Multiple solutions of degenerate perturbed elliptic problems involving a subcritical Sobolev exponent

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This paper is dedicated with admiration to Professor H. Brezis

on the occasion of his 55th birthday anniversary

Abstract. We study the degenerate elliptic equation

$$-\mathrm{div}\,(a(x)\nabla u) + b(x)\,u = K(x)|u|^{p-2}u + g(x) \quad \text{in } \mathbf{R}^N,$$

where $N \geq 2$ and $2 < p < 2^*$. We assume that $a \not\equiv 0$ is a continuous, bounded and nonnegative function, while b and K are positive and essentially bounded in $\mathbf{R}^N$. Under some assumptions on a, b and K, which control the location of zeros of a and the behaviour of a, b and K at infinity we prove that if the perturbation g is sufficiently small then the above problem has at least two distinct solutions in an appropriate weighted Sobolev space. The proof relies essentially on the Ekeland Variational Principle [8] and on the Mountain Pass Theorem without the Palais-Smale condition, established in Brezis-Nirenberg [6], combined with a weighted variant of the Brezis-Lieb Lemma [5], in order to overcome the lack of compactness .
Key words: degenerate elliptic problem, weighted Sobolev space, unbounded domain, perturbation, multiple solutions.

1 Introduction

Perturbations of semilinear elliptic equations and of inequality value problems have been intensively studied in the last two decades. We start with the elementary example

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

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where Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ and $2 < p < 2^*$. Here 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$, if $N \geq 3$, and $2^* = +\infty$, if $N = 2$. A classical result, based on a $\mathbf{Z}_2$ symmetric version of the Mountain Pass Theorem (see Ambrosetti-Rabinowitz [1]), implies that problem (1) has infinitely many solutions in $H_0^1(\Omega)$. A natural question is to see what happens if the above problem is affected by a certain perturbation. Consider the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + g(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

Bahri-Berestycki [3] and Struwe [14] have showed independently that there exists $p_0 < 2^*$ such that for any $g \in L^2(\Omega)$, problem (2) still has infinitely many solutions, provided $2 < p < p_0$. Moreover, Bahri [2] has shown that for any $2 < p < p_0$ there is a dense open set of $g \in H^{-1}(\Omega)$ for which problem (2) possesses infinitely many solutions.

Our aim is to study a perturbation problem, but from another point of vue. More exactly, we will analyse the effect of a small perturbation g in the degenerate semilinear elliptic problem

$$-\mathrm{div}\,(a(x)\nabla u) + b(x)\,u = K(x)|u|^{p-2}u + g(x) \quad \text{in } \mathbf{R}^N, \tag{3}$$

where $N \geq 2$ and $2 < p < 2^*$. Suppose that $a \in C(\mathbf{R}^N)$ and $b,\ K \in L^\infty(\mathbf{R}^N)$ satisfy the hypotheses:

$(A1)$ There exists $R_0 > 0$ such that

$$\{x;\, a(x) = 0\} \subset B(0,R_0) \quad \text{and} \quad \frac{1}{a} \in L^q(B(0,R_0)) \text{ for some } q > \frac{Np}{2N+2p-Np}\,;$$

$(A2)$ $\lim_{|x|\to\infty} a(x) = a(\infty) \in \mathbf{R}_+$ and $0 \leq a(x) \leq a(\infty)$ in $\mathbf{R}^N$;

(B) $\mathrm{ess}\lim_{|x|\to\infty} b(x) = b(\infty) \in \mathbf{R}_+$ and there exists $b_1 > 0$ such that $b_1 \leq b(x) \leq b(\infty)$ a.e. in $\mathbf{R}^N$;

(K) $\mathrm{ess}\lim_{|x|\to\infty} K(x) = K(\infty) \in \mathbf{R}_+$ and $K(x) \geq K(\infty)$ a.e. in $\mathbf{R}^N$;

(M) $\mathrm{meas}\,(\{x \in \mathbf{R}^N;\, b(x) < b(\infty)\} \cup \{x \in \mathbf{R}^N;\, K(x) > K(\infty)\}) > 0$.

The degeneracy hypothesis $(A1)$ is inspired by condition $(A-1)$ introduced in Murthy-Stampacchia [11]. In light of Proposition 1, assumption $(A1)$ should be seen as a "subcritically" condition. Our framework includes degeneracies a that behave like $a(x) \sim |x|^\alpha$ near the origin, with $0 < \alpha < 2N/p + 2 - N$. For the treatment of supercritical degeneracies on bounded domains we refer to Passaseo [12], where several nonexistence results are proven. The full strength of condition $(A2)$ will appear in the proof of Proposition 2. This assumption is taken over from Chabrowski [7] and it will be used in this paper only to check that the hypotheses of [7, Theorem 1] are fulfilled in our situation.

Let $H^1_{a,b}(\mathbf{R}^N)$ be the Sobolev space defined as the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_{a,b}^2 = \int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\, dx.$$

We denote by $\|\cdot\|_{-1}$ the norm of $H_{a,b}^{-1}(\mathbf{R}^N)$ which is the dual space of $H_{a,b}^1(\mathbf{R}^N)$, i. e. $H_{a,b}^{-1}(\mathbf{R}^N) = (H_{a,b}^1(\mathbf{R}^N))^*$. Throughout this work we suppose that $g \in H_{a,b}^{-1}(\mathbf{R}^N) \setminus \{0\}$.

Definition 1 *We say that $u \in H_{a,b}^1(\mathbf{R}^N)$ is a weak solution of (3) if*

$$\int_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(x)uv)\, dx - \int_{\mathbf{R}^N} K(x)|u|^{p-2}uv\, dx - \int_{\mathbf{R}^N} g(x)v\, dx = 0 \quad \textit{for all } v \in C_0^\infty(\mathbf{R}^N)\,.$$

We are concerned in this paper with the study of the degenerate semilinear elliptic equation (3), in other words it is assumed that a vanishes in at least one point in $\mathbf{R}^N$. The main result asserts that if $\|g\|_{-1}$ is sufficiently small then problem (3) possesses at least two solutions. We overcome the lack of compactness of our problem by applying a variant of the Mountain Pass Theorem without the Palais-Smale condition (see Brezis-Nirenberg [6, Theorem 2.2]), combined with a generalization of the Brezis-Lieb Lemma [5, Theorem 1]. We also point out that the study of degenerate elliptic boundary value problems was initiated in Mikhlin [9], [10] and many papers have been devoted in the past decades to the study of several questions related to these problems. We refer only to Murthy-Stampacchia [11], Stredulinsky [13], Passaseo [12] and the references therein.

Taking into account our hypothesis $(A2)$, the continuity of a implies that meas $\{x \in \mathbf{R}^N;\ a(x) < a(\infty)\} > 0$. On the other hand, combining the hypotheses $(A1)$ and $(A2)$ with the continuity of a we obtain that $\inf_{\mathbf{R}^N \setminus B(0,R_0)} a(x) > 0$. According to these comments we see that if a, $K \in C(\mathbf{R}^N)$ satisfy $(A1)$, $(A2)$ and (K) then all the assumptions of Lemma 1 and Theorem 1 in [7] are fulfilled. In virtue of these results, Chabrowski [7] established the existence of a weak solution to problem (3) in the case $g \equiv 0$ and $b \equiv \lambda > 0$. We prove in this paper that if we perturbe the problem studied in Chabrowski's paper such that the perturbation does not exceed some level, then equation (3) has at least two distinct solutions. More precisely, if g is small then there is a local minimum near the origin, while the second solution is obtained as a mountain pass. Assumptions (B), (K) and (M) will be used to deduce the existence of the mountain pass solution, while the existence of a simple solution (the local minimum) will follow without these stronger hypotheses. Results of this type have been originally proven in Tarantello [15], but in a different framework. More precisely, Tarantello considered the non-degenerate ($a \equiv 1$) problem (3) in a bounded domain, and for $p = 2^*$ ($N \geq 3$), $b \equiv 0$, $K \equiv 1$ it is showed that (3) has at least two distinct solutions, provided that $g \not\equiv 0$ is sufficiently "small" in a suitable sense.

Our main result is the following.

Theorem 1 *Assume conditions $(A1)$, $(A2)$, (B), (K) and (M) are fulfilled. Then there exists $C > 0$ such that problem (3) has at least two solutions, for any $g \not\equiv 0$ satisfying $\|g\|_{-1} < C$.*

2 Auxiliary results

Weak solutions of (3) correspond to the critical points of the energy functional

$$J(u) = \frac{1}{2}\int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\,dx - \frac{1}{p}\int\limits_{\mathbf{R}^N} K(x)|u|^p\,dx - \int\limits_{\mathbf{R}^N} g(x)u dx\,, \qquad u \in H^1_{a,b}(\mathbf{R}^N).$$

It is easy to observe that the boundedness of a and b implies that $H^1(\mathbf{R}^N)$ is continuously embedded in $H^1_{a,b}(\mathbf{R}^N)$. Our first result shows that $H^1_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^p(\mathbf{R}^N)$. Using this fact and (K) we conclude that the functional J is well defined.

Proposition 1 *There exists a positive constant $C_p > 0$ such that, for any $u \in H^1_{a,b}(\mathbf{R}^N)$,*

$$\left(\int\limits_{\mathbf{R}^N} |u|^p\,dx\right)^{\frac{1}{p}} \le C_p \left(\int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\,dx\right)^{\frac{1}{2}}.$$

Proof. We follow the method used in the proof of Proposition 2.1 in Passaseo [12] (see also Chabrowski [7]). In view of our hypotheses $(A1)$ and $(A2)$, we may assume, by taking R_0 large enough, that

$$\{x;\, a(x) = 0\} \subset B(0, R_0 - 1) \quad \text{and} \quad \inf_{\mathbf{R}^N\setminus B(0,R_0-1)} a(x) > 0. \tag{4}$$

Choosing q appearing in (A1), we define $r = \frac{2q}{q+1}$. We see that our hypothesis $q > \frac{Np}{2N+2p-Np}$ implies $p < \frac{Nr}{N-r}$, where $1 < r < 2 \le N$. So, by the Sobolev embedding theorem, $W_0^{1,r}(B(0,R_0))$ is continuously embedded in $L^p(B(0,R_0))$. Using this fact, $(A1)$ and Hölder's inequality we find

$$\left(\int\limits_{B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}} \le C_1 \left(\int\limits_{B(0,R_0)} |\nabla u|^r\,dx\right)^{\frac{1}{r}} = C_1 \left(\int\limits_{B(0,R_0)} \frac{1}{a(x)^{\frac{q}{q+1}}} |\nabla u|^r a(x)^{\frac{q}{q+1}}\,dx\right)^{\frac{1}{r}} \le$$
$$C_1 \left(\int\limits_{B(0,R_0)} \frac{1}{a(x)^q}\,dx\right)^{\frac{1}{2q}} \left(\int\limits_{B(0,R_0)} a(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}} \le C_2 \left(\int\limits_{B(0,R_0)} (a(x)|\nabla u|^2 + b(x)u^2)\,dx\right)^{\frac{1}{2}}. \tag{5}$$

Let $\Psi_{R_0-1} \in C^1(\mathbf{R}^N)$ be such that $\Psi_{R_0-1} = 1$ in $\mathbf{R}^N \setminus B(0,R_0)$, $\Psi_{R_0-1} = 0$ on $B(0,R_0-1)$ and $0 \le \Psi_{R_0-1} \le 1$ in $\mathbf{R}^N$. The continuous embedding $H^1(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ and relation (4) imply

$$
\begin{gathered}
\left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}} = \left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0)} |u\Psi_{R_0-1}|^p\,dx\right)^{\frac{1}{p}} \leq \\
\left(\int\limits_{\mathbf{R}^N} |u\Psi_{R_0-1}|^p\,dx\right)^{\frac{1}{p}} \leq C_3\left(\int\limits_{\mathbf{R}^N} (|\nabla(u\Psi_{R_0-1})|^2 + |u\Psi_{R_0-1}|^2)\,dx\right)^{\frac{1}{2}} \leq \\
C_4\left(\int\limits_{\mathbf{R}^N} (|\nabla\Psi_{R_0-1}|^2 u^2 + |\Psi_{R_0-1}|^2|\nabla u|^2 + |\Psi_{R_0-1}|^2 u^2)\,dx\right)^{\frac{1}{2}} \leq \\
C_5\left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0-1)} (|\nabla u|^2 + u^2)\,dx\right)^{\frac{1}{2}} \leq C_6\left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0-1)} (a(x)|\nabla u|^2 + b(x)u^2)\,dx\right)^{\frac{1}{2}},
\end{gathered}
\tag{6}
$$

where C_i with $i = \overline{1,6}$ are some positive constants.

From (5), (6) and the elementary inequality

$$
(a+b)^{1/p} \leq C(p)\,(a^{1/p} + b^{1/p}) \qquad \text{for all } a, b > 0
$$

we obtain

$$
\begin{gathered}
\left(\int\limits_{\mathbf{R}^N} |u|^p\,dx\right)^{\frac{1}{p}} \leq C(p)\left[\left(\int\limits_{B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}} + \left(\int\limits_{\mathbf{R}^N\setminus B(0,R_0)} |u|^p\,dx\right)^{\frac{1}{p}}\right] \leq \\
C_p\left(\int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\,dx\right)^{\frac{1}{2}},
\end{gathered}
$$

for some positive constants $C(p)$ and C_p depending only on p. This completes our proof. $\square$

In this paper we denote by "$\rightharpoonup$" the weak convergence and by "$\to$" the strong convergence, in an arbitrary Banach space X.

Remark 1 Let $\{u_n\}$ be a sequence that converges weakly to some u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Since $\{u_n\}$ is bounded in $H^1_{a,b}(\mathbf{R}^N)$ we see easily that $\{u_n\}$ restricted to $\mathbf{R}^N \setminus B(0,R_0)$ is bounded in $H^1(\mathbf{R}^N \setminus B(0,R_0))$. It also follows from the proof of Proposition 1 that the sequence $\{u_n\}$ restricted to $B(0,R_0)$ is bounded in $W^{1,r}_0(B(0,R_0))$, $p < \frac{Nr}{N-r}$. Therefore, we may assume (up to a subsequence) that

$$
u_n \to u_0 \text{ in } L^p_{\mathrm{loc}}(\mathbf{R}^N) \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N. \tag{7}
$$

Remark 2 If we examine carefully the proof of Proposition 1 we see that it holds in order to conclude that $H^1_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^s(\mathbf{R}^N)$, for every $2 \le s \le p$. If $\{u_n\}$ is a bounded sequence in $H^1_{a,b}(\mathbf{R}^N)$, then using the fact that $H^1_{a,b}(\mathbf{R}^N)$ is a reflexive space and Remark 1 we can assume (passing eventually to subsequences) that

$$u_n \rightharpoonup u_0 \text{ in } H^1_{a,b}(\mathbf{R}^N), \quad u_n \to u_0 \text{ in } L^s_{\mathrm{loc}}(\mathbf{R}^N),\ 2 \le s \le p \quad \text{and} \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N. \tag{8}$$

We define the functionals $I : H^1_{a,b}(\mathbf{R}^N) \to \mathbf{R}$ and $I_\infty : H^1_{a,b}(\mathbf{R}^N) \to \mathbf{R}$ by

$$I(u) = \frac{1}{2}\int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(x)u^2)\,dx - \frac{1}{p}\int\limits_{\mathbf{R}^N} K(x)|u|^p\,dx,$$
$$I_\infty(u) = \frac{1}{2}\int\limits_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\,dx - \frac{1}{p}\int\limits_{\mathbf{R}^N} K(\infty)|u|^p\,dx.$$

A simple calculation shows that $J,\ I,\ I_\infty \in C^1(H^1_{a,b}(\mathbf{R}^N), \mathbf{R})$ and their derivatives are given by

$$\langle J'(u), v\rangle = \int\limits_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(x)uv)\,dx - \int\limits_{\mathbf{R}^N} K(x)|u|^{p-2}uv\,dx - \int\limits_{\mathbf{R}^N} g(x)v\,dx,$$
$$\langle I'(u), v\rangle = \int\limits_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(x)uv)\,dx - \int\limits_{\mathbf{R}^N} K(x)|u|^{p-2}uv\,dx,$$
$$\langle I'_\infty(u), v\rangle = \int\limits_{\mathbf{R}^N} (a(x)\nabla u \cdot \nabla v + b(\infty)uv)\,dx - \int\limits_{\mathbf{R}^N} K(\infty)|u|^{p-2}uv\,dx,$$

for all $u,\ v \in H^1_{a,b}(\mathbf{R}^N)$. We have denoted by $\langle\ ,\ \rangle$ the duality pairing between $H^1_{a,b}(\mathbf{R}^N)$ and $H^{-1}_{a,b}(\mathbf{R}^N)$.

Definition 2 *If F is a C^1 functional on some Banach space X and c is a real number, we say that a sequence $\{u_n\}$ in X is a $(PS)_c$ sequence of F if $F(u_n) \to c$ and $F'(u_n) \to 0$ in X^*.*

We now prove that the weak limit (if exists) of any $(PS)_c$ sequence of J is a solution of problem (3).

Lemma 1 *Let $\{u_n\} \subset H^1_{a,b}(\mathbf{R}^N)$ be a $(PS)_c$ sequence of J for some $c \in \mathbf{R}$. Assume that $\{u_n\}$ converges weakly to some u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Then $J'(u_0) = 0$ i.e. u_0 is a weak solution of problem (3).*

Proof. Consider an arbitrary function $\zeta \in C_0^\infty(\mathbf{R}^N)$ and set $\Omega = \operatorname{supp}\zeta$. Obviously $J'(u_n) \to 0$ in $H^{-1}_{a,b}(\mathbf{R}^N)$ implies $\langle J'(u_n), \zeta\rangle \to 0$ as $n \to \infty$, that is

$$\lim_{n\to\infty}\left(\int_\Omega (a(x)\nabla u_n \cdot \nabla\zeta + b(x)u_n\zeta)\,dx - \int_\Omega K(x)|u_n|^{p-2}u_n\zeta\,dx - \int_\Omega g(x)\zeta\,dx\right) = 0. \tag{9}$$

Since $u_n \rightharpoonup u_0$ in $H^1_{a,b}(\mathbf{R}^N)$ it follows that

$$\lim_{n\to\infty}\int_\Omega (a(x)\nabla u_n\cdot\nabla\zeta + b(x)u_n\zeta)\,dx = \int_\Omega (a(x)\nabla u_0\cdot\nabla\zeta + b(x)u_0\zeta)\,dx. \tag{10}$$

The boundedness of $\{u_n\}$ in $H^1_{a,b}(\mathbf{R}^N)$ and Proposition 1 show that $\{|u_n|^{p-2}u_n\}$ is a bounded sequence in $L^{p/(p-1)}(\mathbf{R}^N)$. Combining this with the convergence $|u_n|^{p-2}u_n \to |u_0|^{p-2}u_0$ a.e. in $\mathbf{R}^N$ (which is a consequence of (7)) we deduce that $|u_0|^{p-2}u_0$ is the weak limit of the sequence $|u_n|^{p-2}u_n$ in $L^{p/(p-1)}(\mathbf{R}^N)$. So

$$\lim_{n\to\infty}\int_\Omega K(x)|u_n|^{p-2}u_n\zeta\,dx = \int_\Omega K(x)|u_0|^{p-2}u_0\zeta\,dx. \tag{11}$$

From (9), (10) and (11) we deduce that

$$\int_\Omega (a(x)\nabla u_0\cdot\nabla\zeta + b(x)u_0\zeta)\,dx - \int_\Omega K(x)|u_0|^{p-2}u_0\zeta\,dx - \int_\Omega g(x)\zeta\,dx = 0.$$

By density, this equality holds for any $\zeta \in H^1_{a,b}(\mathbf{R}^N)$ which means that $J'(u_0) = 0$. The proof of our lemma is complete. $\square$

Brezis and Lieb established in [5, Theorem 1] a subtle refinement of Fatou's Lemma. Our next result is a weighted variant of the Brezis-Lieb Lemma.

Lemma 2 *Let $\{u_n\}$ be a sequence which is weakly convergent to u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Then*

$$\lim_{n\to\infty}\int_{\mathbf{R}^N} K(x)\left(|u_n|^p - |u_n-u_0|^p\right)dx = \int_{\mathbf{R}^N} K(x)|u_0|^p\,dx.$$

Proof. From Proposition 1 and the boundedness of $\{u_n\}$ in $H^1_{a,b}(\mathbf{R}^N)$ we obtain that $\{u_n\}$ is a bounded sequence in $L^p(\mathbf{R}^N)$. For a given $\varepsilon > 0$ we choose $R_\varepsilon > 0$ such that

$$\int_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx < \varepsilon. \tag{12}$$

We have

$$\begin{aligned}
&\Big|\int_{\mathbf{R}^N} K(x)(|u_n|^p - |u_0|^p - |u_n-u_0|^p)\,dx\Big| = \Big|\int_{|x|\le R_\varepsilon} K(x)(|u_n|^p - |u_0|^p)\,dx - \\
&\int_{|x|\le R_\varepsilon} K(x)|u_n-u_0|^p\,dx - \int_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx + \int_{|x|>R_\varepsilon} K(x)(|u_n|^p - |u_n-u_0|^p)\,dx\Big| \le \\
&\Big|\int_{|x|\le R_\varepsilon} K(x)(|u_n|^p - |u_0|^p)\,dx\Big| + \int_{|x|\le R_\varepsilon} K(x)|u_n-u_0|^p\,dx + \int_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx + \\
&\int_{|x|>R_\varepsilon} p\,K(x)|\theta u_0 + (u_n-u_0)|^{p-1}|u_0|\,dx, \quad \text{where} \quad 0\le\theta(x)\le 1.
\end{aligned} \tag{13}$$

From (12) and Hölder's inequality we find

$$\int\limits_{|x|>R_\varepsilon} K(x)|\theta u_0+(u_n-u_0)|^{p-1}|u_0|\,dx \le c \int\limits_{|x|>R_\varepsilon} K(x)(|u_0|^p+|u_n-u_0|^{p-1}|u_0|)\,dx \le$$

$$c\left[\int\limits_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx + \left(\int\limits_{|x|>R_\varepsilon} K(x)|u_n-u_0|^p\,dx\right)^{\frac{p-1}{p}} \left(\int\limits_{|x|>R_\varepsilon} K(x)|u_0|^p\,dx\right)^{\frac{1}{p}}\right] < \tilde{c}(\varepsilon+\varepsilon^{\frac{1}{p}}) \tag{14}$$

for some constants $c, \tilde{c} > 0$ independent of n and ε.
Now, by (7),

$$\lim_{n\to\infty} \int\limits_{|x|\le R_\varepsilon} K(x)(|u_n|^p-|u_0|^p)\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int\limits_{|x|\le R_\varepsilon} K(x)|u_n-u_0|^p\,dx = 0. \tag{15}$$

From (12), (13), (14) and (15) it follows that

$$\limsup_{n\to\infty} \left|\int\limits_{\mathbf{R}^N} K(x)(|u_n|^p-|u_0|^p-|u_n-u_0|^p)\,dx\right| \le (p\,\tilde{c}+1)(\varepsilon+\varepsilon^{\frac{1}{p}})\,.$$

Since $\varepsilon > 0$ is arbitrary we deduce that

$$\lim_{n\to\infty}\left(\int\limits_{\mathbf{R}^N} K(x)|u_n|^p\,dx - \int\limits_{\mathbf{R}^N} K(x)|u_0|^p\,dx - \int\limits_{\mathbf{R}^N} K(x)|u_n-u_0|^p\,dx\right) = 0,$$

which concludes our proof. □

Lemma 3 *Let $\{v_n\} \subset H^1_{a,b}(\mathbf{R}^N)$ be a sequence converging weakly to 0 in $H^1_{a,b}(\mathbf{R}^N)$. Then*

$$\lim_{n\to\infty}[I(v_n)-I_\infty(v_n)] = 0; \tag{16}$$

$$\lim_{n\to\infty}[\langle I'(v_n),v_n\rangle - \langle I'_\infty(v_n),v_n\rangle] = 0. \tag{17}$$

Proof. A simple computation yields

$$I(v_n) = I_\infty(v_n) - \frac{1}{2}\int\limits_{\mathbf{R}^N}(b(\infty)-b(x))v_n^2\,dx - \frac{1}{p}\int\limits_{\mathbf{R}^N}(K(x)-K(\infty))|v_n|^p\,dx.$$

$$\langle I'(v_n),v_n\rangle = \langle I'_\infty(v_n),v_n\rangle - \int\limits_{\mathbf{R}^N}(b(\infty)-b(x))v_n^2\,dx - \int\limits_{\mathbf{R}^N}(K(x)-K(\infty))|v_n|^p\,dx.$$

Let $\varepsilon > 0$ be a positive number. The assumption (K) implies that there exists $R_\varepsilon > 0$ such that

$$|K(x)-K(\infty)| = K(x)-K(\infty) < \varepsilon \quad \text{for a.e. } x \in \mathbf{R}^N \text{ with } |x| > R_\varepsilon.$$

Using this fact we obtain

$$\int_{\mathbf{R}^N} (K(x) - K(\infty))|v_n|^p \, dx = \int_{|x|\le R_\varepsilon} (K(x) - K(\infty))|v_n|^p \, dx + \int_{|x|>R_\varepsilon} (K(x) - K(\infty))|v_n|^p \, dx \le$$
$$(\|K\|_\infty - K(\infty)) \int_{|x|\le R_\varepsilon} |v_n|^p \, dx + \varepsilon \left(\int_{|x|>R_\varepsilon} |v_n|^p \, dx \right).$$

Since $v_n \rightharpoonup 0$ in $H^1_{a,b}(\mathbf{R}^N)$, it follows by Proposition 1 that $\{v_n\}$ is bounded in $L^p(\mathbf{R}^N)$. On the other hand, in virtue of (7), we have that $v_n \to 0$ in $L^p_{\mathrm{loc}}(\mathbf{R}^N)$. Then letting $n \to \infty$ we see that

$$\limsup_{n\to\infty} \int_{\mathbf{R}^N} (K(x) - K(\infty))|v_n|^p \, dx \le C\,\varepsilon$$

for some constant $C > 0$ independent of n and ε. It follows that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (K(x) - K(\infty))|v_n|^p \, dx = 0.$$

To prove (16) and (17) we need only to show that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (b(\infty) - b(x))v_n^2 \, dx = 0. \tag{18}$$

To this end, notice that for any $R > 0$ we have

$$\begin{aligned}\int_{\mathbf{R}^N} (b(\infty) - b(x))v_n^2 \, dx &= \int_{|x|\le R} (b(\infty) - b(x))v_n^2 \, dx + \int_{|x|>R} (b(\infty) - b(x))v_n^2 \, dx \le \\ &(b(\infty) - b_1) \int_{|x|\le R} v_n^2 \, dx + \int_{|x|>R} (b(\infty) - b(x))v_n^2 \, dx.\end{aligned} \tag{19}$$

From (B) we have that for any $\varepsilon > 0$ we can choose $R_\varepsilon > 0$ such that

$$|b(\infty) - b(x)| = b(\infty) - b(x) < \varepsilon \quad \text{for a.e. } x \in \mathbf{R}^N \text{ with } |x| > R_\varepsilon. \tag{20}$$

But, from Remark 2, we know that $H^1_{a,b}(\mathbf{R}^N)$ is continuously embedded in $L^2(\mathbf{R}^N)$ and, by (8), $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbf{R}^N)$. Therefore, using (19) and (20) we deduce the existence of a positive number M, independent of n and ε, such that

$$\limsup_{n\to\infty} \int_{\mathbf{R}^N} (b(\infty) - b(x))v_n^2 \, dx \le M\,\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that (18) is true. $\square$

Lemma 4 *For any $0<\varepsilon<1$ there exist $R=R(\varepsilon)>0$ and $C=C(\varepsilon)>0$ such that for all g with $\|g\|_{-1}<C$, there exists a $(PS)_{c_0}$ sequence of $J(u)$ with $c_0=c_0(R)=\inf\limits_{u\in\overline{B}_R} J(u)$, where $\overline{B}_R=\{u\in H^1_{a,b}(\mathbf{R}^N);\ \|u\|_{a,b}\le R\}$. Moreover, $c_0(R)$ is achieved by some $u_0\in H^1_{a,b}(\mathbf{R}^N)$ with $J'(u_0)=0$.*

Proof. Fix $0<\varepsilon<1$. Then for any $u\in H^1_{a,b}(\mathbf{R}^N)$, by (K) and Young's inequality we have

$$\begin{aligned}
&J(u)=\frac{1}{2}\|u\|^2_{a,b}-\frac{1}{p}\int\limits_{\mathbf{R}^N}K(x)|u|^p\,dx-\int\limits_{\mathbf{R}^N}g(x)\,u\,dx\ge\\
&\frac{1}{2}\|u\|^2_{a,b}-\frac{\|K\|_\infty}{p}\|u\|^p_{L^p(\mathbf{R}^N)}-\|u\|_{a,b}\|g\|_{-1}\ge\\
&\frac{1}{2}\|u\|^2_{a,b}-\frac{\|K\|_\infty}{p}C_0^p\|u\|^p_{a,b}-\left(\frac{\varepsilon^2}{2}\|u\|^2_{a,b}+\frac{1}{2\varepsilon^2}\|g\|^2_{-1}\right)=\\
&\left(\frac{1}{2}-\frac{\varepsilon^2}{2}\right)\|u\|^2_{a,b}-\frac{\|K\|_\infty}{p}C_0^p\|u\|^p_{a,b}-\frac{1}{2\varepsilon^2}\|g\|^2_{-1}\,,
\end{aligned}$$

where $C_0>0$ is a positive constant given by Proposition 1. The above estimate shows the existence of $R=R(\varepsilon)>0$, $C=C(\varepsilon)>0$ and $\delta=\delta(R)>0$ such that $J(u)\mid_{\partial B_R}\ge\delta>0$ for all g with $\|g\|_{-1}\le C$. For example, we can take

$$R(\varepsilon)=\left(\frac{1-\varepsilon^2}{\|K\|_\infty C_0^p}\right)^{\frac{1}{p-2}},\ C(\varepsilon)=\sqrt{M}\varepsilon,\ \delta(R)=\frac{M}{2},\quad\text{where}\quad M=M(R)=\left(\frac{1}{2}-\frac{1}{p}\right)\|K\|_\infty C_0^pR^p.$$

Define $c_0=c_0(R)=\inf\limits_{u\in\overline{B}_R}J(u)$. So, $c_0\le J(0)=0$. The set $\overline{B}_R$ becomes a complete metric space with respect to the distance

$$\mathrm{dist}\,(u,v)=\|u-v\|_{a,b}\quad\text{for any } u,v\in\overline{B}_R.$$

On the other hand, J is lower semi-continuous and bounded from below on $\overline{B}_R$. So, by Ekeland's Variational Principle [8, Theorem 1.1], for any positive integer n there exists u_n with

$$c_0\le J(u_n)\le c_0+\frac{1}{n},\tag{21}$$

$$J(w)\ge J(u_n)-\frac{1}{n}\|u_n-w\|_{a,b}\quad\text{for all } w\in\overline{B}_R.\tag{22}$$

We claim that $\|u_n\|_{a,b}<R$ for n large enough. Indeed, if $\|u_n\|_{a,b}=R$ for infinitely many n, we may assume, without loss of generality, that $\|u_n\|_{a,b}=R$ for all $n\ge1$. It follows that $J(u_n)\ge\delta>0$. Combining this with (21) and letting $n\to\infty$, we have $0\ge c_0\ge\delta>0$ which is a contradiction.

We now prove that $J'(u_n) \to 0$ in $H_{a,b}^{-1}(\mathbf{R}^N)$. Indeed, for any $u \in H_{a,b}^1(\mathbf{R}^N)$ with $\|u\|_{a,b} = 1$, let $w_n = u_n + tu$. For a fixed n, we have $\|w_n\|_{a,b} \le \|u_n\|_{a,b} + t < R$, where $t > 0$ is small enough. Using (22) we obtain

$$J(u_n + tu) \ge J(u_n) - \frac{t}{n}\|u\|_{a,b}$$

that is

$$\frac{J(u_n + tu) - J(u_n)}{t} \ge -\frac{1}{n}\|u\|_{a,b} = -\frac{1}{n}.$$

Letting $t \searrow 0$, we deduce that $\langle J'(u_n), u\rangle \ge -\dfrac{1}{n}$ and a similar argument for $t \nearrow 0$ produces $|\langle J'(u_n), u\rangle| \le \dfrac{1}{n}$ for any $u \in H_{a,b}^1(\mathbf{R}^N)$ with $\|u\|_{a,b} = 1$. So,

$$\|J'(u_n)\|_{-1} = \sup_{\substack{u \in H_{a,b}^1(\mathbf{R}^N) \\ \|u\|_{a,b}=1}} |\langle J'(u_n), u\rangle| \le \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

We have obtained the existence of a $(PS)_{c_0}$ sequence, i.e. a sequence $\{u_n\} \subset H_{a,b}^1(\mathbf{R}^N)$ with

$$J(u_n) \to c_0 \quad \text{and} \quad J'(u_n) \to 0 \text{ in } H_{a,b}^{-1}(\mathbf{R}^N). \tag{23}$$

But $\|u_n\|_{a,b} \le R$, for the fixed R, shows that $\{u_n\}$ converges weakly (up to a subsequence) in $H_{a,b}^1(\mathbf{R}^N)$. Therefore (7), (23) and Lemma 1 imply that, for some $u_0 \in H_{a,b}^1(\mathbf{R}^N)$

$$u_n \rightharpoonup u_0 \text{ in } H_{a,b}^1(\mathbf{R}^N), \quad u_n \to u_0 \text{ a.e. in } \mathbf{R}^N \tag{24}$$

$$J'(u_0) = 0. \tag{25}$$

We prove that $J(u_0) = c_0$. By (23) and (24) we have

$$o(1) = \langle J'(u_n), u_n\rangle = \int_{\mathbf{R}^N} (a(x)|\nabla u_n|^2 + b(x)u_n^2)\,dx - \int_{\mathbf{R}^N} K(x)|u_n|^p\,dx - \int_{\mathbf{R}^N} g(x)u_n\,dx.$$

Therefore

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbf{R}^N} K(x)|u_n|^p\,dx - \frac{1}{2}\int_{\mathbf{R}^N} g(x)u_n\,dx + o(1).$$

By (23), (24), (25) and Fatou's lemma we have

$$c_0 = \liminf_{n\to\infty} J(u_n) \ge \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbf{R}^N} K(x)|u_0|^p\,dx - \frac{1}{2}\int_{\mathbf{R}^N} g(x)u_0\,dx = J(u_0).$$

Since $u_0 \in \overline{B}_R$, it follows that $J(u_0) = c_0$. $\square$

3 Proof of Theorem 1

Set

$$S = \{u \in H^1_{a,b}(\mathbf{R}^N) \setminus \{0\};\ \langle I'_\infty(u), u\rangle = 0\}.$$

We first justify that $S \not\equiv \emptyset$. Indeed, fix $u_0 \in H^1_{a,b}(\mathbf{R}^N) \setminus \{0\}$ and set, for any $\lambda > 0$,

$$\Psi(\lambda) = \langle I'_\infty(\lambda u_0), \lambda u_0\rangle\,.$$

It follows that

$$\Psi(\lambda) = \lambda^2 \left(\int_{\mathbf{R}^N} \left(a(x)|\nabla u_0|^2 + b(\infty)u_0^2\right)\, dx - \lambda^{p-2}\int_{\mathbf{R}^N} K(\infty)|u_0|^p\, dx\right)\,.$$

Our hypotheses imply that $\Psi(\lambda) < 0$ for λ large enough and $\Psi(\lambda) > 0$ for λ sufficiently close to zero. It follows that there exists $\lambda_0 \in (0,\infty)$ such that $\Psi(\lambda_0) = 0$. This means that $\lambda_0 u_0 \in S$.

Proposition 2 *Let $J_\infty = \inf\{I_\infty(u);\ u \in S\}$. Then there exists $\bar{u} \in H^1_{a,b}(\mathbf{R}^N)$ such that*

$$J_\infty = I_\infty(\bar{u}) = \sup_{t\geq 0} I_\infty(t\bar{u}). \tag{26}$$

Proof. We consider the constrained minimization problem

$$m = \inf\left\{\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx;\ u \in H^1_{a,b}(\mathbf{R}^N),\ \int_{\mathbf{R}^N} K(\infty)|u|^p\, dx = 1\right\}. \tag{27}$$

For every $\varphi \in H^1_{a,b}(\mathbf{R}^N) \setminus \{0\}$ let

$$f(t) = I_\infty(t\varphi) = \frac{t^2}{2}\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx - \frac{t^p}{p}\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\, dx.$$

We have

$$f'(t) = t\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx - t^{p-1}\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\, dx,$$

which vanishes for

$$\bar{t}(\varphi) = \left\{\frac{\displaystyle\int_{\mathbf{R}^N} (a(x)|\nabla u|^2 + b(\infty)u^2)\, dx}{\displaystyle\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\, dx}\right\}^{\frac{1}{p-2}}.$$

Hence

$$f(\bar{t}(\varphi)) = I_\infty(\bar{t}(\varphi)\varphi) = \sup_{t\geq 0} I_\infty(t\varphi) = \left(\frac{1}{2}-\frac{1}{p}\right)\left\{\frac{\displaystyle\int_{\mathbf{R}^N}(a(x)|\nabla u|^2 + b(\infty)u^2)\,dx}{\left(\displaystyle\int_{\mathbf{R}^N} K(\infty)|\varphi|^p\,dx\right)^{\frac{2}{p}}}\right\}^{\frac{p}{p-2}}.$$

It follows that

$$\inf_{\varphi\in H^1_{a,b}(\mathbf{R}^N)\setminus\{0\}} \sup_{t\geq 0} I_\infty(t\varphi) = \left(\frac{1}{2}-\frac{1}{p}\right) m^{\frac{p}{p-2}}.$$

We easily observe that for every $u \in S$ we have $\bar{t}(u) = 1$ which implies $I_\infty(u) = \sup_{t\geq 0} I_\infty(tu)$. Let $\{u_n\} \subset H^1_{a,b}(\mathbf{R}^N)$ be a minimizing sequence for problem (27), i.e.,

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (a(x)|\nabla u_n|^2 + b(\infty)u_n^2)\,dx = m \quad \text{and} \quad \int_{\mathbf{R}^N} K(\infty)|u_n|^p\,dx = 1.$$

Then $v_n = m^{\frac{1}{p-2}} u_n$ satisfies

$$(i) \quad I_\infty(v_n) \to \left(\tfrac{1}{2}-\tfrac{1}{p}\right) m^{\frac{p}{p-2}} \qquad \text{as } n\to\infty$$

$$(ii) \quad I'_\infty(v_n) \to 0 \quad \text{in } H^{-1}_{a,b}(\mathbf{R}^N) \quad \text{as } n\to\infty\,.$$

Now, using (B) we get that the minimizing sequence $\{u_n\}$ is bounded in $H^1_{a,b}(\mathbf{R}^N)$ and, by Remark 2, we find $u \in H^1_{a,b}(\mathbf{R}^N)$ such that (up to a subsequence) $u_n \rightharpoonup u$ in $H^1_{a,b}(\mathbf{R}^N)$ and $u_n \to u$ in $L^p_{\rm loc}(\mathbf{R}^N)$. Our hypotheses $(A1)$ and $(A2)$ allow us to apply Lemma 1 and Theorem 1 in [7] in order to find that $u \not\equiv 0$ and u is a solution of problem (27). Letting $\bar{u} = m^{\frac{1}{p-2}}u$, we see that $\bar{u} \in S$ and $I_\infty(\bar{u}) = \left(\dfrac{1}{2}-\dfrac{1}{p}\right) m^{\frac{p}{p-2}}$. We obtain

$$J_\infty = \inf_{u\in S} I_\infty(u) = \inf_{u\in S}\sup_{t\geq 0} I_\infty(tu) \geq \inf_{u\in H^1_{a,b}(\mathbf{R}^N)\setminus\{0\}} \sup_{t\geq 0} I_\infty(tu) = \left(\frac{1}{2}-\frac{1}{p}\right) m^{\frac{p}{p-2}} = I_\infty(\bar{u})$$

which concludes our proof. □

Proposition 3 *Assume $\{u_n\}$ is a $(PS)_c$ sequence of J that converges weakly to u_0 in $H^1_{a,b}(\mathbf{R}^N)$. Then either $\{u_n\}$ converges strongly in $H^1_{a,b}(\mathbf{R}^N)$, or $c \geq J(u_0) + J_\infty$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence and $u_n \rightharpoonup u_0$ in $H^1_{a,b}(\mathbf{R}^N)$ we have

$$J(u_n) = c + o(1) \quad \text{and} \quad \langle J'(u_n), u_n\rangle = o(1)\,. \tag{28}$$

Set $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0 \quad \text{in } H^1_{a,b}(\mathbf{R}^N)$ which implies

$$\int\limits_{\mathbf{R}^N} (a(x)\nabla v_n \nabla u_0 + b(x) v_n u_0)\, dx \to 0 \quad \text{as } n \to \infty\,,$$

$$\int\limits_{\mathbf{R}^N} g(x) v_n\, dx \to 0 \quad \text{as } n \to \infty\,.$$

We rewrite the above relations as

$$\begin{aligned} &\|u_n\|^2_{a,b} = \|u_0\|^2_{a,b} + \|v_n\|^2_{a,b} + o(1)\,, \\ &J(v_n) = I(v_n) + o(1)\,. \end{aligned} \tag{29}$$

From (28), (29), Lemma 1 and Lemma 2 it follows that

$$\begin{aligned} &o(1) + c = J(u_n) = J(u_0) + J(v_n) + o(1) = J(u_0) + I(v_n) + o(1)\,, \\ &o(1) = \langle J'(u_n), u_n\rangle = \langle J'(u_0), u_0\rangle + \langle J'(v_n), v_n\rangle + o(1) = \langle I'(v_n), v_n\rangle + o(1)\,. \end{aligned} \tag{30}$$

If $v_n \to 0$ in $H^1_{a,b}(\mathbf{R}^N)$, then $u_n \to u_0$ in $H^1_{a,b}(\mathbf{R}^N)$ and $J(u_0) = \lim_{n\to\infty} J(u_n) = c$.

If $v_n \not\to 0$ in $H^1_{a,b}(\mathbf{R}^N)$, then combining this with the fact that $v_n \rightharpoonup 0$ in $H^1_{a,b}(\mathbf{R}^N)$ we may assume that $\|v_n\|_{a,b} \to l > 0$. Then (30) and Lemma 3 imply

$$c = J(u_0) + I_\infty(v_n) + o(1) \tag{31}$$

$$\mu_n = \langle I'_\infty(v_n), v_n\rangle = \int\limits_{\mathbf{R}^N} (a(x)|\nabla v_n|^2 + b(\infty) v_n^2)\, dx - \int\limits_{\mathbf{R}^N} K(\infty)|v_n|^p\, dx = \alpha_n - \beta_n, \tag{32}$$

where $\lim_{n\to\infty} \mu_n = 0$, $\alpha_n = \int\limits_{\mathbf{R}^N} (a(x)|\nabla v_n|^2 + b(\infty) v_n^2)\, dx \geq \|v_n\|^2_{a,b}$ and $\beta_n = \int\limits_{\mathbf{R}^N} K(\infty)|v_n|^p\, dx \geq 0$. In virtue of (31), it remains to show that $I_\infty(v_n) \geq J_\infty + o(1)$. For $t > 0$, we have

$$\langle I'_\infty(tv_n), tv_n\rangle = t^2 \int\limits_{\mathbf{R}^N} (a(x)|\nabla v_n|^2 + b(\infty) v_n^2)\, dx - t^p \int\limits_{\mathbf{R}^N} K(\infty)|v_n|^p\, dx.$$

If we prove the existence of a sequence $\{t_n\}$ with $t_n > 0$, $t_n \to 1$ and $\langle I'_\infty(t_n v_n), t_n v_n\rangle = 0$, then

$$I_\infty(v_n) = I_\infty(t_n v_n) + \frac{1-t_n^2}{2}\alpha_n - \frac{1-t_n^p}{p} K(\infty)\|v_n\|^p_{L^p(\mathbf{R}^N)} = I_\infty(t_n v_n) + o(1) \geq J_\infty + o(1)$$

and the conclusion follows. To do this, let $t = 1 + \delta$ with $|\delta|$ small enough and using (32) we obtain

$$\begin{aligned} &\langle I'_\infty(tv_n), tv_n\rangle = (1+\delta)^2\alpha_n - (1+\delta)^p\beta_n = (1+\delta)^2\alpha_n - (1+\delta)^p(\alpha_n - \mu_n) = \\ &\alpha_n(2\delta - p\delta + o(\delta)) + (1+\delta)^p\mu_n = \alpha_n(2-p)\delta + \alpha_n o(\delta) + (1+\delta)^p\mu_n. \end{aligned}$$

Since $\alpha_n \to \bar{l} \geq l^2 > 0$, $\lim_{n\to\infty} \mu_n = 0$ and $p > 2$ then, for n large enough, we can define $\delta_n^+ = \dfrac{2\,|\mu_n|}{\alpha_n(p-2)}$ and $\delta_n^- = \dfrac{-2\,|\mu_n|}{\alpha_n(p-2)}$ which satisfy the following properties

$$\begin{aligned} &\delta_n^+ \searrow 0 \quad \text{and} \quad \langle I'_\infty((1+\delta_n^+)v_n), (1+\delta_n^+)v_n\rangle < 0\,, \\ &\delta_n^- \nearrow 0 \quad \text{and} \quad \langle I'_\infty((1+\delta_n^-)v_n), (1+\delta_n^-)v_n\rangle > 0\,. \end{aligned} \tag{33}$$

From (33) we deduce the existence of $t_n \in (1+\delta_n^-, 1+\delta_n^+)$ such that

$$t_n \to 1 \quad \text{and} \quad \langle I'_\infty(t_n v_n), t_n v_n\rangle = 0.$$

This concludes our proof. □

Let $\bar{u} \in H^1_{a,b}(\mathbf{R}^N)$ be such that (26) holds. We can find $\bar{t} > 0$ such that

$$\begin{aligned} &I(t\bar{u}) < 0 \quad \text{if } t \geq \bar{t} \\ &J(t\bar{u}) < 0 \quad \text{if } t \geq \bar{t} \text{ and } \|g\|_{-1} \leq 1\,. \end{aligned}$$

We put

$$\mathcal{P} = \{\gamma \in C([0,1], H^1_{a,b}(\mathbf{R}^N));\ \gamma(0) = 0, \gamma(1) = \bar{t}\bar{u}\} \tag{34}$$

$$c_g = \inf_{\gamma\in\mathcal{P}} \sup_{u\in\gamma} J(u). \tag{35}$$

Proposition 4 *There exist $R_0 > 0$, $C = C(R_0) > 0$ and $\delta_{R_0} > 0$ such that for all g with $\|g\|_{-1} < C$ we have $J\,|_{\partial B_{R_0}} \geq \delta_{R_0}$ and $c_g < c_0 + J_\infty$, where c_g is given by (35) and $c_0 = \inf_{u\in\overline{B}_{R_0}} J(u)$.*

Proof. By our hypothesis (M) and the definition of I we can assume that $I(t\bar{u}) < I_\infty(t\bar{u})$ for all $t > 0$. A simple computation implies that there exists $t_0 \in (0, \bar{t})$ such that

$$\sup_{t\geq 0} I(t\bar{u}) = I(t_0\bar{u}) < I_\infty(t_0\bar{u}) \leq \sup_{t\geq 0} I_\infty(t\bar{u}) = J_\infty.$$

Then there exists an $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\geq 0} I(t\bar{u}) < J_\infty - \varepsilon_0. \tag{36}$$

For this ε_0, we get the existence of $R_0 = R_0(\varepsilon_0)$ and $C_1 = C_1(\varepsilon_0) = C_1(R_0)$ such that for all g with $\|g\|_{-1} < C_1$ the conclusion of Lemma 4 holds. Moreover, in virtue of its proof, there exists $\delta_{R_0} > 0$ such that $J\,|_{\partial B_{R_0}} \geq \delta_{R_0}$, provided that $\|g\|_{-1} < C_1$. Taking $C_2 = \min\{C_1, \varepsilon_0\sqrt{\varepsilon_0}\}$ we find

$$c_0 = \inf_{u\in\overline{B}_{R_0}} J(u) \geq -\frac{1}{2\varepsilon_0^2}\|g\|_{-1}^2 > -\frac{\varepsilon_0}{2} \quad \text{for all } g \text{ with } \|g\|_{-1} < C_2. \tag{37}$$

If $\|g\|_{-1} < \dfrac{\varepsilon_0}{2\bar{t}\,\|\bar{u}\|_{a,b}}$, then for $u \in \gamma_0 = \{t\bar{t}\,\bar{u};\ 0 \le t \le 1\}$ we have

$$|J(u) - I(u)| = |\int_{\mathbf{R}^N} g(x)u\,dx| \le \bar{t}\,|\int_{\mathbf{R}^N} g(x)\bar{u}\,dx| \le \bar{t}\,\|\bar{u}\|_{a,b}\|g\|_{-1} < \frac{\varepsilon_0}{2}.$$

So, if $\|g\|_{-1} < C = \min\left\{C_2, \dfrac{\varepsilon_0}{2\bar{t}\,\|\bar{u}\|_{a,b}}\right\}$ then for all g with $\|g\|_{-1} < C$ we obtain

$$J(u) < I(u) + \frac{\varepsilon_0}{2} \quad \text{for } u \in \gamma_0$$

and from (34), (36), (37) it follows that

$$c_g = \inf_{\gamma\in\mathcal{P}} \sup_{u\in\gamma} J(u) \le \sup_{u\in\gamma_0} J(u) \le \sup_{u\in\gamma_0} I(u) + \frac{\varepsilon_0}{2} \le \sup_{t\ge 0} I(t\bar{u}) + \frac{\varepsilon_0}{2} < J_\infty - \frac{\varepsilon_0}{2} < J_\infty + c_0.$$

$\square$

Proof of Theorem 1 concluded. Consider $R_0 > 0$, $C = C(R_0) > 0$ and $\delta_{R_0} > 0$ given by Proposition 4 and, in view of its proof, we have that for all g with $\|g\|_{-1} < C$ the conclusion of Lemma 4 is also true. Therefore, we obtain the existence of a solution $u_0 \in H^1_{a,b}(\mathbf{R}^N)$ of (3) such that $J(u_0) = c_0$.

On the other hand, it follows from the Mountain Pass Theorem without the Palais-Smale condition [6, Theorem 2.2] that there is a $(PS)_{c_g}$ sequence $\{u_n\}$ of $J(u)$, that is

$$J(u_n) = c_g + o(1) \quad \text{and} \quad J'(u_n) \to 0 \text{ in } H^{-1}_{a,b}(\mathbf{R}^N)\,.$$

This implies

$$c_g+o(1)+\frac{1}{p}\|J'(u_n)\|_{-1}\|u_n\|_{a,b} \ge J(u_n)-\frac{1}{p}\langle J'(u_n), u_n\rangle \ge \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2_{a,b}-\left(1-\frac{1}{p}\right)\|g\|_{-1}\|u_n\|_{a,b}.$$

Hence $\{u_n\}$ is a bounded sequence in $H^1_{a,b}(\mathbf{R}^N)$ and, passing to a subsequence, we may assume that $u_n \rightharpoonup u_1$ in $H^1_{a,b}(\mathbf{R}^N)$ for some $u_1 \in H^1_{a,b}(\mathbf{R}^N)$. So, by Lemma 1, u_1 is a weak solution of (3).

We prove in what follows that $J(u_0) \ne J(u_1)$. Indeed, by Proposition 3, either $u_n \to u_1$ in $H^1_{a,b}(\mathbf{R}^N)$ which gives

$$J(u_1) = \lim_{n\to\infty} J(u_n) = c_g > 0 \ge c_0 = J(u_0)$$

and the conclusion follows, or

$$c_g = \lim_{n\to\infty} J(u_n) \ge J(u_1) + J_\infty.$$

If we suppose that $J(u_1) = J(u_0) = c_0$, then $c_g \ge c_0 + J_\infty$ which contradicts Proposition 4. This concludes our proof. $\square$

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