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Calculus of Variations

Ground state solutions of scalar field fractional Schrödinger equations

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Abstract In this paper, we study the existence of multiple ground state solutions for a class of parametric fractional Schrödinger equations whose simplest prototype is

$$(-\Delta)^s u + V(x)u = \lambda f(x,u) \quad \text{in } \mathbb{R}^n,$$

where $n > 2$, $(-\Delta)^s$ stands for the fractional Laplace operator of order $s \in (0,1)$, and λ is a positive real parameter. The nonlinear term f is assumed to have a superlinear behaviour at the origin and a sublinear decay at infinity. By using variational methods, we establish the existence of a suitable range of positive eigenvalues for which the problem admits at least two nontrivial solutions in a suitable weighted fractional Sobolev space.

Mathematics Subject Classification 35A15 · 35S15 · 49J35 · 45G05 · 47G20

1 Introduction

In this paper, we study ground state solutions for a nonlinear problem driven by the fractional Laplace operator $(-\Delta)^s$ of order $s \in (0,1)$. Let $\mathcal{S}$ be the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^n)$ functions. For every $u \in \mathcal{S}$, we recall that the fractional Laplace operator acting on u is defined by

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$$(-\Delta)^s u = \mathfrak{F}^{-1}(|\xi|^{2s}(\mathfrak{F}u)(\xi)), \tag{1}$$

where

$$\mathfrak{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx, \quad \xi \in \mathbb{R}^n \tag{2}$$

denotes the Fourier transform of u.

From a probabilistic point of view, the fractional Laplace operator is the infinitesimal generator of a Lévy process, cf. [7]. This operator arises in the description of various phenomena in the applied sciences, such as plasma physics [32], flame propagation [10], finance [16], free boundary obstacle problems [11], Signorini problems [53], Hamilton-Jacobi equation with critical fractional diffusion [54], or phase transitions in the Gamma convergence framework [1].

We start focusing our attention on the fractional Schrödinger equation (briefly fractional NLS) of the form

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - |\psi|^{p-1}\psi \quad \text{in}\, \mathbb{R}^n \times (0, +\infty), \tag{3}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a suitable potential. Here $\psi = \psi(x, t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t (the corresponding probability density is $|\psi|^2$, under a confinement due to the potential V. Equation (3) comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths and was considered for the first time in literature by Laskin [34,35].

When $s = 1$, Eq. (3) gives back the classical Schrödinger equation. In this case, standing wave solutions are of the form

$$\psi(x, t) = e^{-i\omega t} u(x),$$

where ω is a suitable constant and u solves the nonlinear elliptic equation

$$-\Delta u + V(x)u - |u|^{p-1}u = 0. \tag{4}$$

We do not intend to review the huge bibliography of equations like (4), we just emphasize that the potential $V : \mathbb{R}^n \to \mathbb{R}$ has a crucial role concerning the existence and behaviour of solutions. For instance, when V is a positive constant, or V is radially symmetric, it is natural to look for radially symmetric solutions, see [55,58]. On the other hand, after the seminal paper of Rabinowitz [43] where the potential V is assumed to be coercive, several different assumptions are adopted in order to obtain existence and multiplicity results (see [4–6,25,27]).

Contrary to the classical Laplacian case that is widely investigated, the situation seems to be in a developing state when $s \in (0, 1)$ and of an increasing interest (see, for instance, [19,21,24,26]).

In this spirit, in [48], the author looked for standing wave solutions of a more general equation than (3). More precisely, in the cited paper it is studied the following nonlocal fractional equation

$$(-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^n \tag{5}$$

under certain hypotheses on the potential V and the nonlinearity f (see also [49,50]). Moreover, fractional Schrödinger-type problems have been considered in some interesting papers [2,17,18]. In addition, nonlocal fractional equations appear in many fields and a lot of interest has been devoted to this kind of problems and to their concrete applications; see, for instance the seminal papers [12–14] and [3,9,15,22,40,57], as well as the references therein. See also

[42] where the open problem given in [3] was solved. We also mention here, for completeness, some very interesting regularity results for fractional problems proved recently in [29,30].

Motivated by this large interest in the current literature, under suitable conditions on the potential V and exploiting variational methods, we are concerned in the present paper with the study of multiple solutions for the following fractional parametric problem

$$(-\Delta)^s u + V(x)u = \lambda(f(x,u) + \mu g(x,u)), \quad x \in \mathbb{R}^n \ (n \geqslant 3) \tag{6}$$

where we suppose that $V : \mathbb{R}^n \to \mathbb{R}$ and $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are given functions verifying suitable growth conditions, while λ and μ are real parameters.

1.1 Technical assumptions on V

We assume that the potential V satisfies the following hypotheses:

(p_1) $V \in C(\mathbb{R}^n)$ *with* $\inf_{x\in\mathbb{R}^n} V(x) > 0$;
(p_2) *for any* $M > 0$ *there exists* $r_0 > 0$ *such that*:

$$\lim_{|y|\to+\infty} |\{x \in B(y,r_0) : V(x) \leqslant M\}| = 0,$$

where $B(y,r)$ *denotes the open ball in* $\mathbb{R}^n$ *with center* y *and radius* $r > 0$, *and where* $|\cdot|$ *denotes the standard Lebesgue measure in* $\mathbb{R}^n$.

Note that conditions (p_1) and (p_2) are not new in the literature, see for instance, the paper [18].

We also require that $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are two continuous functions for which:

(h_1) *there exist* $W \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $W \not\equiv 0$, *and* $q \in (0,1)$ *such that*

$$\max\{|f(x,t)|, |g(x,t)|\} \leqslant W(x)|t|^q,$$

for each $(x,t) \in \mathbb{R}^n \times \mathbb{R}$;
(h_2) $\lim_{t\to0} \frac{f(x,t)}{t} = \lim_{t\to0} \frac{g(x,t)}{t} = 0$ *uniformly for each* $x \in \mathbb{R}^n$;
(h_3) *there exist* $s_0 \in \mathbb{R}$ *such that*

$$\sup_{\sigma>0}\left(\min_{|x|\leqslant\sigma} \int_0^{s_0} f(x,\tau)\,d\tau\right) > 0.$$

The above conditions are standard assumptions to be satisfied in presence of subcritical terms. Consider the function space

$$E_s^n(V) := \left\{u \in H^s(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\xi|^{2s}|\mathfrak{F}u(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2dx < \infty\right\}. \tag{7}$$

We endow this space with the norm

$$\|u\|_{E_s^n(V)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s}|\mathfrak{F}u(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2dx\right)^{1/2},$$

where $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space of the functions $u \in L^2(\mathbb{R}^n)$ such that

$$\text{the map } (x,y) \longmapsto \frac{u(x)-u(y)}{|x-y|^{\frac{n+2s}{2}}} \text{ is in } L^2\big(\mathbb{R}^{2n}, dxdy\big).$$

On $H^s(\mathbb{R}^n)$ we consider the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u(x)|^2\,dx + \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy\right)^{1/2}. \tag{8}$$

The nonlocal analysis that we perform here in order to use our abstract approach is quite general and successfully exploited for other goals in recent contributions; see [18,19,23,56] for an introduction to this topic and for a list of related references.

By a *weak solution* of problem (6) we mean a function $u \in E^n_s(V)$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} |\xi|^{2s}\mathfrak{F}u(\xi)\mathfrak{F}v(\xi)\,d\xi + \int_{\mathbb{R}^n} V(x)u(x)v(x)\,dx \\ \qquad\qquad \displaystyle= \lambda\int_{\mathbb{R}^n} f(x,u(x))v(x)\,dx + \lambda\mu\int_{\mathbb{R}^n} g(x,u(x))v(x)\,dx, \\ \forall\ v \in E^n_s(V). \end{cases} \tag{9}$$

Let

$$F(x,t) := \int_0^t f(x,\tau)d\tau \quad \text{and} \quad G(x,t) := \int_0^t g(x,\tau)\,d\tau,$$

for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

By direct computation we observe that problem (9) represents the Euler–Lagrange equation of the C^1-functional $\mathcal{J}_{\lambda,\mu} : E^n_s(V) \to \mathbb{R}$ defined as

$$\begin{aligned} \mathcal{J}_{\lambda,\mu}(u) := &\frac{1}{2}\left(\int_{\mathbb{R}^n} |\xi|^{2s}|\mathfrak{F}u(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2dx\right) \\ &-\lambda\int_{\mathbb{R}^n} F(x,u(x))\,dx - \lambda\mu\int_{\mathbb{R}^n} G(x,u(x))\,dx, \end{aligned} \tag{10}$$

for every $u \in E^n_s(V)$.

1.2 The main results

The first result establishes that if the parameters lie in a certain range, then the problem has at least two solutions.

Theorem 1 *Let $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and $V : \mathbb{R}^n \to \mathbb{R}$ be a potential which satisfy* (h$_1$)–(h$_3$), *and* (p$_1$)–(p$_2$), *respectively. Then there exists $\mu_0 > 0$ such that to every $\mu \in [-\mu_0, \mu_0]$ it corresponds a nonempty open interval $\Sigma_\mu \subset (0, +\infty)$ and a number $\kappa_\mu > 0$ for which problem (6) has at least two distinct, nontrivial weak solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ with the property that*

$$\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leqslant \kappa_\mu$$

whenever $\lambda \in \Sigma_\mu$.

The above theorem will be proved by using variational techniques, in particular performing a direct consequence of some general results given in [45,46], which assure the existence of multiple critical points for a functional, under suitable regularity assumptions (see Theorem 5 below).

Furthermore, by using the notations adopted along the paper, we give additional information for the values of

$$\mu_0 := \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{\left|\displaystyle\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right| + 1},$$

where $u_0 \in E_s^n(V)$ is from Lemma 2, and the localization of the interval Σ_μ. More precisely, we show that

$$\Sigma_\mu \subset \left[0, \frac{2\|u_0\|^2_{E_s^n(V)}}{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}\left(1 + \left|\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right|\right)\right],$$

for suitable $\mu \in \mathbb{R}$; see Remark 2 in Sect. 3 for a detailed proof.

From the point of view of the eigenvalues, the counterpart of Theorem 1 is the following.

Theorem 2 *Let $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and $V : \mathbb{R}^n \to \mathbb{R}$ be a potential which satisfy* (h_1)–(h_3)*, and* (p_1)–(p_2)*, respectively. Then there exists $\lambda_0 > 0$ such that to every $\lambda \in (\lambda_0, +\infty)$ it corresponds a nonempty open interval $\Lambda_\lambda \subset \mathbb{R}$ and a number $\kappa_\lambda > 0$ for which (6) has at least two distinct, nontrivial weak solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ with*

$$\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0 < \mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$$

and

$$\max\{\|v_{\lambda\mu}\|_{E_s^n(V)}, \|w_{\lambda\mu}\|_{E_s^n(V)}\} \leqslant \kappa_\lambda$$

whenever $\mu \in \Lambda_\lambda$.

Theorem 2 is based on the classical mountain pass theorem. In such a case

$$\lambda_0 := \frac{\|u_0\|^2_{E_s^n(V)}}{2\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx},$$

and, for every $\lambda > \lambda_0$, setting

$$\mu_\lambda^* := \frac{1}{1 + \displaystyle\int_{\mathbb{R}^n} |G(x, u_0(x))|\, dx}\left(1 - \frac{\lambda_0}{\lambda}\right)\int_{\mathbb{R}^n} F(x, u_0(x))\, dx, \tag{11}$$

we also have that

$$\Lambda_\lambda \equiv (-\mu_\lambda^*, \mu_\lambda^*).$$

Although the two theorems above are completely independent, as a simple byproduct of Theorem 2 we obtain the following result whose conclusion partially goes back to Theorem 1.

Theorem 3 *Let $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and $V : \mathbb{R}^n \to \mathbb{R}$ be a potential which satisfy* (h_1)–(h_3)*, and* (p_1)–(p_2)*, respectively. Then there exists $\overline{\mu} > 0$ such that for every $\mu \in [-\overline{\mu}, \overline{\mu}]$ the set*

$$\Sigma := \{\lambda > 0 : problem\ (6)\ has\ at\ least\ two\ distinct,\ nontrivial\ weak\ solutions\}$$

contains an interval.

We note that the above theorems do not work in general for every $\lambda \in \mathbb{R}$. For instance, consider

$$f(x,t) := \frac{(\sin t)^2}{(1+|x|^\alpha)^\beta}, \quad \forall\,(x,t) \in \mathbb{R}^n \times \mathbb{R}$$

and

$$g(x,t) := \frac{(\arctan t)^2}{(1+|x|^\alpha)^\beta}, \quad \forall\,(x,t) \in \mathbb{R}^n \times \mathbb{R}$$

with $\alpha, \beta > 0$ such that $\alpha\beta > n \geqslant 3$. In such a case, an easy computation shows that the following problem

$$(-\Delta)^s u + V(x)u = \frac{\lambda}{(1+|x|^\alpha)^\beta}((\sin u)^2 + \mu(\arctan u)^2), \quad x \in \mathbb{R}^n \tag{12}$$

possesses only the trivial solution, whenever

$$0 < \lambda < \frac{1}{(1+|\mu|\pi)S_2^2}$$

and μ is arbitrary. Here S_2 denotes the best Sobolev embedding constant of the injection $E_s^n(V) \hookrightarrow L^2(\mathbb{R}^n)$.

More generally, the following non-existence result holds.

Proposition 1 *Let $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and $V : \mathbb{R}^n \to \mathbb{R}$ be a potential which satisfy* (h_2)*, and* (p_1)–(p_2)*, respectively. Assume that*

$$f(x,t) := W(x)h(t),$$

and

$$g(x,s) := W(x)k(t),$$

for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, where h and k are Lipschitz continuous functions with Lipschitz constants $L_h > 0$ and $L_k > 0$, respectively and $W \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfying (h_1)*. Then, for every parameter $\mu \in \mathbb{R}$ and every*

$$0 < \lambda < \frac{1}{\|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu|L_k)S_2^2},$$

the following problem

$$(-\Delta)^s u + V(x)u = \lambda W(x)(h(u) + \mu k(u)), \quad x \in \mathbb{R}^n \tag{13}$$

admits only the trivial solution.

The last part of the paper is dedicated to the existence of multiple weak solutions to the following nonlocal Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda W(x) f(u) + \mu g(x,u), \quad x \in \mathbb{R}^n \tag{14}$$

where

(H) *$V : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying* (p_1) *and* (p_2);
$W \in L^1(\mathbb{R}^n) \cap L^\infty_+(\mathbb{R}^n)$, and

$$\sup_{\sigma>0}\left(\inf_{|x|\leqslant\sigma} W(x)\right) > 0;$$

λ and μ are real parameters.

Assuming that the nonlinear term f is superlinear at zero and sublinear at infinity, the main result ensures that for $\lambda > 0$ large enough, problem (14) admits at least two non-trivial weak solutions, as well as the stability of this problem with respect to an arbitrary subcritical perturbation g of the Schrödinger equation (see Sect. 4).

The paper is organized as follows. In Sect. 2 we give some notations and we recall some properties of the functional space we work in. We also give some tools which will be useful along the paper. In Sect. 3 we study problem (6) and we prove our existence and non-existence results. Finally, in the last section we study the existence of multiple weak solutions to the problem (14).

2 Some preliminaries

The Hilbert space $H^s(\mathbb{R}^n)$ defined in the Introduction can be described by means of the Fourier transform as follows:

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathfrak{F}u(\xi)|^2 \, d\xi < +\infty \right\}.$$

In this case, the inner product and the norm are given by

$$(u, v)_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \mathfrak{F}u(\xi) \mathfrak{F}v(\xi) \, d\xi,$$

for every $u, v \in H^s(\mathbb{R}^n)$, respectively

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathfrak{F}u(\xi)|^2 \, d\xi \right)^{1/2}. \tag{15}$$

In order to give the relationship between the two norms (8) and (15), we recall the definition of the fractional Laplacian operator $(-\Delta)^s$ acting on the rapidly decreasing $C^\infty(\mathbb{R}^n)$ functions (i.e., the space of Schwartz functions $\mathcal{S}$).

Precisely, let $s \in (0, 1)$ and define the operator $(-\Delta)^s : \mathcal{S} \to L^2(\mathbb{R}^n)$ given by

$$(-\Delta)^s u(x) := C(n, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

where $B(x, \varepsilon)$ is the open ball centered at $x \in \mathbb{R}^n$ and radius ε and $C(n, s)$ is the following (positive) normalization constant

$$C(n, s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1}.$$

The operator $(-\Delta)^s$ is a pseudo-differential operator with symbol $|\eta|^{2s}$, with $s \in (0, 1)$, where η denotes the variable in the frequency space.

This nonlocal operator can also be defined by the formula

$$\begin{aligned} (-\Delta)^s u(x) &= C(n, s) \, \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy \\ &:= C(n, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy. \end{aligned}$$

The proof of this formula, as well as the computation of the constant $C(n, s)$, can be found in the book of Landkof [33].

In [23] it is proved that

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)(\xi)), \tag{16}$$

$$[u]^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy = \frac{2}{C(n,s)}\int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}u(\xi)|^2\,d\xi, \tag{17}$$

and

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1}\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}, \tag{18}$$

for every $u \in H^s(\mathbb{R}^n)$.

By (16)–(18) and using the classical Plancherel's formula, the following norms

$$u \mapsto \left(\int_{\mathbb{R}^n} |u(x)|^2\,dx + \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy\right)^{1/2}$$

$$u \mapsto \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s|\mathfrak{F}u(\xi)|^2\,d\xi\right)^{1/2}$$

$$u \mapsto \left(\int_{\mathbb{R}^n} |u(x)|^2\,dx + \int_{\mathbb{R}^n} |\xi|^{2s}|\mathfrak{F}u(\xi)|^2\,d\xi\right)^{1/2}$$

$$u \mapsto \left(\int_{\mathbb{R}^n} |u(x)|^2\,dx + \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2},$$

are equivalent. As a direct consequence of the above remarks, we obtain that the space $E^n_s(V)$ can be defined in other (equivalent) ways.

In order to prove the compactness embedding property given in Proposition 2, the following preparatory results are necessary.

Theorem 4 *Let $s \in (0,1)$ such that $n > 2s$. Then, there exists a positive constant $K = K(n,s)$ such that*

$$\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant K[u]^2_{H^s(\mathbb{R}^n)},$$

for any $u \in H^s(\mathbb{R}^n)$, where the constant

$$2^*_s := \frac{2n}{n-2s}$$

is the so-called fractional critical exponent. Consequently, the space $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ for any $p \in [2, 2^*_s]$.

Moreover, the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is compact for every $p \in [2, 2^*_s)$.

See [23] for a detailed proof.

By using Theorem 4, the following fractional Gagliardo-Nirenberg inequality should be proved.

Lemma 1 *Let $s \in (0,1)$, $p \in [1,\infty)$ and $n > 2s$. Then for every $u \in H^s(\mathbb{R}^n)$ we have*

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant C(n,s,\alpha)[u]^\alpha_{H^s(\mathbb{R}^n)}\|u\|^{1-\alpha}_{L^r(\mathbb{R}^n)}, \tag{19}$$

with

$$\frac{n}{p} = \alpha\frac{n-2s}{2} + (1-\alpha)\frac{n}{r},$$

where $r \geqslant 1$, $\alpha \in [0,1]$ and $C(n,s,\alpha)$ is a positive constant.

Proof The conclusion is trivial if $p = 1$. Indeed, in such a case, it suffices to take $r = 1$ and $\alpha = 0$. Hence, let us suppose that $p > 1$. Since

$$\frac{1}{p} = \frac{\alpha}{2_s^*} + \frac{1-\alpha}{r},$$

by the Hölder inequality and Theorem 4 it follows that

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant \|u\|^{\alpha}_{L^{2_s^*}(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^r(\mathbb{R}^n)} \leqslant C(n, s, \alpha)[u]^{\alpha}_{H^s(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^r(\mathbb{R}^n)}, \tag{20}$$

where we set

$$C(n, s, \alpha) := K(n, s)^{\alpha/2}.$$

This concludes the proof. □

As a byproduct of Theorem 4 and Proposition 1 we obtain the following result that is crucial in the sequel.

Proposition 2 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a potential that satisfies hypotheses* (p_1) *and* (p_2). *Then the embedding*

$$E_s^n(V) \hookrightarrow L^p(\mathbb{R}^n)$$

is compact for every $p \in [2, 2_s^*)$.

Proof By [18], we know that the Hilbert space $E_s^n(V)$ is compactly embedded into $L^2(\mathbb{R}^n)$. Therefore, we only consider the case $p \in (2, 2_s^*)$. In order to do this, we use the fractional Gagliardo-Nirenberg inequality proved in Lemma 1. Hence, let $\{u_j\}_{j\in\mathbb{N}} \subset E_s^n(V)$ be a sequence such that $u_j \rightharpoonup u_0$ in $E_s^n(V)$, i.e. $\{u_j\}_{j\in\mathbb{N}}$ weakly converges to u_0 in $E_s^n(V)$. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $E_s^n(V)$ and, by Lemma 1, for $r = 2$ and

$$\alpha := \frac{(p-2)n}{2sp} \in (0, 1),$$

there exists a constant $C_1(n, s, \alpha) > 0$ such that

$$\|u_j - u_0\|_{L^p(\mathbb{R}^n)} \leqslant C_1(n, s, \alpha)\|u_j - u_0\|^{\alpha}_{E_s^n(V)} \|u_j - u_0\|^{1-\alpha}_{L^2(\mathbb{R}^n)}, \tag{21}$$

for every $p \in (2, 2_s^*)$. Thus, since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $E_s^n(V)$, by (21) we find

$$\|u_j - u_0\|_{L^p(\mathbb{R}^n)} \leqslant C_1(n, s, \alpha)(M + \|u_0\|^{\alpha}_{E_s^n(V)})\|u_j - u_0\|^{1-\alpha}_{L^2(\mathbb{R}^n)} \to 0,$$

i.e. $u_j \to u_0$ in $L^p(\mathbb{R}^n)$. This completes the proof.

Remark 1 By Proposition 2, for any $p \in [2, 2_s^*]$, there exists a positive constant S_p such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant S_p \|u\|_{E_s^n(V)}, \tag{22}$$

for any $u \in E_s^n(V)$.

2.1 Some useful tools

The main tool used along this paper in order to prove our multiplicity result stated in Theorem 1 is given by a direct consequence of some general theorems due to Ricceri [45,46] that we recall in what follows.

Theorem 5 *Let* $(E, \|\cdot\|)$ *be a separable and reflexive real Banach space and let* $\Phi, \Psi : E \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that there exists* $z_0 \in E$ *such that* $\Phi(z_0) = \Psi(z_0) = 0$ *and* $\inf_{z\in E} \Phi(z) \geqslant 0$ *and that there exist* $z_1 \in E$, $\varrho > 0$ *such that*

(i) $\varrho < \Phi(z_1)$;

(ii) $\sup_{\Phi(z)<\varrho} \Psi(z) < \varrho \dfrac{\Psi(z_1)}{\Phi(z_1)}$.

Set

$$\bar{a} := \frac{\zeta \varrho}{\varrho \dfrac{\Psi(z_1)}{\Phi(z_1)} - \sup\limits_{\Phi(z)<\varrho} \Psi(z)},$$

with $\zeta > 1$ *and assume that the functional*

$$J_\lambda(z) := \Phi(z) - \lambda\Psi(z), \quad \forall\, z \in E$$

is sequentially weakly lower semicontinuous, satisfies the (PS) *condition, and*

(iii) $\lim\limits_{\|z\|\to+\infty} J_\lambda(z) = +\infty$,

for every $\lambda \in [0, \bar{a}]$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ *and a number* $\kappa > 0$ *such that for each* $\lambda \in \Lambda$, *the equation* $J'_\lambda(z) = 0$ *admits at least three solutions in* E *having norm less than* κ.

Some details and related topics on the above result can be found in the recent monographs [20,28].

For the sake of completeness, we also recall that the C^1-functional $J_\lambda : E \to \mathbb{R}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ when

$(\mathrm{PS})_c$ *Every sequence* $\{z_j\}_{j\in\mathbb{N}} \subset E$ *such that*

$$J_\lambda(z_j) \to c, \qquad \text{and} \qquad \|J'_\lambda(z_j)\|_{E'} \to 0,$$

as $j \to \infty$, *possesses a convergent subsequence in* E.

Here E' denotes the topological dual of E. Finally, we say that J_λ satisfies the Palais-Smale condition (in short (PS)) if $(\mathrm{PS})_c$ holds for every $c \in \mathbb{R}$.

At this point, let us fix $\sigma > 0$ and $\eta \in \mathbb{R}$. For every $\varepsilon > 0$, define $u_\varepsilon^\eta \in E_s^n(V)$ as follows

$$u_\varepsilon^\eta(x) := \begin{cases} 0 & \text{if } \ x \in \mathbb{R}^n \setminus B(0,\sigma) \\ \dfrac{\eta}{\varepsilon}(\sigma - |x|) & \text{if } \ x \in B(0,\sigma)\setminus B(0,\sigma-\varepsilon) \\ \eta & \text{if } \ x \in B(0,\sigma-\varepsilon), \end{cases} \tag{23}$$

where $B(0, r)$ denotes the n-dimensional open ball centered at the origin and with radius $r > 0$. Note that $u_\varepsilon^\eta \in E_s^n(V)$ by [23, Proposition 2.2 and Lemma 5.1]. Further

$$\|u_\varepsilon^\eta\|_\infty := \max_{x\in\mathbb{R}^n} |u_\varepsilon^\eta(x)| \leqslant |\eta|.$$

This function will be useful in the sequel in the proof of our theorem as well as the next auxiliary results.

Lemma 2 *There exists $u_0 \in E_s^n(V)$ such that*

$$\int_{\mathbb{R}^n} F(x, u_0(x))\, dx > 0. \tag{24}$$

Proof By (h_3) there are $\sigma_0 > 0$ and $s_0 \in \mathbb{R}$ such that

$$\min_{|x|\leqslant\sigma_0} \int_0^{s_0} f(x,\tau)\, d\tau > 0.$$

Fix $\varepsilon \in (0, \sigma_0/2)$ and denote $\omega_0 := \min\limits_{|x|\leqslant\sigma_0} F(x, s_0) > 0$. Further, let $u_\varepsilon^{s_0} \in E_s^n(V)$ be the function obtained by (23) replacing σ with σ_0, as well as η with s_0. We have

$$\begin{aligned}\int_{\mathbb{R}^n} F(x, u_\varepsilon^{s_0}(x))\, dx &= \int_{|x|\leqslant\sigma_0-\varepsilon} F(x, u_\varepsilon^{s_0}(x))\, dx + \int_{|x|\geqslant\sigma_0} F(x, u_\varepsilon^{s_0}(x))\, dx \\ &\quad + \int_{\sigma_0-\varepsilon<|x|<\sigma_0} F(x, u_\varepsilon^{s_0}(x))\, dx.\end{aligned}$$

Hence

$$\int_{\mathbb{R}^n} F(x, u_\varepsilon^{s_0}(x))\, dx \geqslant \omega_0|B_{\sigma_0/2}| - \int_{\sigma_0-\varepsilon<|x|<\sigma_0} |F(x, u_\varepsilon^{s_0}(x))|\, dx,$$

and finally

$$\int_{\mathbb{R}^n} F(x, u_\varepsilon^{s_0}(x))\, dx \geqslant \omega_0|B_{\sigma_0/2}| - \max_{|x|\in[\sigma_0/2,\sigma_0],|t|\leqslant|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|),$$

where $|B_r|$ denotes the Lebesgue measure of the ball $B(0, r)$. Since

$$\max_{|x|\in[\sigma_0/2,\sigma_0],|t|\leqslant|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|) \to 0,$$

as $\varepsilon \to 0^+$, there exists $\varepsilon_0 > 0$ such that

$$\omega_0|B_{\sigma_0/2}| > \max_{|x|\in[\sigma_0/2,\sigma_0],|t|\leqslant|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon_0}|).$$

Thus the function $u_0 := u_{\varepsilon_0}^{s_0} \in E_s^n(V)$ verifies inequality (24). □

Setting

$$\lambda_1^B := \inf_{u\in H_0^1(B(0,\sigma_0))\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(B(0,\sigma_0))}}{\|u\|^2_{L^2(B(0,\sigma_0))}}, \tag{25}$$

the following result holds.

Lemma 3 *Let $u_0 \in E_s^n(V)$ be the function defined in Lemma 2. Then*

$$\|u_0\|^2_{E_s^n(V)} < \pi^{\frac{n}{2}} \frac{\sigma_0^2}{\varepsilon_0^2} \frac{\left(\sigma_0^n - (\sigma_0-\varepsilon_0)^n\right)}{\Gamma\left(1+\dfrac{n}{2}\right)} S_0, \tag{26}$$

where

$$S_0 := \left(1 + \frac{1}{\lambda_1^B}\right) \max\left\{\left(1 + \frac{1}{\lambda_1^B}\right), \|V\|_\infty\right\}.$$

Proof Computing the standard seminorm of the function u_0 in $H^1(\mathbb{R}^n)$, we easily have

$$\begin{aligned}[u_0]^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla u_0(x)|^2\, dx = \frac{\sigma_0^2}{\varepsilon_0^2} \int_{B(0,\sigma_0)\setminus B(0,\sigma_0-\varepsilon_0)} dx \\ &= \frac{\sigma_0^2}{\varepsilon_0^2} \left(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon_0}|\right) \\ &= \pi^{\frac{n}{2}} \frac{\sigma_0^2}{\varepsilon_0^2} \frac{\left(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n\right)}{\Gamma\left(1 + \dfrac{n}{2}\right)}. \end{aligned} \tag{27}$$

Hence, since $s \in (0, 1)$, bearing in mind that

$$\|u_0\|^2_{L^2(\mathbb{R}^n)} = \|\mathfrak{F}u_0\|^2_{L^2(\mathbb{R}^n)},$$

as well as

$$\|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} = \||\xi|\mathfrak{F}u_0\|^2_{L^2(\mathbb{R}^n)},$$

we deduce that

$$\begin{aligned}\|u_0\|^2_{E^n_s(V)} &= \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u_0(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u_0(x)|^2 dx \\ &< \int_{\mathbb{R}^n} (1 + |\xi|^2)|\mathfrak{F}u_0(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u_0(x)|^2 dx \\ &= \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + \|u_0\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} V(x)|u_0(x)|^2 dx. \end{aligned} \tag{28}$$

Thus since $u_0 \in H^1_0(B(0, \sigma_0))$, inequality (28) yields

$$\|u_0\|^2_{E^n_s(V)} < \max\left\{\left(1 + \frac{1}{\lambda_1^B}\right), \|V\|_\infty\right\}\left(1 + \frac{1}{\lambda_1^B}\right)[u_0]^2_{H^1(\mathbb{R}^n)}. \tag{29}$$

Substituting (27) in (29), the conclusion is achieved.

Finally, a standard computation ensures that our assumptions give a natural control on the growth of the nonlinearities f and g, as well as of their potentials F and G, according to the following proposition.

Lemma 4 *Let $p \in (2, 2^*_s)$. For each $\varepsilon > 0$ one has*

$$\max\{|f(x,t)|, |g(x,t)|\} \leqslant \varepsilon|t| + c(\varepsilon)|t|^{p-1} \quad \textit{for every } (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

and

$$\max\{|F(x,t)|, |G(x,t)|\} \leqslant \varepsilon t^2 + c(\varepsilon)|t|^p \quad \textit{for every } (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

for some $c(\varepsilon) > 0$.

2.2 The Palais-Smale condition

Our purpose in what follows is to show that the energy functional $\mathcal{J}_{\lambda,\mu}$ satisfies the Palais-Smale condition.

Lemma 5 *Let $\lambda, \mu \in \mathbb{R}$ be arbitrary fixed. Then every bounded sequence $\{u_j\}_{j\in\mathbb{N}} \subset E_s^n(V)$ such that*

$$\|\mathcal{J}'_{\lambda,\mu}(u_j)\|_{(E_s^n(V))'} := \sup_{\varphi\in E_s^n(V)} \Big\{|\langle \mathcal{J}'_{\lambda,\mu}(u_j), \varphi\rangle| : \|\varphi\|_{E_s^n(V)} = 1\Big\} \to 0, \tag{30}$$

as $j \to +\infty$, contains a strongly convergent subsequence.

Proof Let $\{u_j\}_{j\in\mathbb{N}} \subset E_s^n(V)$ be a bounded sequence such that condition (30) holds. By Proposition 2, due to the compact embedding $E_s^n(V) \hookrightarrow L^p(\mathbb{R}^n)$ for every $p \in (2, 2_s^*)$, we may assume, taking a subsequence if necessary, that $\{u_j\}_{j\in\mathbb{N}}$ converges to u weakly in $E_s^n(V)$ and strongly in $L^p(\mathbb{R}^n)$, for every $p \in (2, 2_s^*)$. Therefore, fixing $p \in (2, 2_s^*)$ and bearing in mind the regularity assumptions on the function W, we have

$$\begin{aligned}
\|u_j - u\|^2_{E_s^n(V)} &= (u, u-u_j)_{E_s^n(V)} + \mathcal{J}'_{\lambda,\mu}(u_j)(u_j - u)\\
&\quad -\lambda \int_{\mathbb{R}^n} f(x, u_j(x))(u-u_j)(x)\,dx\\
&\quad -\lambda\mu \int_{\mathbb{R}^n} g(x, u_j(x))(u-u_j)(x)\,dx\\
&\leqslant (u, u-u_j)_{E_s^n(V)} + \|\mathcal{J}'_{\lambda,\mu}(u_j)\|_{(E_s^n(V))'}\|u_j - u\|_{E_s^n(V)}\\
&\quad +|\lambda|(1+|\mu|)\|W\|_{L^{\frac{p}{p-q-1}}(\mathbb{R}^n)} \|u_j\|^q_{L^p(\mathbb{R}^n)}\|u-u_j\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where q is given in our hypothesis (h_1). We deduce that $\|u_j - u\|_{E_s^n(V)} \to 0$ as $j \to +\infty$. The proof is complete. □

As a direct consequence of the above result, the following compactness property is valid.

Lemma 6 *For every $\lambda, \mu \in \mathbb{R}$, the functional $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded from below on $E_s^n(V)$. Moreover, $\mathcal{J}_{\lambda,\mu}$ satisfies the* (PS) *condition.*

Proof Since $E_s^n(V) \hookrightarrow L^2(\mathbb{R}^n)$, exploiting condition ($\mathrm{h}_1$) we obtain

$$\mathcal{J}_{\lambda,\mu}(u) \geqslant \frac{1}{2}\|u\|^2_{E_s^n(V)} - |\lambda|(1+|\mu|)\|W\|_{L^{\frac{2}{1-q}}(\mathbb{R}^n)} S_2^{q+1}\|u\|^{q+1}_{E_s^n(V)}, \tag{31}$$

for every $u \in E_s^n(V)$. Since $q \in (0,1)$, the functional $\mathcal{J}_{\lambda,\mu}$ is bounded from below in $E_s^n(V)$.

Now, let us prove that the (PS) condition is verified. For this purpose, let $\{u_j\}_{j\in\mathbb{N}} \subset E_s^n(V)$ be a sequence such that

$$\{\mathcal{J}_{\lambda,\mu}(u_j)\}_{j\in\mathbb{N}} \quad \text{is bounded in } E_s^n(V) \tag{32}$$

and for which condition (30) holds. Since the functional $\mathcal{J}_{\lambda,\mu}$ is coercive, we deduce by (32) that the sequence $\{u_j\}_{j\in\mathbb{N}} \subset E_s^n(V)$ is bounded. In conclusion, on account of Lemma 5, the functional $\mathcal{J}_{\lambda,\mu}$ satisfies the compactness (PS) condition. □

3 Proof of the main results

In order to prove Theorem 1 we start with some auxiliary results. For every $\mu \in \mathbb{R}$, let $L_\mu : E_s^n(V) \to \mathbb{R}$ be the functional defined as follows

$$u \longmapsto \int_{\mathbb{R}^n} F(x, u(x))\,dx + \mu \int_{\mathbb{R}^n} G(x, u(x))\,dx.$$

With the above notation we exhibit the following asymptotic property.

Lemma 7 *For every $\mu \in \mathbb{R}$, setting*

$$\chi(\varrho) := \frac{\sup\left\{L_\mu(u) : \|u\|_{E^n_s(V)} < \sqrt{2\varrho}\right\}}{\varrho},$$

we have

$$\lim_{\varrho \to 0^+} \chi(\varrho) = 0.$$

Proof Let us fix arbitrarily $\varepsilon > 0$ and $p \in (2, 2^*_s)$. Now, due to the growth conditions in Lemma 4, since $E^n_s(V) \hookrightarrow L^p(\mathbb{R}^n)$, we easily have

$$L_\mu(u) \leqslant (1 + |\mu|)\left(\varepsilon S_2^2 \|u\|^2_{E^n_s(V)} + c(\varepsilon) S_p^p \|u\|^p_{E^n_s(V)}\right),$$

for every $u \in E^n_s(V)$.
Moreover,

$$0 \leqslant \chi(\varrho) \leqslant (1 + |\mu|)(2\varepsilon S_2^2 + c(\varepsilon) 2^{\frac{p}{2}} S_p^p \varrho^{\frac{p}{2}-1},$$

for every $\varrho > 0$. Thus, when the parameter $\varrho \to 0^+$, since ε is arbitrary, the right-hand side of the above inequality tends to zero. The proof is complete. □

Lemma 8 *The functional $\mathcal{J}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous in $E^n_s(V)$, for every $\lambda, \mu \in \mathbb{R}$.*

Proof The functional

$$u \longmapsto \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2 \, d\xi + \int_{\mathbb{R}^n} V(x) u(x)^2 dx$$

is sequentially weakly lower semicontinuous on $E^n_s(V)$. Thus it is enough to prove that the map L_μ is sequentially weakly continuous on $E^n_s(V)$, for every $\mu \in \mathbb{R}$. On the contrary, let us suppose that $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in $E^n_s(V)$ which converges weakly to $u_\infty \in E^n_s(V)$ and such that the sequence $\left\{L(u_j)\right\}_{j\in\mathbb{N}}$ does not converge to $L(u_\infty)$ as $j \to +\infty$. Therefore there exist a positive constant ε_0 and a subsequence of $\{u_j\}_{j\in\mathbb{N}}$, denoted again by $\{u_j\}_{j\in\mathbb{N}}$, such that

$$0 < \varepsilon_0 \leqslant |L_\mu(u_j) - L_\mu(u_\infty)| \tag{33}$$

for every $j \in \mathbb{N}$, and $u_j \to u$ strongly in $L^p(\mathbb{R}^n)$ for every $p \in (2, 2^*_s)$. Since the functionals

$$\Psi_1(u) := \int_{\mathbb{R}^n} F(x, u(x)) dx, \quad \text{and} \quad \Psi_2(u) := \int_{\mathbb{R}^n} G(x, u(x)) dx,$$

are smooth with derivatives given by

$$\Psi_1'(u)(v) = \int_{\mathbb{R}^n} f(x, u(x)) v(x) dx, \quad \text{and} \quad \Psi_2'(u)(v) = \int_{\mathbb{R}^n} g(x, u(x)) v(x) dx,$$

for every $v \in E^n_s(V)$, the Mean Value Theorem, hypothesis (h_1) and inequality (33) yield

$$\begin{aligned}
0 < \varepsilon_0 &\leqslant |L'_\mu(u + \theta_j(u_j - u))(u_j - u)| \\
&\leqslant (1 + |\mu|) \int_{\mathbb{R}^n} W(x) |u(x) + \theta_j(u_j - u)(x)|^q |(u_j - u)(x)| \, dx \\
&\leqslant (1 + |\mu|) \|W\|_{L^{\frac{p}{p-q-1}}(\mathbb{R}^n)} (\|u\|_{L^p(\mathbb{R}^n)} + \|u_j - u\|_{L^p(\mathbb{R}^n)})^q \|u_j - u\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

for some $\theta_j \in (0, 1)$.
As $j \to +\infty$ the right-hand side of the above inequality tends to zero, which is a contradiction. □

Proof of Theorem 1 Let $u_0 \in E^n_s(V)$ be the function defined in Lemma 2 and fix

$$\mu_0 := \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{\left|\displaystyle\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right| + 1}.$$

We apply Theorem 5 by choosing $E := E^n_s(V)$, $\Phi(u) := \frac{1}{2}\|\cdot\|^2_{E^s_n(V)}$, and $\Psi := L_\mu$ for every fixed $\mu \in [-\mu_0, \mu_0]$. It is clear that

$$\mathcal{J}_{\lambda,\mu} = \Phi - \lambda L_\mu.$$

By direct computations one has

$$L_\mu(u_0) = \int_{\mathbb{R}^n} F(x, u_0(x))\, dx + \mu \int_{\mathbb{R}^n} G(x, u_0(x))\, dx \geqslant \mu_0 > 0, \tag{34}$$

for every $\mu \in [-\mu_0, \mu_0]$. Using (34) and Lemma 7, for every $\mu \in [-\mu_0, \mu_0]$ one has

$$\varrho_\mu < \min\left\{1, \frac{\|u_0\|^2_{E^n_s(V)}}{2}\right\}; \tag{35}$$

$$\frac{\sup\{L_\mu(u) :\ \|u\|_{E^n_s(V)} < \sqrt{2\varrho_\mu}\}}{\varrho_\mu} < \frac{L_\mu(u_0)}{\|u_0\|^2_{E^n_s(V)}}, \tag{36}$$

for some $\varrho_\mu > 0$. Now, let us choose $z_1 := u_0$, $z_0 := 0$, $\zeta := 1 + \varrho_\mu$ and

$$\overline{a} = \overline{a}_\mu = \frac{1 + \varrho_\mu}{2L_\mu(u_0)\|u_0\|^{-2}_{E^n_s(V)} - \chi(\varrho_\mu)},$$

where

$$\chi(\varrho_\mu) := \frac{\sup\left\{L_\mu(u) : \|u\|_{E^n_s(V)} < \sqrt{2\varrho_\mu}\right\}}{\varrho_\mu}.$$

By Lemmas 6 and 8, all the assumptions of Theorem 5 are verified. Then there exists an open interval of parameters $\Sigma_\mu \subset [0, \overline{a}_\mu]$ and a positive constant κ_μ such that, for any $\lambda \in \Sigma_\mu$, the functional $\mathcal{J}_{\lambda,\mu}$ admits at least three distinct critical points $u^i_{\lambda,\mu} \in E^n_s(V)$ (with $i \in \{1, 2, 3\}$), such that $\|u\|_{E^n_s(V)} \leqslant \kappa_\mu$. This concludes the proof of Theorem 1. □

Remark 2 Thanks to (35), (36) and (34), it follows that

$$\begin{aligned}
\overline{a}_\mu < \frac{2\|u_0\|^2_{E^n_s(V)}}{L_\mu(u_0)} &\leqslant \frac{2\|u_0\|^2_{E^n_s(V)}}{\mu_0} \\
&= \frac{2\|u_0\|^2_{E^n_s(V)}}{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}\left(1 + \left|\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right|\right),
\end{aligned}$$

for every parameter $\mu \in [-\mu_0, \mu_0]$. Since the right-hand side does not depend on $\mu \in \mathbb{R}$, we have a uniform estimation of Λ_μ, that is,

$$\Sigma_\mu \subset [0, \overline{a}_\mu] \subset \left[0, \frac{2\|u_0\|^2_{E^n_s(V)}}{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}\left(1 + \left|\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right|\right)\right],$$

for every $\mu \in [-\mu_0, \mu_0]$.

Proof of Theorem 2 Define

$$c_0 := \int_{\mathbb{R}^n} |G(x, u_0(x))| dx \quad \text{and} \quad \lambda_0 := \frac{\|u_0\|^2_{E^n_s(V)}}{2\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx},$$

where $u_0 \in E^n_s(V)$ comes from Lemma 2. Further, for every $\lambda > \lambda_0$, we set

$$\mu^*_\lambda := \frac{1}{1 + c_0}\left(1 - \frac{\lambda_0}{\lambda}\right)\int_{\mathbb{R}^n} F(x, u_0(x))\, dx. \tag{37}$$

□

Remark 3 Note that an explicit estimate of the number λ_0 appearing in the main results can be obtained by using relation (26). More precisely, a direct computation gives

$$\lambda_0 < \frac{1}{2\varepsilon_0^2 \Gamma\left(1 + \frac{n}{2}\right)} \frac{S_0 \sigma_0^2 \pi^{\frac{n}{2}}\left(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n\right)}{\omega_0 |B_{\sigma_0/2}| - \max\limits_{|x| \in [\sigma_0/2, \sigma_0], |t| \leqslant |s_0|} |F(x, t)|(|B_{\sigma_0}| - |B_{\sigma_0 - \varepsilon}|)}.$$

Lemma 9 *Assume that $\lambda > \lambda_0$ and $\mu \in (-\mu^*_\lambda, \mu^*_\lambda)$. Then*

$$\inf_{u \in E^n_s(V)} \mathcal{J}_{\lambda,\mu}(u) < 0.$$

Proof We prove that $\mathcal{J}_{\lambda,\mu}(u_0) < 0$ whenever $\lambda > \lambda_0$ and $|\mu| < \mu^*_\lambda$. Indeed, due to the choice of λ_0 and μ^*_λ, we have

$$\begin{aligned}
\mathcal{J}_{\lambda,\mu}(u_0) &= \frac{1}{2}\|u_0\|^2_{E^n_s(V)} - \lambda \int_{\mathbb{R}^n} F(x, u_0(x))\, dx - \lambda\mu \int_{\mathbb{R}^n} G(x, u_0(x))\, dx \\
&\leqslant (\lambda_0 - \lambda) \int_{\mathbb{R}^n} F(x, u_0(x))\, dx + \lambda|\mu| c_0 \\
&= -\lambda(1 + c_0)\mu^*_\lambda + \lambda|\mu| c_0 < 0.
\end{aligned}$$

Hence

$$\inf_{u \in E^n_s(V)} \mathcal{J}_{\lambda,\mu}(u) \leqslant \mathcal{J}_{\lambda,\mu}(u_0) < 0.$$

The proof is complete. □

Lemma 10 *For every $\lambda > \lambda_0$ and $\mu \in (-\mu^*_\lambda, \mu^*_\lambda)$, the functional $\mathcal{J}_{\lambda,\mu}$ has the mountain pass geometry.*

Proof Let us fix $p \in (2, 2_s^*)$. By Lemma 4 we deduce that for every $\varepsilon > 0$ there exists a positive constant $c(\varepsilon)$ such that

$$\max\left\{\left|\int_{\mathbb{R}^n} F(x,u(x))\,dx\right|, \left|\int_{\mathbb{R}^n} G(x,u(x))\,dx\right|\right\} \\ \leqslant \varepsilon S_2^2 \|u\|^2_{E_s^n(V)} + c(\varepsilon) S_p^p \|u\|^p_{E_s^n(V)},$$

for every $u \in E_s^n(V)$. Thus, it follows that

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u) &\geqslant \frac{1}{2}\|u\|^2_{E_s^n(V)} - \lambda\left|\int_{\mathbb{R}^n} F(x,u(x))\,dx\right| - \lambda|\mu|\left|\int_{\mathbb{R}^n} G(x,u(x))\,dx\right| \\ &\geqslant \left(\frac{1}{2} - \lambda(1+|\mu|)\varepsilon S_2^2\right)\|u\|^2_{E_s^n(V)} - \lambda(1+|\mu|)c(\varepsilon)S_p^p\|u\|^p_{E_s^n(V)},\end{aligned}$$

for every $u \in E_s^n(V)$. Now, set

$$\varepsilon := \frac{1}{4\lambda(1+|\mu|)S_2^2}.$$

Then, by the above inequality, one has

$$\mathcal{J}_{\lambda,\mu}(u) \geqslant \left(\frac{1}{4} - \lambda(1+|\mu|)c(\lambda,\mu)S_p^p \varrho^{p-2}\right)\varrho^2 > 0$$

provided that

$$\|u\|_{E_s^n(V)} = \varrho < \min\left\{(4\lambda(1+|\mu|)c(\lambda,\mu)S_p^p)^{\frac{1}{2-p}}, \|u_0\|_{E_s^n(V)}\right\}, \quad (p > 2)$$

where, for simplicity, we set

$$c(\lambda,\mu) := c\left(\frac{1}{4\lambda(1+|\mu|)S_2^2}\right).$$

By construction $\varrho < \|u_0\|_{E_s^n(V)}$ and, as observed in the proof of Lemma 9, we also have $\mathcal{J}_{\lambda,\mu}(u_0) < 0$. In conclusion, the functional $\mathcal{J}_{\lambda,\mu}(u)$ verifies the mountain pass geometry. □

Proof of Theorem 2 Let us fix $\lambda > \lambda_0$ and $\mu \in (-\mu_\lambda^*, \mu_\lambda^*) \equiv \Lambda_\lambda$. Due to the regularity property of the functional $\mathcal{J}_{\lambda,\mu}$, Lemma 6 ensures that there exists an element $v_{\lambda\mu} \in E_s^n(V)$ such that

$$\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) = \inf_{u \in E_s^n(V)} \mathcal{J}_{\lambda,\mu}(u).$$

Clearly, the function $v_{\lambda\mu}$ is nontrivial. Indeed, by using Lemma 9, we have $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0$. Moreover, arguing as in [44, Theorem 2.2], by Lemma 10 there exists an element $w_{\lambda\mu} \in E_s^n(V)$ such that $\mathcal{J}'_{\lambda,\mu}(w_{\lambda\mu}) = 0$ and

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) \geqslant \left(\frac{1}{4} - \lambda(1+|\mu|)c(\lambda,\mu)S_p^p \varrho^{p-2}\right)\varrho^2 > 0,$$

i.e. $w_{\lambda\mu}$ is also nontrivial. Now, we recall that the mountain pass level $\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$ has the following characterization

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) = \inf_{g \in \Gamma} \max_{t \in [0,1]} \mathcal{J}_{\lambda,\mu}(g(t)), \tag{38}$$

where, as usual, we set

$$\Gamma := \{g \in C([0,1]; E^n_s(V)) : g(0) = 0,\ g(1) = u_0\}.$$

Let $g_0 \in \Gamma$, defined by $g_0(t) := tu_0$, for every $t \in [0,1]$. By using the characterization given in (38), we obtain

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) \leqslant \max_{t\in[0,1]} \mathcal{J}_{\lambda,\mu}(tu_0).$$

Thus

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) \leqslant \frac{1}{2}\|u_0\|^2_{E^n_s(V)} + \lambda M,$$

for every $\mu \in \Lambda_\lambda$, where

$$M := \max_{t\in[0,1]} \left\{ \left| \int_{\mathbb{R}^n} F(x, tu_0(x))\,dx \right| + \mu^*_\lambda \left| \int_{\mathbb{R}^n} G(x, tu_0(x))\,dx \right| \right\}.$$

Moreover, inequality (31) yields

$$\begin{aligned}\|w_{\lambda\mu}\|^2_{E^n_s(V)} &\leqslant 2\lambda(1+\mu^*_\lambda)\|W\|_{L^{\frac{2}{1-q}}(\mathbb{R}^n)} S_2^{q+1}\|w_{\lambda\mu}\|^{q+1}_{E^n_s(V)} \\ &\quad + \|u_0\|^2_{E^n_s(V)} + 2\lambda M,\end{aligned}$$

for every $\mu \in \Lambda_\lambda$.
Bearing in mind that $q+1<2$, there exists a positive constant κ^1_λ such that

$$\|w_{\lambda\mu}\|_{E^n_s(V)} \leqslant \kappa^1_\lambda$$

for every $\mu \in \Lambda_\lambda$.
Further, owing to $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0$ for every $\mu \in \Lambda_\lambda$, arguing as above, there exists $\kappa^2_\lambda > 0$ such that

$$\|v_{\lambda\mu}\|_{E^n_s(V)} \leqslant \kappa^2_\lambda,$$

for every $\mu \in \Lambda_\lambda$. Thus, the proof is complete setting $\kappa_\lambda := \max\{\kappa^1_\lambda, \kappa^2_\lambda\}$. □

Remark 4 By using definition (37) clearly

$$\mu^*_\lambda < \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\,dx}{1+c_0},$$

for every $\lambda > \lambda_0$. On the other hand, since the right-hand side does not depend on the parameter $\lambda \in \mathbb{R}$, we have a uniform estimation of Λ_λ, that is

$$\Lambda_\lambda \subset \left[-\frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\,dx}{1+c_0}, \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\,dx}{1+c_0} \right],$$

for every $\lambda > \lambda_0$.

Proof of Theorem 3 and Proposition 1.

Proof of Theorem 3 Fix $\overline{\lambda} > \lambda_0$, $\gamma \in (0, \overline{\lambda} - \lambda_0)$ and define the positive number

$$\overline{\mu} := \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{1 + c_0} \left(1 - \frac{\lambda_0}{\overline{\lambda} + \gamma}\right) \frac{\overline{\lambda} - \lambda_0 - \gamma}{\overline{\lambda} - \lambda_0 + \gamma}.$$

Let us consider $\mu \in (-\overline{\mu}, \overline{\mu})$. Thus, since $\overline{\mu} < \mu^*_\lambda$, for every $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$, one has that $\mu \in (-\mu^*_\lambda, \mu^*_\lambda) = \Lambda_\lambda$, for every

$$\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma).$$

Now, Theorem 2 ensures that, for each $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$, problem (6) has at least two distinct, nontrivial, weak solutions. Consequently, it follows that

$$(\overline{\lambda} - \gamma, \overline{\lambda} + \gamma) \subset \Sigma.$$

The proof is complete. $\square$

Remark 5 The abstract approach adopted here is employed for nonlocal fractional Schrödinger equations patterned after Problem (1.1) in [31]. Let us note that in our setting the situation is much more delicate with respect to the one treated in the cited paper. Indeed, the fractional framework produces several technical difficulties that we overcome by using an appropriate variational formulation, as well as the embedding properties proved in Lemma 1 and Proposition 2 (see also Lemma 3). Further, concerning the potential term V, assumptions (p_1) and (p_2) are previously used, in the fractional case, in the recent paper [56]. Hence, our multiplicity results represent a nontrivial fractional counterpart of [31, Theorems 2.1–2.3]. We refer to the monograph [39] for more details.

Proof of Theorem 3 Arguing by contradiction, let us assume that there exists a weak solution $\bar{u} \in E^n_s(V)\backslash\{0\}$ of the problem (6), that is,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} |\xi|^{2s} \mathfrak{F}\bar{u}(\xi)\mathfrak{F}v(\xi)\, d\xi + \int_{\mathbb{R}^n} V(x)\bar{u}(x)v(x)\, dx \\ \qquad\qquad = \displaystyle\lambda \int_{\mathbb{R}^n} W(x)h(\bar{u}(x))v(x)\, dx + \lambda\mu \int_{\mathbb{R}^n} W(x)k(\bar{u}(x))v(x)\, dx, \\ \forall\ v \in E^n_s(V). \end{cases} \tag{39}$$

In particular, testing (39) with $v := \bar{u}$, we have

$$\|\bar{u}\|^2_{E^n_s(V)} = \lambda \int_{\mathbb{R}^n} W(x)h(\bar{u}(x))\bar{u}(x)\, dx + \lambda\mu \int_{\Omega} W(x)k(\bar{u}(x))\bar{u}(x)\, dx. \tag{40}$$

Hence, by (40), since h and k are Lipschitz continuous functions with $h(0) = k(0) = 0$, it follows that

$$\begin{aligned} \|\bar{u}\|^2_{E^n_s(V)} &\leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} h(\bar{u}(x))\bar{u}(x)\, dx + \mu \int_{\mathbb{R}^n} k(\bar{u}(x))\bar{u}(x)\, dx \right) \\ &\leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |h(\bar{u}(x))||\bar{u}(x)|\, dx + |\mu| \int_{\mathbb{R}^n} |k(\bar{u}(x))||\bar{u}(x)|\, dx \right) \\ &\leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)} (L_h + |\mu| L_k) \|\bar{u}\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

By the above computations, since $E_s^n(V) \hookrightarrow L^2(\mathbb{R}^n)$, bearing in mind that

$$0 < \lambda < \frac{1}{\|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu|L_k)S_2^2},$$

we obtain

$$\|\bar{u}\|^2_{E_s^n(V)} \leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu|L_k)S_2^2 \|\bar{u}\|^2_{E_s^n(V)} < \|\bar{u}\|^2_{E_s^n(V)},$$

which is a contradiction. In conclusion problem (6) admits only the trivial solution. □

4 A stability property for fractional NLS

In this section we prove a multiplicity result to the problem (14). We assume that the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(t)| \leqslant |t|^q \quad \textit{for some } q \in (0, 1) \quad \textit{and} \quad \textit{every } t \in \mathbb{R}, \tag{41}$$

and

$$\lim_{t \to 0} \frac{f(t)}{t} = 0. \tag{42}$$

Further, we require that the perturbation term g belongs to the class $\mathcal{C}$ of the continuous functions $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

there exist $Z \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $Z \not\equiv 0$, *and* $r \in (0, 1)$ *such that*

$$|g(x, t)| \leqslant Z(x)|t|^r,$$

for each $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

As pointed out in the Introduction, the main result in this section (Theorem 7) ensures that for $\lambda > 0$ large enough, problem (14) admits at least two non-trivial weak solutions, as well as the stability of this problem with respect to an arbitrary subcritical perturbation of the Schrödinger equation.

The key tool will be the following abstract critical point theorem for differentiable functionals (cf. [47, Theorem 2] for a detailed proof).

Theorem 6 *Let E be a separable and reflexive real Banach space, $\Phi : E \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_E$, bounded on each bounded subset of E and whose derivative admits a continuous inverse on E'. Let $J : E \to \mathbb{R}$ be a C^1 functional with compact derivative and assume that Φ has a strict local minimum z_0 with $\Phi(z_0) = J(z_0) = 0$. Finally, assume that*

$$\max\left\{\limsup_{\|z\| \to +\infty} \frac{J(z)}{\Phi(z)}, \limsup_{z \to z_0} \frac{J(z)}{\Phi(z)}\right\} \leqslant 0 \tag{43}$$

and

$$\sup_{z \in E} \min\{\Phi(z), J(z)\} > 0.$$

Set

$$\theta := \inf\left\{\frac{\Phi(z)}{J(z)} : z \in E,\ \min\{\Phi(z), J(z)\} > 0\right\}. \tag{44}$$

Then for each compact interval $[a, b] \subset (\theta, +\infty)$ there exists a number $\varrho > 0$ with the following property:

for every $\lambda \in [a, b]$ *and every* C^1 *functional* $\Psi : E \to \mathbb{R}$ *with compact derivative, there exists* $\tilde{\mu} > 0$ *such that for all* $\mu \in [0, \tilde{\mu}]$*, the equation*

$$\Phi'(z) - \lambda J'(z) - \mu \Psi'(z) = 0 \tag{45}$$

has at least three solutions whose norms are less than ϱ*.*

Here $\mathcal{W}_E$ denotes the class of all functionals $I : E \to \mathbb{R}$ with the following property: *if* $u_j \rightharpoonup u$ *in* E *and*

$$\liminf_{j\to\infty} I(u_j) \leqslant I(u),$$

then $u_j \to u$ *up to a subsequence.*

The main result of this section reads as follows.

Theorem 7 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying (41) and (42). Assume that condition* (H) *hold.*

Set

$$\theta := \frac{1}{2} \inf_{u\in E_s^n(V)} \left\{ \frac{\|u\|^2_{E_s^n(V)}}{\displaystyle\int_{\mathbb{R}^n} W(x)F(u(x))dx} : \int_{\mathbb{R}^n} W(x)F(u(x))dx > 0 \right\}, \tag{46}$$

Then for each compact interval $[a, b] \subset (\theta, +\infty)$*, there exists a number* $\varrho > 0$ *with the following property: for every* $\lambda \in [a, b]$ *and every* $g \in \mathcal{C}$ *there exists* $\tilde{\mu} > 0$ *such that, for each* $\mu \in [0, \tilde{\mu}]$*, problem (14) has at least three weak solutions whose norms in* $E_s^n(V)$ *are less than* ϱ*.*

Sketch of the Proof of Theorem 7 We apply Theorem 5 by choosing $E := E_s^n(V)$ and the functionals Φ and J defined respectively by

$$\Phi(u) := \frac{1}{2}\|u\|^2_{E_n^s(V)},$$

and

$$J(u) := \int_{\mathbb{R}^n} W(x)F(u(x))dx,$$

for every $u \in E_s^n(V)$.

Under our hypotheses direct computations ensure that

$$\lim_{\|u\|_{E_s^n(V)}\to 0} \frac{J(u)}{\|u\|^2_{E_s^n(V)}} = \lim_{\|u\|_{E_s^n(V)}\to +\infty} \frac{J(u)}{\|u\|^2_{E_s^n(V)}} = 0. \tag{47}$$

By (47) it follows that (43) holds true.

Hence, since also all the regularity assumptions on Φ and J are verified, our conclusions are easily achieved.

Remark 6 Collecting the estimates of Lemmas 2 and 3 we obtain a concrete upper bound for the parameter θ in (46). More precisely we have $\theta \in (0, \theta^\star)$, where

$$\theta^\star := \frac{1}{2\varepsilon_0^2 \Gamma\left(1+\frac{n}{2}\right)} \frac{S_0 \sigma_0^2 \pi^{\frac{n}{2}}\left(\sigma_0^n - (\sigma_0-\varepsilon_0)^n\right)}{\omega_0|B_{\sigma_0/2}| - \max\limits_{|x|\in[\sigma_0/2,\sigma_0]} W(x) \max\limits_{|t|\leqslant|s_0|} |F(t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|)}.$$

Indeed, recalling the definition of θ, in view of inequality (26) and Lemma 2 we obtain

$$\begin{aligned}\theta &\leqslant \frac{\|u_0\|^2_{E^n_s(V)}}{2\displaystyle\int_{\mathbb{R}^n} W(x)F(u_0(x))dx}\\ &< \frac{\sigma_0^2}{2\varepsilon_0^2}\frac{\left(\sigma_0^n-(\sigma_0-\varepsilon_0)^n\right)}{\Gamma\left(1+\frac{n}{2}\right)}\frac{S_0\pi^{\frac{n}{2}}}{\displaystyle\int_{\mathbb{R}^n} W(x)F(u_0(x))dx} \leqslant \theta^\star.\end{aligned}$$

Example 1 Fix $r, q \in (0, 1)$ and consider the following problem

$$(-\Delta)^s u + V(x)u = \frac{\lambda|u|^q \sin u + \mu|\sin u|^r}{(1+|x|^n)^2}, \quad x \in \mathbb{R}^n. \tag{48}$$

Owing to Theorem 7, there exists $\theta > 0$ such that for each compact interval $[a, b] \subset (\theta, +\infty)$, there is some $\varrho > 0$ with the following property: for every $\lambda \in [a, b]$ there exists $\tilde{\mu} > 0$ such that for all $\mu \in [0, \tilde{\mu}]$, problem (48) admits at least three weak solutions whose norms in $E^n_s(V)$ are less than ϱ.

Remark 7 For completeness we just mention here that, by using a variational approach similar to the one adopted here, in [36–38] the authors proved the existence and the multiplicity of weak solutions for nonlocal problems involving regional fractional Laplacian operators in a suitable abstract setting previously introduced in [51,52].

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