ON THE GINZBURG-LANDAU ENERGY WITH WEIGHT

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1. Introduction

In a recent book [BBH4], F. Bethuel, H. Brezis and F. Hélein studied the vortices related to the Ginzburg-Landau functional. Similar functionals appear in the study of problems occuring in superconductivity or the theory of superfluids.

In [BBH4], F. Bethuel, H. Brezis and F. Hélein have studied the behavior as $\varepsilon \to 0$ of minimizers u_ε of the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G (1 - \mid u \mid^2)^2$$

in the class of functions

$$H^1_g(G) = \{u \in H^1(G; \mathbf{R}^2);\ u = g \text{ on } \partial G\}\,,$$

where:

a) $\varepsilon > 0$ is a (small) parameter.

b) G is a smooth, simply connected, starshaped domain in $\mathbf{R}^2$.

c) $g : \partial G \to S^1$ is a smooth data with a topological degree $d > 0$.

They obtained the convergence of (u_{ε_n}) in certain topologies to $u_\star$. The function $u_\star$ is a *harmonic map* from $G \setminus \{a_1, ..., a_d\}$ to S^1, and is *canonical,* in the sense that

$$\frac{\partial}{\partial x_1}\left(u_\star \wedge \frac{\partial u_\star}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(u_\star \wedge \frac{\partial u_\star}{\partial x_2}\right) = 0 \quad \text{in} \ \ \mathcal{D}'(G)\,.$$

Recall (see [BBH4]) that a canonical harmonic map $u_\star$ with values in S^1 and singularities $b_1, ..., b_k$ of degrees $d_1, ..., d_k$ may be expressed as

$$u_\star(x) = \left(\frac{x - b_1}{\mid x - b_1 \mid}\right)^{d_1} \cdots \left(\frac{x - b_k}{\mid x - b_k \mid}\right)^{d_k} e^{i\varphi_0(x)}\,,$$

with

$$\Delta\varphi_0 = 0 \qquad \text{in} \ \ G\,.$$

They also defined the notion of renormalized energy $W(b, \overline{d}, g)$ associated to a given configuration $b = (b_1, ..., b_k)$ of distinct points with associated degrees $\overline{d} = (d_1, ..., d_k)$. For simplicity we set $W(b) = W(b, \overline{d}, g)$ when $k = d$ and all the degrees equal $+1$. The expression of the renormalized energy W is given by

$$W(b, \overline{d}, g) = -\pi \sum_{i \neq j} d_i d_j \ \log \mid b_i - b_j \mid + \frac{1}{2}\int_{\partial G} \Phi_0(g \wedge g_\tau) - \pi \sum_{j=1}^{k} d_j R_0(b_j)\ ,$$

where Φ_0 is the unique solution of

(1)
$$\begin{cases} \Delta\Phi_0 = 2\pi \displaystyle\sum_{j=1}^{k} d_j \delta_{b_j}\ , & \text{in } G \\ \dfrac{\partial\Phi_0}{\partial\nu} = g \wedge g_\tau\ , & \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 & \end{cases}$$

and

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^{k} d_j \ \log \mid x - b_j \mid \ .$$

The functional W is also related to the asymptotic behavior of minimizers u_ε as follows:

(2)
$$\lim_{\varepsilon \to 0} \{E_\varepsilon(u_\varepsilon) - \pi d \mid \log \varepsilon \mid\} = \min_{b \in G^d} W(b) + d\gamma\,,$$

where γ is an universal constant, $k = d$, $d_i = +1$ for all i and the configuration $a = (a_1, \cdots, a_d)$ achieves the minimum of W.

We study in this paper a similar problem, related to the Ginzburg-Landau energy with the weight w, that is

$$E_\varepsilon^w(u) = \frac{1}{2} \int_G | \nabla u |^2 + \frac{1}{4\varepsilon^2} \int_G (1 - | u |^2)^2 w \,,$$

with $w \in C^1(\overline{G})$, $w > 0$ in $\overline{G}$. Throughout, u_ε will denote a minimizer of E_ε^w. We mention that u_ε verifies the Ginzburg-Landau equation with weight

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - | u_\varepsilon |^2) w & \text{in } G \\ u_\varepsilon = g \quad \text{on } \partial G \,. \end{cases} \tag{3}$$

Our work is motivated by the Open Problem 2, p. 137 in [BBH4]. We are concerned in this paper with the study of the convergence of minimizers, as well as with the corresponding expression of the renormalized energy. We prove that the behavior of minimizers is of the same type as in the case $w \equiv 1$, the change appearing in the expression of the renormalized energy and, consequently, in the location of singularities of the limit $u_\star$ of u_{ε_n}. In our proof we borrow some of the ideas from Chapter VIII in [BBH4], without relying on the vanishing gradient property that is used there. We then prove a corresponding vanishing gradient property for the configuration of singularities obtained at the limit. In the last section we obtain the new renormalized energy by a variant of the "shrinking holes" method which was developed in [BBH4], Chapter I.

2. The renormalized energy

Theorem 1. *There is a sequence $\varepsilon_n \to 0$ and exactly d points $a_1, ..., a_d$ in G such that*

$$u_{\varepsilon_n} \to u_\star \quad \text{in } H^1_{\text{loc}}(\overline{G} \setminus \{a_1, ..., a_d\}; \mathbf{R}^2) \,,$$

where $u_\star$ is the canonical harmonic map associated to the singularities $a_1, ..., a_d$ of degrees +1 and to the boundary data g.

Moreover, $a = (a_1, \cdots, a_d)$ minimizes the functional

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2} \sum_{j=1}^{d} \log w(b_j) \tag{4}$$

among all configurations $b = (b_1, ..., b_d)$ of d distinct points in G.

In addition, the following holds:

$$\lim_{n\to\infty} \{E^w_{\varepsilon_n}(u_{\varepsilon_n}) - \pi d \mid \log \varepsilon_n \mid\} = W(a) + \frac{\pi}{2}\sum_{j=1}^{d} \log w(a_j) + d\gamma\,, \tag{5}$$

where γ is some universal constant, the same as in (2).

Remark. The functional $\widetilde{W}$ may be regarded as the renormalized energy corresponding to the energy E^w_ε.

Before giving the proof, we shall make some useful notations: given the constants $c, \varepsilon, \eta > 0$, set

$$I^c(\varepsilon,\eta) = \min\{E^c_\varepsilon(u);\; u \in H^1(B_\eta;\mathbf{R}^2) \text{ and } u(x) = \frac{x}{\eta} \text{ on } \partial B_\eta\}\,.$$

Here $B_\eta = B(0,\eta) \subset \mathbf{R}^2$.

For $x \in G$, denote

$$M_\eta(x) = \sup_{B(x,\eta)\cap\overline{G}} w \qquad \text{and} \qquad m_\eta(x) = \inf_{B(x,\eta)\cap\overline{G}} w\,.$$

Note that

$$I^c(\varepsilon,\eta) = I^c(\frac{\varepsilon}{\eta},1) = I^1(\frac{\varepsilon}{\eta\sqrt{c}},1)$$

and

$$I^{c_1}(\varepsilon,\eta) \le I^{c_2}(\varepsilon,\eta)\,,$$

provided $c_1 \le c_2$.

We shall drop the superscript c if it equals 1.

Proof of Theorem 1. The first part of the conclusion may be obtained by adapting the techniques developed in [BBH1], [BBH2], [BBH3], [BBH4] (see also [S]). We shall point out only the main steps that are necessary to prove the convergence:

a) Using the techniques from [S] we find a sequence $\varepsilon_n \to 0$ such that, for each n,

$$\frac{1}{\varepsilon_n^2}\int_G (1- \mid u_{\varepsilon_n} \mid^2)^2 w \le C\,. \tag{6}$$

b) Using the methods developed in [BBH4], Chapters 3-5, we determine the "bad" disks, as well as the fact that their number is uniformly bounded. These techniques allow us to prove the convergence of (u_{ε_n}) weakly in $H^1_{\text{loc}}(G \setminus \{a_1,...,a_k\};\mathbf{R}^2)$ to $u_\star$, which is the canonical harmonic map associated to $a_1,...,a_k$ with some degrees $d_1,...,d_k$ and to the given boundary data.

c) The strong convergence of (u_{ε_n}) in $H^1_{\rm loc}(\overline{G}\setminus\{a_1,...,a_k\};\mathbf{R}^2)$ follows as in [BBH4], Theorem VI.1 with the techniques from [BBH3], Theorem 2, Step 1. Now the local convergence of (u_{ε_n}) in $G\setminus\{a_1,\cdots,a_k\}$ in stronger topologies, say C^2, may be easily obtained by a bootstrap argument in (3). This implies that

$$\frac{1-\mid u_{\varepsilon_n}\mid^2}{\varepsilon_n^2}w\to\mid\nabla u_\star\mid^2\,, \tag{7}$$

uniformly on every compact subset of $G\setminus\{a_1,...,a_k\}$.

d) For each $1\leq j\leq k$, deg $(u_\star,a_j)\neq 0$. Indeed, if not, then as in Step 1 of Theorem 2 [BBH3], the H^1-convergence is extended up to a_j, which becomes a "removable singularity".

e) The fact that all degrees equal $+1$ may be deduced as in Theorem VI.2, [BBH4].

f) The points $a_1,...,a_d$ lie in G. The proof of this fact is similar to the corresponding result in [BBH4].

The proof of the second part of the theorem is divided into 3 steps:

Step 1. *An upper bound for* $E^w_\varepsilon(u_\varepsilon)$.

We shall prove that if $b=(b_j)$ is an arbitrary configuration of d distinct points in G, then there exists $\eta_0>0$ such that, for each $\eta<\eta_0$,

$$E^w_\varepsilon(u_\varepsilon)\leq\sum_{j=1}^d I(\frac{\varepsilon}{\eta\sqrt{M_\eta(b_j)}},1)+W(b)+\pi d\log\frac{1}{\eta}+O(\eta)\quad\text{as }\eta\to 0\,, \tag{8}$$

for $\varepsilon>0$ small enough. Here $O(\eta)$ is a quantity which is bounded by $C\eta$, with C independent of $\eta>0$ small enough.

The idea is to construct a suitable comparison function v_ε. Let $\eta<\eta_0$, where $\eta_0=\max_{j,k}\{\text{dist}\,(b_j,\partial G),\mid b_j-b_k\mid\}$. Applying Theorem I.9 in [BBH4] to the configuration b, we find $\widetilde{u}:G_\eta:=G\setminus\bigcup_{j=1}^d\overline{B(b_j,\eta)}\to S^1$ with $\widetilde{u}=g$ on ∂G and $\alpha_j\in$, $\mid\alpha_j\mid=1$ such that

$$\widetilde{u}=\alpha_j\frac{z-b_j}{\mid z-b_j\mid}\quad\text{on }\ \partial B(b_j,\eta)$$

and

$$\frac{1}{2}\int_{G_\eta}\mid\nabla\widetilde{u}\mid^2=\pi d\log\frac{1}{\eta}+W(b)+O(\eta)\,,\quad\text{as }\eta\to 0\,. \tag{9}$$

We define v_ε as follows: let $v_\varepsilon=\widetilde{u}$ on G_η and, in $B(b_j,\eta)$, let v_ε be a minimizer of E^w_ε on $H^1_h(B(b_j,\eta);\mathbf{R}^2)$, where $h=\widetilde{u}\mid_{\partial B(b_j,\eta)}$. We have the following estimate

$$E^w_\varepsilon(v_\varepsilon\mid_{B(b_j,\eta)})\leq I^{M_\eta(b_j)}(\varepsilon,\eta)=I(\frac{\varepsilon}{\eta\sqrt{M_\eta(b_j)}},1)\,. \tag{10}$$

The desired conclusion follows from (9),(10) and $E^w_\varepsilon(u_\varepsilon) \le E^w_\varepsilon(v_\varepsilon)$.

Step 2. *A lower bound for* $E^w_{\varepsilon_n}(u_{\varepsilon_n})$.

We shall prove that, if $a_1, ..., a_d$ are the singularities of $u_\star$, then given any $\eta > 0$, there is $N_0 = N_0(\eta) \in$ such that, for each $n \ge N_0$,

$$E^w_{\varepsilon_n}(u_{\varepsilon_n}) \ge \sum_{j=1}^{d} I\left(\frac{\varepsilon_n}{\alpha\eta\sqrt{m_{\alpha\eta}(a_j)}}, 1\right) + \pi d \log\frac{1}{\eta} + W(a) + O(\eta)\,. \tag{11}$$

Here $\alpha = 1 + \eta$ and $O(\eta)$ is a quantity with the same behavior as in (8).

Indeed, for a fixed a_j, supposed to be 0, $u_\star$ may be written

$$u_\star = e^{i(\psi+\theta)}\,,$$

where ψ is a smooth harmonic function in a neighbourhood of 0. We may assume, without loss of generality, that $\psi(0) = 0$.

In the annulus $A_{\eta,\alpha\eta} = \{x \in \mathbf{R}^2\ ;\ \eta \le\mid x \mid\le \alpha\eta\}$ the function u_{ε_n} may be written, for n large enough, as

$$u_{\varepsilon_n} = \rho_n e^{i(\psi_n+\theta)}\,,$$

where ψ_n is a smooth function and $0 < \rho_n \le 1$. Define, for $\eta \le r \le \alpha\eta$, the interpolation function

$$v_n(r,\theta) = \frac{r - \eta + \rho_n(\eta,\theta)(\alpha\eta - r)}{\eta(\alpha-1)} \cdot e^{i[\frac{\alpha\eta-r}{\eta(\alpha-1)}\psi_n(\eta,\theta)+\theta]}\,.$$

We have

$$\frac{1}{\varepsilon_n^2}\int_{A_{\eta,\alpha\eta}} (1-\mid v_n\mid^2)^2 w \le \frac{\|w\|_{L^\infty}}{\varepsilon_n^2}\cdot\int_\eta^{\alpha\eta}\frac{r}{\eta}\,(\int_{\partial B_\eta}(1-\mid u_n\mid^2)^2 d\sigma)dr =$$

$$= \|w\|_{L^\infty}\cdot\frac{\alpha+1}{2}\eta^2\int_{\partial B_\eta}\frac{(1-\mid u_n\mid^2)^2}{\varepsilon_n^2}d\sigma \to 0\,,\quad \text{as } n\to\infty\,.$$

This convergence is motivated by (7). We also observe that the convergence of (u_{ε_n}) in $H^1_{\rm loc}(G\setminus\{a_1,...,a_d\};\mathbf{R}^2)$ implies

$$\int_{A_{\eta,\alpha\eta}}\mid\nabla v_n\mid^2 \to \int_{A_{\eta,\alpha\eta}}\mid\nabla v\mid^2\,,\quad \text{as } \eta\to 0\,, \tag{12}$$

where

$$v(\eta,\theta) = e^{i[\frac{\alpha\eta-r}{\eta(\alpha-1)}\psi(\eta,\theta)+\theta]}\,.$$

Thus, we may write, for $n \ge N_1$,

$$E^w_{\varepsilon_n}(v_n\mid_{A_{\eta,\alpha\eta}}) = \frac{1}{2}\int_{A_{\eta,\alpha\eta}}\mid\nabla v\mid^2 + o(1)\,.$$

We prove in what follows that

$$\int_{A_{\eta,\alpha\eta}} | \nabla v |^2 = O(\eta)\,. \tag{13}$$

Indeed, since

$$| \nabla v |^2 = \frac{\psi^2(\eta,\theta)}{\eta^2(\alpha-1)^2} + \frac{1}{r^2}\left[\frac{\alpha\eta - r}{\eta(\alpha-1)}\psi_\theta(\eta,\theta)+1\right]^2$$

and

$$\psi(r,\theta) \le Cr\,, \quad | \psi_r(r,\theta) | \le C\,, \quad | \psi_\theta(r,\theta) | \le Cr\,,$$

the desired conclusion follows by a straightforward calculation.

We obtain

$$E^w_{\varepsilon_n}(v_{\varepsilon_n|B_{(a_j,\eta)}}) \ge I^{m_{\alpha\eta}(a_j)}(\varepsilon_n,\alpha\eta) + O(\eta)\,. \tag{14}$$

On the other hand, by the convergence of (u_{ε_n}) in $H^1_{\rm loc}(G \setminus \{a_1,...,a_d\};\mathbf{R}^2)$ it follows that

$$E^w_{\varepsilon_n}(u_{\varepsilon_n}\, |_{G_\eta}) = \int_{G_\eta} | \nabla u_\star |^2 + O(\eta)\,, \tag{15}$$

for ε_n sufficiently small.

Taking into account (12)-(15) we obtain the desired result.

<u>Step 3</u>. *The final conclusion.*

It follows from [BBH4], Chapter IX that

$$I(\varepsilon,\eta) = \pi \,|\log\frac{\varepsilon}{\eta}\,| + \gamma + o(1) \quad \text{as} \quad \frac{\varepsilon}{\eta} \to 0\,, \tag{16}$$

where the constant γ represents the minimum of the renormalized energy corresponding to the boundary data x in B_1.

From (8) and (11) we obtain

$$W(b) + \frac{\pi}{2}\sum_{j=1}^d \log M_\eta(b_j) - \pi d \log\varepsilon_n + d\gamma + o(1) \ge \tag{17}$$

$$\ge W(a) + \frac{\pi}{2}\sum_{i=1}^d \log m_\eta(a_i) - \pi d\log\varepsilon_n + \pi d\,\log\frac{1}{\eta} - \pi d\,\log\frac{1}{\eta} + d\gamma + o(1)\,,$$

where $o(1)$ stands for a quantity which goes to 0 as $\varepsilon_n \to 0$ for fixed η. Adding $\pi d\,\log\varepsilon_n$ and passing to the limit firstly as $n \to \infty$ and then as $\eta \to 0$, we obtain that $a = (a_1,...,a_d)$ is a global minimum point of $\widetilde{W}$. We also deduce that

$$\lim_{n\to\infty}\{E^w_{\varepsilon_n}(u_{\varepsilon_n}) - \pi d\,|\log\varepsilon_n\,|\} = W(a) + \frac{\pi}{2}\sum_{j=1}^d \log w(a_j) + d\gamma\,.$$

We now generalize another result from [BBH4] concerning the behavior of u_ε.

Theorem 2. *Set*

$$W_n = \frac{1}{4\varepsilon_n^2}\,(1- \mid u_{\varepsilon_n} \mid^2)^2 w\,.$$

Then (W_n) *converges in the weak* $\star$ *topology of* $C(\overline{G})$ *to*

$$W_\star = \frac{\pi}{2}\sum_{j=1}^{d}\delta_{a_j}\,.$$

Proof. The boundedness of (W_n) in $L^1(G)$ follows directly from (6). Hence (up to a subsequence), W_n converges in the sense of measures of $\overline{G}$ to some $W_\star$. With the same techniques as those developed in [BBH3] (Theorem 2) or [BBH4] (Theorem X.3) we can obtain that, for any compact subset K of $\overline{G}\setminus\bigcup_{j=1}^{d}\{a_j\}$,

$$\frac{1}{\varepsilon_n^2}\,\|1- \mid u_{\varepsilon_n} \mid^2\|_{L^\infty(K)} \le C_K\ .$$

Hence

$$\text{supp } W_\star \subset \bigcup_{j=1}^{d}\{a_j\}\,.$$

Therefore

$$W_\star = \sum_{j=1}^{d} m_j\delta_{a_j} \qquad \text{with} \ \ m_j \in \mathbf{R}\,.$$

We now determine m_j using the same methods as in [BBH4]. Fix one of the points a_j (supposed to be 0) and consider $B_R = B(0,R)$ for R small enough so that B_R contains no other point a_i $(i \neq j)$. As in the proof of the Pohozaev identity, multiplying the Ginzburg-Landau equation (3) by $x\cdot\nabla u_\varepsilon$ and integrating on B_R we obtain

$$(18)\qquad \frac{R}{2}\int_{\partial B_R}\mid\frac{\partial u_\varepsilon}{\partial\nu}\mid^2 + \frac{1}{2\varepsilon^2}\int_{B_R}(1-\mid u_\varepsilon\mid^2)^2 w + \frac{1}{4\varepsilon^2}\int_{B_R}(1-\mid u_\varepsilon\mid^2)^2(\nabla w\cdot x) =$$

$$= \frac{R}{2}\int_{\partial B_R}\mid\frac{\partial u_\varepsilon}{\partial\tau}\mid^2 + \frac{R}{4\varepsilon^2}\int_{\partial B_R}(1-\mid u_\varepsilon\mid^2)^2 w\,.$$

Passing to the limit in (18) as $\varepsilon\to 0$ and using the convergence of W_n we find

$$(19)\qquad \frac{R}{2}\int_{\partial B_R}\mid\frac{\partial u_\star}{\partial\nu}\mid^2 + 2m_j = \frac{R}{2}\int_{\partial B_R}\mid\frac{\partial u_\star}{\partial\tau}\mid^2\ .$$

Using now the expression of $u_\star$ around a singularity we deduce that, on ∂B_R,

$$(20)\qquad \left|\frac{\partial u_\star}{\partial \nu}\right|^2 = \left|\frac{\partial \theta}{\partial \nu} + \frac{\partial \psi}{\partial \nu}\right|^2 = \left|\frac{\partial \psi}{\partial \nu}\right|^2 .$$

$$(21)\qquad \left|\frac{\partial u_\star}{\partial \tau}\right|^2 = \left|\frac{\partial \theta}{\partial \tau} + \frac{\partial \psi}{\partial \tau}\right|^2 = \frac{1}{R^2} + \frac{2}{R}\frac{\partial \psi}{\partial \tau} + \left|\frac{\partial \psi}{\partial \tau}\right|^2 .$$

Inserting (20) and (21) into (19) we obtain

$$(22)\qquad \frac{R}{2}\int_{\partial B_R} \left|\frac{\partial \psi}{\partial \nu}\right|^2 + 2m_j = \pi + \frac{R}{2}\int_{\partial B_R} \left|\frac{\partial \psi}{\partial \tau}\right|^2 .$$

On the other hand, multiplying $\Delta \psi = 0$ by $x \cdot \nabla \psi$ and integrating on B_R we find

$$(23)\qquad \frac{R}{2}\int_{\partial B_R} \left|\frac{\partial \psi}{\partial \nu}\right|^2 = \frac{R}{2}\int_{\partial B_R} \left|\frac{\partial \psi}{\partial \tau}\right|^2 .$$

Thus, from (17) and (18) we obtain

$$m_j = \frac{\pi}{2} .$$

□

3. The vanishing gradient property of the renormalized energy with weight

The expression of the renormalized energy $\widetilde{W}$ allows us, by using the results obtained in [BBH4], to give an expression of the vanishing gradient property in the case of a weight.

From (4) it follows that

$$(24)\qquad D\widetilde{W}(b_1, ..., b_d) = DW(b_1, ..., b_d) + \frac{\pi}{2}\left(\frac{\nabla w(b_1)}{w(b_1)}, ..., \frac{\nabla w(b_d)}{w(b_d)}\right),$$

for each configuration $b = (b_1, ..., b_d) \in G^d$.

Recall now Theorem VIII.3 in [BBH4], which gives the expression of the differential of W in an arbitrary configuration of distinct points $b = (b_1, ..., b_d) \in G^d$:

$$(25)\qquad DW(b) = -2\pi\left[\left(\frac{\partial S_1}{\partial x_1}(b_1), \frac{\partial S_1}{\partial x_2}(b_1)\right), ..., \left(\frac{\partial S_d}{\partial x_1}(b_d), \frac{\partial S_d}{\partial x_2}(b_d)\right)\right] =$$

$$= 2\pi\left[\left(-\frac{\partial H_1}{\partial x_2}(b_1), \frac{\partial H_1}{\partial x_1}(b_1)\right), ..., \left(-\frac{\partial H_d}{\partial x_2}(b_d), \frac{\partial H_d}{\partial x_1}(b_d)\right)\right].$$

Here $S_j(x) = \Phi_0(x) - \log | x - b_j |$ in G and Φ_0 the unique solution of

$$\begin{cases} \Delta \Phi_0 = 2\pi \sum_{j=1}^{d} \delta_{b_j} \,, \text{in } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau \,, \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0. \end{cases}$$

The function H_j is harmonic around b_j and is related to $u_\star$ by

$$u_\star(x) = \frac{x - b_j}{| x - b_j |} e^{iH_j(x)} \,, \quad \text{near } b_j \,.$$

Let

$$R_0(x) = S_j(x) - \sum_{i \neq j} \log | x - b_i | \,.$$

Our variant of the vanishing gradient property in [BBH4] (Corollary VIII.1) is:

Theorem 3. *The following properties are equivalent:*
i) $a = (a_1, ..., a_d)$ *is a critical point of the renormalized energy* $\widetilde{W}$.
ii) $\nabla S_j(a_j) = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}$, *for each* j.
iii) $\nabla H_j(a_j) = \frac{1}{4w(a_j)} \left(-\frac{\partial w}{\partial x_2}(a_j), \frac{\partial w}{\partial x_1}(a_j) \right)$, *for each* j.
iv) $\nabla R_0(a_j) + \sum_{i \neq j} \frac{a_j - a_i}{| a_j - a_i |^2} = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}$, *for each* j.

The proof follows by the above considerations and the fact that, for each j,

$$\nabla R_0(x) = \nabla S_j(x) - \sum_{i \neq j} \frac{x - a_i}{| x - a_i |^2} \,.$$

4. Shrinking holes and the renormalized energy with weight

As in [BBH4], Chapter I.4, we may define the renormalized energy by considering a suitable variational problem in a domain with "shrinking holes".

Let, as above, G be a smooth, bounded and simply connected domain in $\mathbf{R}^2$ and let $b_1, ..., b_k$ be distinct points in G. Fix $d_1, ..., d_k \in$ and a smooth data $g : \partial G \to S^1$ of degree $d = d_1 + ... + d_k$. For each $\eta > 0$ small enough, define

$$G_\eta^w = G \setminus \bigcup_{j=1}^{k} \overline{\omega_{j,\eta}} \,,$$

where

$$\omega_{j,\eta} = B\left(b_j, \frac{\eta}{\sqrt{w(b_j)}}\right).$$

Set

$$\mathcal{E}_\eta^w = \{v \in H^1(G_\eta^w; S^1)\,;\ \deg(v, \partial\omega_{j,\eta}) = d_j \quad \text{and} \quad v = g \quad \text{on} \quad \partial G\}\,.$$

We consider the minimization problem

$$\min_{u \in \mathcal{E}_\eta^w} \int_{G_\eta^w} |\nabla u|^2 \ . \tag{26}$$

The following result shows that the renormalized energy $\widetilde{W}$ is what remains in the energy after the singular "core energy" $\pi\, d\ |\log \eta|$ has been removed.

Theorem 4. *We have the following asymptotic estimate:*

$$\frac{1}{2}\int_{G_\eta^w} |\nabla u_\eta|^2 = \pi(\sum_{j=1}^k d_j^2)\ |\log \eta| + \widetilde{W}(b, \overline{d}, g) + O(\eta), \quad \text{as } \eta \to 0\,,$$

where

$$\widetilde{W}(b, \overline{d}, g) = W(b, \overline{d}, g) + \frac{\pi}{2}\left(\sum_{j=1}^k d_j^2 \log w(b_j)\right).$$

Proof. As in [BBH4], Chapter I we associate to (26) the linear problem:

$$\begin{cases} \Delta\Phi_\eta = 0\ , \quad \text{in } G_\eta^w \\ \Phi_\eta = C_j = \text{Const.}\ , \quad \text{on each } \partial\omega_{j,\eta} \\ \displaystyle\int_{\partial\omega_{j,\eta}} \frac{\partial\Phi_\eta}{\partial\nu} = 2\pi d_j\ , \quad \text{for each } j = 1, ..., k \\ \displaystyle\frac{\partial\Phi_\eta}{\partial\nu} = g \wedge g_\tau\ , \quad \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_\eta = 0. \end{cases} \tag{27}$$

With the same techniques as in [BBH4] (see Lemma I.2), one may prove that

$$\|\Phi_\eta - \Phi_0\|_{L^\infty(G_\eta^w)} = O(\eta)\,,$$

where Φ_0 is the unique solution of (1).

Note that the link between Φ_η and an arbitrary solution u_η of (26) is

$$\begin{cases} u_\eta \wedge \dfrac{\partial u_\eta}{\partial x_1} = -\dfrac{\partial\Phi_\eta}{\partial x_2} \quad \text{in } G_\eta^w \\ u_\eta \wedge \dfrac{\partial u_\eta}{\partial x_2} = \dfrac{\partial\Phi_\eta}{\partial x_1} \quad \text{in } G_\eta^w \end{cases} \tag{28}$$

From now on the proof follows the same lines as of Theorem I.7 in [BBH4]. □

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