Uniqueness of the blow-up boundary solution of logistic equations with absorbtion *

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Scientific field: Équations aux dérivées partielles/ *Partial differential equations*

Abstract. Let Ω be a smooth bounded domain in $\mathbf{R}^N$. Assume $f \in C^1[0,\infty)$ is a non-negative function such that $f(u)/u$ is increasing on $(0,\infty)$. Let a be a real number and let $b \geq 0$, $b \not\equiv 0$ be a continuous function such that $b \equiv 0$ on $\partial\Omega$. We study the logistic equation $\Delta u + au = b(x)f(u)$ in Ω. The special feature of this work is the uniqueness of positive solutions blowing-up on $\partial\Omega$, in a general setting that arises in probability theory.

Unicité de la solution explosant au bord pour équations logistiques avec absorption

Résumé. *Soit Ω un domaine borné et régulier de $\mathbf{R}^N$. On suppose que $f \in C^1[0,\infty)$ est une fonction non-negative telle que $f(u)/u$ soit strictement croissante sur $(0,+\infty)$. Soit a un réel et $b \geq 0$, $b \not\equiv 0$, une fonction continue sur $\overline{\Omega}$ telle que $b \equiv 0$ sur $\partial\Omega$. On étudie l'équation logistique $\Delta u + au = b(x)f(u)$ sur Ω. Le but de cette Note est de montrer l'unicité de la solution explosant au bord de Ω dans un contexte général, qui apparaît en théorie des probabilités.*

Version française abrégée. Soit $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ un domaine borné et régulier, a un paramètre réel et $b \in C^{0,\mu}(\overline{\Omega})$, $\mu \in (0,1)$, $b \geq 0$, $b \not\equiv 0$ dans Ω. On considère l'équation logistique

$$\Delta u + au = b(x)f(u) \qquad \text{dans } \Omega, \tag{1}$$

où $f \in C^1[0,\infty)$ satisfait

(A_1) $f \geq 0$ et $f(u)/u$ est strictement croissante sur $(0,+\infty)$.

Soit

$$\Omega_0 := \operatorname{int}\{x \in \Omega : \; b(x) = 0\}$$

*The research of F. Cîrstea was done under the IPRS Programme funded by the Australian Government through DETYA. V. Rădulescu was supported by the P.I.C.S. Research Programme between France and Romania.

et on suppose que $\partial\Omega_0$ est régulier (éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre (avec conditions de Dirichlet) de l'opérateur $(-\Delta)$ dans Ω_0, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$.

On dit que u est une solution *large (explosive)* de (1) si $u \geq 0$ dans Ω et $u(x) \to \infty$ si $d(x) := \mathrm{dist}\,(x, \partial\Omega) \to 0$.

Soit $D > 0$ et $R : [D, \infty) \to (0, +\infty)$ une fonction mesurable. On dit que R a une variation régulière d'indice $\rho \in \mathbf{R}$ (notation: $R \in \mathbf{R}_\rho$) si $\lim_{u\to\infty} R(\xi u)/R(u) = \xi^\rho$, pour chaque $\xi > 0$ (voir [11]).

Soit $\mathcal{K}$ l'ensemble des fonctions $k : (0, \nu) \to (0, +\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\to 0^+} \left(\dfrac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i$, pour $i = \overline{0,1}$.

On démontre le résultat suivant.

THÉORÈME 1. - *Supposons que la fonction f satisfait la condition (A_1) et que f' est une fonction à variation régulière d'indice $\rho \neq 0$. De plus, on suppose que le potentiel b vérifie*

(B) $\quad b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *si* $d(x) \to 0$, *avec* $c > 0$ *et* $k \in \mathcal{K}$.

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'équation (1) admet une unique solution explosive u_a. On a, de plus,

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0,$$

où $\xi_0 = \left(\dfrac{2 + \ell_1 \rho}{c(2+\rho)}\right)^{1/\rho}$ *et la fonction h est définie par*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0, \nu).$$

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ be a smooth bounded domain. Consider the semilinear elliptic equation

$$\Delta u + au = b(x) f(u) \qquad \text{in } \Omega, \tag{1}$$

where a is a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, for some $\mu \in (0,1)$, such that $b \geq 0$, $b \not\equiv 0$ in Ω.

Suppose that $f \in C^1[0, \infty)$ satisfies

(A_1) $\quad f \geq 0$ and $f(u)/u$ is increasing on $(0, \infty)$.

In the study of positive solutions for (1), subject to the homogeneous Dirichlet boundary condition, an important role is played by the zero set (see [1])

$$\Omega_0 := \mathrm{int}\,\{x \in \Omega : \; b(x) = 0\}.$$

We shall assume throughout that Ω_0 is smooth (possibly empty), $\overline{\Omega}_0 \subset \Omega$, and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$.

By a *large (explosive)* solution of (1) we mean a solution u of (1) such that $u \geq 0$ in Ω and $u(x) \to \infty$ as $d(x) := \mathrm{dist}\,(x, \partial\Omega) \to 0$. In [3, 4] we study the existence of large solutions for (1) and also deduce several existence and unicity results for a related problem. Note that any large solution of (1) is *positive* and it can exists only if the Keller-Osserman condition holds (see [4])

$$(A_2) \quad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty\,, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

Let H_∞ define the Dirichlet Laplacian on the set $\Omega_0 \subset \Omega$ as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2\,dx$ with form domain

$$H_D^1(\Omega_0) = \{u \in H_0^1(\Omega) :\ u(x) = 0 \quad \text{for a.e. } x \in \Omega \setminus \Omega_0\}.$$

If $\partial\Omega_0$ satisfies an exterior cone condition, then $H_D^1(\Omega_0)$ coincides with $H_0^1(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$.

The main result in [3] asserts that equation (1) has a large solution iff $a \in (-\infty, \lambda_{\infty,1})$.

The special feature of this paper is the uniqueness of large solutions of (1) in a general framework for f and b, under the restriction $b \equiv 0$ on $\partial\Omega$, inherited from the logistic equation (see [6]).

We start with

DEFINITION 1 ([11]). - A positive measurable function R defined on $[D, \infty)$, for some $D > 0$, is called *regularly varying (at infinity) with index* $q \in \mathbf{R}$, written $R \in \mathbf{R}_q$, if for all $\xi > 0$

$$\lim_{u\to\infty} R(\xi u)/R(u) = \xi^q.$$

When the index of regular variation q is zero, we say that the function is *slowly varying*.

REMARK 1. - *Any function $R \in \mathbf{R}_q$ can be written in terms of a slowly varying function.* Indeed, set $R(u) = u^q L(u)$. From Definition 1 we easily derive that L varies slowly.

The canonical q-varying function is u^q. The functions $\ln(1+u)$, $\ln\ln(e+u)$, $\exp\{(\ln u)^\alpha\}$, $\alpha \in (0,1)$ vary slowly, as well as any measurable function on $[D,\infty)$ with positive limit at infinity.

In what follows L denotes an arbitrary slowly varying function and $D > 0$ a positive number. For details on Properties 1-4 stated below, we refer to Seneta [11] (pp. 7, 18, 53 and 78).

PROPERTY 1. - For any $m > 0$, $u^m L(u) \to \infty$, $\quad u^{-m} L(u) \to 0$ as $u \to \infty$.

PROPERTY 2. - Any positive C^1-function on $[D,\infty)$ satisfying $uL_1'(u)/L_1(u) \to 0$ as $u \to \infty$ is slowly varying. Moreover, if the above limit is $q \in \mathbf{R}$, then $L_1 \in \mathbf{R}_q$.

PROPERTY 3. - Assume $R : [D,\infty) \to (0,\infty)$ is measurable and Lebesgue integrable on each finite subinterval of $[D,\infty)$. Then R varies regularly iff there exists $j \in \mathbf{R}$ such that

$$\lim_{u\to\infty} \frac{u^{j+1} R(u)}{\int_D^u x^j R(x)\,dx} \tag{2}$$

exists and is a positive number, say $a_j + 1$. In this case, $R \in \mathbf{R}_q$ with $q = a_j - j$.

PROPERTY 4 (Karamata Theorem, 1933). - If $R \in \mathbf{R}_q$ is Lebesgue integrable on each finite subinterval of $[D,\infty)$, then the limit defined by (2) is $q + j + 1$, for every $j > -q - 1$.

LEMMA 1. - *Assume (A_1) holds. Then we have the equivalence*

$$a)\ f' \in \mathbf{R}_\rho \iff b)\ \lim_{u\to\infty} uf'(u)/f(u) := \vartheta < \infty \iff c)\ \lim_{u\to\infty} (F/f)'(u) := \gamma > 0.$$

REMARK 2. - *Let a) of Lemma 1 be fulfilled. The following assertions hold*
(*i*) ρ *is* **non-negative**. Indeed, if $\rho < 0$ then Property 1 and Remark 1 would contradict (A_1);
(*ii*) $\gamma = 1/(\rho+2) = 1/(\vartheta+1)$ (see the proof of Lemma 1);
(*iii*) *If* $\rho \neq 0$, *then* (A_2) *holds* (use $\lim_{u\to\infty} f(u)/u^p = \infty$, $\forall p \in (1, 1+\rho)$). *The converse implication is not necessarily true (take* $f(u) = u \ln^4(u+1)$*). However, there are cases when* $\rho = 0$ *and* (A_2) *fails so that (1) has* **no** *large solutions. This is illustrated by* $f(u) = u$ *or* $f(u) = u\ln(u+1)$.

Inspired by the definition of γ, we denote by $\mathcal{K}$ the set of all positive, increasing C^1-functions k defined on $(0,\nu)$, for some $\nu > 0$, which satisfy $\lim_{t\to 0^+} \left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)^{(i)} := \ell_i,\ i = \overline{0,1}$.

It is easy to see that $\ell_0 = 0$ and $\ell_1 \in [0,1]$, for every $k \in \mathcal{K}$. Our next result gives examples of functions $k \in \mathcal{K}$ with $\lim_{t\to 0^+} k(t) = 0$, for *every* $\ell_1 \in [0,1]$.

LEMMA 2. - *Let* $S \in C^1[D,\infty)$ *be such that* $S' \in \mathbf{R}_q$ *with* $q > -1$. *Hence the following hold:*
a) *If* $k(t) = \exp\{-S(1/t)\}$ $\forall t \le 1/D$, *then* $k \in \mathcal{K}$ *with* $\ell_1 = 0$.
b) *If* $k(t) = 1/S(1/t)$ $\forall t \le 1/D$, *then* $k \in \mathcal{K}$ *with* $\ell_1 = 1/(q+2) \in (0,1)$.
c) *If* $k(t) = 1/\ln S(1/t)$ $\forall t \le 1/D$, *then* $k \in \mathcal{K}$ *with* $\ell_1 = 1$.

REMARK 3. - *If* $S \in C^1[D,\infty)$, *then* $S' \in \mathbf{R}_q$ *with* $q > -1$ *iff for some* $m > 0$, $C > 0$ *and* $B > D$ *we have* $S(u) = Cu^m \exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, $\forall u \ge B$, *where* $y \in C[B,\infty)$ *satisfies* $\lim_{u\to\infty} y(u) = 0$. *In this case,* $S' \in \mathbf{R}_q$ *with* $q = m-1$. This is a consequence of Properties 3 and 4.

Our main result is

THEOREM 1. - *Let* (A_1) *hold and* $f' \in \mathbf{R}_\rho$ *with* $\rho > 0$. *Assume* $b \equiv 0$ *on* $\partial\Omega$ *satisfies*

(B) $\quad b(x) = c\,k^2(d(x)) + o(k^2(d(x)))$ *as* $d(x) \to 0$, *for some constant* $c > 0$ *and* $k \in \mathcal{K}$.

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *Eq. (1) admits a unique large solution* u_a. *Moreover,*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \tag{3}$$

where $\xi_0 = \left(\frac{2+\ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$ *and* h *is defined by*

$$\int_{h(t)}^\infty \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,\nu). \tag{4}$$

By Remark 3, the assumption $f' \in \mathbf{R}_\rho$ with $\rho > 0$ holds iff there exist $p > 1$ and $B > 0$ such that $f(u) = Cu^p \exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\}$, for all $u \ge B$ (y as before and $p = \rho+1$). If B is large enough ($y > -\rho$ on $[B,\infty)$), then $f(u)/u$ is increasing on $[B,\infty)$. Thus, to get the whole range of functions f for which our Theorem 1 applies we have only to "paste" a suitable smooth function on $[0,B]$ in accordance with (A_1). A simple way to do this is to define $f(u) = u^p \exp\{\int_0^u \frac{z(t)}{t}\,dt\}$, for all $u \ge 0$, where $z \in C[0,\infty)$ is non-negative such that $\lim_{t\to 0^+} z(t)/t \in [0,\infty)$ and $\lim_{u\to\infty} z(u) = 0$. Clearly, $f(u) = u^p$, $f(u) = u^p \ln(u+1)$, and $f(u) = u^p \arctan u$ ($p > 1$) fall into this category.

Lemma 2 provides a practical method to find functions k which can be considered in the statement of Theorem 1. Here are some examples: $k(t) = \exp\{-1/t^\alpha\}$, $k(t) = \exp\{-\ln(1+\frac{1}{t})/t^\alpha\}$, $k(t) = \exp\{-\left[\arctan\left(\frac{1}{t}\right)\right]/t^\alpha\}$, $k(t) = -1/\ln t$, $k(t) = t^\alpha/\ln(1+\frac{1}{t})$, $k(t) = t^\alpha$, for some $\alpha > 0$.

As we shall see, the uniqueness lies upon the crucial observation (3), which shows that all explosive solutions have the same boundary behaviour. Note that the only case of Theorem 1 studied so far is $f(u) = u^p$ ($p > 1$) and $k(t) = t^\alpha$ ($\alpha > 0$) (see [6]). For related results on the uniqueness of explosive solutions (mainly in the cases $b \equiv 1$ and $a = 0$) we refer to [2, 8, 9, 12].

Proof of Lemma 1. - From Property 4 and Remark 2 (i) we deduce $a) \Longrightarrow b)$ and $\vartheta = \rho + 1$. Conversely, $b) \Longrightarrow a)$ follows by Property 3 since $\vartheta \geq 1$ cf. (A_1).

$b) \Longrightarrow c)$. Indeed, $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = 1 + \vartheta$, which yields $\frac{\vartheta}{1+\vartheta} = \lim_{u\to\infty} \left[1 - \left(\frac{F}{f}\right)'(u)\right] = 1 - \gamma$.

$c) \Longrightarrow b)$. Choose $s_1 > 0$ such that $\left(\frac{F}{f}\right)'(u) \geq \frac{\gamma}{2}$, $\forall u \geq s_1$. So, $\left(\frac{F}{f}\right)(u) \geq \frac{(u-s_1)\gamma}{2} + \left(\frac{F}{f}\right)(s_1)$, $\forall u \geq s_1$. Passing to the limit $u \to \infty$, we find $\lim_{u\to\infty} \frac{F(u)}{f(u)} = \infty$. Thus, $\lim_{u\to\infty} \frac{uf(u)}{F(u)} = \frac{1}{\gamma}$. Since $1 - \gamma := \lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)}$, we obtain $\lim_{u\to\infty} \frac{uf'(u)}{f(u)} = \frac{1-\gamma}{\gamma}$. ■

Proof of Lemma 2. - Since $\lim_{u\to\infty} uS'(u) = \infty$ (cf. Property 1), from Karamata Theorem we deduce $\lim_{u\to\infty} \frac{uS'(u)}{S(u)} = q + 1 > 0$. Therefore, in any of the cases $a)$, $b)$, $c)$, $\lim_{t\to 0^+} k(t) = 0$ and k is an increasing C^1-function on $(0, \nu)$, for $\nu > 0$ sufficiently small.

$a)$ It is clear that $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)\ln k(t)} = \lim_{t\to 0^+} \frac{-S'(1/t)}{tS(1/t)} = -(q+1)$. By l'Hospital's rule, $\ell_0 = \lim_{t\to 0^+} \frac{k(t)}{k'(t)} = 0$ and $\lim_{t\to 0^+} \frac{\left(\int_0^t k(s)\,ds\right)\ln k(t)}{tk(t)} = -\frac{1}{q+1}$. So, $1 - \ell_1 := \lim_{t\to 0^+} \frac{\left(\int_0^t k(s)\,ds\right)k'(t)}{k^2(t)} = 1$.

$b)$ We see that $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \lim_{t\to 0^+} \frac{S'(1/t)}{tS(1/t)} = q + 1$. By l'Hospital's rule, $\ell_0 = 0$ and $\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} = \frac{1}{q+2}$. So, $\ell_1 = 1 - \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} \frac{tk'(t)}{k(t)} = \frac{1}{q+2}$.

$c)$ We have $\lim_{t\to 0^+} \frac{tk'(t)}{k^2(t)} = \lim_{t\to 0^+} \frac{S'(1/t)}{tS(1/t)} = q+1$. By l'Hospital's rule, $\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} = 1$. Thus, $\ell_0 = 0$ and $\ell_1 = 1 - \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{t} \frac{tk'(t)}{k^2(t)} = 1$. ■

Proof of Theorem 1. - Fix $a \in (-\infty, \lambda_{\infty,1})$. By [3, Theorem 1], (1) has at least a large solution.

If we prove that (3) holds for an *arbitrary* large solution u_a of (1), then the uniqueness is a consequence of [3, Lemma 3]. Indeed, if u_1 and u_2 are two arbitrary large solutions of (1), then (3) yields $\lim_{d(x)\to 0^+} \frac{u_1(x)}{u_2(x)} = 1$. Hence, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)u_2(x) \leq u_1(x) \leq (1+\varepsilon)u_2(x), \quad \forall x \in \Omega \text{ with } 0 < d(x) \leq \delta. \tag{5}$$

Choosing eventually a smaller $\delta > 0$, we can assume that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta := \{x \in \Omega : d(x) > \delta\}$.

It is clear that u_1 is a positive solution of the boundary value problem

$$\Delta\phi + a\phi = b(x)f(\phi) \quad \text{in } C_\delta, \qquad \phi = u_1 \quad \text{on } \partial C_\delta. \tag{6}$$

By (A_1) and (5), we see that $\phi^- = (1-\varepsilon)u_2$ (resp., $\phi^+ = (1+\varepsilon)u_2$) is a positive sub-solution (resp., super-solution) of (6). By the sub and super-solutions method, (6) has a positive solution ϕ_1 satisfying $\phi^- \leq \phi_1 \leq \phi^+$ in C_δ. Since $b > 0$ on $\overline{C_\delta} \setminus \overline{\Omega}_0$, by [3, Lemma 3] we derive that (6) has a *unique* positive solution, i.e., $u_1 \equiv \phi_1$ in C_δ. This yields $(1-\varepsilon)u_2(x) \leq u_1(x) \leq (1+\varepsilon)u_2(x)$ in C_δ, so that (5) holds in Ω. Passing to the limit $\varepsilon \to 0^+$, we conclude that $u_1 \equiv u_2$.

In order to prove (3) we state some useful properties about h:

(h_1) $h \in C^2(0,\nu)$, $\lim_{t\to 0^+} h(t) = \infty$ (straightforward from (4)).

(h_2) $\lim_{t\to 0^+} \dfrac{h''(t)}{k^2(t)f(h(t)\xi)} = \dfrac{1}{\xi^{\rho+1}} \dfrac{2+\rho\ell_1}{2+\rho}$, $\forall \xi > 0$ (so, $h'' > 0$ on $(0, 2\delta)$, for $\delta > 0$ small enough).

(h_3) $\lim_{t\to 0^+} h(t)/h''(t) = \lim_{t\to 0^+} h'(t)/h''(t) = 0$.

We check (h_2) for $\xi = 1$ only, since $f \in \mathbf{R}_{\rho+1}$. Clearly, $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t) f(h(t)) \left(1 - 2 \frac{k'(t) \left(\int_0^t k(s)\, ds \right)}{k^2(t)} \frac{\sqrt{F(h(t))}}{f(h(t)) \int_{h(t)}^\infty [F(s)]^{-1/2} ds} \right) \quad \forall t \in (0, \nu). \quad (7)$$

We see that $\lim_{u\to\infty} \sqrt{F(u)}/f(u) = 0$. Thus, from l'Hospital's rule and Lemma 1 we infer that

$$\lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u) \int_u^\infty [F(s)]^{-1/2} ds} = \frac{1}{2} - \gamma = \frac{\rho}{2(\rho+2)}. \quad (8)$$

Using (7) and (8) we derive (h_2) and also

$$\lim_{t\to 0^+} \frac{h'(t)}{h''(t)} = \frac{-2(2+\rho)}{2+\ell_1\rho} \lim_{t\to 0^+} \frac{\int_0^t k(s)\, ds}{k(t)} \lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u) \int_u^\infty [F(s)]^{-1/2} ds} = \frac{-\rho\ell_0}{2+\ell_1\rho} = 0. \quad (9)$$

From (h_1) and (h_2), $\lim_{t\to 0^+} h'(t) = -\infty$. So, l'Hospital's rule and (9) yield $\lim_{t\to 0^+} \frac{h(t)}{h'(t)} = 0$. This and (9) lead to $\lim_{t\to 0^+} \frac{h(t)}{h''(t)} = 0$ which proves (h_3).

Proof of (3). Fix $\varepsilon \in (0, c/2)$. Since $b \equiv 0$ on $\partial\Omega$ and (B) holds, we take $\delta > 0$ so that
(i) $d(x)$ is a C^2-function on the set $\{x \in \mathbf{R}^N : \ d(x) < 2\delta\}$;
(ii) k^2 is increasing on $(0, 2\delta)$;
(iii) $(c-\varepsilon)k^2(d(x)) < b(x) < (c+\varepsilon)k^2(d(x))$, $\forall x \in \Omega$ with $0 < d(x) < 2\delta$;
(iv) $h''(t) > 0$ $\forall t \in (0, 2\delta)$ (from (h_2)).

Let $\sigma \in (0, \delta)$ be arbitrary. We define $\xi^\pm = \left[\frac{2+\ell_1\rho}{(c\mp 2\varepsilon)(2+\rho)} \right]^{1/\rho}$ and $v_\sigma^-(x) = h(d(x)+\sigma)\xi^-$, for all x with $d(x) + \sigma < 2\delta$ resp., $v_\sigma^+(x) = h(d(x)-\sigma)\xi^+$, for all x with $\sigma < d(x) < 2\delta$.

Using *(i)*-*(iv)*, when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| \equiv 1$)

$$\Delta v_\sigma^+ + a v_\sigma^+ - b(x) f(v_\sigma^+) \le \xi^+ h''(d(x)-\sigma) \left(\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)} \Delta d(x) + a \frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} + 1 \right.$$
$$\left. -(c-\varepsilon) \frac{k^2(d(x)-\sigma) f(h(d(x)-\sigma)\xi^+)}{h''(d(x)-\sigma)\xi^+} \right).$$

Similarly, when $d(x) + \sigma < 2\delta$ we find

$$\Delta v_\sigma^- + a v_\sigma^- - b(x) f(v_\sigma^-) \ge \xi^- h''(d(x)+\sigma) \left(\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)} \Delta d(x) + a \frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} + 1 \right.$$
$$\left. -(c+\varepsilon) \frac{k^2(d(x)+\sigma) f(h(d(x)+\sigma)\xi^-)}{h''(d(x)+\sigma)\xi^-} \right).$$

Using (h_2) and (h_3) we see that, by diminishing δ, we can assume

$$\Delta v_\sigma^+(x) + a v_\sigma^+(x) - b(x) f(v_\sigma^+(x)) \le 0 \quad \forall x \text{ with } \sigma < d(x) < 2\delta;$$

$$\Delta v_\sigma^-(x) + a v_\sigma^-(x) - b(x) f(v_\sigma^-(x)) \ge 0 \quad \forall x \text{ with } d(x) + \sigma < 2\delta.$$

Let Ω_1 and Ω_2 be smooth bounded domains such that $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ and the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $\Omega_1 \setminus \overline{\Omega}$ is greater than a. Let $p \in C^{0,\mu}(\overline{\Omega}_2)$ satisfy $0 < p(x) \le b(x)$ for $x \in \Omega \setminus C_{2\delta}$, $p = 0$ on $\overline{\Omega}_1 \setminus \Omega$ and $p > 0$ on $\Omega_2 \setminus \overline{\Omega}_1$. Denote by w a positive large solution of

$$\Delta w + a w = p(x) f(w) \quad \text{in } \Omega_2 \setminus \overline{C}_{2\delta}.$$

The existence of w is ensured by Theorem 1 in [3].

Suppose that u_a is an arbitrary large solution of (1) and let $v := u_a + w$. Then v satisfies

$$\Delta v + av - b(x)f(v) \leq 0 \quad \text{in } \Omega \setminus \overline{C}_{2\delta}.$$

Since $v_{|\partial\Omega} = \infty > v^-_{\sigma|\partial\Omega}$ and $v_{|\partial C_{2\delta}} = \infty > v^-_{\sigma|\partial C_{2\delta}}$, Lemma 1 in [3] implies

$$u_a + w \geq v^-_\sigma \quad \text{on } \Omega \setminus \overline{C}_{2\delta}. \tag{10}$$

Similarly,

$$v^+_\sigma + w \geq u_a \quad \text{on } C_\sigma \setminus \overline{C}_{2\delta}. \tag{11}$$

Letting $\sigma \to 0$ in (10) and (11), we deduce $h(d(x))\xi^+ + 2w \geq u_a + w \geq h(d(x))\xi^-$, for all $x \in \Omega \setminus \overline{C}_{2\delta}$. Since w is uniformly bounded on $\partial\Omega$, we have $\xi^- \leq \liminf_{d(x)\to 0} \dfrac{u_a(x)}{h(d(x))} \leq \limsup_{d(x)\to 0} \dfrac{u_a(x)}{h(d(x))} \leq \xi^+$. Letting $\varepsilon \to 0^+$ we obtain (3). This concludes the proof of Theorem 1. ∎

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