

Critical singular problems on infinite cones

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Abstract

We prove existence results for non autonomous perturbations of singular critical elliptic boundary value problems. The non singular case was treated by Tarantello [11] for bounded domains; here the singular weight allows for unbounded domains as cones and give rise to a different non compactness picture (as was first remarked by Caldirolì and Musina [5]).

Keywords : Singular weights, critical exponent, unbounded domains, Caffarelli-Kohn-Nirenberg inequalities.

1 Introduction

Let Ω be an open set in \mathbb{R}^N , $N \geq 2$ and let $\alpha \in (0, 2)$. For any $\zeta \in C_c^\infty(\Omega)$, define

$$\|\zeta\|_\alpha = \left(\int_\Omega |x|^\alpha |\nabla \zeta|^2 dx \right)^{1/2}.$$

Let $H_0^1(\Omega; |x|^\alpha)$ be the closure of $C_c^\infty(\Omega)$ with respect to the $\|\cdot\|_\alpha$ -norm. It turns out that $H_0^1(\Omega; |x|^\alpha)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_\alpha = \int_\Omega |x|^\alpha \nabla u \cdot \nabla v dx, \quad \forall u, v \in H_0^1(\Omega; |x|^\alpha).$$

If $\Omega = \mathbb{R}^N$ we set $H^1(\mathbb{R}^N; |x|^\alpha) = H_0^1(\mathbb{R}^N; |x|^\alpha)$. We remark that if Ω_1 and Ω_2 are arbitrary open sets in \mathbb{R}^N such that $\Omega_1 \subset \Omega_2$ then $H_0^1(\Omega_1; |x|^\alpha) \hookrightarrow H_0^1(\Omega_2; |x|^\alpha)$, with continuous embedding. We also point out that since we allow the cases $0 \in \overline{\Omega}$ or Ω unbounded then there is no inclusion relationship between $H_0^1(\Omega; |x|^\alpha)$ and the standard Sobolev space $H_0^1(\Omega)$. However, the Caffarelli-Kohn-Nirenberg inequality asserts that $H_0^1(\Omega; |x|^\alpha)$ is continuously embedded in $L^{2_\alpha^*}(\Omega)$, where $2_\alpha^* = 2N/(N-2+\alpha)$. More precisely, there exists $C_\alpha > 0$ such that

$$\left(\int_\Omega |u|^{2_\alpha^*} dx \right)^{1/2_\alpha^*} \leq C_\alpha \left(\int_\Omega |x|^\alpha |\nabla u|^2 dx \right)^{1/2},$$

for any $u \in H_0^1(\Omega; |x|^\alpha)$.

Consider the problem

$$\begin{cases} -\operatorname{div}(|x|^\alpha \nabla u) = |u|^{2_\alpha^*-2} u & \text{in } \Omega, \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

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We observe that degeneracy occurs in (1) if $0 \in \overline{\Omega}$ or if Ω is unbounded. We also point out that if 2_α^* in problem (1) is replaced by a subcritical exponent $p \in [2, 2_\alpha^*)$ then the corresponding equation is characterized by local compactness, and existence results are carried out in an easier way.

Consider the quotient

$$S_\alpha(u; \Omega) = \frac{\int_\Omega |x|^\alpha |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2_\alpha^*} dx\right)^{2/2_\alpha^*}},$$

and denote

$$S_\alpha(\Omega) = \inf_{u \in H_0^1(\Omega; |x|^\alpha) \setminus \{0\}} S_\alpha(u; \Omega). \quad (2)$$

It is obvious that if $u \in H_0^1(\Omega; |x|^\alpha)$ satisfies

$$\int_\Omega |x|^\alpha |\nabla u|^2 dx = S_\alpha(\Omega) \quad \text{and} \quad \int_\Omega |u|^{2_\alpha^*} dx = 1,$$

then the function $U(x) = [S_\alpha(\Omega)]^{1/(2_\alpha^*-2)} u(x)$ is a solution of (1).

Caldirola and Musina [5] studied the critical case and they showed that some concentration phenomena may occur in (1), due to the action of the non compact group of dilations in \mathbb{R}^N . They proved in [5] that if $\alpha \in (0, 2)$ then, in certain cases, $S_\alpha(\Omega)$ is attained in $H_0^1(\Omega; |x|^\alpha)$ by a positive function, so problem (1) has a solution. We point out (see Struwe [10, Theorem III.1.2]) that $S_\alpha(\Omega)$ is never attained in $H_0^1(\Omega)$ in the limiting case $\alpha = 0$ and if $\Omega \neq \mathbb{R}^N$.

Let $H^{-1}(\Omega; |x|^\alpha)$ be the dual space of $H_0^1(\Omega; |x|^\alpha)$ and denote by $\|\cdot\|_{-1}$ the norm in $H^{-1}(\Omega; |x|^\alpha)$. For any $f \in H^{-1}(\Omega; |x|^\alpha)$, consider the perturbed problem

$$\begin{cases} -\operatorname{div}(|x|^\alpha \nabla u) = |u|^{2_\alpha^*-2} u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

We say that a function $u \in H_0^1(\Omega; |x|^\alpha)$ is a solution of problem (3) if u is a critical point of the energy functional

$$J(u) = \frac{1}{2} \int_\Omega |x|^\alpha |\nabla u|^2 dx - \frac{1}{2_\alpha^*} \int_\Omega |u|^{2_\alpha^*} dx - \int_\Omega f u dx.$$

We observe that the Caffarelli-Kohn-Nirenberg inequality ensures that J is well defined on the space $H_0^1(\Omega; |x|^\alpha)$. Moreover, by the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{2_\alpha^*}(\Omega)$, the functional J is Fréchet differentiable on $H_0^1(\Omega; |x|^\alpha)$.

Perturbations of critical semilinear boundary value problems on bounded domains were initially studied by Tarantello in [11]. Our purpose is to prove a corresponding multiplicity result for the degenerate problem (3). Notice that in our case, Ω will be unbounded. We first need some preliminaries. Set

$$s_\alpha^0(\Omega) = \lim_{r \rightarrow 0} S_\alpha(\Omega \cap B_r)$$

and

$$s_\alpha^\infty(\Omega) = \lim_{r \rightarrow \infty} S_\alpha(\Omega \setminus B_r).$$

These limits are well defined because the mappings $r \mapsto S_\alpha(\Omega \cap B_r)$ and $r \mapsto S_\alpha(\Omega \setminus B_r)$ are easily seen to be respectively non increasing and non decreasing.

CONDITION \mathcal{C} . We say that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) satisfies Condition \mathcal{C} provided that Ω is a cone in \mathbb{R}^N , or $\Omega = \mathbb{R}^N$, or

$$S_\alpha(\Omega) < \min\{s_\alpha^0(\Omega), s_\alpha^\infty(\Omega)\}. \quad (4)$$

We recall that $\Omega \subset \mathbb{R}^N$ is a cone if Ω has Lipschitz boundary and if $\lambda x \in \Omega$ for every $\lambda > 0$ and $x \in \Omega$. If Ω is a cone then

$$S_\alpha(\Omega) = s_\alpha^0(\Omega) = s_\alpha^\infty(\Omega),$$

so equality holds in (4) (see [5, Lemma 3.9]). We also point out (see Caldiroli-Musina [5]) the following situations in which property (4) is fulfilled:

- (i) $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 is a cone and Ω_1 is an open bounded set such that $0 \notin \overline{\Omega_1}$;
- (ii) $\Omega = I \times \mathbb{R}^{N-1}$, where $I = \mathbb{R}$, or $I = (0, +\infty)$, or $I = (-\infty, 0)$, or I is bounded and $0 \notin \bar{I}$.

Denote by E_+ the positive cone of $E = H^{-1}(\Omega; |x|^\alpha)$. This means that $f \in E_+$ if and only if $f \neq 0$ and

$$\int_{\Omega} f u dx \geq 0,$$

for any $u \in H_0^1(\Omega; |x|^\alpha)$ such that $u \geq 0$ a.e. in Ω .

Our main result is the following

Theorem 1.1. *Assume that $\alpha \in (0, 2)$ and Ω satisfies Condition \mathcal{C} . Then, for each $g \in E_+$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, problem (3) with $f = \varepsilon g$ has at least two positive solutions.*

Remark 1.2. *a) In the previous theorem, ε_0 can be chosen uniformly for g in a compact subset of E_+ .*

b) The existence of at least two solutions (not necessarily positive) when g belongs to E instead of E_+ is less clear. The sign condition can easily be weakened, but we think the general case should require some additional assumption.

2 The first solution

We first recall that if c is a real number, X is a Banach space and $F : X \rightarrow \mathbb{R}$ is a C^1 -functional then F satisfies condition $(PS)_c$ if any sequence (u_n) in X such that $F(u_n) \rightarrow c$ and $\|F'(u_n)\|_{X^*} \rightarrow 0$ as $n \rightarrow \infty$, is relatively compact. It is obvious that if a Palais-Smale sequence converges strongly, then its limit is a critical point. Our first result shows that if a $(PS)_c$ sequence of J is weakly convergent then its limit is a solution of problem (3).

Lemma 2.1. *Let $(u_n) \subset H_0^1(\Omega; |x|^\alpha)$ be a $(PS)_c$ sequence of J , for some $c \in \mathbb{R}$. Assume that (u_n) converges weakly to some u_0 . Then u_0 is a solution of problem (3).*

Proof. Consider an arbitrary function $\zeta \in C_0^\infty(\Omega)$ and set $\omega = \text{supp}(\zeta)$. Obviously $J'(u_n) \rightarrow 0$ in $H_0^1(\Omega; |x|^\alpha)$ implies $\langle J'(u_n), \zeta \rangle \rightarrow 0$ as $n \rightarrow \infty$, that is

$$\lim_{n \rightarrow \infty} \left(\int_{\omega} |x|^\alpha \nabla u_n \cdot \nabla \zeta dx - \int_{\omega} |u_n|^{2_\alpha^*-2} u_n \zeta dx - \int_{\omega} f \zeta dx \right) = 0. \quad (5)$$

Since $u_n \rightharpoonup u_0$ in $H_0^1(\Omega; |x|^\alpha)$ it follows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |x|^\alpha \nabla u_n \cdot \nabla \zeta \, dx = \int_{\Omega} |x|^\alpha \nabla u_0 \cdot \nabla \zeta \, dx. \quad (6)$$

The boundedness of (u_n) in $H_0^1(\Omega; |x|^\alpha)$ and the Caffarelli-Kohn-Nirenberg inequality imply that $|u_n|^{2_\alpha^*-2} u_n$ is bounded in $L^{2_\alpha^*/(2_\alpha^*-1)}(\Omega; |x|^\alpha)$. Combining this with the convergence (up to a subsequence)

$$|u_n|^{2_\alpha^*-2} u_n \rightharpoonup |u_0|^{2_\alpha^*-2} u_0 \quad \text{a.e. in } \Omega$$

we deduce that $|u_0|^{2_\alpha^*-2} u_0$ is the weak limit of the sequence $|u_n|^{2_\alpha^*-2} u_n$ in the space $L^{2_\alpha^*/(2_\alpha^*-1)}(\Omega; |x|^\alpha)$. So

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{2_\alpha^*-2} u_n \zeta \, dx = \int_{\Omega} |u_0|^{2_\alpha^*-2} u_0 \zeta \, dx. \quad (7)$$

From (5), (6) and (7) we deduce that

$$\int_{\Omega} |x|^\alpha \nabla u_0 \cdot \nabla \zeta \, dx - \int_{\Omega} |u_0|^{2_\alpha^*-2} u_0 \zeta \, dx - \int_{\Omega} f \zeta \, dx = 0.$$

By density, this equality holds for any $\zeta \in H_0^1(\Omega; |x|^\alpha)$ which means that $J'(u_0) = 0$. \square

Lemma 2.2. *There exists $\varepsilon_1 > 0$ such that problem (3) has at least one solution u_0 provided that $f \neq 0$ and $\|f\|_{-1} < \varepsilon_1$. Moreover, u_0 is positive if $f \in E_+$.*

Proof. The idea is to show that there exist $c_0 < 0$ and $R > 0$ such that J has the $(PS)_{c_0}$ property, where

$$c_0 = \inf\{J(u); u \in H_0^1(\Omega; |x|^\alpha) \text{ and } \|u\| \leq R\}. \quad (8)$$

Then we prove that c_0 is achieved by some $u_0 \in H_0^1(\Omega; |x|^\alpha)$ and, furthermore, $J'(u_0) = 0$.

Applying the Caffarelli-Kohn-Nirenberg inequality we have

$$\begin{aligned} J(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{2_\alpha^*} \int_{\Omega} |u|^{2_\alpha^*} \, dx - \int_{\Omega} f u \, dx \geq \\ &\frac{1}{2} \|u\|^2 - \frac{1}{2_\alpha^*} \int_{\Omega} |u|^{2_\alpha^*} \, dx - \|f\|_{-1} \cdot \|u\| \geq \\ &\left(\frac{1}{2} - \frac{\varepsilon^2}{2}\right) \|u\|^2 - C \|u\|^{2_\alpha^*} - C_\varepsilon \|f\|_{-1}^2. \end{aligned}$$

Fixing $\varepsilon \in (0, 1)$ we find $R > 0$, $\varepsilon_1 > 0$ and $\delta > 0$ such that $J(u) \geq \delta$ if $\|u\| = R$ and $\|f\|_{-1} < \varepsilon_1$.

Let c_0 be defined in (8). Since $f \neq 0$, $c_0 < J(0) = 0$. The set

$$\overline{B}_R := \{u \in H_0^1(\Omega; |x|^\alpha); \|u\| \leq R\}$$

becomes a complete metric space with respect to the distance

$$\text{dist}(u, v) = \|u - v\| \quad \text{for any } u, v \in \overline{B}_R.$$

On the other hand, J is lower semi-continuous and bounded from below on \overline{B}_R . So, by Ekeland's variational principle [8, Theorem 1.1], for any positive integer n there exists u_n such that

$$c_0 \leq J(u_n) \leq c_0 + \frac{1}{n}, \quad (9)$$

and

$$J(w) \geq J(u_n) - \frac{1}{n} \|u_n - w\| \quad \text{for all } w \in \overline{B}_R. \quad (10)$$

We claim that $\|u_n\| < R$ for n large enough. Indeed, if $\|u_n\| = R$ for infinitely many n , we may assume, without loss of generality, that $\|u_n\| = R$ for all $n \geq 1$. It follows that $J(u_n) \geq \delta > 0$. Combining this with (9) and letting $n \rightarrow \infty$, we have $0 \geq c_0 \geq \delta > 0$ which is a contradiction.

We now prove that $\|J'(u_n)\|_{-1} \rightarrow 0$. Indeed, for any $u \in H_0^1(\Omega; |x|^\alpha)$ with $\|u\| = 1$, let $w_n = u_n + tu$. For a fixed n , we have $\|w_n\| \leq \|u_n\| + t < R$, where $t > 0$ is small enough. Using (10) we obtain

$$J(u_n + tu) \geq J(u_n) - \frac{t}{n} \|u\|$$

that is

$$\frac{J(u_n + tu) - J(u_n)}{t} \geq -\frac{1}{n} \|u\| = -\frac{1}{n}.$$

Letting $t \searrow 0$, we deduce that $\langle J'(u_n), u \rangle \geq -\frac{1}{n}$ and a similar argument for $t \nearrow 0$ produces $|\langle J'(u_n), u \rangle| \leq \frac{1}{n}$ for any $u \in H_0^1(\Omega; |x|^\alpha)$ with $\|u\| = 1$. So,

$$\|J'(u_n)\|_{-1} = \sup_{\|u\|=1} |\langle J'(u_n), u \rangle| \leq \frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We have obtained the existence of a $(PS)_{c_0}$ sequence, i.e. a sequence $(u_n) \subset H_0^1(\Omega; |x|^\alpha)$ with

$$J(u_n) \rightarrow c_0 \quad \text{and} \quad \|J'(u_n)\|_{-1} \rightarrow 0. \quad (11)$$

But $\|u_n\| \leq R$ shows that (u_n) converges weakly in $H_0^1(\Omega; |x|^\alpha)$, up to a subsequence. Therefore, by (11) and Lemma 2.1 we find that for some $u_0 \in H_0^1(\Omega; |x|^\alpha)$,

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega; |x|^\alpha), \quad u_n \rightarrow u_0 \text{ a.e. in } \mathbb{R}^N \quad (12)$$

and

$$J'(u_0) = 0. \quad (13)$$

We now prove that $J(u_0) = c_0$. By (11) and (12) we have

$$o(1) = \langle J'(u_n), u_n \rangle = \int_{\Omega} |x|^\alpha |\nabla u_n|^2 dx - \int_{\Omega} |u_n|^{2_\alpha^*} dx - \int_{\Omega} f u_n dx.$$

Therefore

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \int_{\Omega} |u_n|^{2_\alpha^*} dx - \left(1 - \frac{1}{2_\alpha^*} \right) \int_{\Omega} f u_n dx + o(1).$$

By (11), (12), (13) and Fatou's lemma we have

$$c_0 = \liminf_{n \rightarrow \infty} J(u_n) \geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \int_{\Omega} |x|^\alpha |u_0|^{2_\alpha^*} dx - \left(1 - \frac{1}{2_\alpha^*} \right) \int_{\Omega} f u_0 dx = J(u_0).$$

Since $u_0 \in \overline{B}_R$, it follows that $J(u_0) = c_0$. If $f \in E_+$, u_0 can be replaced by $|u_0|$, and the proof is complete. \square

3 A priori estimates for the second solution

Set

$$I(u) = \frac{1}{2} \int_{\Omega} |x|^{\alpha} |\nabla u|^2 dx - \frac{1}{2_{\alpha}^*} \int_{\Omega} |u|^{2_{\alpha}^*} dx$$

and denote

$$S = \{u \in H_0^1(\Omega; |x|^{\alpha}) \setminus \{0\}; \langle I'(u), u \rangle = 0\}.$$

We first justify that $S \neq \emptyset$. Indeed, fix $u_0 \in H_0^1(\Omega; |x|^{\alpha}) \setminus \{0\}$ and set, for any $\lambda > 0$,

$$\Psi(\lambda) = \langle I'(\lambda u_0), \lambda u_0 \rangle = \lambda^2 \int_{\Omega} |x|^{\alpha} |\nabla u_0|^2 dx - \lambda^{2_{\alpha}^*} \int_{\Omega} |u_0|^{2_{\alpha}^*} dx.$$

Since $2_{\alpha}^* > 2$, it follows that $\Psi(\lambda) < 0$ for λ large enough and $\Psi(\lambda) > 0$ for λ sufficiently close to zero.

Hence there exists $\lambda_0 \in (0, \infty)$ such that $\Psi(\lambda_0) = 0$. This means that $\lambda_0 u_0 \in S$.

Lemma 3.1. *Let $I_{\infty} = \inf \{I(u); u \in S\}$. Then there exists $\bar{u} \in H_0^1(\Omega; |x|^{\alpha})$ such that*

$$I_{\infty} = I(\bar{u}) = \sup_{t \geq 0} I(t\bar{u}). \quad (14)$$

Proof. We first claim that

$$I_{\infty}(u) = \sup_{t \geq 0} I(tu) \quad \forall u \in S. \quad (15)$$

Indeed, for some fixed $\varphi \in H_0^1(\Omega; |x|^{\alpha}) \setminus \{0\}$, denote

$$f(t) = I(t\varphi) = \frac{t^2}{2} \int_{\Omega} |x|^{\alpha} |\nabla \varphi|^2 dx - \frac{t^{2_{\alpha}^*}}{2_{\alpha}^*} \int_{\Omega} |\varphi|^{2_{\alpha}^*} dx.$$

We have

$$f'(t) = t \int_{\Omega} |x|^{\alpha} |\nabla \varphi|^2 dx - t^{2_{\alpha}^*-1} \int_{\Omega} |\varphi|^{2_{\alpha}^*} dx,$$

which vanishes for

$$t_0 = t_0(\varphi) = \left\{ \frac{\int_{\Omega} |x|^{\alpha} |\nabla \varphi|^2 dx}{\int_{\Omega} |\varphi|^{2_{\alpha}^*} dx} \right\}^{\frac{1}{2_{\alpha}^*-2}}.$$

Hence

$$f(t_0) = I(t_0\varphi) = \sup_{t \geq 0} I(t\varphi) = \frac{2-\alpha}{2N} \left\{ \frac{\int_{\Omega} |x|^{\alpha} |\nabla \varphi|^2 dx}{\left(\int_{\Omega} |\varphi|^{2_{\alpha}^*} dx \right)^{\frac{N-2+\alpha}{N}}} \right\}^{\frac{N}{2-\alpha}}.$$

It follows that

$$\inf_{\varphi \in H_0^1(\Omega; |x|^{\alpha}) \setminus \{0\}} \sup_{t \geq 0} I(t\varphi) = \frac{2-\alpha}{2N} [S_{\alpha}(\Omega)]^{\frac{N}{2-\alpha}}. \quad (16)$$

We now easily observe that for every $u \in S$ we have $t_0(u) = 1$. So, by (16), we find (15).

By Caldiroli-Musina [5], Theorems 2.2 and 3.1, the minimum is achieved in (2) by some function $U \in H_0^1(\Omega; |x|^\alpha)$. We prove in what follows that the function $\bar{u} := [S_\alpha(\Omega)]^{1/(2_\alpha^*-2)}U$ satisfies (14). We first observe that $\bar{u} \in S$ and

$$I(\bar{u}) = \frac{2-\alpha}{2N} [S_\alpha(\Omega)]^{\frac{N}{2-\alpha}}. \quad (17)$$

So, by (15) and (17),

$$I_\infty = \inf_{u \in S} I(u) = \inf_{u \in S} \sup_{t \geq 0} I(tu) \geq \inf_{u \in H_0^1(\Omega; |x|^\alpha) \setminus \{0\}} \sup_{t \geq 0} I(tu) = \frac{2-\alpha}{2N} [S_\alpha(\Omega)]^{\frac{N}{2-\alpha}} = I(\bar{u}),$$

which concludes our proof. \square

Lemma 3.2. *Assume (u_n) is a $(PS)_c$ sequence of J that converges weakly to u_0 in $H_0^1(\Omega; |x|^\alpha)$. Then either (u_n) converges strongly in $H_0^1(\Omega; |x|^\alpha)$, or $c \geq J(u_0) + I_\infty$.*

Proof. Since (u_n) is a $(PS)_c$ sequence and $u_n \rightharpoonup u_0$ in $H_0^1(\Omega; |x|^\alpha)$ we have

$$J(u_n) = c + o(1) \quad \text{and} \quad \langle J'(u_n), u_n \rangle = o(1). \quad (18)$$

Set $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in $H_0^1(\Omega; |x|^\alpha)$ which implies

$$\begin{aligned} \int_{\Omega} |x|^\alpha \nabla v_n \cdot \nabla u_0 \, dx &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ \int_{\Omega} f v_n \, dx &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We rewrite the above relations as

$$\begin{aligned} \|u_n\|^2 &= \|u_0\|^2 + \|v_n\|^2 + o(1), \\ J(v_n) &= I(v_n) + o(1). \end{aligned} \quad (19)$$

The Brezis-Lieb Lemma combined with the Caffarelli-Kohn-Nirenberg Inequality yield

$$\int_{\Omega} (|u_n|^{2_\alpha^*} - |v_n|^{2_\alpha^*}) \, dx = \int_{\Omega} |u_0|^{2_\alpha^*} \, dx + o(1). \quad (20)$$

From (18), (19), (20) and Lemma 2.1 we find

$$\begin{aligned} o(1) + c &= J(u_n) = J(u_0) + J(v_n) + o(1) = J(u_0) + I(v_n) + o(1), \\ o(1) &= \langle J'(u_n), u_n \rangle = \langle J'(u_0), u_0 \rangle + \langle J'(v_n), v_n \rangle + o(1) = \langle I'(v_n), v_n \rangle + o(1). \end{aligned} \quad (21)$$

If $v_n \rightarrow 0$ in $H_0^1(\Omega; |x|^\alpha)$, then $u_n \rightarrow u_0$ in $H_0^1(\Omega; |x|^\alpha)$ and $J(u_0) = \lim_{n \rightarrow \infty} J(u_n) = c$.

If $v_n \not\rightarrow 0$ in $H_0^1(\Omega; |x|^\alpha)$, then combining this with the fact that $v_n \rightharpoonup 0$ in $H_0^1(\Omega; |x|^\alpha)$ we may assume that $\|v_n\| \rightarrow l > 0$. Then, by (21),

$$c = J(u_0) + I(v_n) + o(1) \quad (22)$$

$$\mu_n = \langle I'(v_n), v_n \rangle = \int_{\Omega} |x|^\alpha |\nabla v_n|^2 \, dx - \int_{\Omega} |v_n|^{2_\alpha^*} \, dx = \alpha_n - \beta_n, \quad (23)$$

where $\lim_{n \rightarrow \infty} \mu_n = 0$, $\alpha_n = \int_{\Omega} |x|^\alpha |\nabla v_n|^2 \, dx \geq \|v_n\|^2$ and $\beta_n = \int_{\Omega} |v_n|^{2_\alpha^*} \, dx \geq 0$. In virtue of (22), it remains to show that $I(v_n) \geq I_\infty + o(1)$. For $t > 0$, we have

$$\langle I'(tv_n), tv_n \rangle = t^2 \int_{\Omega} |x|^\alpha |\nabla v_n|^2 \, dx - t^{2_\alpha^*} \int_{\Omega} |v_n|^{2_\alpha^*} \, dx.$$

If we prove the existence of a sequence (t_n) with $t_n \rightarrow 1$ and $\langle I'(t_n v_n), t_n v_n \rangle = 0$ then

$$I(v_n) = I(t_n v_n) + \frac{1 - t_n^2}{2} \alpha_n - \frac{1 - t_n^{2^*}}{2^*} \|v_n\|_{L^{2^*}}^{2^*} = I(t_n v_n) + o(1) \geq I_\infty + o(1)$$

and the conclusion follows. To do this, let $t = 1 + \delta$ with $\delta > 0$ small enough and using (23) we obtain

$$\begin{aligned} \langle I'(t v_n), t v_n \rangle &= (1 + \delta)^2 \alpha_n - (1 + \delta)^{2^*} \beta_n = (1 + \delta)^2 \alpha_n - (1 + \delta)^{2^*} (\alpha_n - \mu_n) = \\ &= \alpha_n (2\delta - 2^* \delta + o(\delta)) + (1 + \delta)^{2^*} \mu_n = \alpha_n (2 - 2^*) \delta + \alpha_n o(\delta) + (1 + \delta)^{2^*} \mu_n. \end{aligned}$$

Since $\alpha_n \rightarrow \bar{l} \geq l^2 > 0$, $\lim_{n \rightarrow \infty} \mu_n = 0$ and $2^* > 2$ then, for n large enough, we can define the sequence $\delta_n = \frac{2|\mu_n|}{\alpha_n(2^* - 2)} > 0$ and $\delta_n \rightarrow 0$. Then

$$\langle I'((1 + \delta_n) v_n), (1 + \delta_n) v_n \rangle < 0 \quad \langle I'((1 - \delta_n) v_n), (1 - \delta_n) v_n \rangle > 0. \quad (24)$$

From (24) we deduce the existence of $t_n \in (1 - \delta_n, 1 + \delta_n)$ such that

$$t_n \rightarrow 1 \quad \text{and} \quad \langle I'(t_n v_n), t_n v_n \rangle = 0.$$

This concludes our proof. \square

Fix $\bar{u} \in H_0^1(\Omega; |x|^\alpha)$ such that (14) holds. Since $2 < 2^*$, there exists $t_0 > 0$ such that

$$\begin{aligned} I(t\bar{u}) &< 0 \quad \text{if } t \geq t_0 \\ J(t\bar{u}) &< 0 \quad \text{if } t \geq t_0. \end{aligned}$$

Set

$$\mathcal{P} = \{\gamma \in C([0, 1], H_0^1(\Omega; |x|^\alpha)); \gamma(0) = 0, \gamma(1) = t_0 \bar{u}\} \quad (25)$$

$$c_1 = \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} J(u). \quad (26)$$

In the next result c_0 , resp. c_1 , are those defined in (8), resp. (26).

Lemma 3.3. *Given $g \in E_+$, $\|g\|_{-1} = 1$, there exist $R > 0$ and $\varepsilon_2 = \varepsilon_2(R) > 0$ such that $c_1 < c_0 + I_\infty$, for all $f = \varepsilon g$ with $\varepsilon \leq \varepsilon_2$.*

Proof. We first remark that

$$I_\infty + c_0 > 0, \quad (27)$$

provided that ε_1 and R given in the proof of Lemma 2.2 are sufficiently small. Indeed, let u_0 be the solution obtained in Lemma 2.2. Then, by Cauchy-Schwarz,

$$\begin{aligned} c_0 &= \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\Omega} |x|^\alpha |\nabla u_0|^2 dx - \left(1 - \frac{1}{2^*} \right) \int_{\Omega} f u_0 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\Omega} |x|^\alpha |\nabla u_0|^2 dx - \left(1 - \frac{1}{2^*} \right) \|f\|_{-1} \cdot \|u_0\|. \end{aligned} \quad (28)$$

Applying the inequality

$$\alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2} \quad \forall \alpha, \beta > 0$$

we find

$$\left(1 - \frac{1}{2_\alpha^*}\right) \|f\|_{-1} \cdot \|u_0\| \leq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right) \|u_0\|^2 + \frac{(N - \alpha + 2)^2}{16N(2 - \alpha)} \|f\|_{-1}^2. \quad (29)$$

So, by (28) and (29),

$$c_0 \geq -\frac{(N - \alpha + 2)^2}{16N(2 - \alpha)} \|f\|_{-1}^2. \quad (30)$$

It follows that the negative number c_0 is close enough to 0 if $\|f\|_{-1}$ is small. But, by Lemma 3.1,

$$I_\infty = \frac{2 - \alpha}{2N} [S_\alpha(\Omega)]^{N/(2-\alpha)} > 0,$$

so (27) follows obviously.

In order to conclude the proof we observe, by the definition of c_1 , that it suffices to show that

$$\sup_{t \geq 0} J(t\bar{u}) < c_0 + I_\infty, \quad (31)$$

if $\|f\|_{-1}$ is sufficiently small.

Next, using (27), the continuity of J and $J(0) = 0$, we obtain some $T_0 > 0$ which is uniform with respect to all f satisfying $0 < \|f\|_{-1} < \varepsilon_1$ such that, for some $\varepsilon' < \varepsilon_1$,

$$c_0 + I_\infty > \sup_{t \in [0, T_0]} J(t\bar{u}),$$

if $\|f\|_{-1} < \varepsilon'$. So, in order to prove (31), it suffices to show that if $\|f\|_{-1}$ is small then

$$c_0 + I_\infty > \sup_{t \geq T_0} J(t\bar{u}). \quad (32)$$

But

$$\begin{aligned} J(t\bar{u}) &= \frac{t^2}{2} \int_{\Omega} |x|^\alpha |\nabla \bar{u}|^2 dx - \frac{t^{2_\alpha^*}}{2_\alpha^*} \int_{\Omega} |\bar{u}|^{2_\alpha^*} dx - t \int_{\Omega} f \bar{u} dx \\ &\leq \frac{t^2}{2} \int_{\Omega} |x|^\alpha |\nabla \bar{u}|^2 dx - \frac{t^{2_\alpha^*}}{2_\alpha^*} \int_{\Omega} |\bar{u}|^{2_\alpha^*} dx - T_0 \int_{\Omega} f \bar{u} dx, \end{aligned}$$

for any $t \geq T_0$. But, by Lemma 3.1,

$$I(\bar{u}) = \frac{2 - \alpha}{2N} [S_\alpha(\Omega)]^{N/(2-\alpha)}.$$

Hence, using an argument similar to that used for proving (28), we find

$$\begin{aligned} \sup_{t \geq T_0} J(t\bar{u}) &\leq \sup_{t \geq T_0} \left(\frac{t^2}{2} \int_{\Omega} |x|^\alpha |\nabla \bar{u}|^2 dx - \frac{t^{2_\alpha^*}}{2_\alpha^*} \int_{\Omega} |\bar{u}|^{2_\alpha^*} dx \right) - T_0 \int_{\Omega} f \bar{u} dx \\ &\leq I_\infty - T_0 \int_{\Omega} f \bar{u} dx < I_\infty + c_0, \end{aligned}$$

if $f = \varepsilon g$ with $\varepsilon \leq \varepsilon''$. Indeed, it follows from (30) that c_0 is quadratic in ε while $\int f \bar{u}$ is linear. Letting $\varepsilon_2 = \min\{\varepsilon', \varepsilon''\}$, we conclude the proof. \square

4 Proof of Theorem 1.1 concluded

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Hence, by Lemma 2.2, we obtain the existence of a positive solution $u_0 \in H_0^1(\Omega; |x|^\alpha)$ of (3) such that $J(u_0) = c_0$.

On the other hand, since $J(|u|) \leq J(u)$ when $f \in E_+$, it follows from the Mountain Pass Theorem without the Palais-Smale condition [3, Theorem 2.2] that there exists a positive $(PS)_{c_1}$ sequence (u_n) of J , that is

$$J(u_n) = c_1 + o(1) \quad \text{and} \quad \|J'(u_n)\|_{-1} \rightarrow 0.$$

This implies

$$\begin{aligned} c_1 + \frac{1}{2_\alpha^*} \|J'(u_n)\|_{-1} \cdot \|u_n\| + o(1) &\geq J(u_n) - \frac{1}{2_\alpha^*} \langle J'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \|u_n\|^2 - \left(1 - \frac{1}{2_\alpha^*} \right) \|f\|_{-1} \cdot \|u_n\|. \end{aligned} \quad (33)$$

Hence $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega; |x|^\alpha)$. So, up to a subsequence, we may assume that $u_n \rightharpoonup u_1 \geq 0$ in $H_0^1(\Omega; |x|^\alpha)$. Lemma 2.1 implies that u_1 is a solution of (3).

We prove in what follows that $u_0 \neq u_1$. For this aim we shall prove that $J(u_0) \neq J(u_1)$. Indeed, by Lemma 3.2, either $u_n \rightarrow u_1$ in $H_0^1(\Omega; |x|^\alpha)$ which gives

$$J(u_1) = \lim_{n \rightarrow \infty} J(u_n) = c_1 > 0 > c_0 = J(u_0)$$

and the conclusion follows, or

$$c_1 = \lim_{n \rightarrow \infty} J(u_n) \geq J(u_1) + I_\infty.$$

If we suppose that $J(u_1) = J(u_0) = c_0$, then $c_1 \geq c_0 + I_\infty$ which contradicts Lemma 3.3. This concludes our proof. \square

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