

Asymptotics for the blow-up boundary solution of the logistic equation with absorption*

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Abstract. Let Ω be a smooth bounded domain in \mathbf{R}^N . Assume that $f \geq 0$ is a C^1 -function on $[0, \infty)$ such that $f(u)/u$ is increasing on $(0, +\infty)$. Let a be a real number and let $b \geq 0$, $b \not\equiv 0$ be a continuous function such that $b \equiv 0$ on $\partial\Omega$. The purpose of this Note is to establish the asymptotic behaviour of the unique positive solution of the logistic problem $\Delta u + au = b(x)f(u)$ in Ω , subject to the singular boundary condition $u(x) \rightarrow +\infty$ as $\text{dist}(x, \partial\Omega) \rightarrow 0$. Our analysis is based on the Karamata regular variation theory.

Comportement asymptotique de la solution explosant au bord de l'équation logistique avec absorption

Résumé. Soit Ω un domaine borné et régulier de \mathbf{R}^N . On suppose que $0 \leq f \in C^1[0, \infty)$ est telle que $f(u)/u$ soit strictement croissante sur $(0, +\infty)$. Soit a un réel et $b \geq 0$, $b \not\equiv 0$, une fonction continue sur $\overline{\Omega}$ telle que $b \equiv 0$ sur $\partial\Omega$. Dans cette Note on établit le comportement asymptotique de l'unique solution positive du problème logistique $\Delta u + au = b(x)f(u)$ sur Ω avec la donnée au bord singulière $u(x) \rightarrow +\infty$ si $\text{dist}(x, \partial\Omega) \rightarrow 0$. Notre analyse porte sur la théorie de la variation régulière de Karamata.

Version française abrégée. Soit $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) un domaine borné et régulier, a un paramètre réel et $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$, $b \geq 0$ dans Ω . On considère le problème logistique avec explosion au bord

$$\Delta u + au = b(x)f(u) \quad \text{dans } \Omega, \quad u(x) \rightarrow +\infty \quad \text{si } d(x) := \text{dist}(x, \partial\Omega) \rightarrow 0, \quad (1)$$

où $0 \leq f \in C^1[0, \infty)$ satisfait la condition de Keller–Osserman (voir [6, 7]) et telle que $f(u)/u$ soit strictement croissante sur $(0, +\infty)$. Soit $\Omega_0 := \text{int} \{x \in \Omega : b(x) = 0\}$. On suppose que $\partial\Omega_0$ est régulier (éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre de l'opérateur $(-\Delta)$ dans $H_0^1(\Omega_0)$, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$. Dans [2] on

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montre que le problème (1) admet une solution positive u_a si et seulement si $a < \lambda_{\infty,1}$. L'unicité de la solution u_a est établie dans [1]. Soit \mathcal{K} l'ensemble des fonctions $k : (0, \nu) \rightarrow (0, \infty)$ (pour un certain ν), de classe C^1 , croissantes, telles que $\lim_{t \searrow 0} (\int_0^t k(s) ds / k(t))^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

Soit RV_q ($q \in \mathbf{R}$) l'ensemble des fonctions positives et mesurables $Z : [A, \infty) \rightarrow \mathbf{R}$ (avec $A > 0$) telles que $\lim_{u \rightarrow \infty} Z(\xi u) / Z(u) = \xi^q$, $\forall \xi > 0$. On désigne par NRV_q la classe des fonctions f définies par $f(u) = Cu^q \exp \{ \int_B^u \phi(t) / t dt \}$, $\forall u \geq B > 0$, où $C > 0$ et $\phi \in C[B, \infty)$ satisfait $\lim_{t \rightarrow \infty} \phi(t) = 0$. Supposons que $0 \leq f \in C^1[0, \infty) \cap NRV_{\rho+1}$ ($\rho > 0$) est telle que $f(u)/u$ soit strictement croissante sur $(0, \infty)$ et que $b \equiv 0$ sur $\partial\Omega$ vérifie $b(x) = k^2(d)(1 + o(1))$ si $d(x) \rightarrow 0$, avec $k \in \mathcal{K}$. Alors, pour chaque $a < \lambda_{\infty,1}$, le problème (1) admet une unique solution positive u_a (voir [1]). Le but de cette Note est d'établir la vitesse d'explosion au bord de la solution u_a .

Pour chaque $\zeta > 0$, soit

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{array}{l} k : \quad k(u^{-1}) = d_0 u [\Lambda(u)]^{-1} \exp \left[- \int_{d_1}^u (s \Lambda(s))^{-1} ds \right] \quad (u \geq d_1), \quad 0 < \Lambda \in C^1[d_1, \infty), \\ \lim_{u \rightarrow \infty} \Lambda(u) = \lim_{u \rightarrow \infty} u \Lambda'(u) = 0, \quad \lim_{u \rightarrow \infty} u^{\zeta+1} \Lambda'(u) = \ell_* \in \mathbf{R}, \quad d_0, d_1 > 0 \end{array} \right\}.$$

On a $\mathcal{R}_{0,\zeta} \subset \mathcal{K}$. De plus, si $k \in \mathcal{R}_{0,\zeta}$ alors $\ell_1 = 0$ et $\lim_{t \rightarrow 0} k(t) = 0$.

On définit les classes $\mathcal{F}_{\rho\eta} = \{f \in NRV_{\rho+1}(\rho > 0) : \phi \in RV_\eta \text{ ou } -\phi \in RV_\eta\}$, si $\eta \in (-\rho - 2, 0]$ et $\mathcal{F}_{\rho 0, \tau} = \{f \in \mathcal{F}_{\rho 0} : \lim_{u \rightarrow \infty} (\ln u)^\tau \phi(u) = \ell^* \in \mathbf{R}\}$, pour $\tau \in (0, \infty)$.

On démontre le résultat suivant.

THÉORÈME 1. - *On suppose que $b(x) = k^2(d)(1 + \tilde{c}d^\theta + o(d^\theta))$ si $d(x) \rightarrow 0$ (avec $\theta > 0$, $\tilde{c} \in \mathbf{R}$), où $k \in \mathcal{R}_{0,\zeta}$. Soit $0 \leq f \in C^1[0, \infty)$ telle que $f(u)/u$ soit strictement croissante sur $(0, \infty)$. De plus, on suppose que f satisfait l'un des cas suivants de croissance à l'infini:*

- (i) $f(u) = Cu^{\rho+1}$ dans un voisinage de l'infini;
- (ii) $f \in \mathcal{F}_{\rho\eta}$ avec $\eta \neq 0$;
- (iii) $f \in \mathcal{F}_{\rho 0, \tau_1}$ avec $\tau_1 = \varpi / \zeta$, où $\varpi = \min\{\theta, \zeta\}$.

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'unique solution positive u_a du problème (1) satisfait

$$u_a(x) = \xi_0 h(d)(1 + \chi d^\varpi + o(d^\varpi)) \quad \text{si } d(x) \rightarrow 0,$$

où $\xi_0 = [2(2 + \rho)^{-1}]^{1/\rho}$ et h est définie par $\int_{h(t)}^\infty [2F(s)]^{-1/2} ds = \int_0^t k(s) ds$, pour $t > 0$ suffisamment petit. L'expression de χ est donnée par

$$\chi = \begin{cases} -(1 + \zeta)\ell_*(2\zeta)^{-1} \text{Heaviside}(\theta - \zeta) - \tilde{c}\rho^{-1} \text{Heaviside}(\zeta - \theta) = \chi_1 & \text{dans les cas (i) et (ii)} \\ \chi_1 - \ell^* \rho^{-1} (-\rho \ell_*/2)^{\tau_1} (1/(\rho + 2) + \ln \xi_0) & \text{pour le cas (iii).} \end{cases}$$

Notons que le seul cas lié à ce résultat et correspondant à la situation particulière $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$, $k(t) = ct^\alpha \in \mathcal{K}$ (avec $c, \alpha > 0$), $\theta = 1$, a été étudié dans [4]. Dans ce travail, les deux premiers termes du développement asymptotique de u_a autour de $\partial\Omega$ tiennent compte de $d(x)$ ainsi que de la courbure moyenne H de $\partial\Omega$. Dans notre résultat on enlève la restriction $b > 0$ dans Ω et on garde la condition $b \equiv 0$ sur $\partial\Omega$, comme restriction naturelle héritée du problème logistique (voir [4]). De plus, on raffine la vitesse d'explosion de u_a pour une large classe de potentiels b , avec $\theta > 0$ quelconque et k appartenant à un ensemble très riche de fonctions.

Let $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) be a smooth bounded domain. Consider the blow-up logistic problem

$$\Delta u + au = b(x)f(u) \text{ in } \Omega, \quad u(x) \rightarrow +\infty \text{ as } d(x) := \text{dist}(x, \partial\Omega) \rightarrow 0, \quad (1)$$

where $f \in C^1[0, \infty)$, a is a real parameter and $0 \neq b \in C^{0,\mu}(\overline{\Omega})$ (for some $\mu \in (0, 1)$) satisfies $b \geq 0$ in Ω . Suppose that the absorption term f fulfills both

(A) $f \geq 0$ and $f(u)/u$ is increasing on $(0, \infty)$

and the Keller–Osseman condition (see [6, 7]) $\int_1^\infty [F(t)]^{-1/2} dt < \infty$, where $F(t) = \int_0^t f(s) ds$.

Assume throughout that $\Omega_0 \subset\subset \Omega$ satisfies the exterior cone condition (possibly, $\Omega_0 = \emptyset$) and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$, where $\Omega_0 := \text{int}\{x \in \Omega : b(x) = 0\}$. Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(\Omega_0)$. Set $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$. Under the above assumptions, we have proved in [2] that (1) has a positive solution u_a if and only if $a < \lambda_{\infty,1}$. Moreover, the uniqueness of u_a is studied in [1]. Denote by \mathcal{K} the set of all positive increasing C^1 -functions k defined on $(0, \nu)$, for some $\nu > 0$, which satisfy $\lim_{t \searrow 0} (\int_0^t k(s) ds / k(t))^{(i)} := \ell_i$, $i \in \overline{0,1}$. We have $\ell_0 = 0$ and $\ell_1 \in [0, 1]$.

Let us now recall some basic definitions related to the Karamata regular variation theory (see [5, 8]). Let RV_q ($q \in \mathbf{R}$) be the set of all positive measurable functions $Z : [A, \infty) \rightarrow \mathbf{R}$ (for some $A > 0$) satisfying $\lim_{u \rightarrow \infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. Define by NRV_q the class of functions f in the form $f(u) = Cu^q \exp\{\int_B^u \phi(t)/t dt\}$, $\forall u \geq B > 0$, where $C > 0$ is a constant and $\phi \in C[B, \infty)$ satisfies $\lim_{t \rightarrow \infty} \phi(t) = 0$. The Karamata Representation Theorem shows that $NRV_q \subset RV_q$.

If $f \in NRV_{\rho+1}$ ($\rho > 0$) satisfies (A) and $b \equiv 0$ on $\partial\Omega$ such that $b(x) = k^2(d)(1 + o(1))$ as $d(x) \rightarrow 0$, for some $k \in \mathcal{K}$, then for any $a \in (-\infty, \lambda_{\infty,1})$, there is a unique positive solution u_a of Eq. (1). Note that the Keller–Osseman condition is automatically fulfilled. Moreover, we have $\lim_{u \rightarrow \infty} \Xi(u) = \lim_{u \rightarrow \infty} [F(u)]^{1/2} [f(u) \int_u^\infty (F(s))^{-1/2} ds]^{-1} = \rho[2(\rho+2)]^{-1}$ (see [1]).

We have seen in [1] that the uniqueness of u_a is essentially based on the same boundary behaviour shown by any positive solution of (1). The purpose of this Note is to refine the blow-up rate of u_a near $\partial\Omega$ by giving the second term in the expansion of u_a near $\partial\Omega$. This is a more subtle question which represents the goal of more recent literature (see [4] and the references therein). The approach we give is very general and, as a novelty, it relies on the theory of regular variation instituted in the 30's by Karamata and subsequently developed by himself and many others (see [5, 8]). For any $\zeta > 0$, set $\mathcal{K}_{0,\zeta}$ the subset of \mathcal{K} with $\ell_1 = 0$ and $\lim_{t \searrow 0} t^{-\zeta} (\int_0^t k(s) ds / k(t))' := L_\star \in \mathbf{R}$. It can be proven that $\mathcal{K}_{0,\zeta} \equiv \mathcal{R}_{0,\zeta}$, where

$$\mathcal{R}_{0,\zeta} = \left\{ k : \begin{array}{l} k(u^{-1}) = d_0 u [\Lambda(u)]^{-1} \exp \left[- \int_{d_1}^u (s \Lambda(s))^{-1} ds \right] \quad (u \geq d_1), \quad 0 < \Lambda \in C^1[d_1, \infty), \\ \lim_{u \rightarrow \infty} \Lambda(u) = \lim_{u \rightarrow \infty} u \Lambda'(u) = 0, \quad \lim_{u \rightarrow \infty} u^{\zeta+1} \Lambda'(u) = \ell_\star \in \mathbf{R}, \quad d_0, d_1 > 0 \end{array} \right\}.$$

Moreover, ℓ_\star and L_\star are connected by $L_\star = -(1 + \zeta)\ell_\star/\zeta$ (see [3] for details). Define

$$\begin{aligned} \mathcal{F}_{\rho\eta} &= \{f \in NRV_{\rho+1} \ (\rho > 0) : \phi \in RV_\eta \text{ or } -\phi \in RV_\eta\}, \quad \eta \in (-\rho - 2, 0]; \\ \mathcal{F}_{\rho 0, \tau} &= \{f \in \mathcal{F}_{\rho 0} : \lim_{u \rightarrow \infty} (\ln u)^\tau \phi(u) = \ell^\star \in \mathbf{R}\}, \quad \tau \in (0, \infty). \end{aligned}$$

Our main result establishes the following asymptotic estimate.

THEOREM 1. - Assume that

$$b(x) = k^2(d)(1 + \tilde{c}d^\theta + o(d^\theta)) \text{ if } d(x) \rightarrow 0, \text{ where } k \in \mathcal{R}_{0,\zeta}, \theta > 0, \tilde{c} \in \mathbf{R}. \quad (2)$$

Suppose that f fulfills (A) and one of the following growth conditions at infinity:

(i) $f(u) = Cu^{\rho+1}$ in a neighbourhood of infinity;

- (ii) $f \in \mathcal{F}_{\rho\eta}$ with $\eta \neq 0$;
 - (iii) $f \in \mathcal{F}_{\rho 0, \tau_1}$ with $\tau_1 = \varpi/\zeta$, where $\varpi = \min\{\theta, \zeta\}$.
- Then, for any $a \in (-\infty, \lambda_{\infty, 1})$, the unique positive solution u_a of (1) satisfies

$$u_a(x) = \xi_0 h(d)(1 + \chi d^\varpi + o(d^\varpi)) \quad \text{if } d(x) \rightarrow 0, \quad \text{where } \xi_0 = [2(2 + \rho)^{-1}]^{1/\rho} \quad (3)$$

and h is defined by $\int_{h(t)}^\infty [2F(s)]^{-1/2} ds = \int_0^t k(s) ds$, for $t > 0$ small enough. The expression of χ is

$$\chi = \begin{cases} -(1 + \zeta)\ell_*(2\zeta)^{-1} \text{Heaviside}(\theta - \zeta) - \bar{c}\rho^{-1} \text{Heaviside}(\zeta - \theta) := \chi_1 & \text{if (i) or (ii) holds} \\ \chi_1 - \ell^* \rho^{-1} (-\rho\ell_*/2)^{\tau_1} [1/(\rho + 2) + \ln \xi_0] & \text{if } f \text{ obeys (iii).} \end{cases}$$

Note that the only case related, in same way, to our Theorem 1 corresponds to $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$ on $[0, \infty)$, $k(t) = ct^\alpha \in \mathcal{K}$ (where $c, \alpha > 0$), $\theta = 1$ in (2), being studied in [4]. There, the two-term asymptotic expansion of u_a near $\partial\Omega$ ($a \in \mathbf{R}$ since $\lambda_{\infty, 1} = \infty$) involves both the distance function $d(x)$ and the mean curvature H of $\partial\Omega$. However, the blow-up rate of u_a we present in Theorem 1 is of a different nature since the class $\mathcal{R}_{0, \zeta}$ does not include $k(t) = ct^\alpha$.

Our main result contributes to the knowledge in some new directions. More precisely, the blow-up rate of the unique positive solution u_a of (1) (found in [1]) is here refined

(a) on the maximal interval $(-\infty, \lambda_{\infty, 1})$ for the parameter a , which is in connection with an appropriate semilinear eigenvalue problem; thus, the condition $b > 0$ in Ω (which appears in [4]) is removed by defining the set Ω_0 , but we maintain $b \equiv 0$ on $\partial\Omega$ since this is a *natural* restriction inherited from the logistic problem (see [4] for details).

(b) when b satisfies (2), where θ is *any* positive number and k belongs to a very rich class of functions, namely $\mathcal{R}_{0, \zeta}$. The equivalence $\mathcal{R}_{0, \zeta} \equiv \mathcal{K}_{0, \zeta}$ shows the connection to the larger class \mathcal{K} (introduced in [1]) for which the uniqueness of u_a holds. In addition, the explicit form of $k \in \mathcal{R}_{0, \zeta}$ shows us how to build $k \in \mathcal{K}_{0, \zeta}$.

(c) for a wide class of functions $f \in NRV_{\rho+1}$ where either $\phi \equiv 0$ (case (i)) or ϕ (resp., $-\phi$) belongs to RV_η with $\eta \in (-\rho - 2, 0]$ (cases (ii) and (iii)). Therefore, the theory of regular variation plays a key role in understanding the general framework and the approach as well.

Proof of Theorem 1. - We first state two auxiliary results (see [3] for their proofs).

LEMMA 1. - Assume (2) and $f \in NRV_{\rho+1}$ satisfies (A). Then h has the following properties:

- (i) $h \in C^2(0, \nu)$, $\lim_{t \searrow 0} h(t) = \infty$ and $\lim_{t \searrow 0} h'(t) = -\infty$;
- (ii) $\lim_{t \searrow 0} h''(t)/[k^2(t)f(h(t)\xi)] = (2 + \rho\ell_1)/[\xi^{\rho+1}(2 + \rho)]$, $\forall \xi > 0$;
- (iii) $\lim_{t \searrow 0} h(t)/h''(t) = \lim_{t \searrow 0} h'(t)/h''(t) = \lim_{t \searrow 0} h(t)/h'(t) = 0$;
- (iv) $\lim_{t \searrow 0} h'(t)/[th''(t)] = -\rho\ell_1/(2 + \rho\ell_1)$ and $\lim_{t \searrow 0} h(t)/[t^2 h''(t)] = \rho^2 \ell_1^2/[2(2 + \rho\ell_1)]$;
- (v) $\lim_{t \searrow 0} h(t)/[th'(t)] = \lim_{t \searrow 0} [\ln t]/[\ln h(t)] = -\rho\ell_1/2$;
- (vi) If $\ell_1 = 0$, then $\lim_{t \searrow 0} t^j h(t) = \infty$, for all $j > 0$;
- (vii) $\lim_{t \searrow 0} 1/[t^\zeta \ln h(t)] = -\rho\ell_*/2$ and $\lim_{t \searrow 0} h'(t)/[t^{\zeta+1} h''(t)] = \rho\ell_*/(2\zeta)$, $\forall k \in \mathcal{R}_{0, \zeta}$.

Let $\tau > 0$ be arbitrary. For any $u > 0$, define $T_{1, \tau}(u) = \{\rho/[2(\rho + 2)] - \Xi(u)\}(\ln u)^\tau$ and $T_{2, \tau}(u) = \{f(\xi_0 u)/[\xi_0 f(u)] - \xi_0^\rho\}(\ln u)^\tau$. Note that if $f(u) = Cu^{\rho+1}$, for u in a neighbourhood V_∞ of infinity, then $T_{1, \tau}(u) = T_{2, \tau}(u) = 0$ for each $u \in V_\infty$.

LEMMA 2. - Assume (A) and $f \in \mathcal{F}_{\rho\eta}$. The following hold:

- (i) If $f \in \mathcal{F}_{\rho 0, \tau}$, then $\lim_{u \rightarrow \infty} T_{1, \tau}(u) = -\ell^*/(\rho + 2)^2$ and $\lim_{u \rightarrow \infty} T_{2, \tau}(u) = \xi_0^\rho \ell^* \ln \xi_0$.
- (ii) If $f \in \mathcal{F}_{\rho\eta}$ with $\eta \neq 0$, then $\lim_{u \rightarrow \infty} T_{1, \tau}(u) = \lim_{u \rightarrow \infty} T_{2, \tau}(u) = 0$.

Fix $\varepsilon \in (0, 1/2)$. We can find $\delta > 0$ such that $d(x)$ is of class C^2 on $\{x \in \mathbf{R}^N : d(x) < \delta\}$, k is nondecreasing on $(0, \delta)$, and $h'(t) < 0 < h''(t)$ for all $t \in (0, \delta)$ (see [1] for details). A straightforward computation shows that $\lim_{t \searrow 0} t^{1-\theta} k'(t)/k(t) = \infty$, for every $\theta > 0$. Using now (2), it follows that we can diminish $\delta > 0$ such that $k^2(t) [1 + (\tilde{c} - \varepsilon)t^\theta]$ is increasing on $(0, \delta)$ and

$$1 + (\tilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\tilde{c} + \varepsilon)d^\theta, \quad \forall x \in \Omega \text{ with } d \in (0, \delta). \quad (4)$$

We define $u^\pm(x) = \xi_0 h(d)(1 + \chi_\varepsilon^\pm d^\varpi)$, with $d \in (0, \delta)$, where $\chi_\varepsilon^\pm = \chi \pm \varepsilon [1 + \text{Heaviside}(\zeta - \theta)]/\rho$. Take $\delta > 0$ small enough such that $u^\pm(x) > 0$, for each $x \in \Omega$ with $d \in (0, \delta)$. By the Lagrange mean value theorem, we obtain $f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm d^\varpi h(d) f'(\Upsilon^\pm(d))$, where $\Upsilon^\pm(d) = \xi_0 h(d)(1 + \lambda^\pm(d) \chi_\varepsilon^\pm d^\varpi)$, for some $\lambda^\pm(d) \in [0, 1]$. We claim that

$$\lim_{d \searrow 0} f(\Upsilon^\pm(d))/f(\xi_0 h(d)) = 1. \quad (5)$$

Fix $\sigma \in (0, 1)$ and $M > 0$ such that $|\chi_\varepsilon^\pm| < M$. Choose $\mu^* > 0$ so that $|(1 \pm Mt)^{\rho+1} - 1| < \sigma/2$, for all $t \in (0, 2\mu^*)$. Let $\mu_\star \in (0, (\mu^*)^{1/\varpi})$ be such that, for every $x \in \Omega$ with $d \in (0, \mu_\star)$

$$|f(\xi_0 h(d)(1 \pm M\mu_\star))/f(\xi_0 h(d)) - (1 \pm M\mu_\star)^{\rho+1}| < \sigma/2.$$

Hence, $1 - \sigma < (1 - M\mu_\star)^{\rho+1} - \sigma/2 < f(\Upsilon^\pm(d))/f(\xi_0 h(d)) < (1 + M\mu_\star)^{\rho+1} + \sigma/2 < 1 + \sigma$, for every $x \in \Omega$ with $d \in (0, \mu_\star)$. This proves (5).

Step 1. There exists $\delta_1 \in (0, \delta)$ so that $\Delta u^+ + au^+ - k^2(d)[1 + (\tilde{c} - \varepsilon)d^\theta]f(u^+) \leq 0$, $\forall x \in \Omega$ with $d \in (0, \delta_1)$ and $\Delta u^- + au^- - k^2(d)[1 + (\tilde{c} + \varepsilon)d^\theta]f(u^-) \geq 0$, $\forall x \in \Omega$ with $d \in (0, \delta_1)$.

Indeed, for every $x \in \Omega$ with $d \in (0, \delta)$, we have

$$\begin{aligned} & \Delta u^\pm + au^\pm - k^2(d) [1 + (\tilde{c} \mp \varepsilon)d^\theta] f(u^\pm) \\ &= \xi_0 d^\varpi h''(d) \left[a \chi_\varepsilon^\pm \frac{h(d)}{h''(d)} + \chi_\varepsilon^\pm \Delta d \frac{h'(d)}{h''(d)} + 2\varpi \chi_\varepsilon^\pm \frac{h'(d)}{d h''(d)} + \varpi \chi_\varepsilon^\pm \Delta d \frac{h(d)}{d h''(d)} \right. \\ & \quad \left. + \varpi(\varpi - 1) \chi_\varepsilon^\pm \frac{h(d)}{d^2 h''(d)} + \Delta d \frac{h'(d)}{d^\varpi h''(d)} + \frac{a h(d)}{d^\varpi h''(d)} + \sum_{j=1}^4 \mathcal{S}_j^\pm(d) \right] \end{aligned} \quad (6)$$

where, for any $t \in (0, \delta)$, we denote

$$\begin{aligned} \mathcal{S}_1^\pm(t) &= (-\tilde{c} \pm \varepsilon) t^{\theta-\varpi} k^2(t) f(\xi_0 h(t)) / [\xi_0 h''(t)], \quad \mathcal{S}_2^\pm(t) = \chi_\varepsilon^\pm (1 - k^2(t) h(t) f'(\Upsilon^\pm(t)) / h''(t)), \\ \mathcal{S}_3^\pm(t) &= (-\tilde{c} \pm \varepsilon) \chi_\varepsilon^\pm t^\theta k^2(t) h(t) f'(\Upsilon^\pm(t)) / h''(t), \quad \mathcal{S}_4^\pm(t) = t^{-\varpi} (1 - k^2(t) f(\xi_0 h(t)) / [\xi_0 h''(t)]). \end{aligned}$$

By Lemma 1 (ii), we find $\lim_{t \searrow 0} k^2(t) f(\xi_0 h(t)) / [\xi_0 h''(t)]^{-1} = 1$, which yields $\lim_{t \searrow 0} \mathcal{S}_1^\pm(t) = (-\tilde{c} \pm \varepsilon) \text{Heaviside}(\zeta - \theta)$. Using [1, Lemma 1] and (5), we obtain $\lim_{t \searrow 0} k^2(t) h(t) f'(\Upsilon^\pm(t)) / h''(t) = \rho + 1$. Hence, $\lim_{t \searrow 0} \mathcal{S}_2^\pm(t) = -\rho \chi_\varepsilon^\pm$ and $\lim_{t \searrow 0} \mathcal{S}_3^\pm(t) = 0$.

Using the expression of h'' , we derive $\mathcal{S}_4^\pm(t) = \frac{k^2(t) f(h(t))}{h''(t)} \sum_{i=1}^3 \mathcal{S}_{4,i}(t)$, $\forall t \in (0, \delta)$, where we denote $\mathcal{S}_{4,1}(t) = 2 \frac{\Xi(h(t))}{t^\varpi} (\int_0^t k(s) ds / k(t))'$, $\mathcal{S}_{4,2}(t) = 2 \frac{T_{1,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}$ and $\mathcal{S}_{4,3}(t) = -\frac{T_{2,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}$.

Since $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$, we find $\lim_{t \searrow 0} \mathcal{S}_{4,1}(t) = -(1 + \zeta) \rho \ell_\star \zeta^{-1} (\rho + 2)^{-1} \text{Heaviside}(\theta - \zeta)$.

Cases (i), (ii). By Lemma 1 (vii) and Lemma 2 (ii), we find $\lim_{t \searrow 0} \mathcal{S}_{4,2}(t) = \lim_{t \searrow 0} \mathcal{S}_{4,3}(t) = 0$. In view of Lemma 1 (ii), we derive that $\lim_{t \searrow 0} \mathcal{S}_4^\pm(t) = -(1 + \zeta) \rho \ell_\star (2\zeta)^{-1} \text{Heaviside}(\theta - \zeta)$.

Case (iii). By Lemma 1 (vii) and Lemma 2 (i), $\lim_{t \searrow 0} \mathcal{S}_{4,2}(t) = -2\ell^\star (\rho + 2)^{-2} (-\rho \ell_\star / 2)^{\tau_1}$ and $\lim_{t \searrow 0} \mathcal{S}_{4,3}(t) = -2\ell^\star (\rho + 2)^{-1} (-\rho \ell_\star / 2)^{\tau_1} \ln \xi_0$. Using Lemma 1 (ii) once more, we arrive at $\lim_{t \searrow 0} \mathcal{S}_4^\pm(t) = -(1 + \zeta) \rho \ell_\star (2\zeta)^{-1} \text{Heaviside}(\theta - \zeta) - \ell^\star (-\rho \ell_\star / 2)^{\tau_1} [1/(\rho + 2) + \ln \xi_0]$.

Note that in each of the cases (i)–(iii), the definition of χ_ε^\pm yields $\lim_{t \searrow 0} \sum_{j=1}^4 \mathcal{S}_j^\pm(t) = -\varepsilon < 0$ and $\lim_{t \searrow 0} \sum_{j=1}^4 \mathcal{S}_j^-(t) = \varepsilon > 0$. By Lemma 1 (vii), $\lim_{t \searrow 0} \frac{h'(t)}{t^\varpi h''(t)} = 0$. But $\lim_{t \searrow 0} \frac{h(t)}{h'(t)} = 0$, so $\lim_{t \searrow 0} \frac{h(t)}{t^\varpi h''(t)} = 0$. Thus, using Lemma 1 [(iii), (iv)], relation (6) concludes our Step 1.

Step 2. There exists $M^+, \delta^+ > 0$ such that $u_a(x) \leq u^+(x) + M^+$, for all $x \in \Omega$ with $0 < d < \delta^+$.

Define $(0, \infty) \ni u \mapsto \Psi_x(u) = au - b(x)f(u)$, $\forall x$ with $d \in (0, \delta_1)$. Clearly, $\Psi_x(u)$ is decreasing when $a \leq 0$. Suppose $a \in (0, \lambda_{\infty,1})$. Obviously, $f(t)/t : (0, \infty) \rightarrow (f'(0), \infty)$ is bijective. Let $\delta_2 \in (0, \delta_1)$ be such that $b(x) < 1$, $\forall x$ with $d \in (0, \delta_2)$. Let u_x define the unique positive solution of $b(x)f(u)/u = a + f'(0)$, $\forall x$ with $d \in (0, \delta_2)$. Hence, for any x with $d \in (0, \delta_2)$, $u \rightarrow \Psi_x(u)$ is decreasing on (u_x, ∞) . But $\lim_{d(x) \searrow 0} \frac{b(x)f(u^+(x))}{u^+(x)} = +\infty$ (use $\lim_{d(x) \searrow 0} u^+(x)/h(d) = \xi_0$, (A) and Lemma 1 [(ii) and (iii)]). So, for δ_2 small enough, $u^+(x) > u_x$, $\forall x$ with $d \in (0, \delta_2)$.

Fix $\sigma \in (0, \delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega : \sigma < d(x) < \delta_2/2\}$. We define $u_\sigma^*(x) = u^+(d - \sigma, s) + M^+$, where (d, s) are the local coordinates of $x \in \mathcal{N}_\sigma$. We choose $M^+ > 0$ large enough to have $u_\sigma^*(\delta_2/2, s) \geq u_a(\delta_2/2, s)$, $\forall \sigma \in (0, \delta_2/4)$ and $\forall s \in \partial\Omega$. Using (4) and Step 1, we find

$$\begin{aligned} -\Delta u_\sigma^*(x) &\geq au^+(d - \sigma, s) - [1 + (\tilde{c} - \varepsilon)(d - \sigma)^\theta]k^2(d - \sigma)f(u^+(d - \sigma, s)) \\ &\geq au^+(d - \sigma, s) - [1 + (\tilde{c} - \varepsilon)d^\theta]k^2(d)f(u^+(d - \sigma, s)) \geq \Psi_x(u^+(d - \sigma, s)) \\ &\geq \Psi_x(u_\sigma^*) = au_\sigma^*(x) - b(x)f(u_\sigma^*(x)) \quad \text{in } \mathcal{N}_\sigma. \end{aligned}$$

Thus, by [2, Lemma 1], $u_a \leq u_\sigma^*$ in \mathcal{N}_σ , $\forall \sigma \in (0, \delta_2/4)$. Letting $\sigma \rightarrow 0$, we have proved Step 2.

Step 3. There exists $M^-, \delta^- > 0$ such that $u_a(x) \geq u^-(x) - M^-$, for all $x \in \Omega$ with $0 < d < \delta^-$.

For every $r \in (0, \delta)$, define $\Omega_r = \{x \in \Omega : 0 < d(x) < r\}$. We will prove that for $\lambda > 0$ sufficiently small, $\lambda u^-(x) \leq u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$. Indeed, fix arbitrarily $\sigma \in (0, \delta_2/4)$. Define $v_\sigma^*(x) = \lambda u^-(d + \sigma, s)$, for $x = (d, s) \in \Omega_{\delta_2/2}$. We choose $\lambda \in (0, 1)$ small enough such that $v_\sigma^*(\delta_2/4, s) \leq u_a(\delta_2/4, s)$, $\forall \sigma \in (0, \delta_2/4)$, $\forall s \in \partial\Omega$. Using (4), Step 1 and (A), we find

$$\begin{aligned} \Delta v_\sigma^*(x) + av_\sigma^*(x) &\geq \lambda k^2(d + \sigma)[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta]f(u^-(d + \sigma, s)) \\ &\geq k^2(d)[1 + (\tilde{c} + \varepsilon)d^\theta]f(\lambda u^-(d + \sigma, s)) \geq bf(v_\sigma^*), \end{aligned}$$

for all $x = (d, s) \in \Omega_{\delta_2/4}$, that is v_σ^* is a subsolution of $\Delta u + au = b(x)f(u)$ in $\Omega_{\delta_2/4}$. By [2, Lemma 1], we conclude that $v_\sigma^* \leq u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \rightarrow 0$, we find $\lambda u^-(x) \leq u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$.

Since $\lim_{d \searrow 0} u^-(x)/h(d) = \xi_0$, by using (A) and Lemma 1 [(ii), (iii)], we can easily obtain $\lim_{d \searrow 0} k^2(d)f(\lambda^2 u^-(x))/u^-(x) = \infty$. So, there exists $\tilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)[1 + (\tilde{c} + \varepsilon)d^\theta]f(\lambda^2 u^-)/u^- \geq \lambda^2|a|, \quad \forall x \in \Omega \text{ with } 0 < d \leq \tilde{\delta}. \quad (7)$$

By Lemma 1 [(i) and (v)], we deduce that $u^-(x)$ decreases with d when $d \in (0, \tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished). Choose $\delta_* \in (0, \tilde{\delta})$, close enough to $\tilde{\delta}$, such that

$$h(\delta_*)(1 + \chi_\varepsilon^- \delta_*^\varpi)/[h(\tilde{\delta})(1 + \chi_\varepsilon^- \tilde{\delta}^\varpi)] < 1 + \lambda. \quad (8)$$

For each $\sigma \in (0, \tilde{\delta} - \delta_*)$, we define $z_\sigma(x) = u^-(d + \sigma, s) - (1 - \lambda)u^-(\delta_*, s)$. We prove that z_σ is a subsolution of $\Delta u + au = b(x)f(u)$ in Ω_{δ_*} . Using (8), $z_\sigma(x) \geq u^-(\tilde{\delta}, s) - (1 - \lambda)u^-(\delta_*, s) > 0$ $\forall x = (d, s) \in \Omega_{\delta_*}$. By (4) and Step 1, z_σ is a subsolution of $\Delta u + au = b(x)f(u)$ in Ω_{δ_*} if

$$k^2(d + \sigma)[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta] [f(u^-(d + \sigma, s)) - f(z_\sigma(d, s))] \geq a(1 - \lambda)u^-(\delta_*, s), \quad (9)$$

for all $(d, s) \in \Omega_{\delta_*}$. Applying the Lagrange mean value theorem and (A), we infer that (9) is a consequence of $k^2(d + \sigma)[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta] f(z_\sigma(d, s))/z_\sigma(d, s) \geq |a|$, $\forall (d, s) \in \Omega_{\delta_*}$. This inequality holds by virtue of (7), (8) and the decreasing character of u^- with d .

On the other hand, $z_\sigma(\delta_*, s) \leq \lambda u^-(\delta_*, s) \leq u_a(x)$, $\forall x = (\delta_*, s) \in \Omega$. Clearly, $\limsup_{d \rightarrow 0} (z_\sigma - u_a)(x) = -\infty$ and $b > 0$ in Ω_{δ_*} . Thus, by [2, Lemma 1], $z_\sigma \leq u_a$ in Ω_{δ_*} , $\forall \sigma \in (0, \tilde{\delta} - \delta_*)$. Letting $\sigma \rightarrow 0$, we conclude the assertion of Step 3.

By Steps 2 and 3, $\chi_\varepsilon^+ \geq \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} - M^+ / [\xi_0 d^\varpi h(d)] \forall x \in \Omega$ with $d \in (0, \delta^+)$ and $\chi_\varepsilon^- \leq \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} + M^- / [\xi_0 d^\varpi h(d)] \forall x \in \Omega$ with $d \in (0, \delta^-)$. Passing to the limit as $d \rightarrow 0$ and using Lemma 1 (vi), we obtain $\chi_\varepsilon^- \leq \liminf_{d \rightarrow 0} \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi}$ and $\limsup_{d \rightarrow 0} \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} \leq \chi_\varepsilon^+$. Letting $\varepsilon \rightarrow 0$, we conclude our proof. ■

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