev space

$$H_0^{\alpha/2}(\Omega) := \left\{ u = \sum_{j=1}^{\infty} a_j\varphi_j \in L^2(\Omega) : \sum_{j=1}^{\infty} a_j^2 \lambda_j^{\alpha/2} < +\infty \right\},$$

endowed by the norm

$$\|u\|_{H_0^{\alpha/2}(\Omega)} := \left(\sum_{j=1}^{\infty} a_j^2 \lambda_j^{\alpha/2}\right)^{1/2},$$

and has the following form:

$$(-\Delta)^{\alpha/2} u = \sum_{j=1}^{\infty} a_j \lambda_j^{\alpha/2} \varphi_j.$$

2.2. The extension problem. Associated to the bounded domain Ω, let us consider the cylinder

$$\mathcal{C}_\Omega := \{(x,y) : x \in \Omega,\, y \in \mathbb{R}_+\} \subset \mathbb{R}_+^{N+1},$$

and denote by $\partial_L \mathcal{C}_\Omega := \partial\Omega \times \mathbb{R}_+$ its lateral boundary.

For a function $u \in H_0^{\alpha/2}(\Omega)$, define the α-harmonic extension $\mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution of the problem

$$\begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla \mathrm{E}_\alpha(u)) = 0 & \text{in } \mathcal{C}_\Omega, \\ \mathrm{E}_\alpha(u) = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \mathrm{Tr}(\mathrm{E}_\alpha(u)) = u & \text{on } \Omega, \end{cases} \tag{2.3}$$

where the trace operator $\mathrm{Tr} : X_0^\alpha(\mathcal{C}_\Omega) \to L^2(\Omega)$ is given by

$$\mathrm{Tr}(\mathrm{E}_\alpha(u)) := \mathrm{E}_\alpha(u)(\cdot, 0).$$

The extension function $\mathrm{E}_\alpha(u)$ belongs to the Hilbert space

$$X_0^\alpha(\mathcal{C}_\Omega) := \left\{ w \in L^2(\mathcal{C}_\Omega) : w = 0 \text{ on } \partial_L \mathcal{C}_\Omega,\, \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w(x,y)|^2 \, dxdy < +\infty \right\},$$

with the standard norm

$$\|w\|_{X_0^\alpha(\mathcal{C}_\Omega)} := \left(\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w(x,y)|^2 \, dxdy \right)^{1/2},$$

where the normalization constant κ_α is given by

$$\kappa_\alpha := \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}.$$

Introducing this constant we have that the extension operator

$$\mathrm{E}_\alpha : H_0^{\alpha/2}(\Omega) \to X_0^\alpha(\mathcal{C}_\Omega)$$

is an isometry, i.e.,

$$\|\mathrm{E}_\alpha(u)\|_{X_0^\alpha(\mathcal{C}_\Omega)} = \|u\|_{H_0^{\alpha/2}(\Omega)},$$

for every $u \in H_0^{\alpha/2}(\Omega)$.

Further, we have the following trace inequality:

$$\|\mathrm{Tr}(w)\|_{H_0^{\alpha/2}(\Omega)} \leq \|w\|_{X_0^\alpha(\mathcal{C}_\Omega)},$$

for every $w \in X_0^\alpha(\mathcal{C}_\Omega)$.

By using the α-extension $\mathrm{E}_\alpha(u) \in X_0^\alpha(\mathcal{C}_\Omega)$ of the function $u \in H_0^{\alpha/2}(\Omega)$, we can define the fractional operator $(-\Delta)^{\alpha/2}$ in Ω, acting on u, as follows;

$$(-\Delta)^{\alpha/2}u(x) := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial \mathrm{E}_\alpha(u)}{\partial y}(x,y).$$

2.3. Weak solutions. Let $f : [0,+\infty[\to [0,+\infty[$ be a continuous function with $f(0)=0$ and such that, for some $a>0$, the map $h :]0,+\infty[\to [0,+\infty[$ defined by

$$h(\xi) := \frac{F(\xi)}{\xi^2}$$

is nonincreasing in the real interval $]0,a]$, where

$$F(\xi) := \int_0^\xi f(t)dt,$$

for each $\xi \in [0,+\infty[$. Put

$$\bar{a} := \sup\{\eta > 0 : h \text{ is nonincreasing in }]0,\eta]\} \in]0,+\infty].$$

Assume that $\bar{a} = +\infty$. In such a case, we say that a positive function $u = \mathrm{Tr}(w) \in H_0^{\alpha/2}(\Omega)$ is a **weak solution** of (S_λ) if $w \in X_0^\alpha(\mathcal{C}_\Omega)$ weakly solves

$$(\widehat{S}_\lambda) \qquad \begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y) = \lambda f(u) & \text{on } \Omega, \end{cases}$$

i.e.,

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla\varphi\rangle dxdy = \lambda \int_\Omega f(\mathrm{Tr}(w)(x))\mathrm{Tr}(\varphi)(x)dx,$$

for every $\varphi \in X_0^\alpha(\mathcal{C}_\Omega)$.

On the contrary, if $\bar{a} < \infty$, a positive function $u = \mathrm{Tr}(w) \in H_0^{\alpha/2}(\Omega)$ is a weak solution of (S_λ) if $u \leq \bar{a}$ and $w \in X_0^\alpha(\mathcal{C}_\Omega)$ weakly solves $(\widehat{S}_\lambda)$.

Finally, set

$$\lambda_{1,\alpha} := \kappa_\alpha \inf_{w\in X_0^\alpha(\mathcal{C}_\Omega)\setminus\{0\}} \frac{\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w(x,y)|^2\,dxdy}{\int_\Omega (\mathrm{Tr}(w)(x))^2 dx},$$

the first positive eigenvalue of the linear problem

$$(L_\lambda) \qquad \begin{cases} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y) = \lambda w & \text{on } \Omega, \end{cases}$$

and $\varphi_\alpha \in X_0^\alpha(\mathcal{C}_\Omega)$ the corresponding eigenfunction. It should be stated that $\lambda_{1,\alpha}$ is none other than the first eigenvalue of the Dirichlet Laplacian on Ω, raised to the power $\alpha/2$.

Finally, we point out that, by using the above notation, more precise information on the interval J_r that appears in assumption ($\mathrm{h}_2^\star$) can be achieved. Precisely, in the sequel we will prove that condition (h_1) is equivalent to the following:

(h_2) *f is subcritical with $\lim_{\xi\to 0^+} h(\xi) > 0$ and for each $r > 0$, there exists $\varepsilon_r > 0$ such that, for every*

$$\lambda \in \left]\frac{\lambda_{1,\alpha}}{2\lim_{\xi\to 0^+} h(\xi)}, \frac{\lambda_{1,\alpha}}{2\lim_{\xi\to 0^+} h(\xi)} + \varepsilon_r\right[,$$

the problem (S_λ) has a weak solution $u_\lambda \in H_0^{\alpha/2}(\Omega)$, satisfying

$$\|u_\lambda\|_{H_0^{\alpha/2}(\Omega)} < r.$$

Of course if $\lim_{\xi\to 0^+} h(\xi) = +\infty$, the above condition assumes the simple form

(h_2') *for each $r>0$, there exists $\varepsilon_r>0$ such that, for every $\lambda \in]0, \varepsilon_r[$, the problem (S_λ) has a weak solution $u_\lambda \in H_0^{\alpha/2}(\Omega)$, satisfying $\|u_\lambda\|_{H_0^{\alpha/2}(\Omega)} < r$.*

3. Proof of the Main Result

3.1. Part I: (h_1) $\Rightarrow$ (h_2). We divide the proof into two steps:

(1) $\bar{a} = +\infty$;
(2) $\bar{a} < +\infty$.

Step (1). We assume that h is nonincreasing in the half-line $]0, +\infty[$. Set

$$\lim_{\xi\to 0^+} h(\xi) = \sigma_1 \in (0, +\infty],$$

and

$$\lim_{\xi\to +\infty} h(\xi) = \sigma_2.$$

Clearly $\sigma_2 \geq 0$ and, since condition (h_1) holds, it follows that $\sigma_1 > \sigma_2$. Therefore, one has

$$I := \left] \frac{\lambda_{1,\alpha}}{2\sigma_1}, \frac{\lambda_{1,\alpha}}{2\sigma_2} \right[\neq \emptyset.$$

Now consider the nonlocal extended problem $(\widehat{S}_\lambda)$ and let us show that, for every $\lambda \in I$, the problem $(\widehat{S}_\lambda)$ has a nontrivial weak solution in the Hilbert space $X_0^\alpha(\mathcal{C}_\Omega)$.

To this end, we first extend f to the whole real axis by putting $f(t) = 0$ for each $t \in]-\infty, 0[$. After that, fix $\lambda \in I$ and define the functional

$$\mathcal{J}_\lambda(w) := \frac{\kappa_\alpha}{2} \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w(x,y)|^2 \, dxdy - \lambda \int_\Omega F(\mathrm{Tr}(w)(x)) dx, \tag{3.1}$$

for every $w \in X_0^\alpha(\mathcal{C}_\Omega)$.

The nontrivial critical points of $\mathcal{J}_\lambda$ are exactly the nontrivial weak solutions of the problem $(\widehat{S}_\lambda)$. Further, by using the Maximum Principle [10], if $w \in X_0^\alpha(\mathcal{C}_\Omega)$ is a weak solution of $(\widehat{S}_\lambda)$, then $\mathrm{Tr}(w) \in H_0^{\alpha/2}(\Omega)$ is strictly positive in Ω and solves problem (S_λ).

Now, since h is nonincreasing in $]0, +\infty[$, it is easy to infer that f has sublinear growth at $+\infty$. Indeed, from the definition of σ_2, we can find two constants $\varrho > \sigma_2$ and $\sigma > 0$ such that

$$F(\xi) \leq \varrho \xi^2 + \sigma, \tag{3.2}$$

for every $\xi \in \mathbb{R}$.

By using the growth condition (3.2) and since $\xi f(\xi) \leq 2F(\xi)$ for all $\xi \in \mathbb{R}$ (bearing in mind that h is nonincreasing in $]0, +\infty[$ and $f \equiv 0$ in $]-\infty, 0[$), we have

$$\xi f(\xi) \leq 2F(\xi) \leq 2\varrho\xi^2 + 2\sigma, \quad \forall\, \xi \geq 0.$$

Thus

$$f(\xi) \leq 2\varrho|\xi| + \frac{2\sigma}{|\xi|}, \quad \forall\, \xi \in \mathbb{R} \setminus \{0\}.$$

Hence, fixing $\xi_0 > 0$, by using the above inequality it follows that

$$f(\xi) \leq 2\varrho|\xi| + \frac{2\sigma}{\xi_0}, \quad \forall\, |\xi| \geq \xi_0.$$

In conclusion, one has

$$f(\xi) \leq 2\varrho|\xi| + \gamma, \quad \forall\, \xi \in \mathbb{R} \tag{3.3}$$

where

$$\gamma := \max_{|\xi|\leq\xi_0} f(\xi) + \frac{2\sigma}{\xi_0}.$$

Since $\lambda \in I$, we clearly have

$$\lambda < \frac{\lambda_{1,\alpha}}{2\sigma_2}.$$

Therefore, we can fix $\varrho \in]\sigma_1, \sigma_2[$ such that

$$\frac{\lambda_{1,\alpha}}{2\sigma_1} < \lambda < \frac{\lambda_{1,\alpha}}{2\varrho}. \tag{3.4}$$

Hence $\mathcal{J}_\lambda$ is well defined and of class C^1 in $X_0^\alpha(\mathcal{C}_\Omega)$. Moreover, the functional $\mathcal{J}_\lambda$ is weakly lower semicontinuous on $X_0^\alpha(\mathcal{C}_\Omega)$. Indeed, the application

$$w \mapsto \int_\Omega F(\mathrm{Tr}(w)(x))dx$$

is continuous in the weak topology of $X_0^\alpha(\mathcal{C}_\Omega)$.

We prove this regularity result as follows. Let $\{w_j\}_{j\in\mathbb{N}}$ be a sequence in $X_0^\alpha(\mathcal{C}_\Omega)$ such that $w_j \rightharpoonup w$ weakly in $X_0^\alpha(\mathcal{C}_\Omega)$. Then, by using Sobolev embedding results and [8, Theorem IV.9], up to a subsequence, $\{w_j\}_{j\in\mathbb{N}}$ converges to w strongly in $L^\nu(\Omega)$ and almost everywhere (a.e.) in Ω as $j \to +\infty$, and it is dominated by some function $h_\nu \in L^\nu(\Omega)$, i.e.,

$$|\mathrm{Tr}(w_j)(x))| \leq h_\nu(x) \quad \text{a.e. } x \in \Omega \text{ for any } j \in \mathbb{N} \tag{3.5}$$

for any $\nu \in [1, 2^*_\alpha)$, where

$$2^*_\alpha := \frac{2n}{n-\alpha}$$

denotes the critical (fractional) Sobolev exponent.

Then, by the continuity of F and (3.2) it follows that

$$F(\mathrm{Tr}(w_j)(x)) \to F(\mathrm{Tr}(w)(x)) \quad \text{a.e. } x \in \Omega$$

as $j \to \infty$ and

$$|F(\mathrm{Tr}(w_j)(x))| \leq (\varrho \mathrm{Tr}(w_j)(x)^2 + \sigma) \leq (\varrho h_2(x)^2 + \sigma) \in L^1(\Omega)$$

a.e. $x \in \Omega$ and for any $j \in \mathbb{N}$.

Hence, by applying the Lebesgue Dominated Convergence Theorem in $L^1(\Omega)$, we have that

$$\int_\Omega F(\mathrm{Tr}(w_j)(x))\,dx \to \int_\Omega F(\mathrm{Tr}(w)(x))\,dx$$

as $j \to \infty$, that is the map

$$w \mapsto \int_\Omega F(\mathrm{Tr}(w)(x))dx$$

is continuous from $X_0^\alpha(\mathcal{C}_\Omega)$ with the weak topology to $\mathbb{R}$.

On the other hand, the map

$$w \mapsto \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} := \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w(x,y)|^2 \, dxdy$$

is lower semicontinuous in the weak topology of $X_0^\alpha(\mathcal{C}_\Omega)$. Hence the functional $\mathcal{J}_\lambda$ is lower semicontinuous in the weak topology of $X_0^\alpha(\mathcal{C}_\Omega)$.

Now, by (3.2), for some positive constant β independent of λ, for instance $\beta := \sigma\lambda_{1,\alpha}/2\varrho$, we have

$$\lambda \int_\Omega F(\mathrm{Tr}(w)(x))dx \leq \lambda\varrho \int_\Omega \mathrm{Tr}(w)(x)^2 dx + \beta \leq \frac{\lambda\varrho}{\lambda_{1,\alpha}} \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \beta$$

and

$$\mathcal{J}_\lambda(w) \geq \Big(\frac{1}{2} - \frac{\lambda\varrho}{\lambda_{1,\alpha}}\Big) \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \beta \tag{3.6}$$

for every $w \in X_0^\alpha(\mathcal{C}_\Omega)$.

In view of (3.4), by using (3.6) it follows that

$$\mathcal{J}_\lambda(w) \to +\infty, \tag{3.7}$$

as $\|w\|_{X_0^\alpha(\mathcal{C}_\Omega)} \to +\infty$.

Now observe that $\mathrm{Tr}(\varphi_{1,\alpha}) \in H_0^{\alpha/2}(\Omega)$ and is positive in Ω. Moreover, one has $\mathrm{Tr}(\varphi_\alpha) \in C^1(\bar{\Omega})$. Let us put

$$\bar{\varphi}_\alpha = \max_{x \in \bar{\Omega}} \mathrm{Tr}(\varphi_\alpha)(x).$$

Observe that, thanks to (h_1), for every $t > 0$ we have

$$h(t\mathrm{Tr}(\varphi_\alpha)(x)) > h(t\bar{\varphi}_\alpha), \quad \forall\, x \in \Omega_0$$

where $\Omega_0 \subseteq \Omega$ is a set of positive Lebesgue measure.

Thus we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\varphi_\alpha) =& \frac{t^2}{2}\|\varphi_\alpha\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda \int_\Omega h(t\mathrm{Tr}(\varphi_\alpha)(x))(t\mathrm{Tr}(\varphi_\alpha)(x))^2 dx \\
<& \frac{t^2}{2}\|\varphi_\alpha\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda t^2 h(t\bar{\varphi}_\alpha) \int_\Omega (\mathrm{Tr}(\varphi_\alpha)(x))^2 dx \qquad (3.8)\\
=& t^2 \|\varphi_\alpha\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} \Big(\frac{1}{2} - \frac{\lambda}{\lambda_{1,\alpha}} h(t\bar{\varphi}_\alpha)\Big).
\end{aligned}$$

Now

$$\lim_{\zeta\to 0^+} h(\zeta) = \lim_{\zeta\to 0^+} \frac{F(\zeta)}{\zeta^2} = \sigma_1,$$

where $\zeta := t^2\bar{\varphi}_\alpha$. If $\sigma_1 < +\infty$, for every $\varepsilon > 0$ there exists a positive constant δ_ε such that, for every $\zeta \in (0, \delta_\varepsilon]$, one has

$$\frac{F(\zeta)}{\zeta^2} < \sigma_1 + \varepsilon.$$

On the other hand, since

$$\frac{\lambda_{1,\alpha}}{2\lambda} < \sigma_1,$$

there exist $\bar{\varepsilon} > 0$ and a positive $\delta_{\bar{\varepsilon}}$ such that

$$\frac{\lambda_{1,\alpha}}{2\lambda} < \sigma_1 - \bar{\varepsilon} < \sigma_1,$$

and

$$\frac{F(\zeta)}{\zeta^2} > \sigma_1 - \bar{\varepsilon} > \frac{\lambda_{1,\alpha}}{2\lambda}$$

for every $\zeta \in (0, \delta_{\bar{\varepsilon}}]$. On the other hand, if $\sigma_1 = +\infty$, one has

$$\frac{F(\zeta)}{\zeta^2} > \frac{\lambda_{1,\alpha}}{2\lambda}$$

for ζ sufficiently small. Hence there exists $\bar{t} > 0$ such that

$$h(\bar{t}\bar{\varphi}_\alpha) := \frac{F(\bar{t}^2\bar{\varphi}_\alpha)}{\bar{t}^2\bar{\varphi}_\alpha} > \frac{\lambda_{1,\alpha}}{2\lambda}.$$

Consequently

$$\inf_{w\in X_0^\alpha(\mathcal{C}_\Omega)} \mathcal{J}_\lambda(w) < 0$$

which, together with the weak continuity of $\mathcal{J}_\lambda$ and the coercivity (3.7), yields the existence of a nontrivial global minimum $w_\lambda \in X_0^\alpha(\mathcal{C}_\Omega)$ for the functional $\mathcal{J}_\lambda$.

As noted above, w_λ is a solution of the problem $(\widehat{S}_\lambda)$. We claim that

$$\lim_{\lambda\to\mu_0^+} \|w_\lambda\|_{X_0^\alpha(\mathcal{C}_\Omega)} = 0 \tag{3.9}$$

where, from now on, for simplicity, we set

$$\mu_0 := \frac{\lambda_{1,\alpha}}{2\sigma_1} \quad (\mu_0 := 0 \text{ if } \sigma_1 = +\infty).$$

This, of course, completes the proof of (h_2) in the case in which h is nonincreasing in $]0,+\infty[$. To prove (3.9), let us take

$$\{\lambda_j\}_{j\in\mathbb{N}} \subset \left]\frac{\lambda_{1,\alpha}}{2\sigma_1}, \frac{\lambda_{1,\alpha}}{2\varrho}\right[,$$

where ϱ is as above, and $\{\lambda_j\}_{j\in\mathbb{N}}$ is a real sequence such that

$$\lim_{j\to\infty} \lambda_j = \frac{\lambda_{1,\alpha}}{2\sigma_1}.$$

For each $j \in \mathbb{N}$ we have $\mathcal{J}_{\lambda_j}(w_{\lambda_j}) < 0$. Hence, in view of (3.6), we can write

$$\|w_{\lambda_j}\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} < \frac{\beta}{(\frac{1}{2} - \frac{\lambda_j \varrho}{\lambda_{1,\alpha}})}.$$

Observing that

$$\lim_{\lambda_j\to\mu_0^+} \frac{\beta}{(\frac{1}{2} - \frac{\lambda_j \varrho}{\lambda_{1,\alpha}})} = \frac{\beta}{(\frac{1}{2} - \frac{\varrho}{2\sigma_1})} \in]0,+\infty[,$$

we infer that the sequence $\{w_{\lambda_j}\}_{j\in\mathbb{N}}$ is bounded in $X_0^\alpha(\mathcal{C}_\Omega)$.

Thus, up to a subsequence, $w_j \rightharpoonup w_\infty$ weakly in $X_0^\alpha(\mathcal{C}_\Omega)$ and

$$\mathrm{Tr}(w_{\lambda_j}) \to w_\infty$$

strongly in $L^\nu(\Omega)$ for every $\nu \in [1, 2^*_\alpha[$.

We claim that $w_\infty = 0$. Indeed, arguing by contradiction, assume that $w_\infty \neq 0$ in $X_0^\alpha(\mathcal{C}_\Omega)$.

Now note that, for each $\varphi \in X_0^\alpha(\mathcal{C}_\Omega)$ and $j \in \mathbb{N}$, one has

$$\begin{aligned} 0 =& \mathcal{J}'_{\lambda_j}(w_{\lambda_j})(\varphi) \\ =& \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w_{\lambda_j}, \nabla\varphi\rangle dxdy - \lambda_j \int_\Omega f(\mathrm{Tr}(w_{\lambda_j})(x))\mathrm{Tr}(\varphi)(x)dx. \end{aligned} \tag{3.10}$$

Assume that σ_1 is finite (the case $\sigma_1 = +\infty$ is similar). Taking into account inequality (3.3), and since $w_j \rightharpoonup w_\infty$ weakly in $X_0^\alpha(\mathcal{C}_\Omega)$ and $\mathrm{Tr}(w_{\lambda_j}) \to w_\infty$ strongly in $L^1(\Omega)$, passing to the limit in (3.10) we have

$$\begin{aligned} 0 =& \mathcal{J}'_{\mu_0}(w_\infty)(\varphi) \\ :=& \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w_\infty, \nabla\varphi\rangle dxdy \quad - \frac{\lambda_{1,\alpha}}{2\sigma_1} \int_\Omega f(\mathrm{Tr}(w_\infty)(x))\mathrm{Tr}(\varphi)(x)dx, \end{aligned} \tag{3.11}$$

for every $\varphi \in X_0^\alpha(\mathcal{C}_\Omega)$.

Therefore, w_∞ is a nontrivial critical point of $\mathcal{J}_{\mu_0}$, that is w_∞ is a weak solution of the nonlocal problem $(\widehat{S}_{\mu_0})$ and again by the Maximum Principle, $\mathrm{Tr}(w_\infty) > 0$ in Ω.

Testing equation (3.11) with $\varphi = w_\infty$, by using inequality $\xi f(\xi) \leq 2F(\xi)$ we obtain

$$\begin{aligned}0 =&\|w_\infty\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{\lambda_{1,\alpha}}{2\sigma_1}\int_\Omega f(\mathrm{Tr}(w_\infty)(x))\mathrm{Tr}(w_\infty)(x)dx\\ \geq&\|w_\infty\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{\lambda_{1,\alpha}}{\sigma_1}\int_\Omega F(\mathrm{Tr}(w_\infty)(x))dx\\ =&\|w_\infty\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{\lambda_{1,\alpha}}{\sigma_1}\int_\Omega h(\mathrm{Tr}(w_\infty)(x))(\mathrm{Tr}(w_\infty)(x))^2dx.\end{aligned} \tag{3.12}$$

Taking into account (h_1), relation (3.12) yields

$$\begin{aligned}0 =&\|w_\infty\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{\lambda_{1,\alpha}}{\sigma_1}\int_\Omega h(\mathrm{Tr}(w_\infty)(x))(\mathrm{Tr}(w_\infty)(x))^2dx\\ >&\|w_\infty\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda_{1,\alpha}\int_\Omega(\mathrm{Tr}(w_\infty)(x))^2dx,\end{aligned}$$

that is

$$\lambda_{1,\alpha} > \kappa_\alpha\frac{\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_\infty(x,y)|^2\,dxdy}{\int_\Omega(\mathrm{Tr}(w_\infty)(x))^2dx},$$

in contradiction to the definition of $\lambda_{1,\alpha}$. Therefore, we must have $w_\infty = 0$.

Choosing $\varphi = w_{\lambda_j}$ in (3.11), we have

$$\|w_{\lambda_j}\|^2 = \lambda_j\int_\Omega f(\mathrm{Tr}(w_{\lambda_j})(x))\mathrm{Tr}(w_{\lambda_j})(x)dx$$

for each $j \in \mathbb{N}$.

Now, note that, by (3.3) and $\mathrm{Tr}(w_{\lambda_j}) \to 0$ strongly in $L^2(\Omega)$, the right-hand side in the previous equality converges to 0 as $j \to +\infty$.

Thus

$$\lim_{j\to\infty}\|w_{\lambda_j}\|_{X_0^\alpha(\mathcal{C}_\Omega)} = 0,$$

and the limit (3.9) is proved.

Finally, $u_\lambda := \mathrm{Tr}(w_\lambda) \in H_0^{\alpha/2}(\Omega)$ solves (S_λ) and, by the following trace inequality

$$\|u_\lambda\|_{H_0^{\alpha/2}(\Omega)} \leq \|w_\lambda\|_{X_0^\alpha(\mathcal{C}_\Omega)},$$

relation (3.9) yields

$$\lim_{\lambda\to\mu_0^+}\|u_\lambda\|_{H_0^{\alpha/2}(\Omega)} = 0. \tag{3.13}$$

Step (2). Let us assume that $\bar{a} < +\infty$ and observe that $h'(\bar{a}) = 0$. Thus the function $h_0 :]0, +\infty[\to \mathbb{R}$ given by

$$h_0(\xi) := \begin{cases} h(\xi) & \text{if } \xi \in]0, \bar{a}], = \\ h(\bar{a}) & \text{if } \xi \in]\bar{a}, +\infty[\end{cases}$$

is of class C^1 in $]0, +\infty[$.

Denote

$$F_0(\xi) := \begin{cases} 0 & \text{if } \xi \in] -\infty, 0], \\ h_0(\xi)\xi^2 & \text{if } \xi \in]0, +\infty[. \end{cases}$$

Then F_0 is a C^1 function and $F_0(\xi) = F(\xi)$ for every $\xi \in] - \infty, \bar{a}]$. Of course F_0 satisfies condition (h_1) and the function

$$\xi \mapsto \frac{F_0(\xi)}{\xi^2}$$

is nonincreasing in $]0, +\infty[$.

Now consider the problem

$$(S_\lambda^0) \qquad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda f_0(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$f_0(t) := F_0'(t) = \begin{cases} 0 & \text{if } t \in] -\infty, 0], \\ f(t) & \text{if } t \in]0, \bar{a}], \\ 2h_0(\bar{a})t & \text{if } t \in]\bar{a}, +\infty[. \end{cases}$$

By using Step (1), for any $r > 0$, we can find an open interval

$$J :=]\mu_0, \mu_0 + \varepsilon_0[\quad (\varepsilon_0 > 0)$$

such that, for every $\lambda \in J$, there exists a weak solution $u_\lambda \in H_0^{\alpha/2}(\Omega)$ of (S_λ^0) satisfying $\|u_\lambda\|_{H_0^{\alpha/2}(\Omega)} < r$. Moreover, condition (3.13) holds.

Fix $q > \frac{N}{\alpha}$. We note that, by [18, Theorem 4.1 and Remark 4.1], there exists a positive constant M_q such that for every $g \in L^q(\Omega)$ and every solution $u \in H_0^{\alpha/2}(\Omega)$ of the problem

$$(S_\lambda^g) \qquad \begin{cases} (-\Delta)^{\alpha/2} u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

one has $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq M_q \|g\|_q. \tag{3.14}$$

Now the function f_0 has sublinear growth at ∞ and, since $f_0(0) = 0$, we can find a positive constant ϑ such that

$$f_0(t) \leq \vartheta |t| + \frac{(\mu_0 + \varepsilon_0)^{-1}\bar{a}}{2M_q |\Omega|^{1/q}}, \tag{3.15}$$

for every $t \in \mathbb{R}$.

Therefore, let $w_\lambda \in X_0^\alpha(\mathcal{C}_\Omega)$ such that $\mathrm{Tr}(w_\lambda) = u_\lambda \in H_0^{\alpha/2}(\Omega)$. Since

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla w_\lambda, \nabla \varphi \rangle dxdy = \lambda \int_\Omega f_0(u_\lambda(x)) \mathrm{Tr}(\varphi)(x) dx, \tag{3.16}$$

for every $\varphi \in X_0^\alpha(\mathcal{C}_\Omega)$, one has that $u_\lambda \in H_0^{\alpha/2}(\Omega)$ is a weak solution of the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = \lambda f_0(u_\lambda(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By (3.14) and (3.15) we have

$$\|u_\lambda\|_\infty \leq \lambda \vartheta M_q \|u_\lambda\|_q + \lambda \lambda_0^{-1} \frac{\bar{a}}{2} \leq \lambda_0 \vartheta M_q \|u_\lambda\|_q + \frac{\bar{a}}{2}, \tag{3.17}$$

for every $\lambda \in J$.

Fix $\mu \in]0,1[$ such that $q\mu < 2^*_\alpha$ and let us prove that

$$\|u_\lambda\|_q \leq \|u_\lambda\|_\infty^{1-\mu} \|u_\lambda\|_{\mu q}^\mu. \tag{3.18}$$

Indeed

$$\begin{aligned} \|u_\lambda\|_q := & \left(\int_\Omega |u_\lambda(x)|^q dx \right)^{1/q} \\ = & \left(\int_\Omega |u_\lambda(x)|^{q(1-\mu)} |u_\lambda(x)|^{q\mu} dx \right)^{\mu/(q\mu)} \\ \leq & \left(\int_\Omega \|u_\lambda\|_\infty^{q(1-\mu)} |u_\lambda(x)|^{q\mu} dx \right)^{\mu/(q\mu)} \\ = & \|u_\lambda\|_\infty^{1-\mu} \|u_\lambda\|_{\mu q}^\mu. \end{aligned}$$

By (3.18) and using the Sobolev embedding $H_0^{\alpha/2}(\Omega) \hookrightarrow L^{q\mu}(\Omega)$, one has

$$\|u_\lambda\|_q \leq \|u_\lambda\|_\infty^{1-\mu} \|u_\lambda\|_{\mu q}^\mu \leq c_{q\mu} \|u_\lambda\|_\infty^{1-\mu} \|u_\lambda\|_{H_0^{\alpha/2}(\Omega)}^\mu,$$

for some positive constant $c_{q\mu}$.

Using this inequality in (3.17), we obtain

$$\|u_\lambda\|_\infty \leq k\|u_\lambda\|_\infty^{1-\mu}\|u_\lambda\|^\mu_{H_0^{\alpha/2}(\Omega)} + \frac{\bar{a}}{2} \tag{3.19}$$

for some positive constant k.

From (3.13) and (3.19), we infer that

$$\lim_{\lambda\to\mu_0^+} \|u_\lambda\|_\infty \leq \frac{\bar{a}}{2}.$$

This means that there exists some $\varepsilon_1 \in]0, \varepsilon_0[$ such that $u_\lambda(x) \leq \bar{a}$ for a.e. $x \in \Omega$ and every

$$\lambda \in J' := \,]\mu_0, \mu_0 + \varepsilon_1[\,.$$

In conclusion, for every $\lambda \in J'$, $u_\lambda \in H_0^{\alpha/2}(\Omega)$ is a weak solution of the problem (S_λ) and satisfies

$$\|u_\lambda\|_{H_0^{\alpha/2}(\Omega)} < r$$

observing that $J' \subset J$. The proof is complete.

3.2. Part II: $(\mathrm{h}_1) \Leftarrow (\mathrm{h}_2)$. In order to prove our result we argue by contradiction. Hence, assume that there exist two positive constant b and c such that

$$F(\xi) = c\xi^2$$

for every $\xi \in [0, b]$. Consequently

$$f(\xi) = 2c\xi$$

for every $\xi \in [0, b]$.

Let $\{r_j\}_{j\in\mathbb{N}} \subset]0, +\infty[$ be a sequence such that $\lim_{j\to\infty} r_j = 0$. Then, for every $j \in \mathbb{N}$, there exists $\varepsilon_j > 0$ such that, for every

$$\lambda \in J_j :=]\mu_0, \mu_0 + \varepsilon_j[,$$

the problem

$$(S_\lambda^*) \qquad \begin{cases} (-\Delta)^{\alpha/2}u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive weak solution $u_{\lambda,j} \in H_0^{\alpha/2}(\Omega)$ satisfying $\|u_{\lambda,j}\|_{H_0^{\alpha/2}(\Omega)} < r_j$.

In particular, we have

$$\lim_{j\to\infty} \sup_{\lambda\in J_j} \|u_{\lambda,j}\|_{H_0^{\alpha/2}(\Omega)} = 0. \tag{3.20}$$

Now, arguing as in the previous subsection, we can find a positive constant k (independent of j and λ) and μ sufficiently small such that

$$\|u_{\lambda,j}\|_\infty \leq k\|u_{\lambda,j}\|_\infty^{1-\mu}\|u_{\lambda,j}\|^{\mu}_{H_0^{\alpha/2}(\Omega)} + \frac{b}{2}$$

for every $\lambda \in J_j$ and $j \in \mathbb{N}$.

From (3.20) and the previous inequality, we infer that

$$\lim_{j\to\infty} \sup_{\lambda\in J_j} \|u_{\lambda,j}\|_\infty \leq \frac{b}{2}.$$

In particular, we can fix $j_0 \in \mathbb{N}$ such that

$$\|u_{\lambda,j_0}\|_\infty \leq b,$$

for every $\lambda \in J_{j_0}$.

Consequently, for every

$$\lambda \in \,]\mu_0, \mu_0 + \varepsilon_{j_0}[\,,$$

the problem (S^*_λ) admits a weak solution $u_{\lambda,j_0} \in H_0^{\alpha/2}(\Omega)$.

This is absurd, since the restriction of problem (S^*_λ) has a solution only for countably many positive values of the parameter λ.

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