Perturbations of nonsmooth symmetric nonlinear eigenvalue problems

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Abstract. We consider a symmetric semilinear boundary value problem having infinitely many solutions. We prove that, if we perturb this problem in a non-symmetric way, then the number of solutions goes to infinity as the perturbation tends to zero. The growth conditions on the nonlinearities do not ensure the smoothness of the associated functional.

Perturbations des problèmes non-linéaires aux valeurs propres symétriques non réguliers

Résumé. *On considère un problème semi-linéaire symétrique avec une infinité de solutions. On montre que, si l'on perturbe ce problème d'une manière non-symétrique, alors le nombre de solutions devient de plus en plus grand lorsque la perturbation tend vers zéro. Les conditions de croissance sur les nonlinéarités ne garantissent pas la régularité de la fonctionnelle associée.*

Version française abrégée

Soit $\Omega \subset \mathbf{R}^N$ un ouvert borné. Pour $r > 0$ fixé arbitrairement on considère le problème suivant: trouver $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ tel que

$$(1) \qquad \begin{cases} f(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda f(x,u) \quad \text{dans } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,. \end{cases}$$

On suppose que $f : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory avec les propriétés suivantes:

$(f1)$ $f(x,-s) = -f(x,s)$, p.p. sur Ω et pour chaque $s \in \mathbf{R}$;

$(f2)$ ils existent $a \in L^1(\Omega)$, $b \in \mathbf{R}$ et $0 \le p < \frac{2N}{N-2}$ (si $N > 2$) tels que

$$0 < sf(x,s) \le a(x) + b|s|^p\,, \qquad F(x,s) \le a(x) + b|s|^p\,,$$

p.p. sur Ω et pour chaque $s \in \mathbf{R} \setminus \{0\}$, où $F(x,s) = \int_0^s f(x,t)dt$;

$(f3)$ $\sup\limits_{|s| \le t} |f(x,s)| \in L^1_{loc}(\Omega)$, pour chaque $t > 0$.

THÉORÈME 1. - *Supposons que les conditions $(f1)-(f3)$ soient satisfaites. Alors le problème (1) admet une suite $(\pm u_n, \lambda_n)$ de solutions distinctes.*

Ensuite notre objectif est d'analyser le problème perturbé

$$(2) \qquad \begin{cases} f(x,u),\ g(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda\,(f(x,u) + g(x,u)) \quad \text{dans } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{cases}$$

où $g : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Carathéodory qui n'est pas nécessairement impaire par rapport à la seconde variable. On suppose quand même que g satisfait

$(g1)$ $0 < sg(x,s) \le a(x) + b|s|^p$ p.p. sur Ω et pour chaque $s \in \mathbf{R} \setminus \{0\}$;

$(g2)$ $\sup\limits_{|s| \le t} |g(x,s)| \in L^1_{loc}(\Omega)$, pour chaque $t > 0$;

$(g3)$ $G(x,s) \le C_g\,(1+|s|^p)$, p.p. sur Ω et pour chaque $s \in \mathbf{R}$, avec $C_g > 0$, où $G(x,s) = \int_0^s g(x,t)dt$.

On démontre que le nombre de solutions du problème perturbé (2) devient de plus en plus grand si la perturbation est assez petite, dans un sens précisé ultérieurement. Plus précisement, on a

THÉORÈME 2. - *Supposons que les conditions $(f1)-(f3)$ et $(g1)-(g3)$ soient satisfaites. Alors, pour chaque entier $n \ge 1$, il existe $\varepsilon_n > 0$ tel que le problème (2) admet au moins n solutions distinctes si g est une fonction telle que la condition $(g3)$ soit satisfaite pour $C_g = \varepsilon_n$.*

La preuve des Théorèmes 1 et 2 repose sur un argument variationnel. D'abord on pose

$$S_r = \left\{ u \in H^1_0(\Omega) : \int_\Omega |Du|^2 dx = r^2 \right\}$$

et on étudie les points critiques sur S_r de la fonctionnelle continue et paire $I : H^1_0(\Omega) \to \mathbf{R}$ définie par

$$I(u) = -\int_\Omega F(x,u)dx\,.$$

REMARQUE 1. - *Si* $(f2)$, $(f3)$ *sont remplacées par la condition standarde* $0 < sf(x,s) \leq a_1(x)|s| + b|s|^p$ *avec* $a_1 \in L^{\frac{2N}{N+2}}(\Omega)$, *alors* I *est de classe* C^1 *et le Théorème 1 se trouve dans [8, Theorem 8.17]. Avec notres hypothèses,* f *peut avoir la forme* $f(x,s) = \alpha(x)\gamma(s)$ *avec* $\alpha \in L^1(\Omega)$, $\alpha \geq 0$, $\gamma \in C_c(\mathbf{R})$, γ *impaire et* $s\gamma(s) \geq 0$ *pour chaque* $s \in \mathbf{R}$. *Dans ce cas là,* I *est bien sûr continue, mais pas localement Lipschitz.*

REMARQUE 2. - *Lorsque* f *et* g *satisfont la condition standarde qu'on vient de mentionner, résultats du type du Théorème 2 sont bien classiques (voir par example Krasnoselskii [7]). Des résultats de perturbation, plutôt différents des nôtres, où le problème perturbé admet encore une infinité de solutions, peuvent être trovés dans [8, 9]. Dans un cadre non régulier, un résultat dans la ligne du Théorème 2 a été démontré dans [4] lorsque* f *et* g *satisfont la condition standarde, mais la fonction* u *est contrainte par un obstacle, de sorte que l'équation se transforme dans une inéquation variationnelle.*

Let $\Omega \subset \mathbf{R}^N$ be a bounded open set. For some fixed $r > 0$, consider the problem: find $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ such that

$$(1) \qquad \begin{cases} f(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda f(x,u) \quad \text{in } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{cases}$$

where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that the following conditions hold:

$(f1)$ $f(x,-s) = -f(x,s)$, for a.e. $x \in \Omega$ and every $s \in \mathbf{R}$;

$(f2)$ there exist $a \in L^1(\Omega)$, $b \in \mathbf{R}$ and $0 \leq p < \frac{2N}{N-2}$ (if $N > 2$) such that

$$0 < sf(x,s) \leq a(x) + b|s|^p\,, \qquad F(x,s) \leq a(x) + b|s|^p\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbf{R} \setminus \{0\}$, where $F(x,s) = \int_0^s f(x,t)dt$;

$(f3)$ $\sup\limits_{|s|\leq t} |f(x,s)| \in L^1_{loc}(\Omega)$, for every $t > 0$.

We notice that, if $N = 1$, then in condition $(f2)$ the term $b|s|^p$ can be substituted by any continuous function $\varphi(s)$ of s, while, if $N = 2$, the same term can be substituted by $\exp(\varphi(s))$, with $\varphi(s)s^{-2} \to 0$ as $|s| \to \infty$.

THEOREM 1. - *Assume that hypotheses* $(f1) - (f3)$ *hold. Then Problem (1) admits a sequence* $(\pm u_n, \lambda_n)$ *of distinct solutions.*

Then we want to study what happens when the energy functional is subjected to a perturbation which destroys the symmetry.

Consider the problem: find $(u, \lambda) \in H_0^1(\Omega) \times \mathbf{R}$ such that

$$
(2) \qquad \begin{cases} f(x,u),\, g(x,u) \in L^1_{loc}(\Omega)\,, \\ -\Delta u = \lambda\,(f(x,u) + g(x,u)) \quad \text{in } \mathcal{D}'(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^2 dx = r^2\,, \end{cases}
$$

where $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. We make no symmetry assumption on g, but we impose only

$(g1)$ $0 < sg(x,s) \le a(x) + b|s|^p$ for a.e. $x \in \Omega$ and every $s \in \mathbf{R} \setminus \{0\}$;

$(g2)$ $\sup\limits_{|s|\le t} |g(x,s)| \in L^1_{loc}(\Omega)$, for every $t > 0$;

$(g3)$ $G(x,s) \le C_g\,(1 + |s|^p)$, for a.e. $x \in \Omega$ and every $s \in \mathbf{R}$, for some $C_g > 0$, where $G(x,s) = \int_0^s g(x,t)dt$.

Our second result shows that the number of solutions of Problem (2) becomes greater and greater, as the perturbation tends to zero. More precisely we have

THEOREM 2. - *Assume that hypotheses $(f1) - (f3)$ and $(g1) - (g3)$ hold. Then, for every positive integer n, there exists $\varepsilon_n > 0$ such that Problem (2) admits at least n distinct solutions, provided that $(g3)$ holds for $C_g = \varepsilon_n$.*

We will prove Theorems 1 and 2 by a variational argument. First we set

$$
S_r = \left\{ u \in H_0^1(\Omega) : \int_\Omega |Du|^2 dx = r^2 \right\}
$$

and we study the critical points on S_r of the even continuous functional $I : H_0^1(\Omega) \to \mathbf{R}$ defined by

$$
I(u) = -\int_\Omega F(x,u)dx\,.
$$

REMARK 1. - *If $(f2)$, $(f3)$ are substituted by the more standard condition $0 < sf(x,s) \le a_1(x)|s| + b|s|^p$ with $a_1 \in L^{\frac{2N}{N+2}}(\Omega)$, then I is of class C^1 and Theorem 1 can be found in [8, Theorem 8.17]. Under our assumptions, f could have the form $f(x,s) = \alpha(x)\gamma(s)$ with $\alpha \in L^1(\Omega)$, $\alpha \ge 0$, $\gamma \in C_c(\mathbf{R})$, γ odd and $s\gamma(s) \ge 0$ for any $s \in \mathbf{R}$. In such a case, I is clearly continuous, but not locally Lipschitz.*

REMARK 2. - *When f and g are subjected to the standard condition we have mentioned, results like Theorem 2 go back to Krasnoselskii [7]. For perturbation results, quite different from ours, where the perturbed problem still has infinitely many solutions, we refer the reader to [8, 9]. In a nonsmooth setting, a result in the line of Theorem 2 has been proved in [4] when f and g satisfy the standard condition, but the function u is subjected to an obstacle, so that the equation becomes a variational inequality.*

From $(f2)$ it easily follows that $I(u) < 0$ and that $\sup I_r(u) = 0$, where $I_r = I_{|S_r}$.

Since I is only continuous, we will apply the nonsmooth techniques developed in [1, 3, 4, 5]. In the following, we will adopt the notations of such papers.

LEMMA 1. - *The following facts hold:*

(a) *if* $u \in S_r$ *satisfies* $|dI_r|(u) < +\infty$, *then* $f(x,u) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ *and there exists* $\mu \in \mathbf{R}$ *such that*

$$\|\mu\Delta u + f(x,u)\|_{H^{-1}} \leq |dI_r|(u);$$

(b) *the functional* I_r *satisfies* $(PS)_c$ *for any* $c < 0$;

(c) *if* $u \in S_r$ *is a critical point of* I_r, *then there exists* $\lambda > 0$ *such that* (u, λ) *is a solution of Problem (1).*

Proof. -
(a) Set also

$$I_{r,est}(w) = \begin{cases} I(w) & \text{if } w \in S_r\,, \\ \\ +\infty & \text{if } w \in H^1_0(\Omega) \setminus S_r\,. \end{cases}$$

Then it is immediately seen that $|dI_{r,est}|(u) = |dI_r|(u)$, where we are using the weak slope introduced in [5] (see also [1, Definition 2.1]). By [1, Theorem 4.13] there exists $\alpha \in \partial I_{r,est}(u)$ with $\|\alpha\|_{H^{-1}} \leq |dI_{r,est}|(u)$, where ∂ stands for the subdifferential introduced in [1, Definition 4.1]. Taking into account $(f2)$, we deduce from [6, Theorem 3.3] that

$$I^0(u;0) \leq 0\,, \qquad I^0(u;2u) \leq -2\int_\Omega f(x,u)u\,dx < +\infty\,.$$

Actually, the same proof shows a stronger fact, namely that

$$\overline{I}^0(u;0) \leq 0\,, \qquad \overline{I}^0(u;2u) \leq -2\int_\Omega f(x,u)u\,dx < +\infty\,.$$

Therefore we can apply [1, Corollary 5.10], obtaining $\beta \in \partial I(u)$ and $\mu \in \mathbf{R}$ with $\alpha = \beta - \mu\Delta u$. From [6, Theorems 3.3 and 2.25] we conclude that $f(x,u) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ and $\beta = -f(x,u)$. Then (a) easily follows.
(b) Let $c < 0$ and let (u_n) be a $(PS)_c$-sequence for I_r. By the previous point, we have $f(x,u_n) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ and there exists a sequence (μ_n) in $\mathbf{R}$ with

$$\|\mu_n\Delta u_n + f(x,u_n)\|_{H^{-1}} \to 0\,.$$

Up to a subsequence, (u_n) is convergent to some u weakly in $H^1_0(\Omega)$ and a.e. From $(f2)$ it follows $I(u) = c < 0$, hence $u \neq 0$. Again by $(f2)$ and Lebesgue's Theorem, we deduce that

$$0 < \int_\Omega f(x,u)u\,dx = \lim_n \int_\Omega f(x,u_n)u_n\,dx = \lim_n \mu_n \int_\Omega |Du_n|^2 dx\,.$$

Therefore, up to a further subsequence, (μ_n) is convergent to some $\mu > 0$ and

$$\left\|\Delta u_n + \frac{1}{\mu} f(x,u_n)\right\|_{H^{-1}} \to 0\,.$$

From [6, Lemma 4.8] we deduce that (u_n) is precompact in $H^1_0(\Omega)$ and (b) follows.
(c) Arguing as in (b), we find that $f(x,u) \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$ and that there exists $\mu > 0$ with $\mu\Delta u + f(x,u) = 0$. Then the assertion easily follows. ∎

LEMMA 2. - *There exists a sequence (b_n) of essential values of I_r strictly increasing to* 0.

Proof. - We will adapt some arguments from [4] to our concrete situation. Let $\psi :]-\infty, 0[\to \mathbf{R}$ be an increasing diffeomorphism. From Lemma 1 it follows that $\psi \circ I_r$ satisfies $(PS)_c$ for every $c \in \mathbf{R}$. Then by [2, Theorem 1.4.13] we have that $\{u \in S_r : \psi \circ I_r(u) \leq b\}$ has finite genus for every $b \in \mathbf{R}$. If (c_n) is the sequence defined as in [4, Theorem 2.12] with respect to $\psi \circ I_r$, it follows that $c_n \to +\infty$ as $n \to \infty$. Therefore there exists a sequence (b'_n) of essential values of $\psi \circ I_r$ strictly increasing to $+\infty$. Then $b_n = \psi^{-1}(b'_n)$ has the required properties. ■

Proof of Theorem 1. - Combining Lemma 1 with [4, Theorem 2.10], we deduce that each b_n is a critical value of I_r. Again from Lemma 1 we conclude that there exists a sequence $(\pm u_n, \lambda_n)$ of solutions of Problem 1 with $I(u_n) = b_n$ strictly increasing to 0. ■

Now we introduce the continuous functional $J_r : S_r \to \mathbf{R}$ defined by

$$J(u) = I(u) - \int_\Omega G(x,u)dx\,.$$

LEMMA 3. - *For every $\eta > 0$, there exists $\varepsilon > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, provided that $(g3)$ holds for $C_g = \varepsilon$.*

Proof. - By Sobolev inclusions, we have

$$0 \leq I_r(u) - J_r(u) = \int_\Omega G(x,u)dx \leq C_g \int_\Omega (1 + |u|^p)dx < \eta, \qquad \text{for any } u \in S_r,$$

if g is chosen as in the hypothesis. ■

Proof of Theorem 2. - As in the proof of Theorem 1, let us consider a strictly increasing sequence (b_n) of essential values of I_r such that $b_n \to 0$ as $n \to \infty$. Given $n \geq 1$, take some $\delta > 0$ with $b_n + \delta < 0$ and $2(b_j - b_{j-1}) < \delta$ for $j = 2, \ldots, n$. We apply [4, Theorem 2.6] to I_r and J_r. So, for any $j = 1, \ldots, n$, there exists $\eta_j > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta_j$ implies the existence of an essential value $c_j \in]b_j - \delta, b_j + \delta[$ of J_r. We now apply Lemma 3 for $\eta = \min\{\eta_1, \ldots, \eta_n\}$. Thus we obtain $\varepsilon_n > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, if $(g3)$ holds with $C_g = \varepsilon_n$. It follows that J_r has at least n distinct essential values $c_1, \ldots, c_n$ in the interval $]-\infty, 0[$.

Now Lemma 1 can be clearly adapted to the functional J_r. Then we find $u_1, \ldots, u_n \in S_r$ and $\lambda_1, \ldots, \lambda_n > 0$ such that each (u_j, λ_j) is a solution of Problem 2 with $J_r(u_j) = c_j$. ■

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