Existence and Uniqueness of Positive Solutions to a Semilinear Elliptic Problem in $\mathbf{R}^N$

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Let $p \in C^{0,\alpha}_{loc}(\mathbf{R}^N)$ with $p > 0$ and let $f \in C^1((0,\infty),(0,\infty))$ be such that $\lim_{u \searrow 0} f(u)/u = +\infty$, f is bounded at infinity and the mapping $u \longmapsto f(u)/(u+\beta)$ is decreasing on $(0,\infty)$, for some $\beta > 0$. We prove that the problem $-\Delta u = p(x)f(u)$ in $\mathbf{R}^N$, $N > 2$, has a unique positive $C^{2,\alpha}_{loc}(\mathbf{R}^N)$ solution which vanishes at infinity provided $\int_0^\infty r\Phi(r)dr < \infty$, where $\Phi(r) = \max\{p(x); |x| = r\}$. Furthermore, it is showed that this condition is nearly optimal. Our results extend previous works by Lair-Shaker and Zhang, while the proofs are based on two theorems on bounded domains, due to Brezis-Oswald and Crandall-Rabinowitz-Tartar.

1 Introduction

Consider the problem

$$\begin{cases} -\Delta u = p(x)f(u) \quad \text{in} \quad \mathbf{R}^N \\ u > 0 \quad \text{in } \mathbf{R}^N \\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty\,, \end{cases} \tag{1}$$

where $N > 2$ and the function p satisfies the following hypotheses:
(p1) $p \in C^{0,\alpha}_{\rm loc}(\mathbf{R}^N)$ for some $\alpha \in (0,1)$;
(p2) $p > 0$ in $\mathbf{R}^N$.

This problem has been intensively studied in the case where $f(u) = u^{-\gamma}$, with $\gamma > 0$. For instance, in the case of a bounded domain $\Omega \subset \mathbf{R}^N$, Lazer and McKenna proved in [7] that the problem

$$-\Delta u = p(x)u^{-\gamma}\,, \qquad \text{in } \Omega$$

has a unique classical solution if p is a sufficiently smooth function which is positive on $\overline{\Omega}$. The existence of entire positive solutions on $\mathbf{R}^N$ for $\gamma \in (0,1)$ and under certain additional hypotheses has been established in Edelson [4] and in Kusano-Swanson [5]. For instance, Edelson proved the existence of a solution provided that

$$\int_1^\infty r^{N-1+\lambda(N-2)}\Phi(r)dr < \infty\,,$$

for some $\lambda \in (0,1)$, where $\Phi(r) = \max_{|x|=r} p(x)$. This result is generalized for any $\gamma > 0$ via the sub and super solutions method in Shaker [8] or by other methods in Dalmasso [3]. Lair and Shaker continued in [6] the study of (1) for $f(u) = u^{-\gamma}$, $\gamma > 0$. They proved the existence of a solution under the hypothesis

(p3) $\int\limits_0^\infty r \cdot \Phi(r)dr < \infty$, where $\Phi(r) = \max\limits_{|x|=r} p(x)$.

Zhang studied in [9] the case of a nonlinearity $f \in C^1((0,\infty),(0,\infty))$ which decreases on $(0,\infty)$ and satisfying $\lim_{u\searrow 0} f(u) = +\infty$.

Our aim is to extend the results of Lair, Shaker and Zhang for the case of a nonlinearity which is not necessarily decreasing on $(0,\infty)$. More exactly, let $f : (0,\infty) \to (0,\infty)$ be a C^1 function which satisfies the following assumptions:

(f1) there exists $\beta > 0$ such that the mapping $u \longmapsto \dfrac{f(u)}{u+\beta}$ is decreasing on $(0,\infty)$;

(f2) $\lim\limits_{u\searrow 0} \dfrac{f(u)}{u} = +\infty$ and f is bounded in a neighbourhood of $+\infty$.

Our main result is the following:

Theorem 1 *Under hypotheses (f1), (f2), (p1)-(p3), the problem (1) has a unique positive global solution* $u \in C^{2,\alpha}_{\rm loc}(\mathbf{R}^N)$.

Theorem 1 shows that (p3) is sufficient for the existence of the unique solution to the problem (1). The following result shows that condition (p3) is nearly necessary.

Theorem 2 *Suppose p is a positive radial function which is continuous on $\mathbf{R}^N$ and satisfies*

$$\int\limits_0^\infty rp(r)dr = \infty.$$

Then the problem (1) has no positive radial solution.

2 Uniqueness

Suppose u and v are arbitrary solutions of the problem (1). Let us show that $u \le v$ or, equivalently, $\ln(u(x)+\beta) \le \ln(v(x)+\beta)$, for any $x \in \mathbf{R}^N$. Assume the contrary. Since we have

$$\lim_{|x|\to\infty} (\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0,$$

we deduce that $\max_{\mathbf{R}^N} (\ln(u(x)+\beta) - \ln(v(x)+\beta))$ exists and is positive. At that point, say x_0, we have

$$\nabla(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta)) = 0,$$

so

$$\frac{1}{u(x_0)+\beta}\cdot\nabla u(x_0)=\frac{1}{v(x_0)+\beta}\cdot\nabla v(x_0)\,. \tag{2}$$

By (f1) we obtain

$$\frac{f(u(x_0))}{u(x_0)+\beta}<\frac{f(v(x_0))}{v(x_0)+\beta}\,. \tag{3}$$

So, by (2) and (3),

$$\begin{aligned}
&0\geq\Delta\left(\ln(u(x_0)+\beta)-\ln(v(x_0)+\beta)\right)=\frac{1}{u(x_0)+\beta}\cdot\Delta u(x_0)-\frac{1}{v(x_0)+\beta}\cdot\Delta v(x_0)-\\
&\frac{1}{(u(x_0)+\beta)^2}\cdot|\nabla u(x_0)|^2+\frac{1}{(v(x_0)+\beta)^2}\cdot|\nabla v(x_0)|^2=\\
&\frac{1}{u(x_0)+\beta}\,\Delta u(x_0)-\frac{1}{v(x_0)+\beta}\,\Delta v(x_0)=\\
&-p(x_0)\left(\frac{f(u(x_0))}{u(x_0)+\beta}-\frac{f(v(x_0))}{v(x_0)+\beta}\right)>0\,,
\end{aligned}$$

which is a contradiction. Hence $u\leq v$. A similar argument can be made to produce $v\leq u$, forcing $u=v$.

3 Existence

We first show that our hypothesis (f1) implies that $\lim\limits_{u\searrow 0}f(u)$ exists, finite or $+\infty$. Indeed, since $\frac{f(u)}{u+\beta}$ is decreasing, there exists $L:=\lim\limits_{u\searrow 0}\frac{f(u)}{u+\beta}\in(0,+\infty]$. It follows that $\lim\limits_{u\searrow 0}f(u)=L\beta$.

In order to prove the existence of a solution to (1), we need to employ a corresponding result by Brezis and Oswald (see [1]) for bounded domains. They considered the problem

$$\begin{cases}-\Delta u=g(x,u) & \text{in } \Omega\\ u\geq 0,\quad u\not\equiv 0 & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\,,\end{cases} \tag{4}$$

where $\Omega\subset\mathbf{R}^N$ is a bounded domain with smooth boundary and $g(x,u):\Omega\times[0,\infty)\to\mathbf{R}$.

Assume that

$$\begin{cases}\text{for a.e. } x\in\Omega \text{ the function } u\to g(x,u) \text{ is continuous on } [0,\infty)\\ \text{and the function } u\to g(x,u)/u \text{ is decreasing on } (0,\infty)\,;\end{cases} \tag{5}$$

$$\text{for each } u\geq 0 \text{ the function } x\to g(x,u) \text{ belongs to } L^\infty(\Omega)\,; \tag{6}$$

$$\exists C > 0 \text{ such that } g(x,u) \le C(u+1) \text{ a.e. } x \in \Omega, \quad \forall\, u \ge 0. \tag{7}$$

Set

$$a_0(x) = \lim_{u \searrow 0} g(x,u)/u \qquad \text{and} \qquad a_\infty(x) = \lim_{u\to\infty} g(x,u)/u\,,$$

so that $-\infty < a_0(x) \le +\infty$ and $-\infty \le a_\infty(x) < +\infty$.

Under these hypotheses on g, Brezis and Oswald proved in [1] that there is at most one solution of (4). Moreover, a solution of (4) exists if and only if

$$\lambda_1(-\Delta - a_0(x)) < 0 \tag{8}$$

and

$$\lambda_1(-\Delta - a_\infty(x)) > 0, \tag{9}$$

where $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of the operator $-\Delta - a(x)$ with zero Dirichlet condition. The precise meaning of $\lambda_1(-\Delta - a(x))$ is

$$\lambda_1(-\Delta - a(x)) = \inf_{\varphi \in H_0^1,\ \|\varphi\|_2 = 1} \left(\int |\nabla \varphi|^2 - \int_{[\varphi \neq 0]} a\varphi^2 \right).$$

Note that $\displaystyle\int_{[\varphi\neq 0]} a\varphi^2$ makes sense if $a(x)$ is any measurable function such that either $a(x) \le C$ or $a(x) \ge -C$ a.e. on Ω.

Let us consider the problem

$$\begin{cases} -\Delta u_k = p(x) f(u_k)\,, \text{ if } \mid x \mid < k \\ u_k(x) = 0\,, \quad \text{if } \mid x \mid = k\,. \end{cases} \tag{10}$$

The following two distinct situations may occur:

Case 1: f is bounded on $(0,\infty)$.

In this case, as we have initially observed, there exists and it is finite $\lim_{u\searrow 0} f(u)$, so f can be extended by continuity at the origin.

In order to obtain a solution to the problem (10), it is enough to verify that the hypotheses of the Brezis-Oswald theorem are fulfilled. Obviously, (5) and (6) hold. Now, using (p1), (p2) and the fact that f is bounded, we easily deduce that (7) is satisfied. We observe that $a_0(x) = \lim_{u\searrow 0} \dfrac{p(x)f(u)}{u} = +\infty$ and $a_\infty(x) = \lim_{u\to+\infty} \dfrac{p(x)f(u)}{u} = 0$. Then (8) and (9) are also fulfilled. Thus by Theorem 1 in [1] the problem (10) has a unique solution u_k which, by the maximum principle, is positive in $|x| < k$.

Case 2: $\lim_{u\searrow 0} f(u) = +\infty$.

We will apply the method of sub and supersolutions in order to find a solution to the problem (10). We first observe that 0 is a subsolution for this problem.

We construct in what follows a positive supersolution. By the boundedness of f in a neighbourhood of $+\infty$, there exists $A > 0$ such that $f(u) \leq A$, for any $u \in (1, +\infty)$. Let $f_0 : (0, 1] \to (0, +\infty)$ be a continuous nonincreasing function such that $f_0 \geq f$ on $(0, 1]$. We can assume without loss of generality that $f_0(1) = A$. Set

$$g(u) = \begin{cases} f_0(u)\,, & \text{if } 0 < u \leq 1 \\ A\,, & \text{if } u > 1\,. \end{cases}$$

Then g is a continuous nonincreasing function on $(0, +\infty)$. Let $h : (0, \infty) \to (0, \infty)$ be a C^1 nonincreasing function such that $h \geq g$. Thus, by Theorem 1.1 in [2] the problem

$$\begin{cases} -\Delta U = p(x)h(U) & \text{if } \mid x \mid < k \\ U = 0, & \text{if } \mid x \mid = k \end{cases}$$

has a positive solution. Now, since $h \geq f$ on $(0, +\infty)$, it follows that U is supersolution for the problem (10).

In both cases studied above we define $u_k = 0$ for $|x| > k$. Using a maximum principle argument as already done above for proving the uniqueness, we can show that $u_k \leq u_{k+1}$ on $\mathbf{R}^N$.

We now prove the existence of a positive function $v \in C^2(\mathbf{R}^N)$ for which $u_k \leq v$ on $\mathbf{R}^N$. As in [6] we construct first a positive radially symmetric function w such that $-\Delta w = \Phi(r)$ $(r = \mid x \mid)$ on $\mathbf{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. We obtain

$$w(r) = K - \int_0^r \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)d\sigma d\zeta\,,$$

where

$$K = \int_0^\infty \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\,d\sigma\,d\zeta\,, \tag{11}$$

provided the integral is finite. Integration by parts gives

$$\begin{aligned} &\int_0^r \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\,d\sigma\,d\zeta = -(n-2)^{-1} \int_0^r \frac{d}{d\zeta}\zeta^{2-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\,d\sigma\,d\zeta = \\ &(n-2)^{-1}\left(-r^{2-n}\int_0^r \sigma^{n-1}\Phi(\sigma)\,d\sigma + \int_0^r \zeta\Phi(\zeta)\,d\zeta\right)\,. \end{aligned} \tag{12}$$

Now, using L'Hopital's rule, we evaluate the limit of the right side of (12) as $r \to \infty$. We have

$$\lim_{r\to\infty}\left(-r^{2-n}\int_0^r \sigma^{n-1}\Phi(\sigma)\,d\sigma + \int_0^r \zeta\Phi(\zeta)\,d\zeta\right) =$$

$$= \lim_{r\to\infty}\frac{-\int_0^r \sigma^{n-1}\Phi(\sigma)\,d\sigma + r^{n-2}\int_0^r \zeta\Phi(\zeta)\,d\zeta}{r^{n-2}} =$$

$$= \lim_{r\to\infty}\int_0^r \zeta\Phi(\zeta)\,d\zeta = \int_0^\infty \zeta\Phi(\zeta)\,d\zeta < \infty.$$

Then we obtain $K = \dfrac{1}{n-2}\cdot\displaystyle\int_0^\infty \zeta\Phi(\zeta)\,d\zeta < \infty$.

Clearly, we have

$$w(r) < \frac{1}{n-2}\cdot\int_0^\infty \zeta\Phi(\zeta)d\zeta \quad \forall r > 0.$$

Let v be a positive function such that $w(r) = \dfrac{1}{c}\cdot\displaystyle\int_0^{v(r)}\frac{t}{f(t)}dt$, where $c > 0$ will be chosen such that $Kc \le \displaystyle\int_0^c \frac{t}{f(t)}dt$.

We prove that we can find $c > 0$ with this property.

By our hypothesis (f2) we obtain that $\displaystyle\lim_{x\to\infty}\int_0^x \frac{t}{f(t)}dt = +\infty$. Now using L'Hopital's rule we have

$$\lim_{x\to\infty}\frac{\int_0^x \frac{t}{f(t)}dt}{x} = \lim_{x\to\infty}\frac{x}{f(x)} = +\infty\,.$$

From this we deduce that there exists $x_1 > 0$ such that $\displaystyle\int_0^x \frac{t}{f(t)}dt \ge Kx$ for all $x \ge x_1$. It follows that for any $c \ge x_1$ we have $Kc \le \displaystyle\int_0^c \frac{t}{f(t)}dt$.

But w is a decreasing function, and this implies that v is a decreasing function too. Then $\displaystyle\int_0^{v(r)}\frac{t}{f(t)}dt \le \int_0^{v(0)}\frac{t}{f(t)}dt = c\cdot w(0) = c\cdot K \le \int_0^c \frac{t}{f(t)}dt$.
It follows that $v(r) \le c$ for all $r > 0$.

From $w(r) \to 0$ as $r \to \infty$ we deduce that $v(r) \to 0$ as $r \to \infty$.

By the choice of v we have

$$\nabla w = \frac{1}{c} \cdot \frac{v}{f(v)} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{c} \frac{v}{f(v)} \Delta v + \frac{1}{c} \left(\frac{v}{f(v)} \right)' |\nabla v|^2. \tag{13}$$

The hypothesis $u \longmapsto \dfrac{f(u)}{u+\beta}$ is a decreasing function on $(0, \infty)$ implies that $u \longmapsto \dfrac{f(u)}{u}$ is a decreasing function on $(0, \infty)$. From (13) we deduce that

$$\Delta v < c \frac{f(v)}{v} \Delta w = -c \frac{f(v)}{v} \Phi(r) \leq -f(v) \Phi(r) \,. \tag{14}$$

By (10) and (14) and using in an essential manner the hypothesis (f1), as already done for proving the uniqueness, we obtain that $u_k \leq v$ for $|x| \leq k$ and, hence, for all $\mathbf{R}^N$.

Now we have a bounded increasing sequence

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v \,,$$

with v vanishing at infinity. Thus there exists a function, say $u \leq v$ such that $u_k \to u$ pointwise in $\mathbf{R}^N$.

Now, using the same argument as in [6], it is easy to prove that $u \in C^{2,\alpha}_{loc}(\mathbf{R}^N)$ and thus u is a classical solution of the problem (1).

4 Proof of Theorem 2

Suppose (1) has such a solution, $u(r)$. Then

$$u''(r) + \frac{n-1}{r} u'(r) = -f(u(r)) p(r).$$

We put $\ln(u(r)+1) = \tilde{u}(r) > 0$ for all $r > 0$.

$$\Delta \tilde{u}(r) = \frac{1}{u(r)+1} \Delta u(r) - \frac{1}{(u(r)+1)^2} |\nabla u|^2.$$

Then $\tilde{u}(r)$ satisfies

$$\tilde{u}'' + \frac{n-1}{r} \tilde{u}' + \frac{1}{(u(r)+1)^2} |\nabla u|^2 = -\frac{f(u(r))}{u(r)+1} p(r). \tag{15}$$

Multiplying equation (15) by r^{n-1} and integrating on $(0, \zeta)$ yield

$$\tilde{u}'(\zeta) \zeta^{n-1} + \int_0^\zeta \frac{\sigma^{n-1}}{(u(\sigma)+1)^2} |\nabla u|^2 d\sigma = -\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1} p(\sigma) \sigma^{n-1} d\sigma. \tag{16}$$

Now we multiply (16) by ζ^{1-n} and integrate over $(0,r)$. Hence

$$\tilde{u}(r)-\tilde{u}(0)+\int_0^r \zeta^{1-n}\int_0^\zeta \frac{\sigma^{n-1}}{(u(\sigma)+1)^2}|\nabla u|^2 d\sigma\, d\zeta =$$

$$=-\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta.$$

We observe that $\tilde{u}(r)<\tilde{u}(0)\ \forall r>0$ implies $u(r)<u(0)\ \forall r>0$.

If $\beta \geq 1$ then the function $u \longmapsto \dfrac{f(u)}{u+1}$ is decreasing on $(0,\infty)$. This implies

$$\frac{f(u(\sigma))}{u(\sigma)+1} > \frac{f(u(0))}{u(0)+1}. \tag{17}$$

Since $\tilde{u}$ is positive, we have

$$\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta \leq \tilde{u}(0) \quad \text{for all } r>0\,.$$

Substituting (17) into this expression we obtain

$$\int_0^r \zeta^{1-n}\int_0^\zeta p(\sigma)\sigma^{n-1}d\sigma\, d\zeta \leq \frac{u(0)+1}{f(u(0))}\tilde{u}(0)<\infty.$$

We can use integration by parts and L'Hopital's rule (as we did in proving that the integral in (11) is finite) to rewrite this as

$$\frac{1}{n-2}\lim_{r\to\infty}\int_0^r tp(t)dt \leq \frac{u(0)+1}{f(u(0))}\tilde{u}(0)<\infty\,.$$

contradicting the hypothesis.

If $\beta<1$ then the function $u \longmapsto \dfrac{u+\beta}{u+1}$ is increasing on $(0,\infty)$. In this case we have

$$\tilde{u}(0) > \int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta =$$

$$=\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+\beta}\cdot\frac{u(\sigma)+\beta}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta \geq$$

$$\geq \frac{f(u(0))}{u(0)+\beta}\beta \int_0^r \zeta^{1-n} \int_0^\zeta p(\sigma)\sigma^{n-1} d\sigma\, d\zeta$$

which implies

$$\int_0^r \zeta^{1-n} \int_0^\zeta p(\sigma)\sigma^{n-1} d\sigma\, d\zeta < \frac{\tilde{u}(0)(u(0)+\beta)}{\beta \cdot f(u(0))} < \infty \quad \text{for all } r > 0.$$

We obtain again that

$$\frac{1}{n-2} \lim_{r\to\infty} \int_0^r tp(t)dt \leq \frac{u(0)+\beta}{\beta \cdot f(u(0))}\tilde{u}(0) < \infty$$

contradicting the hypothesis.

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