Nontrivial solutions for a multivalued problem with strong resonance

Vicenţiu D. RĂDULESCU
Department of Mathematics, University of Craiova, 1100 Craiova, Romania

The Mountain-Pass Theorem of Ambrosetti and Rabinowitz ([1]) and the Saddle Point Theorem of Rabinowitz ([21]) are very important tools in the critical point theory of C^1-functionals. That is why it is natural to ask us what happens if the functional fails to be differentiable. The first who considered such a case were Aubin and Clarke ([6]) and Chang ([12]), who gave suitable variants of the Mountain-Pass Theorem for locally Lipschitz functionals which are defined on reflexive Banach spaces. For this aim they replaced the usual gradient with a generalized one, which was firstly defined by Clarke ([13], [14]). As observed by Brezis ([12], p. 114), these abstract critical point theorems remain valid in non-reflexive Banach spaces.

We apply some of these results to solve a multivalued problem with strong resonance at infinity. We remark that it is not natural to consider nonlinearities which are strongly resonant at $+\infty$, but which may not be strongly resonant at $-\infty$. The literature is very rich in resonant problems, the first who studied such problems, however in the smooth case, being Landesman and Lazer ([18]). They found sufficient conditions for the existence of solutions for some singlevalued equations with Dirichlet conditions. These problems, that arise frequently in mechanics, were thereafter intensively studied and many applications to concrete situations were given.

1 Abstract framework

Let X be a real Banach space and let $f : X \to \mathbf{R}$ be a locally Lipschitz function. For each $x, v \in X$, we define the *generalized directional derivative* of f at x in the direction v as

$$f^0(x,v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y+\lambda v) - f(y)}{\lambda} \,.$$

The generalized gradient (the Clarke subdifferential) of f at the point x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*;\ f^0(x,v) \geq \langle x^*, v\rangle,\ \text{for all } v \in X\}\,.$$

We also define the lower semi-continuous function

$$\lambda(x) = \min\{\|x^*\|;\ x^* \in \partial f(x)\}\,.$$

For further properties of these notions we refer to [12, 13, 14].

We say that a point $x \in X$ is a critical point of f provided that $0 \in \partial f(x)$, that is $f^0(x,v) \geq 0$ for every $v \in X$. If c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short $(PS)_c$) if any sequence $(x_n)_n$ in X with the properties $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} \lambda(x_n) = 0$ is relatively compact.

We shall use in this paper the following result, which is an imediate consequence of the Mountain Pass Theorem proved in [12].

THEOREM 1. *Let $f : X \to \mathbf{R}$ be a locally Lipschitzian function. Suppose that $f(0) = 0$ and there is some $v \in X \setminus \{0\}$ such that $f(v) \leq 0$. Moreover, assume that f satisfies the following geometric hypothesis: there exist $0 < R < \|v\|$ and $\alpha > 0$ such that, for each $u \in X$ with $\|u\| = R$, we have $f(u) \geq \alpha$.*

Let $\mathcal{P}$ be the family of all continuous paths $p : [0,1] \to X$ that join 0 to v and

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} f(p(t)) \ .$$

Then there exists a sequence (x_n) in X such that:

(i) $\lim_{n\to\infty} f(x_n) = c$;

(ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Moreover, if f satisfies $(PS)_c$ then c is a critical value of f.

The following Saddle Point type result generalizes the Rabinowitz's Theorem ([21]). Its proof is an easy exercise and is left to the reader.

THEOREM 2. *Let $f : X \to \mathbf{R}$ be a locally Lipschitzian function. Assume that $X = Y \oplus Z$, where Z is a finite dimensional subspace of X and for some $z_0 \in Z$ there exists $R > \|z_0\|$ such that*

$$\inf_{y\in Y} f(y+z_0) > \max\{f(z);\ z \in Z,\ \|z\| = R\} \ ,$$

Let

$$K = \{z \in Z;\ \|z\| \leq R\}$$

and

$$\mathcal{P} = \{p \in C(K,X);\ p(x) = x \ \text{if} \ \|x\| = R\} \ .$$

If c is defined as in Theorem 1 and f satisfies $(PS)_c$, then c is a critical value of f.

2 Main results

Let M be a m-dimensional smooth compact Riemann manifold, possibly with smooth boundary ∂M. Particularly, M can be any open bounded smooth subset of $\mathbf{R}^m$. We shall consider the following multivalued elliptic problem

$$(P) \qquad \begin{cases} -\Delta_M u(x) - \lambda_1 u(x) \in [\,\underline{f}(u(x)), \overline{f}(u(x))\,] & \text{a.e. } x \in M \\ u = 0 & \text{on } \partial M \\ u \not\equiv 0 & \end{cases}$$

where:

i) Δ_M is the Laplace-Beltrami operator on M.

ii) λ_1 is the first eigenvalue of $-\Delta_M$ in $H_0^1(M)$.

iii) $f \in L^\infty(\mathbf{R})$.

iv) $\underline{f}(t) = \lim_{\varepsilon \searrow 0} essinf \{f(s);\ |t-s| < \varepsilon\}$

$\overline{f}(t) = \lim_{\varepsilon \searrow 0} esssup \{f(s);\ |t-s| < \varepsilon\}$.

As proved in [12], the functions $\underline{f}$ and $\overline{f}$ are measurable on $\mathbf{R}$ and, if

$$F(t) = \int_0^t f(s)ds \ ,$$

then the Clarke subdifferential of F is given by

$$\partial F(t) = [\,\underline{f}(t), \overline{f}(t)\,] \quad \text{a.e. } t \in \mathbf{R} \ .$$

Let $(g_{ij}(x))_{i,j}$ define the metric on M. We consider on $H_0^1(M)$ the locally Lipschitz functional $\varphi = \varphi_1 - \varphi_2$, where

$$\varphi_1(u) = \frac{1}{2}\int_M (\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \lambda_1 u^2\ )dx \quad \text{and} \quad \varphi_2(u) = \int_M F(u)dx \ .$$

By a solution of the problem (P) we shall mean any critical point of the energetic functional φ.

Denote

$$f(\pm\infty) = ess \lim_{t\to\pm\infty} f(t) \qquad \text{and} \qquad F(\pm\infty) = \lim_{t\to\pm\infty} F(t) \ .$$

Our basic hypothesis on f will be

$$(f1) \qquad f(+\infty) = F(+\infty) = 0 \ ,$$

which makes the problem (P) a Landesman-Lazer type one, with strong resonance at $+\infty$.

The following formulates a sufficient condition for the existence of solutions of our problem:

THEOREM A. *Assume that f satisfies (f1) and either*

$$(F1) \qquad F(-\infty) = -\infty$$

or $-\infty < F(-\infty) \le 0$ and there exists $\eta > 0$ such that

$$(F2) \qquad F \quad \textit{is non-negative on } (0,\eta) \textit{ or } (-\eta, 0)$$

Then the problem (P) has at least one solution.

For positive values of $F(-\infty)$ it is necessary to impose additional restrictions to f. Our variant for this case is

THEOREM B. *Assume (f1) and $0 < F(-\infty) < +\infty$.*

Then the problem (P) has at least one solution provided the following conditions are satisfied:

$$f(-\infty) = 0$$

and

$$F(t) \leq \frac{\lambda_2 - \lambda_1}{2} t^2 \qquad \textit{for each} \;\; t \in \mathbf{R} \; .$$

For the proof of Theorem A we shall make use of the following non-smooth variants of Lemmas 6 and 7 in [15] (see also [3] for Lemma 1) which can be obtained in the same manner:

LEMMA 1. *Assume $f \in L^\infty(\mathbf{R})$ and there exist $F(\pm\infty) \in \overline{\mathbf{R}}$. Moreover, suppose that*
(i) $f(+\infty) = 0$ if $F(+\infty)$ is finite;
and
(ii) $f(-\infty) = 0$ if $F(-\infty)$ is finite.
Then

$$\mathbf{R} \setminus \{a \cdot \mathit{meas}\,(M); \; a = -F(\pm\infty)\} \subset \{c \in \mathbf{R}; \; \varphi \quad \textit{satisfies} \quad (PS)_c\}$$

LEMMA 2. *Assume f satisfies (f1). Then φ satisfies $(PS)_c$, whenever $c \neq 0$ and $c < -F(-\infty) \cdot \mathit{meas}(M)$.*

Here meas (M) denotes the Riemannian measure of M.

Proof of Theorem A. We shall develop some of the ideas used in [26]. There are two distinct situations:

Case 1. $F(-\infty)$ is finite, that is $-\infty < F(-\infty) \leq 0$. In this case, φ is bounded from below since

$$\varphi(u) = \frac{1}{2} \int_M \Big(\sum_{i,j} g_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \lambda_1 u^2 \Big) dx - \int_M F(u) dx$$

and, by our hypothesis on $F(-\infty)$,

$$\sup_{u \in H_0^1(M)} \int_M F(u) dx < +\infty \; .$$

Therefore,

$$-\infty < a := \inf_{u \in H_0^1(M)} \varphi(u) \leq 0 = \varphi(0) \; .$$

Choose c small enough in order to have $F(ce_1) < 0$ (note that c may be taken positive if $F > 0$ in $(0, \eta)$ and negative if $F < 0$ in $(-\eta, 0)$). Here $e_1 > 0$ denotes the first eigenfunction of $-\Delta_M$ in $H_0^1(M)$. Hence $\varphi(ce_1) < 0$, so $a < 0$. It follows now from Lemma 2 that φ satisfies $(\mathrm{PS})_a$. The proof ends in this case by applying Theorem 1.

Case 2. $F(-\infty) = -\infty$. Then, by Lemma 1, φ satisfies $(PS)_c$ for each $c \neq 0$.

Let V be the orthogonal complement of the space spanned by e_1 with respect to $H_0^1(M)$, that is

$$H_0^1(M) = Sp\;\{e_1\} \oplus V \; .$$

For fixed $t_0 > 0$, denote

$$V_0 = \{t_0 e_1 + v;\ v \in V\} \qquad \text{and} \qquad a_0 = \inf_{v \in V_0} \varphi(v)\ .$$

Note that φ is coercive on V. Indeed, if $v \in V$, then

$$\varphi(v) \geq \frac{1}{2}(1 - \frac{\lambda_1}{\lambda_2})\|v\|^2_{H^1_0} - \int_M F(v) \to +\infty \quad \text{as } \|v\|_{H^1_0} \to +\infty,$$

because the first term has a quadratic growth at infinity (t_0 being fixed), while $\int_M F(v)$ is uniformly bounded (in v), in view of the behaviour of F near $\pm\infty$. Thus, a_0 is attained, because of the coercivity of φ on V. From the boundedness of φ on $H^1_0(M)$ it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

Again, there are two posibilities:

(i) $a < 0$. In this case, by Lemma 2, φ satisfies $(\mathrm{PS})_a$. Hence $a < 0$ is a critical value of φ.

(ii) $a = 0 \leq a_0$. Then, either $a_0 = 0$ or $a_0 > 0$. In the first case, as we have already remarked, a_0 is attained. Thus, there is some $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v)\ .$$

Hence, $u = t_0 e_1 + v \in H^1_0(M) \setminus \{0\}$ is a critical point of φ, that is a solution of (P).

If $a_0 > 0$, notice that φ satisfies $(\mathrm{PS})_b$ for each $b \neq 0$. Since $\lim_{t \to +\infty} \varphi(te_1) = 0$, we may apply Theorem 2 to conclude that φ has a critical value $c \geq a_0 > 0$. ■

PROOF OF THEOREM B. If V has the same signification as above, let

$$V_+ = \{te_1 + v;\ t > 0,\ v \in V\}\ .$$

It will be sufficient to show that the functional φ has a non-zero critical point. To do this, we shall make use of two different arguments.

If $u = te_1 + v \in V_+$ then

$$\varphi(u) = \frac{1}{2}\int_M (\,|\nabla v|^2 - \lambda_1 v^2\,) - \int_M F(te_1 + v)\ .$$

In view of the boundedness of F it follows that

$$-\infty < a_+ := \inf_{u \in V_+} \varphi(u) \leq 0\ .$$

We analyse two distinct situations:

Case 1. $a_+ = 0$.

To prove that φ has a critical point, we use the same arguments as in the proof of Theorem A (the second case). More precisely, for some fixed $t_0 > 0$ we define at the same way V_0 and a_0. Obviously, $a_0 \geq 0 = a_+$, since $V_0 \subset V_+$. The proof follows from now on the same ideas as in Case 2 of Theorem A, by considering the two distinct situations $a_0 > 0$ and $a_0 = 0$.

Case 2. $a_+ < 0$.

Let $u_n = t_n e_1 + v_n$ be a minimizing sequence of φ in V_+. We observe that the sequences $(u_n)_n$ and $(v_n)_n$ are bounded. Indeed, this is essentially a compactness condition and may be proved in a similar way to Lemma 1. It follows that there exists $w \in \overline{V}_+$, such that, going eventually to a subsequence,

$$u_n \rightharpoonup w \quad \text{weakly in } H_0^1(M) \ .$$

$$u_n \to w \quad \text{strongly in } L^2(M) \ .$$

$$u_n \to w \quad \text{a.e.}$$

Applying the Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \varphi_2(u_n) = \varphi_2(w) \ .$$

On the other hand,

$$\varphi(w) \leq \liminf_{n\to\infty} \varphi_1(u_n) - \lim_{n\to\infty} \varphi_2(u_n) = \liminf_{n\to\infty} \varphi(u_n) = a_+ \ .$$

It follows that, necessarily, $\varphi(w) = a_+ < 0$. Since the boundary of V_+ is V and

$$\inf_{u\in V} \varphi(u) = 0 \ ,$$

we conclude that w is a local minimum of φ on V_+ and $w \in V_+$. ∎

References

[1] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. **14** (1973), 349-381.

[2] D. Arcoya, Periodic Solutions of Hamiltonian Systems with Strong Resonance at Infinity, Diff. Int. Eqns. **3** (1990), 909-921.

[3] D. Arcoya and A. Cañada, Critical Point Theorems and Applications to Nonlinear Boundary Value Problems, Nonlinear Analysis-TMA **14** (1990), 393-411.

[4] D. Arcoya and A. Cañada, The Dual Variational Principle and Discontinuous Elliptic Problems with Strong Resonance at Infinity, Nonlinear Analysis-TMA, **15** (1990), 1145-1154.

[5] D. Arcoya and D.G. Costa, Nontrivial Solutions for a Strongly Resonant Problem, Diff. Int. Eqns., to appear.

[6] J.P. Aubin and F.H. Clarke, Shadow Prices and Duality for a Class of Optimal Control Problems, SIAM J. Control and Optimization **17** (1979), 567-586.

[7] T. Aubin, Nonlinear Analysis on Manifolds. Monge-Ampère Equations (Springer-Verlag, 1982).

[8] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, Nonlinear Analysis-TMA **7** (1983), 981-1012.

[9] H. Brezis, J.M. Coron and L. Nirenberg, Free vibrations for a nonlinear wave equation and a theorem of Rabinowitz, Comm. Pure Appl. Math. **33** (1980), 667-689.

[10] A. Capozzi, D. Lupo and S. Solimini, Double Resonance in Semilinear Elliptic Problems, Comm. Part. Diff. Eqns. **6** (1991), 91-120.

[11] M. Choulli, R. Deville and A. Rhandi, A general mountain pass principle for nondifferentiable functions, Revista de Matematicas Aplicadas (Chile) **13** (1992), 45-58.

[12] K.C. Chang, Variational methods for non-differentiable functionals and its applications to partial differential equations, J. Math. Anal. Appl. **80** (1981), 102-129.

[13] F.H. Clarke, Generalized gradients and applications, Trans. Amer. Math. Soc. **205** (1975), 247-262.

[14] F.H. Clarke, Generalized gradients of Lipschitz functionals, Adv. in Math. **40** (1981), 52-67.

[15] D.G. Costa and E.A. Silva, The Palais-Smale Condition versus Coercivity, Nonlinear Analysis-TMA **16** (1991), 371-381.

[16] I. Ekeland, On the variational principle, J. Math. Anal. Appl. **47** (1974), 324-353.

[17] P. Hess, Nonlinear perturbations of linear elliptic and parabolic problems at resonance, Ann. Sc. Norm. Sup. Pisa **5** (1978), 527-537.

[18] E.A. Landesman and A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. **19** (1976), 609-623.

[19] D. Lupo and S. Solimini, A Note on a Resonance Problem, Proc. Soc. Edinb. **102A** (1986), 1-7.

[20] P. Mironescu and V. Rădulescu, A multiplicity theorem for locally Lipschitz periodic functionals, J. Math. Anal. Appl., to appear.

[21] P.H. Rabinowitz, Some critical point theorems and applications to semilinear elliptic partial differential equations, Ann. Sc. Norm. Sup. Pisa **2** (1978), 215-223.

[22] V. Rădulescu, Mountain Pass theorems for non-differentiable functions and applications, Proc. Japan Acad. **69A** (1993), 193-198.

[23] M. Schechter, Nonlinear Elliptic Boundary Value Problems at Strong Resonance, Amer. J. Math. **112** (1990), 439-460.

[24] S. Solimini, On the Solvability of Some Elliptic Partial Differential Equations with the Linear Part at Resonance, J. Math. Anal. Appl. **117** (1986), 138-152.

[25] K. Thews, Nontrivial solutions of elliptic equations at resonance, Proc. Soc. Edinb. **85A** (1980), 119-129.

[26] J.R. Ward, A Boundary Value Problem with a Periodic Nonlinearity, Nonlinear Analysis-TMA **10** (1986), 207-213.