# A MULTIPLICITY THEOREM FOR LOCALLY LIPSCHITZ PERIODIC FUNCTIONALS

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**Abstract.** We prove in this paper a multiplicity theorem of the Ljusternik-Schnirelmann type for locally Lipschitz periodic functionals and related results. The key argument in our proofs is the Ekeland's Variational Principle and a non-smooth Pseudo-Gradient Lemma. As application of these abstract results we solve a non-linear setvalued elliptic problem.

## Introduction

In PDE, two important tools for proving existence of solutions are the Mountain-Pass Theorem of Ambrosetti and Rabinowitz (and its various generalizations) and the Ljusternik-Schnirelmann Theorem. These results apply to the case when the solutions of the given problem are critical points of an appropriate functional of energy $f$, which is supposed to be real and $C^1$, or even differentiable, on a real Banach space $X$. One may ask what happens if $f$, which often is associated to the original equation in a canonical way, fails to be differentiable. In this case the gradient of $f$ must be replaced by a generalized one, in a sense which is to be defined.

The first approach is due to Chang [8] and Aubin and Clarke [2], who considered the case of a locally Lipschitz function $f$. For such functions, Clarke [11] defined a generalized gradient, which coincides to the usual ones if $f$ is differentiable or convex. Still denoting this generalized gradient by $\partial f$, critical points of $f$ are all points $x$ such that $0 \in \partial f(x)$. In this setting, Chang [8] proved a version of the Mountain Pass Lemma, in the case when $X$ is reflexive. For this aim, he used a "Lipschitz version" of the Deformation Lemma. The same result was used for the proof of the Ljusternik-Schnirelmann Theorem in the Lipschitz case. As observed by Brézis, the reflexivity assumption on $X$ is not necessary.

Our main result is a multiplicity theorem for locally Lipschitz periodic functionals, their set of periods being a discrete subgroup of the space where they are defined. This result can be regarded as a Ljusternik-Schnirelmann type theorem for non-differentiable functionals.

After recalling a well known theorem due to Choulli, Deville and Rhandi [9] and giving some consequences of this Mountain Pass type theorem for locally Lipschitz functionals, we present the connection between their theorem and our main result by solving a non-smooth problem that generalizes the forced-pendulum equation.

Following [8], authors usually impose measurability conditions to some *a priori* unknown functions in order to be able to find $\partial f$. We first show that these conditions are automatically fullfilled and then we prove the existence of critical points, which are shown to be solutions of a multivalued PDE.

## 1. The theoretical setting

Throughout, $X$ will be a real Banach space. Let $X^*$ be its dual and $\langle x^*, x\rangle$, for $x \in X$, $x \in X^*$, denote the duality pairing between $X^*$ and $X$. We say that a function $f : X \to \mathbf{R}$ is locally Lipschitz ($f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$) if, for each $x \in X$, there is a neighbourhood $V$ of $x$ and a constant $k = k(V)$ depending on $V$ such that

$$|f(y) - f(z)| \leq k\|y - z\| \quad ,$$

for each $y, z \in V$.

We recall in what follows the definition of the Clarke subdifferential and some of its most important properties (see [10] for details).

For each $x, v \in X$, we define the generalized directional derivative at $x$ in the direction $v$ of a given $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda} \quad .$$

Then $f^0(x, v)$ is a finite number and $|f^0(x, v)| \leq k\|v\|$. The mapping $v \longmapsto f^0(x, v)$ is positively homogeneous and subadditive, hence convex continuous. The generalized gradient (the Clarke subdifferential) of $f$ at $x$ is the subset $\partial f(x)$ of $X^*$ defined by

$$\partial f(x) = \{x^* \in X^*; \;\; f^0(x, v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\} \quad .$$

If $f$ is convex, $\partial f(x)$ coincides with the subdifferential of $f$ at $x$ in the sense of convex analysis.

The fundamental properties of the Clarke subdifferential are:

a) For each $x \in X$, $\partial f(x)$ is a nonempty convex weak-$\star$ compact subset of $X^*$.

b) For each $x, v \in X$, we have

$$f^0(x, v) = \max\{\langle x^*, v\rangle; \; x^* \in \partial f(x)\} \quad .$$

c) The set-valued mapping $x \longmapsto \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X, \varepsilon > 0, v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\|x - x_0\| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

d) The function $f^0(\cdot,\cdot)$ is upper semi-continuous.

e) If $f$ achieves a local minimum or maximum at $x$, then $0 \in \partial f(x)$.

f) The function

$$\lambda(x) = \min_{x^* \in \partial f(x)} \|x^*\|$$

exists and is lower semi-continuous.

g) Lebourg's Mean Value Theorem: If $x$ and $y$ are distinct points in $X$, then there is a point $z$ in the open segment between $x$ and $y$ such that

$$f(y) - f(x) \in \langle\, \partial f(z), y - x\rangle \ .$$

**Definition 1**. *A point $u \in X$ is said to be a critical point of $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ if $0 \in \partial f(u)$, namely $f^0(u, v) \geq 0$ for every $v \in X$. A real number $c$ is called a critical value of $f$ if there is a critical point $u \in X$ such that $f(u) = c$.*

**Definition 2**. *If $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ and $c$ is a real number, we say that $f$ satisfies the Palais-Smale condition at the level $c$ (in short $(PS)_c$) if any sequence $(x_n)$ in $X$ with the properties $\lim\limits_{n\to\infty} f(x_n) = c$ and $\lim\limits_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence. The function $f$ is said to satisfy the Palais-Smale condition (in short (PS)) if each sequence $(x_n)$ in $X$ such that $(f(x_n))$ is bounded and $\lim\limits_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence.*

Let $Z$ be a discrete subgroup of $X$, that is

$$\inf_{z \in Z\setminus\{0\}} \|z\| > 0 \ \ .$$

A function $f : X \to \mathbf{R}$ is said to be *Z-periodic* if $f(x + z) = f(x)$, for every $x \in X$ and $z \in Z$.

If $f \in \mathrm{Lip}_{loc}(X, \mathbf{R})$ is $Z$-periodic, then $x \longmapsto f^0(x, v)$ is $Z$-periodic, for all $v \in X$ and $\partial f$ is $Z$-invariant, that is $\partial f(x + z) = \partial f(x)$, for every $x \in X$ and $z \in Z$. These implies that $\lambda$ inherits the $Z$-periodicity property.

If $\pi : X \to X/Z$ is the canonical surjection and $x$ is a critical point of $f$, then $\pi^{-1}(\pi(x))$ contains only critical points. Such a set is called a *critical orbit* of $f$. Note that $X/Z$ is a complete metric space endowed with the metric

$$d(\pi(x), \pi(y)) = \inf_{z \in Z} \|x - y - z\| \ \ .$$

**Definition 3**. *A locally Lipschitz $Z$-periodic function $f : X \to \mathbf{R}$ is said to satisfy the $(PS)_Z$ - condition provided that, for each sequence $(x_n)$ in $X$ such that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, then $(\pi(x_n))$ is relatively compact in $X/Z$. If $c$ is a real number, then $f$ is* *said to satisfy the $(PS)_{Z,c}$ - condition if, for any sequence $(x_n)$ in $X$ such that $f(x_n) \to c$ and $\lambda(x_n) \to 0$, there is a convergent subsequence of $(\pi(x_n))$.*

Denote Cr $(f, c)$ the set of critical points of the locally Lipschitz function $f : X \to \mathbf{R}$ at the level $c \in \mathbf{R}$, that is

$$\mathrm{Cr}\ (f, c) = \{x \in X;\ f(x) = c \ \text{ and } \ \lambda(x) = 0\} \ .$$

## 2. The main result

**Theorem 1.** *Let $f : X \to \mathbf{R}$ be a bounded below locally Lipschitz $Z$-periodic function with the $(PS)_Z$-property. Then $f$ has at least $n+1$ distinct critical orbits, where $n$ is the dimension of the vector space generated by the discrete subgroup $Z$.*

Before beginning the proof, we shall recall the notion of *category* and some of its properties, which will be required by the proof of the main result.

A topological space $X$ is said to be *contractible* if the identity of $X$ is homotopical to a constant map, that is there exist $u_0 \in X$ and a continuous map $F : [0,1] \times X \to X$ such that

$$F(0, \cdot) = \mathrm{id}_X \quad \text{and} \quad F(1, \cdot) = u_0 \ .$$

A subset $M$ of $X$ is said to be *contractible in* $X$ if there exist $u_0 \in X$ and a continuous map $F : [0,1] \times M \to X$ such that

$$F(0, \cdot) = \mathrm{id}_M \quad \text{and} \quad F(1, \cdot) = u_0 \ .$$

If $A$ is a subset of $X$, we define the category of $A$ in $X$ as follows:

$\mathrm{Cat}_X(A) = 0$, if $A = \emptyset$ .

$\mathrm{Cat}_X(A) = n$, if $n$ is the smallest integer such that $A$ can be covered by $n$ closed sets which are contractible in $X$.

$\mathrm{Cat}_X(A) = \infty$, otherwise.

**Lemma 1.** *Let $A$ and $B$ subsets of $X$. Then the following hold:*

*i) If $A \subset B$, then $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.*

*ii) $\mathrm{Cat}_X(A \cup B) \le \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$*

*iii) Let $h : [0,1] \times A \to X$ be a continuous mapping such that $h(0, x) = x$ for every $x \in A$. If $A$ is closed and $B = h(1, A)$, then $\mathrm{Cat}_X\ (A) \le \mathrm{Cat}_X\ (B)$*

*iv) If $n$ is the dimension of the vector space generated by the discrete group $Z$, then, for each $1 \le i \le n+1$, the set*

$$\mathcal{A}_i = \{A \subset X;\ A \ \text{ is compact and } \ \mathrm{Cat}_{\pi(X)}\pi(A) \ge i\}$$

*is nonempty. Obviously, $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \supset \mathcal{A}_{n+1}$ .*

The only nontrivial part is *iv)* , which can be found in [19].

The following two Lemmas are proved in [26].

**Lemma 2**. *For each $1 \leq j \leq n+1$, the space $\mathcal{A}_i$ endowed with the Hausdorff metric*

$$\delta(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B)\ ,\ \sup_{b\in B} \operatorname{dist}(b,A)\}$$

*is a complete metric space.*

**Lemma 3**. *If $1 \leq i \leq n+1$ and $f \in C(X,\mathbf{R})$, then the function $\eta : \mathcal{A}_i \to \mathbf{R}$ defined by*

$$\eta(A) = \max_{x\in A} f(x)$$

*is lower semi-continuous.*

If $n$ is the dimension of the vector space generated by the discrete group $Z$, one sets for each $1 \leq i \leq n+1$

$$c_i = \inf_{A\in\mathcal{A}_i} \eta(A) \quad .$$

For each $c \in \mathbf{R}$ we denote $[f \leq c] = \{x \in X;\ f(x) \leq c\}$.

## 3. Proof of Theorem 1

It follows from Lemma 1 *iv)* and the lower boundedness of $f$ that

$$-\infty < c_1 \leq c_2 \leq ... \leq c_{n+1} < +\infty \quad .$$

It is sufficient to show that, if $1 \leq i \leq j \leq n+1$ and $c_i = c_j = c$, then the set $\operatorname{Cr}(f,c)$ contains at least $j-i+1$ distinct critical orbits. We argue by contradiction and suppose that, for some $i \leq j$, $\operatorname{Cr}(f,c)$ has $k \leq j-i$ distinct critical orbits, generated by $x_1, ..., x_k \in X$. We construct first an open neighbourhood of $\operatorname{Cr}(f,c)$ of the form

$$V_r = \bigcup_{l=1}^{k} \bigcup_{z\in Z} B(x_l + z, r) \quad .$$

Moreover, we may suppose that $r > 0$ is chosen such that $\pi$ is one-to-one on $\overline{B}(x_l, 2r)$. This condition ensures that $\operatorname{Cat}_{\pi(X)}(\pi(\overline{B}(x_l, 2r))) = 1$, for each $l = 1, ..., k$. Here $V_r = \emptyset$ if $k = 0$.

<u>*Step 1.*</u> We prove that there exists $0 < \varepsilon < \min\{\frac{1}{4}, r\}$ such that, for each $x \in [c-\varepsilon \leq f \leq c+\varepsilon] \setminus V_r$, one has

$$\lambda(x) > \sqrt{\varepsilon} \quad . \tag{1}$$

Indeed, if not, there is a sequence $(x_m)$ in $X \setminus V_r$ such that, for each $m \geq 1$,

$$c - \frac{1}{m} \leq f(x_m) \leq c + \frac{1}{m} \quad \text{and} \quad \lambda(x_m) \leq \frac{1}{\sqrt{m}} \quad .$$

Since $f$ satisfies $(PS)_Z$ , it follows that, up to a subsequence, $\pi(x_m) \to \pi(x)$ as $m \to \infty$, for some $x \in X \setminus V_r$. By the $Z$-periodicity of $f$ and $\lambda$, we can assume that $x_m \to x$ as $m \to \infty$. The continuity of $f$ and the lower semi-continuity of $\lambda$ imply $f(x) = c$ and $\lambda(x) = 0$, which is a contradiction, since $x \in X \setminus V_r$.

*Step 2.* For $\varepsilon$ found above and according to the definition of $c_j$, there exists $A \in \mathcal{A}_j$ such that

$$\max_{x \in A} f(x) < c + \varepsilon^2 \quad .$$

Setting $B = A \setminus V_{2r}$, we get by Lemma 1 that

$$j \leq \mathrm{Cat}_{\pi(X)}(\pi(A)) \leq \mathrm{Cat}_{\pi(X)}(\pi(B) \cup \pi(\overline{V}_{2r})) \leq$$

$$\leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + \mathrm{Cat}_{\pi(X)}(\pi(\overline{V}_{2r})) \leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + k \leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + j - i \ .$$

Hence, $\mathrm{Cat}_{\pi(X)}(\pi(B)) \geq i$, that is $B \in \mathcal{A}_i$.

*Step 3.* For $\varepsilon$ and $B$ as above we apply the Ekeland's Principle to the functional $\eta$ defined in Lemma 3. It follows that there exists $C \in \mathcal{A}_i$ such that, for each $D \in \mathcal{A}_i$, $D \neq C$,

$$\eta(C) \leq \eta(B) \leq \eta(A) \leq c + \varepsilon^2 \quad ,$$

$$\delta(B, C) \leq \varepsilon \quad ,$$

$$\eta(D) > \eta(C) - \varepsilon\delta(C, D) \quad . \tag{2}$$

Since $B \cap V_{2r} = \emptyset$ and $\delta(B, C) \leq \varepsilon < r$, it follows that $C \cap V_r = \emptyset$. In particular, the set $F = [c - \varepsilon \leq f] \cap C$ is contained in $[c - \varepsilon \leq f \leq c + \varepsilon]$ and $F \cap V_r = \emptyset$.

**Lemma 4**. *Let $M$ be a compact metric space and let $\varphi : M \to 2^{X^*}$ be a set-valued mapping which is upper semi-continuous (in the sense of c)) and with weak-$\star$ compact convex values. For $t \in M$ denote*

$$\gamma(t) = \inf\{\|x^*\|;\ x^* \in \varphi(t)\}$$

*and*

$$\gamma = \inf_{t \in M} \gamma(t) \quad .$$

*Then, given $\varepsilon > 0$, there exists a continuous function $v : M \to X$ such that for all $t \in M$ and $x^* \in \varphi(t)$,*

$$\|v(t)\| \leq 1 \text{ and } \langle x^*, v(t) \rangle \geq \gamma - \varepsilon \quad .$$

**Proof of Lemma.** We may suppose $\gamma > 0$ and $0 < \varepsilon < \gamma$. If $B_r$ denotes the open ball in $X^*$ centered at 0 with radius $r$, then, for each $t \in M$, one has

$$B_{\gamma-\frac{\varepsilon}{2}} \cap \varphi(t) = \emptyset \quad .$$

Since $\varphi(t)$ and $B_{\gamma-\frac{\varepsilon}{2}}$ are convex, weak-$\star$ compact and disjoint, it follows from the Theorem 3.4 in [24], applied to the space $(X^*,\ \sigma(X^*, X))$ and from the fact that the dual space of the above one is $X$, that:

for every $t \in M$, there is some $v_t \in X$, $\|v_t\| = 1$ such that

$$\langle \xi, v_t \rangle \leq \langle x^*, v_t \rangle \quad ,$$

for each $\xi \in B_{\gamma-\frac{\varepsilon}{2}}$ and $x^* \in \varphi(t)$. Therefore, for each $x^* \in \varphi(t)$,

$$\langle x^*, v_t \rangle \geq \sup_{\xi \in B_{\gamma-\frac{\varepsilon}{2}}} \langle \xi, v_t \rangle = \gamma - \frac{\varepsilon}{2} \quad .$$

Because of the upper semi-continuity of $\varphi$, there is an open neighbourhood $V(t)$ of $t$ such that, for each $t' \in V(t)$ and each $x^* \in \varphi(t')$,

$$\langle x^*, v_t \rangle > \gamma - \varepsilon \quad .$$

Since $M$ is compact and $M = \bigcup_{t \in M} V(t)$, we can find a finite subcovering $\{V_1, ..., V_n\}$ of $M$. Let $v_1, ..., v_n$ be on the unit sphere of $X$ such that $\langle x^*, v_i \rangle > \gamma - \varepsilon$, for all $1 \leq i \leq n$, $t \in V_i$ and $x^* \in \varphi(t)$.

If $\rho_i(t) = \mathrm{dist}(t, \partial V_i)$, define

$$\zeta_i(t) = \frac{\rho_i(t)}{\sum_{j=1}^n \rho_j(t)} \quad \text{and} \quad v(t) = \sum_{i=1}^n \zeta_i(t) v_i \quad .$$

The function $v$ is the desired mapping. □

Applying Lemma 4 to $\varphi = \partial f$ on $F$, we find a continuous map $v : F \to X$ such that, for all $x \in F$ and $x^* \in \partial f(x)$,

$$\|v(x)\| \leq 1 \quad \text{and} \quad \langle x^*, v(x) \rangle \geq \inf_{x \in F} \lambda(x) - \varepsilon \geq \inf_{x \in C} \lambda(x) - \varepsilon \geq \sqrt{\varepsilon} - \varepsilon \quad ,$$

where the last inequality is justified by (1).

It follows that, for each $x \in F$ and $x^* \in \partial f(x)$,

$$f^0(x, -v(x)) = \max_{x^* \in \partial f(x)} \langle x^*, -v(x) \rangle = -\min_{x^* \in \partial f(x)} \langle x^*, v(x) \rangle \leq \varepsilon - \sqrt{\varepsilon} < -\varepsilon,$$

from our choice of $\varepsilon$.

From the upper semi-continuity of $f^0$ and the compactness of $F$, there exists $\delta > 0$ such that if $x \in F$, $y \in X$, $\|y - x\| \leq \delta$, then

$$f^0(y, -v(x)) < -\varepsilon \quad . \tag{3}$$

Since $C \cap \mathrm{Cr}(f, c) = \emptyset$ and $C$ is compact, while $\mathrm{Cr}(f, c)$ is closed, there is a continuous extension $w : X \to X$ of $v$ such that $w|_{\mathrm{Cr}(f,c)} = 0$ and $\|w(x)\| \leq 1$, for all $x \in X$.

Let $\alpha : X \to [0, 1]$ be a continuous $Z$-periodic function such that $\alpha = 1$ on $[f \geq c]$ and $\alpha = 0$ on $[f \leq c - \varepsilon]$. Let $h : [0, 1] \times X \to X$ be the continuous mapping defined by

$$h(t, x) = x - t\delta\alpha(x)w(x) \quad .$$

If $D = h(1, C)$, it follows from Lemma 1 that

$$\mathrm{Cat}_{\pi(X)}(\pi(D)) \geq \mathrm{Cat}_{\pi(X)}(\pi(C)) \geq i$$

which shows that $D \in \mathcal{A}_i$, since $D$ is compact.

*Step 4.* By Lebourg's mean value Theorem we get that, for each $x \in X$, there exists $\theta \in (0, 1)$ such that

$$f(h(1, x)) - f(h(0, x)) \in \langle \partial f(h(\theta, x)), -\delta\alpha(x)w(x) \rangle \quad .$$

Hence, there is some $x^* \in \partial f(h(\theta, x))$ such that

$$f(h(1, x)) - f(h(0, x)) = \alpha(x)\langle x^*, -\delta w(x) \rangle \quad .$$

It follows by (3) that, if $x \in F$, then

$$f(h(1, x)) - f(h(0, x)) = \delta\alpha(x)\langle x^*, -w(x) \rangle \leq \tag{4}$$

$$\leq \delta\alpha(x)f^0(x - \theta\delta\alpha(x)w(x), -v(x)\rangle \leq -\varepsilon\delta\alpha(x) \quad .$$

It follows that, for each $x \in C$,

$$f(h(1, x)) \leq f(x) \quad .$$

Let $x_0 \in C$ be such that $f(h(1, x_0)) = \eta(D)$. Hence,

$$c \leq f(h(1, x_0)) \leq f(x_0) \quad .$$

By the definition of $\alpha$ and $F$, it follows that $\alpha(x_0) = 1$ and $x_0 \in F$. Therefore, by (4), we get

$$f(h(1, x_0)) - f(x_0) \leq -\varepsilon\delta \quad .$$

Thus,

$$\eta(D) + \varepsilon\delta \le f(x_0) \le \eta(C) \quad . \tag{5}$$

Taking into account the definition of $D$, it follows that

$$\delta(C, D) \le \delta \quad .$$

Therefore,

$$\eta(D) + \varepsilon\delta(C, D) \le \eta(C) \quad ,$$

so that (2) implies $C = D$, which contradicts (5). □

## 4. A multivalued generalized version of the forced-pendulum problem

As an application of the above results, we shall study the periodic multivalued problem of the forced-pendulum

$$\begin{cases} x''(t) + f(t) \in [\underline{g}(x(t)), \overline{g}(x(t))] \ , \ \text{a.e. } t \in (0,1) \\ x(0) = x(1) \ , \end{cases} \tag{6}$$

where:

$$f \in L^p(0,1) \qquad \text{for some } \ p > 1 \quad , \tag{7}$$

$$g \in L^\infty(\mathbf{R}), \ g(u+T) = g(u) \qquad \text{for some } \ T > 0, \ \text{a.e. } \ u \in \mathbf{R} \quad , \tag{8}$$

$$\underline{g}(u) = \lim_{\varepsilon \searrow 0} \operatorname{essinf}\{g(u); \ |u - v| < \varepsilon\} \quad \overline{g}(u) = \lim_{\varepsilon \searrow 0} \operatorname{esssup}\{g(u); \ |u - v| < \varepsilon\} \quad ,$$

$$\int_0^T g(u)du = \int_0^1 f(t)dt = 0 \tag{9}$$

We shall prove

**Theorem 2.** *If $f, g$ are as above, then the problem (6) has at least two solutions in*

$$X := H^1_p(0,1) = \{x \in H^1(0,1); \ \ x(0) = x(1)\} \quad ,$$

*which are distinct in the sense that their difference is not an integer multiple of $T$.*

Define the functional $\psi$ in $L^\infty(0,1)$ by

$$\psi(x) = \int_0^1 (\int_0^{x(s)} g(u)du)ds \quad .$$

It is obvious that $\psi$ is a Lipschitz map on $L^\infty(0,1)$.

Let $G(u) = \displaystyle\int_0^u g(v)dv$.

The following results show that the description of $\partial\psi$ given in [8] holds without further assumptions on $g$.

**Lemma 5.** *Let $g$ be a locally bounded measurable function defined on* $\mathbf{R}$ *and $\underline{g}$, $\overline{g}$ as above. Then the Clarke subdifferential of $G$ is given by*

$$\partial\, G(u) = [\underline{g}(u), \overline{g}(u)] \qquad u \in \mathbf{R} \ .$$

**Proof.** The required equality is equivalent to $G^0(u,1) = \overline{g}(u)$ and $G^0(u,-1) = \underline{g}(u)$.

As a matter of facts, examining the definitions of $G^0$, $\overline{g}$ and $\underline{g}$, it follows that $\underline{g}(u) = -(\overline{-g})(u)$ and $G^0(u,-1) = -(-G)^0(u,1)$, so that the second required equality is equivalent to the first one.

Now the inequality $G^0(u,1) \le \overline{g}(u)$ can be found in [8], so we have only to prove that $G^0(u,1) \ge \overline{g}(u)$. Suppose that $G^0(u,1) = \overline{g}(u) - \varepsilon$ for some $\varepsilon > 0$. Let $\delta > 0$ be such that

$$\frac{G(\tau+\lambda) - G(\tau)}{\lambda} < \overline{g}(u) - \frac{\varepsilon}{2} \ ,$$

if $|\tau - u| < \delta$ and $0 < \lambda < \delta$. Then

$$\frac{1}{\lambda}\int_\tau^{\tau+\lambda} g(s)ds < \overline{g}(u) - \frac{\varepsilon}{2} \qquad \text{if} \ \ |\tau - u| < \delta, \ \lambda > 0 \tag{10}$$

We claim that there exist $\lambda_n \searrow 0$ such that

$$\frac{1}{\lambda_n}\int_\tau^{\tau+\lambda_n} g(s)ds \to g(\tau) \qquad a.e. \ \ \tau \in (u-\delta, u+\delta) \ . \tag{11}$$

Suppose for the moment that (11) has already been proved. Now (10) and (11) show that

$$g(\tau) \le \overline{g}(u) - \frac{\varepsilon}{2} \qquad \text{if} \ \ \tau \in (u-\delta, u+\delta) \ \ ,$$

so we obtain the contradictory inequalities

$$\overline{g}(u) \le \operatorname{esssup}\{g(s);\ s \in [u-\delta, u+\delta]\} \le \overline{g}(u) - \frac{\varepsilon}{2} \ .$$

All it remains to be proved is (11). Note that we may cut $g$ in order to suppose that $g \in L^\infty \cap L^1$. Then (11) is nothing that the classical fact that for each $\varphi \in L^1(\mathbf{R})$,

$$T_\lambda(\varphi) \to \varphi \qquad \text{as} \ \ \lambda \searrow 0 \tag{12}$$

where

$$T_\lambda\varphi(u) = \frac{1}{\lambda}\int_u^{u+\lambda} \varphi(s)ds \qquad \text{for} \ \ \lambda > 0, \ u \in \mathbf{R}, \ \varphi \in L^1(\mathbf{R}) \ .$$

Indeed, it can be easily seen that $T_\lambda$ is linear and continuous in $L^1(\mathbf{R})$ and $\lim\limits_{\lambda \searrow 0} T_\lambda \varphi = \varphi$ in $\mathcal{D}(\mathbf{R})$ for $\varphi \in \mathcal{D}(\mathbf{R})$. Now (14) follows by a density argument. $\square$

Returning to our problem, it follows by Theorem 2.1. in [8] that

$$\partial \psi_{|H_0^1(\Omega)}(x) \subset \partial \psi(x) \tag{13}$$

In order to obtain information on $\partial\psi$, we shall need an improvement of the Theorem 2.1. in [8].

**Theorem 3.** *If $x \in L^\infty(0,1)$, then*

$$\partial\psi(x)(t) \subset [\underline{g}(x(t)), \overline{g}(x(t))] \quad a.e. \;\; t \in (0,1) \;\; ,$$

*in the sense that if $w \in \partial\psi(x)$ then*

$$\underline{g}(x(t)) \leq w(t) \leq \overline{g}(x(t)) \qquad a.e. \;\; t \in (0,1) \tag{14}$$

**Proof.** Let $h$ be a Borel function such that $h = g$ a.e. on $\mathbf{R}$. It follows that the set

$$A = \{t \in (0,1) \; ; \;\; \underline{g}(x(t)) \neq \underline{h}(x(t))\}$$

is a null set. (A similar reasoning can be done for $\overline{g}$ and $\overline{h}$).

Therefore we may suppose that $g$ is a Borel function. We would like to deal with $\displaystyle\int_0^1 \overline{g}(x(t))dt$, so we have to prove that $\overline{g}$ is a Borel function.

**Lemma 6.** *Let $g : \mathbf{R} \to \mathbf{R}$ be a locally bounded Borel function. Then $\overline{g}$ is a Borel function.*

**Proof of Lemma.** Since the requirement is local, we may suppose that $g$ is bounded by 1, for example, and it is nonnegative. Since

$$g = \lim_{n\to\infty} \lim_{m\to\infty} g_{m,n}$$

where

$$g_{m,n}(x,t) = \Big( \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} |g^m(x,s)| ds \Big)^{\frac{1}{m}}$$

it suffices to prove that $g_{m,n}$ is Borel.

Let

$$\mathcal{M} = \{g : \Omega \times \mathbf{R} \to \mathbf{R}; \;\; |g| \leq 1 \;\text{ and }\; g \;\text{ is a Borel function}\}$$

$$\mathcal{N} = \{g \in \mathcal{M}; \;\; g_{m,n} \;\text{ is a Borel function}\}$$

It is known (see [3], p. 178) that $\mathcal{M}$ is the smallest set of functions having the following properties:

i) $\{g \in C(\Omega \times \mathbf{R}, \mathbf{R}); \quad |g| \leq 1\} \subset \mathcal{M}$

ii) $g^{(k)} \xrightarrow{k} g$ imply $g \in \mathcal{M}$. Note that here we have an "each point" convergence.

Since $\mathcal{N}$ contains obviously the continuous functions and ii) is also true for $\mathcal{N}$, by the Dominated Convergence Theorem, it follows that $\mathcal{M} = \mathcal{N}$. $\square$

**Proof of Theorem 3 continued.** Let $v \in L^\infty(\Omega), \ v \geq 0$. Then, for suitable $\lambda_i \searrow 0$ and $h_i \to 0$ in $L^{p+1}(\Omega)$ one has

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_\Omega \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds\ dx$$

We may suppose that $h_i \to 0$ a.e., so that

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_{[v>0]} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds\ dx \leq$$

$$\leq \int_{[v>0]} (\limsup_{i\to\infty} \frac{1}{\{\lambda_i} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds)dx \leq$$

$$\leq \int_{[v>0]} \overline{g}(x,u(x))v(x)dx$$

so that

$$\psi^0(u,v) \leq \int_{[v>0]} \overline{g}(x,u(x))v(x)dx \tag{15}$$

for such $v$.

Suppose now that (21) is false, that is, for example, there exist $\varepsilon > 0$, a set $E$ with $|E| > 0$ and $w \in \partial\psi(u)$ such that

$$w(x) \geq \overline{g}(x,u(x)) + \varepsilon \qquad \text{on } E \tag{16}$$

Now (15) with $v = \mathbf{1}_E$ shows that

$$\langle w, v\rangle = \int_E w \leq \psi^0(u,v) \leq \int_E \overline{g}(x,u(x))dx$$

which contradicts (16). $\square$

**Proof of Theorem 2.** Define on the space $X = H^1_{per}(0,1)$ the locally Lipschitz function

$$\varphi(x) = \frac{1}{2}\int_0^1 x'^2(t)dt - \int_0^1 f(t)x(t)dt + \int_0^1 G(x(t))dt.$$

The critical points of $\varphi$ are solutions of (6). Indeed, it is obvious that

$$\partial\varphi(x) = x'' + f - \partial\psi_{|H^1_{per}(0,1)}(x) \quad \text{in} \quad H^{-1}(0,1)$$

If $x_0$ is a critical point of $\varphi$ it follows that there exists $w \in \partial\psi_| H^1_{per}(0,1)(x_0)$ such that

$$x'' + f = w \quad \text{in} \quad H^{-1}(0,1)$$

Since $\varphi(x+T) = \varphi(x)$, we are going to use the Theorem 1. All we have to do is to verify the $(PS)_{Z,c}$ condition, for each $c$, and to prove that (6) has a solution $x_0$ that minimizes $\varphi$ on $H^1_{per}(0,1)$. Note first that every $x \in H^1_{per}(0,1)$ can be written

$$x(t) = \int_0^1 x(s)ds + \overline{x}(t)dt \quad \text{with } \overline{x} \in H^1_0(0,1).$$

Hence, by the Poincaré's inequality,

$$\varphi(x) = \frac{1}{2}\int_0^1 \overline{x}'^2(t)dt - \int_0^1 f(t)\overline{x}(t)dt + \int_0^1 G(x(t))dt$$

$$\geq \frac{1}{2}\|\overline{x}'^2\|^2_{L^2} - \|f\|_{L^p}\cdot\|\overline{x}\|_{L^{p'}} - \|G\|_{L^\infty}$$

$$\geq \frac{1}{2}\|\overline{x}'^2\|^2_{L^2} - C\,\|f\|_{L^p}\cdot\|\overline{x}'\|_{L^2} - \|G\|_{L^\infty} \to \infty \quad \text{as } \|\overline{x}\|_{H^1} \to \infty\,,$$

where $p'$ denotes the conjugated exponent of $p$.

We verify in what follows the $(PS)_{Z,c}$ condition, for each $c$. Let $(x_n) \subset X$ be such that

$$\varphi(x_n) \to c \tag{17}$$

$$\lambda(x_n) \to 0. \tag{18}$$

Let $w_n \in \partial\varphi(x_n) \subset L^\infty(0,1)$ (because $\underline{g} \circ x_n \leq w_n \leq \overline{g} \circ x_n$ and $\underline{g}, \overline{g} \in L^\infty(\mathbf{R})$) be such that

$$\lambda(x_n) = x_n'' + f - w_n \to 0 \quad \text{in} \quad H^{-1}(0,1)$$

Then, multiplying (18) by $x_n$ we get

$$\int_0^1 (x_n')^2 - \int_0^1 f x_n + \int_0^1 w_n x_n = o(1)\|x_n\|_{H^1_p}$$

and, by (17),

$$-\frac{1}{2}\int_0^1 (x_n')^2 + \int_0^1 f x_n - \int_0^1 G(x_n), \to c$$

so that there exist positive constants $C_1, C_2$ such that

$$\int_0^1 (x_n')^2 \leq C_1 + C_2 \|x_n\|_{H_p^1}$$

Note that $G$ is also $T$-periodic; hence it is bounded.

Replacing $x_n$ by $x_n + kT$ for a suitable integer $k$, we may suppose that

$$x_n(0) \in [0, T]$$

so that $(x_n)$ is bounded in $H_p^1$.

Let $x \in H_p^1$ be such that, up to a subsequence, $x_n \rightharpoonup x$ and $x_n(0) \to x(0)$. Then

$$\int_0^1 (x_n')^2 = \langle -x_n'' - f + w_n, x_n - x \rangle + \int_0^1 w_n(x_n - x) -$$

$$- \int_0^1 f(x_n - x) + \int_0^1 x_n' x' \to \int_0^1 x'^2$$

because $x_n \to x$ in $L^{p'}$, where $p'$ is the conjugated exponent of $p$. It follows that $x_n \to x$ in $H_p^1$. $\square$

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