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Standing waves for the pseudo-relativistic Hartree equation with Berestycki-Lions nonlinearity

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Abstract

We study the following class of pseudo-relativistic Hartree equations

$$\sqrt{-\varepsilon^2\Delta + m^2}u + V(x)u = \varepsilon^{\mu-N}\big(|x|^{-\mu} * F(u)\big)f(u) \quad \text{in } \mathbb{R}^N,$$

where the nonlinearity satisfies general hypotheses of Berestycki-Lions type. By using the method of penalization arguments, we prove the existence of a family of localized positive solutions that concentrate at the local minimum points of the indefinite potential $V(x)$, as $\varepsilon \to 0$.

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Keywords: Standing waves; Pseudo-relativistic Hartree equation; Concentration; Hardy-Littlewood-Sobolev inequality; Regularity; Penalization

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Contents

1. Introduction and main results

In this paper, we are concerned with the qualitative and asymptotic analysis of solutions for the following pseudo-relativistic Hartree equation with general nonlinearity

$$\sqrt{-\varepsilon^2\Delta+m^2}\,u+V(x)u=\varepsilon^{\mu-N}\big(|x|^{-\mu}*F(u)\big)f(u)\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N\geq 3$, $\mu\in(0,2)$, $m>0$ is a physical constant, ε is the semiclassical parameter $0<\varepsilon\leq 1$, a dimensionless scaled Planck constant (all other physical constants are rescaled to be 1), V is a bounded external potential, and $F(u)=\int_0^u f(\tau)d\tau\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$. Here, the pseudo-differential operator $\sqrt{-\varepsilon^2\Delta+m^2}$ is simply defined in Fourier variables by the symbol $\sqrt{\varepsilon^2|\xi|^2+m^2}$, see Lieb and Loss [30]. To the best of our knowledge, the study of pseudo-relativistic Hartree problems was initiated by Coti Zelati and Nolasco [14,15].

Replacing $u(x)$ by $u(\varepsilon x)$, we observe that equation (1.1) is equivalent to the following problem

$$\sqrt{-\Delta+m^2}\,u+V_\varepsilon(x)u=(|x|^{-\mu}*F(u))f(u),\ \ \text{in } \mathbb{R}^N, \tag{1.2}$$

where $V_\varepsilon(x)=V(\varepsilon x)$.

We refer to Lieb and Yau [32] who studied problem (1.2) in the autonomous case $V(x)=1-m$ if $N=3$, $\varepsilon=\mu=1$, and $f(u)=u$. In this case, they proved the existence of solutions provided that $M<M_c$, where M_c is the Chandrasekhar limit mass, which is a prediction on the maximum mass of a white dwarf star. A white dwarf star is the final stage in the evolution of a star whose mass is not too high, see Chandrasekhar [10]. More precisely, Lieb and Yau [32] proved the existence in $H^{1/2}(\mathbb{R}^3)$ of a radial, real-valued nonnegative minimizer (ground state) of the associated energy with given fixed "mass-charge" $M=\int_{\mathbb{R}^3}u^2dx<M_c$.

We observe that problem (1.2) is driven by the fractional Laplace operator, which is intensively studied in relationship with the infinitesimal generators of Lévy stable diffusion processes. We also point out that the exponent $1/2$ of this nonlocal operator in problem (1.2) corresponds to the pseudo-relativistic Hartree equation.

To describe the boson stars in mean-field theory [21,28], a reasonable model is to study the nonlinear mean-field equation called the pseudo-relativistic Hartree equation defined by

$$i\partial_t\psi = (\sqrt{-\Delta + m^2} - m)\psi + W(x)\psi - (|x|^{-1} * |\psi|^2)\psi \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $\psi(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a complex-valued wave field, $m > 0$ is a physical constant and W is a bounded external potential in $\mathbb{R}^3$. The study of solitary wave type solutions $\psi(t,x) = e^{it\lambda}u(x)$ (where $\lambda > 0$) for equation (1.3) leads to investigating the pseudo-relativistic Hartree equation with Coulomb kernel

$$\sqrt{-\Delta + m^2}\, u + V(x)u = (|x|^{-1} * |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Problem (1.4) can be interpreted as a system of N spinless, identical bosons with two-body interactions governed by the Coulomb potential. These bosons are also subject to a time-independent external potential $V(x)$, see Fröhlich and Lenzmann [22] for more details. In the particular case $V(x) = -m$, problem (1.4) was studied by Elgart and Schlein [20] as an effective dynamical description for an N-body quantum system of relativistic bosons with a two-body interaction given by Newtonian gravity. This leads to a Chandrasekhar type theory of boson stars.

Equation (1.4) has attracted a great deal of attention in theoretical and numerical astrophysics over the past years. If $V(x)$ is a constant potential, Lenzmann [27] proved the uniqueness of ground states for pseudo-relativistic Hartree equations (1.4) and he also obtained local and global well-posedness for semi-relativistic Hartree equations of critical type in [26]. In a recent paper, Du and Yang [19] considered the critical Hartree equation and they classified the solutions of the problem.

By using variational methods and some new variational identities involving the half Laplacian, Mugnai [36] proved several existence and non existence results of solitary waves for a class of nonlinear pseudo-relativistic Hartree equations with general nonlinearities. Coti Zelati and Nolasco [15] obtained the existence of ground states for nonlinear pseudo-relativistic Schrödinger equations. The authors [41] investigated the existence and asymptotic behavior of the solutions for the critical pseudo-relativistic Hartree equation. For recent progress in this field, we may refer to [11,14,22–25] and the references therein.

For the study of semi-classical analysis of the Hartree equations we would like to mention the papers [1,2,9,12,13,34,37,40]. In [13], Cingolani and Tanaka studied the existence and multiplicity of semi-classical states for the nonlinear Choquard equation with the general Berestycki-Lions type assumptions. They developed a new variational approach and showed the existence of a family of solutions concentrating, as $\varepsilon \to 0$, to a local minima of potential function. In [18] the authors studied the existence of semiclassical solutions for the critical Choquard equations with critical frequency. In particular, Cingolani and Secchi [12] studied the semi-classical limit for the pseudo-relativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}\, u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Under some proper conditions on m, p and V, they proved the existence of semi-classical solutions that concentrate at the minimum points of the potential V by means of a variational approach introduced in [5]. The operator $\sqrt{-\varepsilon^2\Delta + m^2}$ is a nonlocal operator in $\mathbb{R}^N$ that can be realized through a local problem in $\mathbb{R}^N \times (0,\infty)$. We describe this construction in what follows. For any function $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$, there is a unique function $v \in H^1(\mathbb{R}^{N+1}_+)$ (here, $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\}$) such that

$$\begin{cases} -\varepsilon^2\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ v(x,0) = u(x) & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases} \tag{1.5}$$

Setting

$$T_\varepsilon u(x) = -\varepsilon \frac{\partial v}{\partial y}(x,0),$$

we obtain the equation

$$\begin{cases} -\varepsilon^2\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w(x,0) = T_\varepsilon u(x) & \text{on } \mathbb{R}^N \end{cases}$$

with the solution $w(x,y) = -\varepsilon\frac{\partial v}{\partial y}(x,y)$. From (1.5) we have that

$$T_\varepsilon(T_\varepsilon u)(x) = -\varepsilon\frac{\partial w}{\partial y}(x,0) = \varepsilon^2\frac{\partial^2 v}{\partial y^2}(x,0) = (-\varepsilon^2\Delta_x v + m^2 v)(x,0)$$

and hence $T_\varepsilon^2 = (-\varepsilon^2\Delta_x + m^2)$. Thus, the operator T_ε that maps the Dirichlet-type data u to the Neumann-type data $-\varepsilon\frac{\partial v}{\partial y}(x,0)$ is actually $\sqrt{-\varepsilon^2\Delta + m^2}$. In this way, for equation (1.1), we will study the following mixed value boundary problem:

$$\begin{cases} -\varepsilon^2\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\varepsilon\dfrac{\partial v}{\partial y} = \varepsilon^{\mu-N}(|x|^{-\mu} * F(v(x,0)))f(v(x,0)) - V(x)v(x,0) & \text{on } \mathbb{R}^N. \end{cases} \tag{1.6}$$

We may refer the readers to [8,12,15] for more details about the fractional operator.

To treat the convolution part, we need to recall the Hardy-Littlewood-Sobolev inequality.

Proposition 1.1. *(Hardy-Littlewood-Sobolev inequality, [30]). Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}dxdy \le C(t,N,\mu,r)|f|_t|h|_r. \tag{1.7}$$

If $t = r = 2N/(2N-\mu)$, then

$$C(t,N,\mu,r) = C(N,\mu) = \pi^{\frac{\mu}{2}}\frac{\Gamma(\frac{N}{2}-\frac{\mu}{2})}{\Gamma(N-\frac{\mu}{2})}\left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{-1+\frac{\mu}{N}}.$$

From the Hardy-Littlewood-Sobolev inequality, for any $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$, the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^q|u(z)|^q}{|x-z|^\mu}dxdz$$

is well defined if

$$\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-1}.$$

That is why the exponent $\frac{2N-\mu}{N}$ will be called the lower critical exponent and the exponent $2^*_\mu = \frac{2N-\mu}{N-1}$ will be called the upper critical exponent.

In the sequel, we will assume that the potential function $V(x)$ satisfies the following condition:

(V) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ is a bounded function such that $V_{\min} = \inf_{\mathbb{R}^N} V > -m$ and there exists a bounded open set $O \in \mathbb{R}^N$ with the property that

$$V_0 = \inf_O V < \min_{\partial O} V.$$

Set $\mathcal{M} = \{x \in O : V(x) = V_0\}$.

We assume that the nonlinearity $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies the general Berestycki-Lions type assumptions [3]:

(f_1) There exists $C > 0$ such that for every $t \in \mathbb{R}$,

$$|tf(t)| \leq C(|t|^2 + |t|^{\frac{2N-\mu}{N-1}});$$

(f_2) Let $F : t \in \mathbb{R} \mapsto \int_0^t f(\tau)d\tau$ and suppose that

$$\lim_{t\to 0} \frac{F(t)}{|t|^2} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{F(t)}{|t|^{\frac{2N-\mu}{N-1}}} = 0;$$

(f_3) There exists $t_0 \in \mathbb{R}$ such that $F(t_0) \neq 0$.

The first main result of this paper establishes the following qualitative and asymptotic properties.

Theorem 1.2. *Suppose that $N \geq 3$ and $0 < \mu < 2$. If $V(x)$ satisfies assumption (V) and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies $(f_1) - (f_3)$ and $f(t) = 0$ for $t \leq 0$, then, for sufficiently small $\varepsilon > 0$, problem (1.1) admits a positive solution $u_\varepsilon \in H^{\frac{1}{2}}(\mathbb{R}^N)$, which satisfies*

(i) there exists a local maximum points x_ε of u_ε such that

$$\lim_{\varepsilon\to 0} \text{dist}\,(x_\varepsilon, \mathcal{M}) = 0,$$

and $w_\varepsilon(x) \equiv u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges (up to a subsequence) uniformly to a least energy solution of

$$\sqrt{-\Delta + m^2}\, u + V_0 u = (|x|^{-\mu} * F(u)) f(u), \quad u \in H^{\frac{1}{2}}(\mathbb{R}^N);$$

(ii) $u_\varepsilon(x) \le C\exp(-\frac{c}{\varepsilon}|x-x_\varepsilon|)$ *for some* $c, C > 0$.

In order to study multi-peak solutions for the pseudo-relativistic Hartree equation, we assume that the potential function V satisfies the following hypothesis:

(V') $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ is a bounded function such that $V_{\min} = \inf_{\mathbb{R}^N} V > -m$ and there exist bounded disjoint open sets O^i, $i = 1, 2, \cdots, k$, such that for any $i \in \{1, 2, \cdots, k\}$,

$$m_i \equiv \inf_{x\in O^i} V(x) < \min_{x\in\partial O^i} V(x).$$

Let $\mathcal{M}' = \cup_{i=1}^k \mathcal{M}^i$ and $O' = \cup_{i=1}^k O^i$.

The main result in this case is the following.

Theorem 1.3. *Suppose that* $\mu \in (0,2)$*,* $N \ge 3$*,* (V')*,* $f(t) = 0$ *for* $t \le 0$ *and* $(f_1)-(f_3)$*. Then, for sufficiently small* $\varepsilon > 0$*, problem* (1.1) *admits a positive solution* u_ε*, which satisfies the following properties:*

(i) there exist k *local maximum points* $x_\varepsilon^i \in O^i$ *of* u_ε *such that*

$$\lim_{\varepsilon\to 0}\max_{1\le i\le k} \operatorname{dist}(x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and $w_\varepsilon(x) \equiv u_\varepsilon(\varepsilon x + x_\varepsilon^i)$ *converges (up to a subsequence) uniformly to a least energy solution of*

$$\sqrt{-\Delta+m^2}\,u + m_i u = (|x|^{-\mu} * F(u))f(u), \quad u \in H^{\frac{1}{2}}(\mathbb{R}^N);$$

(ii) $u_\varepsilon(x) \le C\exp(-\frac{c}{\varepsilon}\min_{1\le i\le k}|x-x_\varepsilon^i|)$ *for some* $c, C > 0$.

To prove the main results, we need to study the following the semi-relativistic Hartree equation with constant potential, which plays the role of limit problem for equation (1.2), that is,

$$\begin{cases} \sqrt{-\Delta+m^2}\,u + au = \big(|x|^{-\mu} * F(u)\big)f(u) & \text{in } \mathbb{R}^N, \\ u \in H^{\frac{1}{2}}(\mathbb{R}^N), \end{cases} \tag{1.8}$$

where $a > -m$. Equation (1.8) appears in the study of models of stellar collapse, such as neutron stars. The typical neutron kinetic energy is high, so it must be treated relativistically, see Lieb [29], Lieb and Thirring [31], and Lieb and Yau [32].

We can reformulate the nonlocal problem (1.8) as the following local Neumann problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial v}{\partial y} = (|x|^{-\mu} * F(v(x,0)))f(v(x,0)) - av(x,0) & \text{on } \mathbb{R}^N. \end{cases} \tag{1.9}$$

The natural working space for problem (1.9) is the Sobolev space

$$H^1(\mathbb{R}^{N+1}_+) := \Big\{ v \in L^2(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 dxdy < \infty \Big\},$$

equipped with norm

$$\|v\|^2 = \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + v^2) dxdy.$$

For all $q \in [1, \infty]$, we denote by $|\cdot|_q$ the norm in the space $L^q(\mathbb{R}^N)$ and by $\|\cdot\|_q$ the norm in the space $L^q(\mathbb{R}^{N+1}_+)$. We denote positive constants by $C, C_1, C_2, C_3, \dots$.

By [38], we know that traces of functions $H^1(\mathbb{R}^{N+1}_+)$ are in $H^{\frac{1}{2}}(\mathbb{R}^N)$ and that every function in $H^{\frac{1}{2}}(\mathbb{R}^N)$ is the trace of some function in $H^1(\mathbb{R}^{N+1}_+)$. Let $\gamma : H^1(\mathbb{R}^{N+1}_+) \to H^{\frac{1}{2}}(\mathbb{R}^N)$ be the linear operator that associates the trace $\gamma(v) \in H^{\frac{1}{2}}(\mathbb{R}^N)$ of the function $v \in H^1(\mathbb{R}^{N+1}_+)$. Moreover, we know from [14] that for any $v \in H^1(\mathbb{R}^{N+1}_+)$,

$$|\gamma(v)|_s \le c_s \|v\|, \tag{1.10}$$

where $2 \le s \le 2^* := \frac{2N}{N-1}$.

We will look for solutions to equation (1.9) as critical points of the Euler functional $J_a : H^1(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ defined by

$$J_a(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy + \frac{a}{2} \int_{\mathbb{R}^N} \gamma(v)^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(\gamma(v)(x)) F(\gamma(v)(z))}{|x-z|^\mu} dxdz.$$

Clearly, J_a is well defined on $H^1(\mathbb{R}^{N+1}_+)$ and belongs to $\mathcal{C}^1$. A function v_0 is called a ground state of problem (1.9) if

$$J_a(v_0) = E_a := \inf \Big\{ J_a(v) : v \in H^1(\mathbb{R}^{N+1}_+) \backslash \{0\} \text{ is a critical point of (1.9)} \Big\}.$$

Theorem 1.4. *Assume that $a > -m$, $N \ge 3$ and $\mu \in (0, 2)$. If $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies (f_1), (f_2) and (f_3), then equation (1.9) has at least one ground state solution v and $v \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N \times [0, +\infty)) \cap \mathcal{C}^2(\mathbb{R}^{N+1}_+)$ is a classical solution.*

The paper is organized as follows. In Section 2, we study the nonlocal problem (1.9) and prove the existence, regularity and exponential decay of ground states for the nonlocal problem. In Section 3, we prove Theorem 1.2 by means of a variational approach introduced in [5]. In Section 4, by using the method of penalization argument, we construct a positive solution having multiple concentration regions which concentrate at the minimum points of the potential V.

2. An autonomous problem

In this Section, we prove the existence, regularity and exponential decay of ground states for the nonlocal problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial v}{\partial y} = (|x|^{-\mu} * F(v(x,0)))f(v(x,0)) - av(x,0) & \text{on } \mathbb{R}^N. \end{cases}$$

2.1. Regularity of solutions and Pohožaev identity

In this subsection we are going to show that the solutions for equation (1.9) possess some regularity properties, which will be used to prove a Pohožaev identity for the pseudo-relativistic Hartree equation. We adapt the method of Brezis and Kato [4] and obtain the regularity of the weak solutions by using the Morse iteration method. Similar to the arguments developed in [35, 41] for the nonlocal linear equations dominated by the Laplacian, we have the following estimate lemma for fractional equations with convolution parts.

Lemma 2.1. *Let $N \geq 3$, $\mu \in (0, N)$ and $\theta \in (0, N)$. If $H, K \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^N) + L^{\frac{2N}{N+1-\mu}}(\mathbb{R}^N)$, $(1 - \frac{\mu}{N}) < \theta < (1 + \frac{\mu}{N})$, then for any $\varepsilon > 0$, there exists $C_{\varepsilon,\theta} \in \mathbb{R}$ such that for every $v \in H^1(\mathbb{R}^{N+1}_+)$,*

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * (H|\gamma(v)|^\theta))K|\gamma(v)|^{2-\theta} dx \leq \varepsilon^2 \int_{\mathbb{R}^{N+1}_+} |\nabla v(x,y)|^2 dx + C_{\varepsilon,\theta} \int_{\mathbb{R}^N} |\gamma(v)|^2 dx.$$

The regularity property can be stated as follows.

Proposition 2.2. *Let v be a critical point for the functional J_a on $H^1(\mathbb{R}^{N+1}_+)$, then $\gamma(v) \in L^q(\mathbb{R}^N)$ for all $q \in [2, +\infty]$ and $v \in L^\infty(\mathbb{R}^{N+1}_+)$.*

Proof. Let us define the truncation $v_\tau \colon \mathbb{R}^{N+1}_+ \to \mathbb{R}$, for $\tau > 0$ large,

$$v_\tau(x,y) = \begin{cases} -\tau & \text{if } v \leq -\tau, \\ v(x,y) & \text{if } -\tau < v < \tau, \\ \tau & \text{if } v \geq \tau. \end{cases} \tag{2.1}$$

Since $|v_\tau|^{s-2}v_\tau \in H^1(\mathbb{R}^{N+1}_+)$ for $s \geq 2$ and v is a critical point for the functional J_a, taking $|v_\tau|^{s-2}v_\tau$ as a test function, we obtain

$$\frac{4(s-1)}{s^2}\int_{\mathbb{R}^{N+1}_+}(|\nabla(v_\tau(x,y))^{\frac{s}{2}}|^2+m^2||v_\tau(x,y)|^{\frac{s}{2}}|^2)dxdy$$
$$\leq\int_{\mathbb{R}^{N+1}_+}(\nabla(v_\tau(x,y))\nabla(|v_\tau(x,y)|^{s-2}v_\tau(x,y))+m^2|v_\tau(x,y)|^{s-2}v_\tau(x,y)v(x,y))dxdy$$
$$=\int_{\mathbb{R}^N}(|x|^{-\mu}*F(\gamma(v)))f(\gamma(v))|\gamma(v_\tau)|^{s-2}\gamma(v_\tau)dx-a\int_{\mathbb{R}^N}|\gamma(v_\tau)|^{s-2}\gamma(v_\tau)\gamma(v)dx.$$

If $2\leq s<\frac{2N}{N-\mu}$, using Lemma 2.1 with $\theta=\frac{2}{s}$, there exists $C>0$ such that

$$\int_{\mathbb{R}^N}(|x|^{-\mu}*F(\gamma(v_\tau)))f(\gamma(v_\tau))|\gamma(v_\tau)|^{s-2}\gamma(v_\tau)dx$$
$$\leq\frac{2(s-1)}{s^2}\int_{\mathbb{R}^{N+1}_+}(|\nabla(v_\tau(x,y))^{\frac{s}{2}}|^2+m^2||v_\tau(x,y)|^{\frac{s}{2}}|^2)dxdy+C\int_{\mathbb{R}^N}||\gamma(v_\tau)|^{\frac{s}{2}}|^2dx.$$

Since $|v_\tau|\leq|v|$, we have

$$\frac{2(s-1)}{s^2}\int_{\mathbb{R}^{N+1}_+}(|\nabla(v_\tau(x,y))^{\frac{s}{2}}|^2+m^2||v_\tau(x,y)|^{\frac{s}{2}}|^2)dxdy$$
$$\leq C\int_{\mathbb{R}^N}|\gamma(v)|^sdx+\int_{A_\tau}(|x|^{-\mu}*|f(\gamma(v))|\gamma(v)|^{s-1}|)|F(\gamma(v))|dx,$$

where $A_\tau=\{x\in\mathbb{R}^N:|v|>\tau\}$. Since $2\leq s<\frac{2N}{N-\mu}$, applying the Hardy-Littlewood-Sobolev inequality again, we obtain

$$\int_{A_\tau}(|x|^{-\mu}*|f(\gamma(v))|\gamma(v)|^{s-1}|)|F(\gamma(v))|dx$$
$$\leq C\Big(\int_{\mathbb{R}^N}|f(\gamma(v))|\gamma(v)|^{s-1}|^rdx\Big)^{\frac{1}{r}}\Big(\int_{A_\tau}|F(\gamma(v))|^ldx\Big)^{\frac{1}{l}},$$

with $\frac{1}{r}=1+\frac{N-\mu}{2N}-\frac{1}{s}$ and $\frac{1}{l}=\frac{N-\mu}{2N}+\frac{1}{s}$. By Hölder's inequality, if $\gamma(v)\in L^s(\mathbb{R}^N)$, then $f(\gamma(v_\tau))|\gamma(v_\tau)|^{s-1}\in L^r(\mathbb{R}^N)$ and $F(\gamma(v))\in L^l(\mathbb{R}^N)$, hence by Lebesgue's dominated convergence theorem,

$$\lim_{\tau\to\infty}\int_{A_\tau}(|x|^{-\mu}*|f(\gamma(v))|\gamma(v)|^{s-1}|)|F(\gamma(v))|dx=0.$$

On the other hand, by (1.10), we know that there exists a constant C such that

$$\Big(\int_{\mathbb{R}^N} |\gamma(v_\tau)|^{\frac{sN}{N-1}}dx\Big)^{\frac{N-1}{N}} \le C\int_{\mathbb{R}^N} |\gamma(v)|^s dx.$$

Letting $\tau \to \infty$ we conclude that $\gamma(v) \in L^{\frac{sN}{N-1}}(\mathbb{R}^N)$. By iterating over s a finite number of times we cover the range $s \in \Big[2, \frac{2N}{N-\mu}\Big)$. So we can get $\gamma(v) \in L^s(\mathbb{R}^N)$ for every $s \in \Big[2, \frac{2N^2}{(N-\mu)(N-1)}\Big)$. Using (f_1), we know $F(\gamma(v)) \in L^q(\mathbb{R}^N)$ for every $q \in \Big[\frac{2N}{2N-\mu}, \frac{2N^2}{(N-\mu)(2N-\mu)}\Big)$. Since $\frac{2N}{2N-\mu} < \frac{N}{N-\mu} < \frac{2N^2}{(N-\mu)(2N-\mu)}$, we have

$$U(x) := |x|^{-\mu} * F(\gamma(v)) \in L^\infty(\mathbb{R}^N). \tag{2.2}$$

We claim that $\gamma(v) \in L^p(\mathbb{R}^N)$ for any $p \in [2, +\infty]$. In fact, since $v \in H^1(\mathbb{R}^{N+1}_+)$ is a critical point such that

$$\int_{\mathbb{R}^{N+1}_+} (\nabla v \nabla \varphi + m^2 v\varphi)dxdy + a\int_{\mathbb{R}^N} \gamma(v)\gamma(\varphi)dx = \int_{\mathbb{R}^N} U(x)f(\gamma(v))\gamma(\varphi)dx, \tag{2.3}$$

for every $\varphi \in H^1(\mathbb{R}^{N+1}_+)$, where $U(x)$ is defined in (2.2). For $T > 0$, we denote

$$v_T = \min\{v_+, T\},$$

where $v_+ = \max\{0, v\}$. Since for $\beta > 0$, $|v_T|^{2\beta}v \in H^1(\mathbb{R}^{N+1}_+)$, take it as a test function in (2.3), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} (\nabla v \nabla(|v_T|^{2\beta}v) + m^2|v_T|^{2\beta}v^2)dxdy \\
&\quad = \int_{\mathbb{R}^{N+1}_+} |v_T|^{2\beta}|\nabla v|^2 dxdy + 2\beta\int_{\{v\le T\}} v_T^{2\beta}|\nabla v|^2dxdy + \int_{\mathbb{R}^{N+1}_+} m^2v^2|v_T|^{2\beta}dxdy \\
&\quad = \int_{\mathbb{R}^N} U(x)f(\gamma(v))|\gamma(v_T)|^{2\beta}\gamma(v)dx - a\int_{\mathbb{R}^N} |\gamma(v_T)|^{2\beta}\gamma(v)^2dx.
\end{aligned}$$

Noticing that

$$\int_{\mathbb{R}^{N+1}_+} |\nabla(|v_T|^\beta v)|^2dxdy = \int_{\mathbb{R}^{N+1}_+} |v_T|^{2\beta}|\nabla v|^2dxdy + \big(2\beta+\beta^2\big)\int_{\{v\le T\}} v_T^{2\beta}|\nabla v|^2dxdy,$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} (|\nabla(|v_T|^\beta v)|^2 + ||v_T|^\beta v|^2)dxdy \\
&= \int_{\mathbb{R}^{N+1}_+} |v_T|^{2\beta}|\nabla v|^2 dxdy + \left(2\beta+\beta^2\right)\int_{\{v\le T\}} v_T^{2\beta}|\nabla v|^2 dxdy + \int_{\mathbb{R}^{N+1}_+} ||v_T|^\beta v|^2 dxdy \\
&\le C_\beta \int_{\mathbb{R}^N} U(x) f(\gamma(v))|\gamma(v_T)|^{2\beta}\gamma(v)dx - aC_\beta \int_{\mathbb{R}^N} |\gamma(v_T)|^{2\beta}\gamma(v)^2 dx,
\end{aligned}$$

where $C_\beta = \max\{\frac{1}{m^2}, 1+\frac{\beta}{2}\}$. By assumption (f_1), we have

$$f(\gamma(v))\gamma(v) \le (C + |\gamma(v)|^{\frac{2-\mu}{N-1}})\gamma(v)^2.$$

Since $U(x) \in L^\infty(\mathbb{R}^N)$ and $\gamma(v) \in L^p(\mathbb{R}^N)$ for every $p \in [2, \frac{2N^2}{(N-\mu)(N-1)})$, we know that, for some constant C_1 and function $g \in L^N(\mathbb{R}^N)$, $g \ge 0$ and independent of T and p,

$$U(x) f(\gamma(v))|\gamma(v_T)|^{2\beta}\gamma(v) \le (C_1 + g)|\gamma(v_T)|^{2\beta}\gamma(v)^2.$$

So we have that

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}_+} (|\nabla(|v_T|^\beta v)|^2 + ||v_T|^\beta v|^2)dxdy \le C_2 C_\beta \int_{\mathbb{R}^N} |\gamma(v_T)|^{2\beta}\gamma(v)^2 dx \\
+ C_\beta \int_{\mathbb{R}^N} g|\gamma(v_T)|^{2\beta}\gamma(v)^2 dx, \qquad (2.4)
\end{aligned}$$

and, using Fatou's lemma and the monotone convergence theorem, we can pass to the limit as $T \to \infty$ and get

$$\int_{\mathbb{R}^{N+1}_+} (|\nabla(v_+^{1+\beta})|^2 + |v_+^{1+\beta}|^2)dxdy \le C_2 C_\beta \int_{\mathbb{R}^N} |\gamma(v_+)|^{2(\beta+1)}dx + C_\beta \int_{\mathbb{R}^N} g|\gamma(v_+)|^{2(\beta+1)}dx.$$

For any $M > 0$, let $A_1 = \{g \le M\}$, $A_2 = \{g > M\}$. Since

$$\begin{aligned}
\int_{\mathbb{R}^N} g|\gamma(v_+)|^{2(\beta+1)}dx &= \int_{A_1} g|\gamma(v_+)|^{2(\beta+1)}dx + \int_{A_2} g|\gamma(v_+)|^{2(\beta+1)}dx \\
&\le M\int_{A_1} |\gamma(v_+)|^{2(\beta+1)}dx + \Big(\int_{A_2} g^N dx\Big)^{\frac{1}{N}} \Big(\int_{A_2} |\gamma(v_+)|^{(\beta+1)\frac{2N}{N-1}}dx\Big)^{\frac{N-1}{N}} \\
&\le M|\gamma(v_+)^{(\beta+1)}|_2^2 + \epsilon(M)|\gamma(v_+)^{(\beta+1)}|_{2^*}^2,
\end{aligned}$$

we deduce that

$$\|v_+^{(\beta+1)}\|^2 \le C_\beta(C_2+M)|\gamma(v_+)^{(\beta+1)}|_2^2 + C_\beta\epsilon(M)|\gamma(v_+)^{(\beta+1)}|_{2^*}^2.$$

Using (1.10) and taking M large enough such that $C_\beta c_{2^*}^2\epsilon(M) \le \frac{1}{2}$, we obtain

$$|\gamma(v_+)^{(\beta+1)}|_{2^*}^2 \le 2C_\beta c_{2^*}^2(C_2+M)|\gamma(v_+)^{(\beta+1)}|_2^2.$$

Now a bootstrap argument starting with $\beta+1=\frac{N}{N-1}$ shows that $\gamma(v_+)\in L^p(\mathbb{R}^N)$ for any $p\in[2,+\infty)$. Similarly, we can obtain $\gamma(v_-)\in L^p(\mathbb{R}^N)$ for any $p\in[2,+\infty)$ and hence the same property holds for $\gamma(v)$.

Since $\gamma(v_+)\in L^p(\mathbb{R}^N)$ for any $p\in[2,+\infty)$, repeating the arguments in (2.4), we obtain that there exist some constant C_1 and a function $g\in L^{2N}(\mathbb{R}^N)$, $g\ge 0$ and independent of T and β such that

$$\int_{\mathbb{R}_+^{N+1}} (|\nabla(|v_T|^\beta v)|^2 + ||v_T|^\beta v|^2)dxdy \le C_1C_\beta\int_{\mathbb{R}^N}|\gamma(v_T)|^{2\beta}\gamma(v)^2dx + C_\beta\int_{\mathbb{R}^N} g|\gamma(v_T)|^{2\beta}\gamma(v)^2dx.$$

Using Fatou's lemma and the monotone convergence theorem, we can pass to the limit as $T\to\infty$ to get

$$\int_{\mathbb{R}_+^{N+1}} (|\nabla(v_+^{1+\beta})|^2 + |v_+^{1+\beta}|^2)dxdy \le C_1C_\beta\int_{\mathbb{R}^N}|\gamma(v_+)|^{2(\beta+1)}dx + C_\beta\int_{\mathbb{R}^N} g|\gamma(v_+)|^{2(\beta+1)}dx.$$

Using Young's inequality, we see

$$\int_{\mathbb{R}^N} g|\gamma(v_+)|^{2(\beta+1)}dx \le |g|_{2N}|(\gamma(v_+))^{\beta+1}|_2|\gamma(v_+)^{\beta+1}|_{2^*} \le |g|_{2N}(\lambda|\gamma(v_+)^{\beta+1}|_2^2 + \frac{1}{\lambda}|\gamma(v_+)^{\beta+1}|_{2^*}^2).$$

Therefore,

$$\int_{\mathbb{R}_+^{N+1}} (|\nabla(v_+^{1+\beta})|^2 + |v_+^{1+\beta}|^2)dxdy \le C_\beta(C_1+|g|_{2N}\lambda)|\gamma(v_+)^{\beta+1}|_2^2 + \frac{C_\beta|g|_{2N}}{\lambda}|\gamma(v_+)^{\beta+1}|_{2^*}^2. \tag{2.5}$$

Using (1.10) and taking λ large enough such that $\frac{C_\beta|g|_{2N}}{\lambda}c_{2^*}^2=\frac{1}{2}$, we obtain

$$|\gamma(v_+)^{\beta+1}|_{2^*}^2 \le 2c_{2^*}^2C_\beta(C_1+|g|_{2N}\lambda)|\gamma(v_+)^{\beta+1}|_2^2 = C_2C_\beta|\gamma(v_+)^{\beta+1}|_2^2.$$

Since $C_2C_\beta \le C_2(m^{-2}+1+\beta)^2 \le M_0^2e^{2\sqrt{1+\beta}}$ for a positive constant M_0, we know that

$$|\gamma(v_+)|_{2^*(\beta+1)} \leq M_0^{\frac{1}{\beta+1}} e^{\frac{1}{\sqrt{1+\beta}}} |\gamma(v_+)|_{2(\beta+1)}.$$

Start with $\beta_0 = 0$, $2(\beta_{n+1}+1) = 2^*(\beta_n + 1)$, an iteration shows

$$|\gamma(v_+)|_{2^*(\beta_n+1)} \leq M_0^{\sum_{i=0}^{n} \frac{1}{\beta_i+1}} e^{\sum_{i=0}^{n} \frac{1}{\sqrt{\beta_i+1}}} |\gamma(v_+)|_{2(\beta_0+1)}.$$

Since $\beta_n + 1 = (\frac{2^*}{2})^n = (\frac{N}{N-1})^n$, we can get that

$$\sum_{i=0}^{\infty} \frac{1}{\beta_i + 1} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{\sqrt{\beta_i + 1}} < \infty$$

and from this we deduce that

$$|\gamma(v_+)|_\infty = \lim_{n\to\infty} |\gamma(v_+)|_{2^*(\beta_n+1)} < \infty.$$

Thus, $\gamma(v_+) \in L^\infty(\mathbb{R}^N)$. Clearly, the same is true for $\gamma(v_-)$ and hence for $\gamma(v)$. We can use the fact that $|\gamma(v_+)|_p \leq C_3 < +\infty$ for all p in (2.5) (with $\lambda = 1$) to deduce that, for all $\beta > 0$,

$$\|v_+^{\beta+1}\|^2 \leq C_\beta (C_1 + |g|_{2N}) C_3^{2(\beta+1)} + C_\beta |g|_{2N} C_3^{2(\beta+1)}.$$

Since by Sobolev's embedding we have

$$\|v_+\|_{2^\star(\beta+1)}^{2(\beta+1)} = \|v_+^{\beta+1}\|_{2^\star} \leq c_{2^\star} \|v_+^{\beta+1}\|,$$

where $2^\star := \frac{2(N+1)}{N-1}$, we deduce from the above inequality that

$$\|v_+\|_{2^\star(\beta+1)}^{\beta+1} = \|v_+^{\beta+1}\|_{2^\star} \leq \widetilde{C} C_\beta C_3^{2(\beta+1)}$$

for a positive constant $\widetilde{C}$. Since the right-hand side of the last inequality is uniformly bounded for all $\beta > 0$, we can get that $v_+ \in L^\infty(\mathbb{R}_+^{N+1})$ as before. Similarly, we can see $v_- \in L^\infty(\mathbb{R}_+^{N+1})$ and hence the same property holds for v. □

The following property is established in [15, Proposition 2.9].

Lemma 2.3. *Suppose that $v \in H^1(\mathbb{R}_+^{N+1}) \cap L^\infty(\mathbb{R}_+^{N+1})$ is a weak solution of*

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\dfrac{\partial v}{\partial y} = g(x) & \text{on } \mathbb{R}^N, \end{cases} \tag{2.6}$$

where $g \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty]$.

Then $v \in C^{0,\alpha}(\mathbb{R}^N \times [0, +\infty)) \cap W^{1,q}(\mathbb{R}^N \times (0, R))$ for any $q \in [2, +\infty)$ and $R > 0$. If, in addition, $g \in C^\alpha(\mathbb{R}^N)$ then $v \in C^{1,\alpha}(\mathbb{R}^N \times [0, +\infty)) \cap C^2(\mathbb{R}_+^{N+1})$ is a classical solution of (2.6).

Proposition 2.4. *Assume that $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, then the weak solution $v \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N \times [0, +\infty)) \cap \mathcal{C}^2(\mathbb{R}^{N+1}_+)$ is a classical solution and satisfies*

$$\frac{N-1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v(x,y)|^2 dxdy + \frac{(N+1)m^2}{2} \int_{\mathbb{R}^{N+1}_+} |v(x,y)|^2 dxdy + \frac{Na}{2} \int_{\mathbb{R}^N} |\gamma(v)|^2 dx = \frac{2N-\mu}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(v)))F(\gamma(v))dx.$$

Proof. We know from Proposition 2.2 that $\gamma(v) \in L^p(\mathbb{R}^N)$ for any $p \in [2, +\infty]$, consequently

$$h := -\gamma(v) + (|x|^{-\mu} * F(\gamma(v)))f(\gamma(v)) \in L^p(\mathbb{R}^N)$$

for any $p \in [2, +\infty]$. From Lemma 2.3 we then deduce that $\gamma(v) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ and then that $h \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$. Again Lemma 2.3 tells us that $v \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N \times [0, +\infty)) \cap \mathcal{C}^2(\mathbb{R}^{N+1}_+)$ is a classical solution.

We denote $\mathbb{D} = \{z = (x, y) \in \mathbb{R}^{N+1}_+ : |z| \leq 1\}$. Fix $\varphi \in \mathcal{C}^1_0(\mathbb{R}^{N+1}_+)$ such that $\varphi = 1$ on $\mathbb{D}$ and $\varphi_\lambda := \varphi(\lambda x, \lambda y)$. Then, for $\lambda \in (0, \infty)$ and $z \in \mathbb{R}^{N+1}_+$, the function w_λ defined by

$$w_\lambda(x, y) = \varphi_\lambda(x, y) \cdot \nabla v(x, y)$$

can be used as a test function. It follows that

$$\int_{\mathbb{R}^{N+1}_+} \nabla v(x,y)\nabla w_\lambda(x,y)dxdy + m^2 \int_{\mathbb{R}^{N+1}_+} v(x,y)w_\lambda(x,y)dxdy + a\int_{\mathbb{R}^N} \gamma(v)\gamma(w_\lambda)dx = \int_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v))\big)f(\gamma(v))\gamma(w_\lambda)dx.$$

As in [17, Theorem 6.1], we know that

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^{N+1}_+} \nabla v(x,y)\nabla w_\lambda(x,y)dxdy = -\frac{N-1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v(x,y)|^2 dxdy,$$

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^{N+1}_+} v(x,y)w_\lambda(x,y)dxdy = -\frac{N+1}{2} \int_{\mathbb{R}^{N+1}_+} |v(x,y)|^2 dxdy,$$

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} \gamma(v)\gamma(w_\lambda)dx = -\frac{N}{2} \int_{\mathbb{R}^N} |\gamma(v)|^2 dx$$

and

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v))\big)f(\gamma(v))\gamma(w_\lambda)dx = -\frac{2N-\mu}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(v)))F(\gamma(v))dx.$$

The conclusion follows by combining the above equalities. □

2.2. Existence of ground states

It is convenient to show that the functional J_a satisfies the mountain-pass geometry.

Lemma 2.5. *Assume that $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies (f_1), (f_2) and (f_3), then*

(i). There exist $\rho > 0$ and $\beta > 0$ such that $J_a|_{\mathcal{B}} \geq \beta$ for all $v \in \mathcal{B} = \{v \in H^1(\mathbb{R}^{N+1}_+) : \|v\| = \rho\}$.
(ii). There exists $v \in H^1(\mathbb{R}^{N+1}_+)$ such that $\|v\| > \rho$ and $J_a(v) < 0$, where ρ is given in (i).

Proof. (i) By the growth assumption (f_1), Proposition 1.1 and (1.10), for every $v \in H^1(\mathbb{R}^{N+1}_+)$, there holds

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \Big(|x|^{-\mu} * F(\gamma(v))\Big) F(\gamma(v)) dx &\leq C_1 \Big(\int\limits_{\mathbb{R}^N} |F(\gamma(v))|^{\frac{2N}{2N-\mu}} dx \Big)^{\frac{2N-\mu}{N}} \\ &\leq C_2 \Big(\int\limits_{\mathbb{R}^N} (|\gamma(v)|^{\frac{4N}{2N-\mu}} + |\gamma(v)|^{\frac{2N}{N-1}}) dx \Big)^{\frac{2N-\mu}{N}} \\ &\leq C_3 \Big(\|v\|^4 + \|v\|^{\frac{2(2N-\mu)}{N-1}} \Big),\end{aligned}$$

and so

$$J_a(u) \geq \frac{1}{2} \int\limits_{\mathbb{R}^{N+1}_+} \Big(|\nabla v(x,y)|^2 + m^2 v(x,y)^2\Big) dxdy + \frac{a}{2} \int\limits_{\mathbb{R}^N} \gamma(v)^2 dx - \frac{C_3}{2} \Big(\|v\|^4 + \|v\|^{\frac{2(2N-\mu)}{N-1}} \Big).$$

Note that $\frac{2(2N-\mu)}{N-1} > 4 > 2$, hence we can choose ρ sufficiently small and $\beta > 0$, such that $J_a|_{\mathcal{B}} \geq \beta$ for all $u \in \mathcal{B} = \{u \in H^1(\mathbb{R}^{N+1}_+) : \|u\| = \rho\}$.

(ii) From assumption (f_3), we can choose $t_0 \in \mathbb{R}$ such that $F(t_0) \neq 0$. Let $w = t_0 \chi_{B_1}$, we get

$$\int\limits_{\mathbb{R}^N} \Big(|x|^{-\mu} * F(\gamma(w))\Big) F(\gamma(w)) dx = F(t_0)^2 \int\limits_{B_1} \int\limits_{B_1} |x - z|^{-\mu} dxdz > 0,$$

where B_1 is the open ball centered at the origin with radius 1 in $\mathbb{R}^N$. Since $H^{\frac{1}{2}}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-1}}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \Big(|x|^{-\mu} * F(\gamma(v))\Big) F(\gamma(v)) dx$ is continuous in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-1}}(\mathbb{R}^N)$, we know there exists $v \in H^1(\mathbb{R}^{N+1}_+)$ such that

$$\int\limits_{\mathbb{R}^N} \Big(|x|^{-\mu} * F(\gamma(v))\Big) F(\gamma(v)) dx > 0.$$

For $\tau > 0$, $x \in \mathbb{R}^N$ and $y \in \mathbb{R}_+$, let $v_\tau(x,y) = v(\frac{x}{\tau}, \frac{y}{\tau})$, we find that, for $\tau > 0$,

$$\begin{aligned} J_a(v_\tau) = \frac{\tau^{N-1}}{2} \int\limits_{\mathbb{R}^{N+1}_+} |\nabla v(x,y)|^2 dxdy + \frac{m^2 \tau^{N+1}}{2} \int\limits_{\mathbb{R}^{N+1}_+} |v(x,y)|^2 dxdy \\ + \frac{\tau^N}{2} \int\limits_{\mathbb{R}^N} \gamma(v)^2 dx - \frac{\tau^{2N-\mu}}{2} \int\limits_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v))\big) F(\gamma(v)) dx. \end{aligned} \tag{2.7}$$

Since $0 < \mu < 2$, we can obtain, for $\tau > 0$ large enough, $J_a(v_\tau) < 0$. By the proof of (i), we also know $\|v_\tau\| > \rho$. So, the assertion follows by taking $v = v_\tau$, with τ sufficiently large. □

In order to prove the existence of weak solutions, we will apply the mountain pass theorem [39]. Define the mountain pass level

$$E_a^\star := \inf_{\iota \in \Lambda} \sup_{t \in [0,1]} J_a(\iota(t)), \tag{2.8}$$

with the set of admissible paths defined by

$$\Lambda = \Big\{ \iota \in \mathcal{C}([0,1]; H^1(\mathbb{R}^{N+1}_+)) : \iota(0) = 0, J_a(\iota(1)) < 0 \Big\}.$$

It is convenient to define Pohožaev functional $P : H^1(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ for $v \in H^1(\mathbb{R}^{N+1}_+)$ by

$$\begin{aligned} P(v) = \frac{N-1}{2} \int\limits_{\mathbb{R}^{N+1}_+} |\nabla v(x,y)|^2 dxdy + \frac{m^2(N+1)}{2} \int\limits_{\mathbb{R}^{N+1}_+} |v(x,y)|^2 dxdy \\ + \frac{aN}{2} \int\limits_{\mathbb{R}^N} \gamma(v)^2 dx - \frac{2N-\mu}{2} \int\limits_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v))\big) F(\gamma(v)) dx. \end{aligned}$$

In order to construct a Pohožaev-Palais-Smale sequence, for $\sigma \in \mathbb{R}$, $v \in H^1(\mathbb{R}^{N+1}_+)$, $x \in \mathbb{R}^N$ and $y \in \mathbb{R}_+$, we define the map $\Phi : \mathbb{R} \times H^1(\mathbb{R}^{N+1}_+) \to H^1(\mathbb{R}^{N+1}_+)$ by

$$\Phi(\sigma, v)(x,y) = v(e^{-\sigma} x, e^{-\sigma} y).$$

Then the functional $J_a \circ \Phi$ is computed as

$$\begin{aligned} J_a(\Phi(\sigma, v)) = \frac{e^{(N-1)\sigma}}{2} \int\limits_{\mathbb{R}^{N+1}_+} |\nabla v(x,y)|^2 dxdy + \frac{m^2 e^{(N+1)\sigma}}{2} \int\limits_{\mathbb{R}^{N+1}_+} |v(x,y)|^2 dxdy \\ + \frac{a e^{N\sigma}}{2} \int\limits_{\mathbb{R}^N} \gamma(v)^2 dx - \frac{e^{(2N-\mu)\sigma}}{2} \int\limits_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v))\big) F(\gamma(v)) dx. \end{aligned}$$

Define the family of paths

$$\tilde{\Lambda} = \{\iota \in \mathcal{C}([0,1]; \mathbb{R} \times H^1(\mathbb{R}^{N+1}_+)) : \iota(0) = (0,0) \text{ and } J_a \circ \Phi(\iota(1)) < 0\}$$

and notice that $\Lambda = \{\Phi \circ \iota : \iota \in \tilde{\Lambda}\}$. It follows that the mountain pass levels of J_a and $J_a \circ \Phi$ coincide, that is,

$$E_a^\star = \inf_{\iota \in \tilde{\Lambda}} \sup_{t \in [0,1]} (J_a \circ \Phi)(\iota(t)).$$

From Lemma 2.5 and the min-max characterization of the value $E_a^\star$, we obtain $0 < E_a^\star < \infty$. Using condition (f_1), we know that $J_a \circ \Phi$ is continuous and Fréchet-differentiable on $\mathbb{R} \times H^1(\mathbb{R}^{N+1}_+)$.

Applying [39, Theorem 2.9] and Lemma 2.5, there exists a sequence $\{(\sigma_n, w_n)\}$ in $\mathbb{R} \times H^1(\mathbb{R}^{N+1}_+)$ such that as $n \to \infty$

$$(J_a \circ \Phi)(\sigma_n, w_n) \to E_a^\star \quad \text{and} \quad (J_a \circ \Phi)'(\sigma_n, w_n) \to 0 \quad \text{in } (\mathbb{R} \times H^1(\mathbb{R}^{N+1}_+))^*.$$

Since for every $(h, w) \in \mathbb{R} \times H^1(\mathbb{R}^{N+1}_+)$,

$$(J_a \circ \Phi)'(\sigma_n, w_n)[h, w] = J_a'(\Phi(\sigma_n, w_n))[\Phi(\sigma_n, w)] + P(\Phi(\sigma_n, w_n))h.$$

We take $v_n = \Phi(\sigma_n, w_n)$, then as $n \to \infty$,

$$J_a(v_n) \to E_a^\star > 0, \quad J_a'(v_n) \to 0, \quad P(v_n) \to 0. \tag{2.9}$$

Now we are ready to obtain a nontrivial solution from this special sequence by applying a version of Lions' concentration-compactness principle for the fractional Laplacian, see [17].

Lemma 2.6. *Let $\{u_n\}$ be a bounded sequence in $H^{\frac{1}{2}}(\mathbb{R}^N)$. Assuming that for some $\sigma > 0$ and $2 \le q < 2^*$ we have*

$$\sup_{x \in \mathbb{R}^N} \int_{B_\sigma(x)} |u_n|^q dx \to 0, \ \textit{as} \ n \to \infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

Lemma 2.7. *Suppose that $(f_1) - (f_3)$, then equation (1.9) has at least one nontrivial solution.*

Proof. By direct computation, we obtain, for any $n \in \mathbb{N}$,

$$J_a(v_n) - \frac{1}{2N-\mu} P(v_n) = \frac{N-\mu+1}{2(2N-\mu)} \int_{\mathbb{R}^{N+1}_+} (|\nabla v_n(x,y)|^2 + m^2 |v_n(x,y)|^2) dx dy$$

$$+ \frac{a(N-\mu)}{2(2N-\mu)} \int_{\mathbb{R}^N} \gamma(v_n)^2 dx.$$

By (2.9), it is easy to see that the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$. Moreover, we claim that there exist $\sigma, \delta > 0$ and a sequence $\{x_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_\sigma(x_n)} \gamma(v_n)^2 dx \ge \delta.$$

If the above claim does not hold for $\{v_n\}$, by Lemma 2.6, we must have that

$$\gamma(v_n) \to 0 \text{ in } L^r(\mathbb{R}^N) \text{ for } 2 < r < 2^*. \tag{2.10}$$

Fix $2 < q < 2^*$, from assumption (f_2), for any $\xi > 0$ there is $C_\xi > 0$ such that

$$|F(s)|^{\frac{2N}{2N-\mu}} \le \xi(|s|^{\frac{4N}{2N-\mu}} + |s|^{\frac{2N}{N-1}}) + C_\xi |s|^q \quad \forall s \ge 0.$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$ and hence, by the Sobolev embedding, in $L^{\frac{2N}{N-1}}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |F(\gamma(v_n))|^{\frac{2N}{2N-\mu}} dx \le C_1\xi + C_2 \int_{\mathbb{R}^N} |\gamma(v_n)|^q dx.$$

Since $\xi > 0$ is arbitrary and (2.10), we have

$$\int_{\mathbb{R}^N} |F(\gamma(v_n))|^{\frac{2N}{2N-\mu}} dx \to 0,$$

as $n \to \infty$. It follows from the Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v_n))\big) f(\gamma(v_n))\gamma(v_n) dx \\
&\quad \le C_1 \Big(\int_{\mathbb{R}^N} |F(\gamma(v_n))|^{\frac{2N}{2N-\mu}} dx \Big)^{\frac{2N-\mu}{2N}} \Big(\int_{\mathbb{R}^N} |f(\gamma(v_n))\gamma(v_n)|^{\frac{2N}{2N-\mu}} dx \Big)^{\frac{2N-\mu}{2N}} \to 0,
\end{aligned}$$

which leads to $\|v_n\| \to 0$. This contradicts with (2.9) and so the claim is proved. And so, up to translation, we may assume that

$$\liminf_{n\to\infty} \int_{B_\sigma(0)} \gamma(v_n)^2 dx \ge \delta.$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$, there exists $v_0 \in H^1(\mathbb{R}^{N+1}_+)$, $v_0 \neq 0$, such that, up to a subsequence, v_n converges to v_0 weakly in $H^1(\mathbb{R}^{N+1}_+)$ and v_n converges to v_0 a.e. in $\mathbb{R}^{N+1}_+$. From (f_1), we know the sequence $\{F(\gamma(v_n))\}$ is bounded in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. Since F is continuous, we have $F(\gamma(v_n))$ converges to $F(\gamma(v_0))$ a.e. in $\mathbb{R}^N$. So,

$$F(\gamma(v_n)) \rightharpoonup F(\gamma(v_0)) \ \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

By the Hardy-Littlewood-Sobolev inequality, $|x|^{-\mu}$ defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$. Thus we know that

$$|x|^{-\mu} * (F(\gamma(v_n)) \rightharpoonup |x|^{-\mu} * (F(\gamma(v_0))) \ \text{ in } L^{\frac{2N}{\mu}}(\mathbb{R}^N).$$

On the other hand, applying condition (f_1), we can get, for every $p \in [1, \frac{2N}{N+1-\mu})$,

$$f(\gamma(v_n)) \to f(\gamma(v_0)) \ \text{ in } L^p_{loc}(\mathbb{R}^N).$$

We conclude that, for every $p \in [1, \frac{2N}{N+1})$

$$(|x|^{-\mu} * F(\gamma(v_n)))f(\gamma(v_n)) \rightharpoonup (|x|^{-\mu} * F(\gamma(v_0)))f(\gamma(v_0)) \ \text{ in } L^p(\mathbb{R}^N).$$

In particular, for every $\varphi \in H^1(\mathbb{R}^{N+1}_+)$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle J_a'(v_n), \varphi \rangle \\ &= \lim_{n\to\infty} \Big(\int\limits_{\mathbb{R}^{N+1}_+} (\nabla v_n \nabla \varphi + m^2 v_n \varphi) dxdy + a \int\limits_{\mathbb{R}^N} \gamma(v_n)\gamma(\varphi) dx \\ &\quad - \int\limits_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(v_n))) f(\gamma(v_n))\gamma(\varphi) dx \Big) \\ &= \int\limits_{\mathbb{R}^{N+1}_+} (\nabla v_0 \nabla \varphi + m^2 v_0 \varphi) dxdy + a \int\limits_{\mathbb{R}^N} \gamma(v_0)\gamma(\varphi) dx \\ &\quad - \int\limits_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(v_0))) f(\gamma(v_0))\gamma(\varphi) dx \\ &= \langle J_a'(v_0), \varphi \rangle . \end{aligned}$$

We conclude that v_0 is a weak solution of equation (1.9). □

Proof of Theorem 1.4. The weak lower-semicontinuity of the norm and the Pohožaev identity in Proposition 2.4 imply that

$$\begin{aligned}
J_a(v_0) &= J_a(v_0) - \frac{P(v_0)}{2N-\mu} \\
&= \frac{N-\mu+1}{2(2N-\mu)} \int\limits_{\mathbb{R}^{N+1}_+} (|\nabla v_0(x,y)|^2 + m^2|v_0(x,y)|^2)dxdy + \frac{a(N-\mu)}{2(2N-\mu)} \int\limits_{\mathbb{R}^N} \gamma(v_0)^2 dx \\
&\leq \liminf_{n\to\infty} \Big(\frac{N-\mu+1}{2(2N-\mu)} \int\limits_{\mathbb{R}^{N+1}_+} |\nabla v_n(x,y)|^2 + m^2|v_n(x,y)|^2 dxdy \\
&\quad + \frac{a(N-\mu)}{2(2N-\mu)} \int\limits_{\mathbb{R}^N} \gamma(v_n)^2 dx \Big) \\
&= \liminf_{n\to\infty} \Big[J_a(v_n) - \frac{P(v_n)}{2N-\mu} \Big] = \liminf_{n\to\infty} J_a(v_n) = E_a^\star.
\end{aligned} \tag{2.11}$$

Since v_0 is a critical point for functional J_a and by the definition of the ground state energy level E_a, we have $J_a(v_0) \geq E_a$. So, we can get $E_a \leq E_a^\star$.

We define the path $\iota : [0, \infty) \to H^1(\mathbb{R}^{N+1}_+)$ by

$$\iota(\tau)(x,y) = \begin{cases} v_0(\frac{x}{\tau}, \frac{y}{\tau}) & \text{if } \tau > 0, \\ 0 & \text{if } \tau = 0. \end{cases}$$

Since the function ι is continuous on $(0, \infty)$, we have, for every $\tau > 0$,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{N+1}_+} |\nabla \iota(\tau)(x,y)|^2 dxdy + m^2 \int\limits_{\mathbb{R}^{N+1}_+} |\iota(\tau)(x,y)|^2 dxdy + a \int\limits_{\mathbb{R}^N} |\gamma(\iota(\tau))|^2 dx \\
&= \tau^{N-1} \int\limits_{\mathbb{R}^{N+1}_+} |\nabla v_0(x,y)|^2 dxdy + m^2 \tau^{N+1} \int\limits_{\mathbb{R}^{N+1}_+} |v_0(x,y)|^2 dxdy + a\tau^N \int\limits_{\mathbb{R}^N} |\gamma(v_0)|^2 dx,
\end{aligned}$$

which implies that ι is continuous at 0.

By the Pohožaev identity in Proposition 2.4, we have

$$J_a(\iota(\tau)) = \frac{\tau^{N-1}}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla v_0(x,y)|^2 dxdy + \frac{m^2\tau^{N+1}}{2}\int_{\mathbb{R}^{N+1}_+} |v_0(x,y)|^2 dxdy$$
$$+ \frac{a\tau^N}{2}\int_{\mathbb{R}^N} |\gamma(v_0)|^2 dx - \frac{\tau^{2N-\mu}}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(v_0)))F(\gamma(v_0))dxdy$$
$$= \Big(\frac{\tau^{N-1}}{2} - \frac{(N-1)\tau^{2N-\mu}}{2(2N-\mu)}\Big)\int_{\mathbb{R}^{N+1}_+} |\nabla v_0(x,y)|^2 dxdy + m^2\Big(\frac{\tau^{N+1}}{2} - \frac{(N+1)\tau^{2N-\mu}}{2(2N-\mu)}\Big)$$
$$\times \int_{\mathbb{R}^{N+1}_+} |v_0(x,y)|^2 dxdy + a\Big(\frac{\tau^N}{2} - \frac{N\tau^{2N-\mu}}{2(2N-\mu)}\Big)\int_{\mathbb{R}^N} |\gamma(v_0)|^2 dx.$$

It is easy to see that $J_a(\iota(\tau))$ achieves the strict global maximum at 1, that is, for every $\tau \in [0,1)\cup(1,\infty)$, $J_a(\iota(\tau)) < J_a(v_0)$. Then after a suitable change of variable, for every $t_0 \in (0,1)$, there exists a path $\bar{\iota} \in \mathcal{C}([0,1]; H^1(\mathbb{R}^{N+1}_+))$ such that $\bar{\iota} \in \Lambda$, $\bar{\iota}(t_0) = v_0$ and

$$J_a(\bar{\iota}(t)) < J_a(v_0), \qquad \forall t \in [0,t_0)\cup(t_0,1].$$

Let $w_0 \in H^1(\mathbb{R}^{N+1}_+)\backslash\{0\}$ be another solution of (1.9) such that $J_a(w_0) \le J_a(v_0)$. If we lift w_0 to a path and recall the definition of $E_a^\star$, we conclude that $J_a(v_0) \le E_a^\star \le J_a(w_0)$. Thus, we have proved that $J_a(v_0) = J_a(w_0) = E_a = E_a^\star$, and this concludes the proof of Theorem 1.4. □

Remark 2.8. Denote by

$$S_a = \{v \in H^1(\mathbb{R}^{N+1}_+) : J_a(v) = E_a \text{ and } v \text{ is a ground state of (1.9)}\}$$

the set of groundstates of (1.9). Then S_a is compact in $H^1(\mathbb{R}^{N+1}_+)$.

In fact, for every $v \in S_a$, we have

$$J_a(v) = \frac{N-\mu+1}{2(2N-\mu)}\int_{\mathbb{R}^{N+1}_+} (|\nabla v(x,y)|^2 + m^2|v(x,y)|^2)dxdy + \frac{N-\mu}{2(2N-\mu)}\int_{\mathbb{R}^N} \gamma(v)^2 dx.$$

Thus, for every $\{v_n\} \subset S_a$, up to a subsequence and translations, we can assume that $v_n \rightharpoonup v$. Since equality holds in (2.11),

$$J_a(v) = \frac{N-\mu+1}{2(2N-\mu)}\int_{\mathbb{R}^{N+1}_+} (|\nabla v(x,y)|^2 + m^2|v(x,y)|^2)dxdy + \frac{N-\mu}{2(2N-\mu)}\int_{\mathbb{R}^N} \gamma(v)^2 dx$$
$$= \liminf_{n\to\infty}\Big(\frac{N-\mu+1}{2(2N-\mu)}\int_{\mathbb{R}^{N+1}_+} (|\nabla v_n(x,y)|^2 + m^2|v_n(x,y)|^2)dxdy$$

$$+\frac{N-\mu}{2(2N-\mu)}\int_{\mathbb{R}^N}\gamma(v_n)^2dx\Big),$$

and hence $\{v_n\}$ converges strongly to v in $H^1(\mathbb{R}^{N+1}_+)$.

2.3. Exponential decay of the ground states

Now, we took some ideas from [15] to prove the decay of the ground state solutions to (1.9).

Lemma 2.9. *Let $v \in S_a$ be the ground state solution obtained in Theorem 1.4. If $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfies (f_1) and $f(t)=0$ for $t \le 0$, then for any $\sigma \in (-a,m)\cap[0,+\infty)$, there exists $C>0$ such that*

$$0 < v(x,y) \le Ce^{-(m-\sigma)\sqrt{|x|^2+y^2}}e^{-\sigma y}$$

for any $(x,y)\in\mathbb{R}^N\times[0,+\infty)$.

Proof. Since $f(t)=0$ for $t\le 0$, we see that $v \ge 0$. By Proposition 2.4, we know $v\in\mathcal{C}^{1,\alpha}(\mathbb{R}^N\times[0,+\infty))\cap\mathcal{C}^2(\mathbb{R}^{N+1}_+)$. The strict positivity of v follows immediately from the maximum principle: since $v\ge 0$, if $v(\overline{x},\overline{y})=0$, then $\overline{y}=0$. From the equation we deduce that $\frac{\partial v}{\partial \nu}(\overline{x},0)=0$ and we reach a contradiction applying the Hopf lemma.

Following [15], for any $R>0$ we denote

$$B^+_R=\{(x,y)\in\mathbb{R}^{N+1}_+ : \sqrt{|x|^2+y^2}<R\},$$
$$\Omega^+_R=\{(x,y)\in\mathbb{R}^{N+1}_+ : \sqrt{|x|^2+y^2}>R\},$$
$$\Gamma_R=\{(x,0)\in\partial\mathbb{R}^{N+1}_+ : |x|\ge R\}$$

and define

$$f_R(x,y)=C_Re^{-\sigma y}e^{-(m-\sigma)\sqrt{|x|^2+y^2}}\quad\text{for } (x,y)\in\overline{\Omega^+_R},$$

where the positive constants C_R and $\sigma\in(-a,m)\cap[0,+\infty)$ will be chosen later on. A simple computation shows that

$$\Delta f_R=\Big(\sigma^2+(m-\sigma)^2+\frac{2\sigma(m-\sigma)y}{\sqrt{|x|^2+y^2}}-\frac{N(m-\sigma)}{\sqrt{|x|^2+y^2}}\Big)f_R.$$

Thus, for R large enough, we have

$$\begin{cases}-\Delta f_R+m^2f_R\ge 0 & \text{in } \Omega^+_R,\\ -\dfrac{\partial f_R}{\partial y}=\dfrac{\partial f_R}{\partial \eta}=\sigma f_R & \text{on } \Gamma^+_R.\end{cases}$$

We now define

$$w(x, y) = f_R(x, y) - v(x, y)$$

for any $(x, y) \in \overline{\Omega}_R^+$. We clearly have $-\Delta w + m^2 w \geq 0$ in Ω_R^+. Choosing

$$C_R = e^{mR} \max_{\partial B_R^+} v,$$

we also have $w(x, y) \geq 0$ on ∂B_R^+ and $w(x, y) \to 0$ when $|x| + y \to \infty$.

We claim that $w(x, y) \geq 0$ in $\overline{\Omega}_R^+$. Supposing the contrary, let us assume that $\inf_{\overline{\Omega}_R^+} w(x, y) < 0$. By the strong maximum principle, there exist $(x_0, 0) \in \Gamma_R^+$ such that $w(x_0, 0) = \inf_{\overline{\Omega}_R^+} w(x, y) < w(x, y)$ for all $(x, y) \in \Omega_R^+$. Defining

$$\rho(x, y) = w(x, y)e^{\lambda y}$$

for some $\lambda \in (-a, m) \cap [0, +\infty)$, a straightforward calculation shows that

$$-\Delta w + m^2 w = e^{-\lambda y}(\Delta\rho + 2\lambda\partial_y\rho + (m^2 - \lambda^2)\rho).$$

Since $-\Delta w + m^2 w \geq 0$, we conclude that $\Delta\rho + 2\lambda\partial_y\rho + (m^2 - \lambda^2)\rho \geq 0$. Another application of the strong maximum principle yields

$$\rho(x_0, 0) = \inf_{\Gamma_R^+} \rho = \inf_{\Gamma_R^+} w = w(x_0, 0) < 0.$$

By Hopf's lemma we have $\frac{\partial\rho}{\partial\eta}(x_0, 0) < 0$, hence

$$-\frac{\partial\rho}{\partial y}(x_0, 0) < 0. \tag{2.12}$$

Since $\frac{\partial\rho}{\partial y} = \frac{\partial w}{\partial y}e^{\lambda y} + \lambda w e^{\lambda y}$, we conclude that

$$\frac{\partial\rho}{\partial y}(x_0, 0) = \frac{\partial w}{\partial y}(x_0, 0) + \lambda w(x_0, 0)$$

and so

$$\begin{aligned}
-\frac{\partial\rho}{\partial y}(x_0, 0) &= -\frac{\partial f_R}{\partial y}(x_0, 0) + \frac{\partial v}{\partial y}(x_0, 0) - \lambda f_R(x_0, 0) + \lambda v(x_0, 0) \\
&= (\sigma - \lambda) f_R(x_0, 0) + (a + \lambda)v(x_0, 0) - \big(|x|^{-\mu} * F(v(x_0, 0))\big) f(v(x_0, 0)).
\end{aligned}$$

By Proposition 2.2, we know that $|x|^{-\mu} * F(v(x, 0)) \in L^\infty(\mathbb{R}^N)$. Combining this with hypothesis (f_1), we have

$$\big(|x|^{-\mu} * F(v(x_0, 0))\big) f(v(x_0, 0)) \to 0 \tag{2.13}$$

as $|x_0| \to \infty$.

Now, choosing $\sigma = \lambda$, since $\lambda \in (-a, m) \cap [0, +\infty)$ (so that, the last term in the above inequality is non-negative), the positiveness of $(a + \lambda)v(x_0, 0)$ and (2.13) guarantee that

$$-\frac{\partial \rho}{\partial y}(x_0, 0) \geq 0,$$

thus reaching a contradiction with (2.12). It follows that

$$0 < v(x, y) \leq f_R(x, y) = C_R e^{-\sigma y} e^{-(m-\sigma)\sqrt{|x|^2+y^2}} \text{ for } (x, y) \in \overline{\Omega}_R^+.$$

The proof is now complete. □

3. Penalization arguments: single peak solutions

To study problem (1.1), it suffices to investigate equation (1.2).

For any set $B \subset \mathbb{R}^N$ and $\varepsilon > 0$, we define $B_\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in B\}$ and $B^\delta \equiv \{x \in \mathbb{R}^N : \operatorname{dist}(x, B) \leq \delta\}$.

For $v \in H^1(\mathbb{R}^{N+1}_+)$, let

$$P_\varepsilon(v) = \frac{1}{2} \int\limits_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy + \frac{1}{2} \int\limits_{\mathbb{R}^N} V_\varepsilon \gamma(v)^2 dx$$
$$- \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(|x|^{-\mu} * F(\gamma(v))\right) F(\gamma(v)) dx.$$

Fixing an arbitrary $\alpha > 0$ and $v \in H^1(\mathbb{R}^{N+1}_+)$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in O_\varepsilon, \\ \varepsilon^{-\alpha}, & \text{if } x \in \mathbb{R}^N \setminus O_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(v) = \left(\int\limits_{\mathbb{R}^N} \chi_\varepsilon \gamma(v)^2 dx - 1 \right)_+^2.$$

Let $\Gamma_\varepsilon : H^1(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ be given by

$$\Gamma_\varepsilon(v) = P_\varepsilon(v) + Q_\varepsilon(v).$$

It is standard to check that $\Gamma_\varepsilon \in \mathcal{C}^1(H^1(\mathbb{R}^{N+1}_+))$. To look for solutions of problem (1.2) that concentrate in O as $\varepsilon \to 0$, we shall search for critical points of Γ_ε such that Q_ε is zero. The functional Q_ε was firstly introduced in [7] and it will play the role of a penalization to force the concentration phenomena to occur inside O.

Now, we will construct a set of approximate solutions of (1.2). Let

$$\delta = \frac{1}{10}\,\mathrm{dist}\,(\mathcal{M}, \mathbb{R}^N \backslash O).$$

We fix $\beta \in (0, \delta)$ and a cut-off $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+^{N+1})$ such that $0 \le \varphi \le 1, \varphi(x) = 1$ for $|x| + y \le \beta$ and $\varphi(x) = 0$ for $|x| + y \ge 2\beta$. Let $\varphi_\varepsilon(x, y) = \varphi(\varepsilon x, \varepsilon y), x \in \mathbb{R}^N$, $y \in \mathbb{R}_+$ and for some $x_0 \in \mathcal{M}^\beta$ and $U \in S_{V_0}$, we define

$$U_\varepsilon^{x_0}(x, y) = \varphi_\varepsilon \left(x - \frac{x_0}{\varepsilon}, y\right) U \left(x - \frac{x_0}{\varepsilon}, y\right),$$

where S_{V_0} is the set of least energy solutions of (1.9) with $a = V_0$. As in [5], we will find a solution near the set

$$X_\varepsilon = \{U_\varepsilon^{x_0} \mid x_0 \in \mathcal{M}^\beta, U \in S_{V_0}\}$$

for sufficiently small $\varepsilon > 0$. Choosing some fixed $U \in S_{V_0}$ and $x_0 \in \mathcal{M}$, we define

$$W_{\varepsilon,t}(x, y) \equiv (\varphi_\varepsilon U_t)(x - \frac{x_0}{\varepsilon}, y), \quad \text{where } U_t(x, y) = U(\frac{x}{t}, y).$$

Lemma 3.1. *There exist $T > 0$, such that for $\varepsilon > 0$ small enough, $\Gamma_\varepsilon(W_{\varepsilon,T}) < -2$.*

Proof. By a direct calculation, we get $\Gamma_\varepsilon(W_{\varepsilon,t}) = P_\varepsilon(W_{\varepsilon,t})$ for any $t > 0$,

$$\begin{aligned} J_{V_0}(U_t) = \frac{t^{N-2}}{2} \int\limits_{\mathbb{R}_+^{N+1}} |\nabla U|^2 dxdy + \frac{m^2 t^N}{2} \int\limits_{\mathbb{R}_+^{N+1}} |U|^2 dxdy + \frac{t^N}{2} V_0 \int\limits_{\mathbb{R}^N} |\gamma(U)|^2 dx \\ - \frac{t^{2N-\mu}}{2} \int\limits_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(U)))F(\gamma(U))dx. \end{aligned}$$

Then there exists $T > 1$ such that $J_{V_0}(U_T) < -2$ for $t > T$. Noticing that

$$P_\varepsilon(W_{\varepsilon,t}) = J_{V_0}(W_{\varepsilon,t}) + \frac{1}{2} \int\limits_{\mathbb{R}^N} (V_\varepsilon(x) - V_0)\gamma(W_{\varepsilon,t})^2 dx,$$

we have

$$P_\varepsilon(W_{\varepsilon,t}) = J_{V_0}(U_t) + O(\varepsilon),$$

by the decay property of U in Lemma 2.9. We conclude that $\Gamma_\varepsilon(W_{\varepsilon,T}) < -2$ for $\varepsilon > 0$ small. □

Let $\iota_\varepsilon(t)(x,y) = W_{\varepsilon,t}(x,y)$ for $t>0$. Due to $\lim_{t\to 0} W_{\varepsilon,t} = 0$, let $\iota_\varepsilon(0)=0$. We define a min-max value C_ε by

$$C_\varepsilon = \inf_{\iota\in\Phi_\varepsilon}\max_{s\in[0,T]} \Gamma_\varepsilon(\iota(s)),$$

where $\Phi_\varepsilon = \{\iota \in \mathcal{C}([0,T], H^1(\mathbb{R}^{N+1}_+)) : \iota(0)=0, \iota(T)=\iota_\varepsilon(T)\}$. Similarly to Propositions 2 and 3 in [5], we have the following property.

Proposition 3.2. *We have*

$$\lim_{\varepsilon\to 0} C_\varepsilon = E_{V_0},$$

where E_{V_0} is the least energy of (1.9) with $a = V_0$.

Now define

$$\Gamma_\varepsilon^\alpha := \{v \in H^1(\mathbb{R}^{N+1}_+) : \Gamma_\varepsilon(v) \le \alpha\}$$

and for a set $A \subset H^1(\mathbb{R}^{N+1}_+)$ and $\alpha>0$, let

$$A^\alpha := \{v\in H^1(\mathbb{R}^{N+1}_+) : \inf_{w\in A}\|v-w\| \le \alpha\}.$$

In the following, we will construct a special PS-sequence of Γ_ε, which is localized in some neighborhood X_ε^d of X_ε.

Proposition 3.3. *Let $\{\varepsilon_j\}$ with $\lim_{j\to\infty}\varepsilon_j = 0$, $\{v_{\varepsilon_j}\}\subset X^d_{\varepsilon_j}$ be such that*

$$\lim_{j\to\infty}\Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \le E_{V_0} \ \ \textit{and} \ \ \lim_{j\to\infty}\Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0. \tag{3.1}$$

Then for sufficiently small $d>0$, there exist, up to a subsequence, $\{x_j\}\subset\mathbb{R}^N$, $x_0\in\mathcal{M}$, and $U\in S_{V_0}$ such that

$$\lim_{j\to\infty}\left|\varepsilon_j x_j - x_0\right| = 0,$$

and

$$\lim_{j\to\infty}\left\|v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot - x_j,\cdot)U(\cdot - x_j,\cdot)\right\| = 0.$$

Proof. Without confusion, we write ε for ε_j. Since S_{V_0} is compact, then there exist $Z\in S_{V_0}$, $x_\varepsilon \in (\mathcal{M})^\beta$, $x_0\in(\mathcal{M})^\beta$, $\lim_{\varepsilon\to0} x_\varepsilon = x_0$, such that up to a subsequence, denoted still by $\{v_\varepsilon\}$ satisfying that for sufficiently small $\varepsilon>0$,

$$\left\|v_\varepsilon - \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon},\cdot\right) Z\left(\cdot - \frac{x_\varepsilon}{\varepsilon},\cdot\right)\right\| \le 2d. \tag{3.2}$$

Set $v_{1,\varepsilon}(x,y)=\varphi_\varepsilon\left(x-\frac{x_\varepsilon}{\varepsilon},y\right)v_\varepsilon$, $v_{2,\varepsilon}(x,y)=v_\varepsilon(x,y)-v_{1,\varepsilon}(x,y)$.

Step 1. We claim that

$$\Gamma_\varepsilon(v_\varepsilon)\geq\Gamma_\varepsilon(v_{1,\varepsilon})+\Gamma_\varepsilon(v_{2,\varepsilon})+O(\varepsilon). \tag{3.3}$$

Suppose that there exist $\overline{x}_\varepsilon\in B\left(\frac{x_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon}\right)\setminus B\left(\frac{x_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon}\right)$ and $R>0$, such that

$$\varliminf_{\varepsilon\to0}\int\limits_{B(\overline{x}_\varepsilon,R)}\gamma(v_\varepsilon)^2dx>0. \tag{3.4}$$

Let $\overline{W}_\varepsilon(x,y)=v_\varepsilon(x+\overline{x}_\varepsilon,y)$, then we get

$$\varliminf_{\varepsilon\to0}\int\limits_{B(0,R)}\gamma(\overline{W}_\varepsilon)^2dx>0. \tag{3.5}$$

Since $\varepsilon\overline{x}_\varepsilon\in B(x_\varepsilon,2\beta)\setminus B(x_\varepsilon,\beta)$, by taking a subsequence, we can assume $\varepsilon\overline{x}_\varepsilon\to x^0\in\overline{B(x_0,2\beta)\setminus B(x_0,\beta)}$. From (3.2), one has $\{\overline{W}_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$. Without loss of generality, we assume that $\overline{W}_\varepsilon\rightharpoonup\overline{W}$ weakly in $H^1(\mathbb{R}^{N+1}_+)$ and $\gamma(\overline{W}_\varepsilon)\to\gamma(\overline{W})$ strongly in $L^q_{loc}(\mathbb{R}^N)$ for $q\in[2,2^*)$. Clearly, (3.5) implies that $\overline{W}\neq0$ and from (3.1) we get that $\overline{W}$ is a nontrivial solution of

$$\begin{cases}-\Delta\overline{W}+m^2\overline{W}=0 & \text{in }\mathbb{R}^{N+1}_+,\\ -\dfrac{\partial\overline{W}}{\partial y}=(|x|^{-\mu}*F(\gamma(\overline{W})))f(\gamma(\overline{W}))-V(x^0)\gamma(\overline{W}) & \text{on }\mathbb{R}^N.\end{cases} \tag{3.6}$$

Once choosing R large enough, we deduce by the weak convergence that

$$\begin{aligned}&\varliminf_{\varepsilon\to0}\Big(\int\limits_{B(\overline{x}_\varepsilon,R)\times(0,+\infty)}(|\nabla v_\varepsilon|^2+m^2|v_\varepsilon|^2)dxdy+\int\limits_{B(x_\varepsilon,R)}V_\varepsilon(x)\gamma(v_\varepsilon)^2dx\Big)\\ &=\varliminf_{\varepsilon\to0}\Big(\int\limits_{B(0,R)\times(0,+\infty)}(|\nabla\overline{W}_\varepsilon|^2+m^2|\overline{W}_\varepsilon|^2)dxdy+\int\limits_{B(0,R)}V_\varepsilon(x+\overline{x}_\varepsilon)\gamma(\overline{W}_\varepsilon)^2dx\Big)\\ &\geq\int\limits_{B(0,R)\times(0,+\infty)}(|\nabla\overline{W}|^2+m^2|\overline{W}|^2)dxdy+\int\limits_{B(0,R)}V(x^0)\gamma(\overline{W})^2dx\\ &\geq\frac12\int\limits_{\mathbb{R}^{N+1}_+}(|\nabla\overline{W}|^2+m^2|\overline{W}|^2)dxdy+\frac12\int\limits_{\mathbb{R}^N}V(x^0)\gamma(\overline{W})^2dx.\end{aligned} \tag{3.7}$$

By Theorem 1.4, E_a is a mountain pass value. One can get E_a as being strictly increasing for $a>0$. Then

$$J_{V(x^0)}(\overline{W})\geq E_{V(x^0)}\geq E_{V_0},\quad\text{since }V(x^0)\geq V_0.$$

Thus by (3.7) and the Pohozǎev identity, we get

$$\begin{aligned}
&\varliminf_{\varepsilon\to 0}\Big(\int\limits_{B(\overline{x}_\varepsilon,R)\times(0,+\infty)}(|\nabla v_\varepsilon|^2+m^2|v_\varepsilon|^2)dxdy+\int\limits_{B(x_\varepsilon,R)}V_\varepsilon(x)\gamma(v_\varepsilon)^2dx\Big)\\
&\geq\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+}(|\nabla\overline{W}|^2+m^2|\overline{W}|^2)dxdy+\frac{1}{2}\int\limits_{\mathbb{R}^N}V(x^0)\gamma(\overline{W})^2dx\\
&\geq\frac{2N-\mu}{N-\mu+2}J_{V(x^0)}(\overline{W})\geq\frac{2N-\mu}{N-\mu+2}E_{V_0}.
\end{aligned}$$

Combining this with the exponential decay at infinity of Z and the fact that $x_0\neq x^0$, we get a contradiction with (3.2) by taking $d>0$ sufficiently small. Then, it follows from [33, Lemma I.1] that, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int\limits_{B\left(\frac{x_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon}\right)\setminus B\left(\frac{x_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon}\right)}|\gamma(v_\varepsilon)|^q dx=0,\quad\text{for any } 2<q<2^*. \tag{3.8}$$

Then, by (22) in [40], we know that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N}(I_\alpha * F(\gamma(v_\varepsilon)))F(\gamma(v_\varepsilon))dx=&\int\limits_{\mathbb{R}^N}(|x|^{-\mu}*F(\gamma(v_{1,\varepsilon})))F(\gamma(v_{1,\varepsilon}))dx\\
&+\int\limits_{\mathbb{R}^N}(|x|^{-\mu}*F(\gamma(v_{2,\varepsilon})))F(\gamma(v_{2,\varepsilon}))dx+o_\varepsilon(1).
\end{aligned}\tag{3.9}$$

Thus, it is easy to see

$$\begin{aligned}
\Gamma_\varepsilon(v_\varepsilon)=&\Gamma_\varepsilon(v_{1,\varepsilon})+\Gamma_\varepsilon(v_{2,\varepsilon})\\
&+\int\limits_{B\left(\frac{x_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon}\right)\setminus B\left(\frac{x_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon}\right)\times(0,+\infty)}\varphi_\varepsilon\Big(x-\frac{x_\varepsilon}{\varepsilon},y\Big)\Big(1-\varphi_\varepsilon\Big(x-\frac{x_\varepsilon}{\varepsilon},y\Big)\Big)|\nabla v_\varepsilon|^2dxdy\\
&+\int\limits_{B\left(\frac{x_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon}\right)\setminus B\left(\frac{x_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon}\right)}V_\varepsilon(x)\gamma\Big(\varphi_\varepsilon\Big(x-\frac{x_\varepsilon}{\varepsilon},y\Big)\Big)\Big(1-\gamma\Big(\varphi_\varepsilon\Big(x-\frac{x_\varepsilon}{\varepsilon},y\Big)\Big)\Big)\gamma(v_\varepsilon)^2dx\\
&-\int\limits_{\mathbb{R}^N}\Big((|x|^{-\mu}*F(\gamma(v_\varepsilon)))F(\gamma(v_\varepsilon))-(|x|^{-\mu}*F(\gamma(v_{1,\varepsilon})))F(\gamma(v_{1,\varepsilon}))\\
&-(|x|^{-\mu}*F(\gamma(v_{2,\varepsilon})))F(\gamma(v_{2,\varepsilon}))\Big)dx+o_\varepsilon(1)
\end{aligned}$$

as $\varepsilon\to 0$. Therefore, we deduce that (3.3) holds true.

Step 2. We claim that for $d, \varepsilon > 0$ small enough,

$$\Gamma_\varepsilon(v_{2,\varepsilon}) \geq c_0 \|v_{2,\varepsilon}\|^2 \tag{3.10}$$

for some $c_0 > 0$.

By (f_2), for any $\rho > 0$ there exists $b > 0$ (depending on ρ) such that $|F(t)| \leq \rho t^2 + b|t|^{\frac{2N-\mu}{N-1}}$ for $t \in \mathbb{R}$. Then by the Hardy-Littlewood-Sobolev inequality and Sobolev's inequality,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(v_{2,\varepsilon})))F(\gamma(v_{2,\varepsilon}))dx \right| \\
&\leq C(N,\mu) \left(\int_{\mathbb{R}^N} |F(\gamma(v_{2,\varepsilon}))|^{\frac{2N}{2N-\mu}} dx \right)^{\frac{2N-\mu}{N}} \\
&\leq C_1 \left[\int_{\mathbb{R}^N} (\rho^{\frac{2N}{2N-\mu}} |\gamma(v_{2,\varepsilon})|^{\frac{4N}{2N-\mu}} + b^{\frac{2N}{2N-\mu}} |\gamma(v_{2,\varepsilon})|^{\frac{2N}{N-1}})dx \right]^{\frac{2N-\mu}{N}} \\
&\leq C_2 \left(\rho^2 \|v_{2,\varepsilon}\|^4 + b^2 \|v_{2,\varepsilon}\|^{\frac{2(2N-\mu)}{N-1}} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Gamma_\varepsilon(v_{2,\varepsilon}) \geq P_\varepsilon(v_{2,\varepsilon}) &\geq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla v_{2,\varepsilon}|^2 + m^2 v_{2,\varepsilon}^2)dxdy + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon \gamma(v_{2,\varepsilon})^2 dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \big(|x|^{-\mu} * F(\gamma(v_{2,\varepsilon}))\big) F(\gamma(v_{2,\varepsilon}))dx \\
&\geq C_3 \|v_{2,\varepsilon}\|^2 - \frac{1}{2} C_2 \left(\rho^2 \|v_{2,\varepsilon}\|^4 + b^2 \|v_{2,\varepsilon}\|^{\frac{2(2N-\mu)}{N-1}} \right) + o_\varepsilon(1).
\end{aligned} \tag{3.11}$$

Since $2 < 4 < \frac{2(2N-\mu)}{N-1}$, taking $d, \varepsilon > 0$ small enough, we can deduce from (3.2) and (3.11) that $\Gamma_\varepsilon(v_{2,\varepsilon}) \geq c_0 \|v_{2,\varepsilon}\|^2$ for some $c_0 > 0$.

Step 3. We define

$$v_{1,\varepsilon}^1(x,y) = \begin{cases} v_{1,\varepsilon}(x,y), & x \in O_\varepsilon, \\ 0, & x \notin O_\varepsilon, \end{cases} \tag{3.12}$$

and set $W_\varepsilon(x,y) = v_{1,\varepsilon}^1 \left(x + \frac{x_\varepsilon}{\varepsilon}, y\right)$. We can proceed as before and conclude that, up to a subsequence, we have as $\varepsilon \to 0$,

$$W_\varepsilon \rightharpoonup W \ \text{ weakly in } H^1(\mathbb{R}^{N+1}_+),$$

and W is a solution of

$$\begin{cases} -\Delta W + m^2 W = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial W}{\partial y} = (|x|^{-\mu} * F(\gamma(W)))f(\gamma(W)) - V(\overline{x})\gamma(W) & \text{on } \mathbb{R}^N. \end{cases} \tag{3.13}$$

In the following, we prove that $W_\varepsilon \to W$ strongly in $H^1(\mathbb{R}^{N+1}_+)$. As before, assume the existence of a radius $R > 0$ and of a sequence $x'_\varepsilon \in \mathbb{R}^N$ such that $x'_\varepsilon \in B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{2\beta}{\varepsilon}\right)$,

$$\varliminf_{\varepsilon\to 0} |x'_\varepsilon - \frac{x_\varepsilon}{\varepsilon}| = 0$$

and

$$\varliminf_{\varepsilon\to 0} \int_{B(x'_\varepsilon, R)} |\gamma(v^1_{1,\varepsilon})|^2 dx > 0. \tag{3.14}$$

Without loss of generality, we can assume that $\varepsilon x'_\varepsilon \to x' \in O$ as $\varepsilon \to 0$. Define $\widetilde{W}_\varepsilon(x, y) = W_\varepsilon(x + x'_\varepsilon, y)$, then up to a subsequence, as $\varepsilon \to 0$,

$$\widetilde{W}_\varepsilon \rightharpoonup \widetilde{W} \not\equiv 0 \text{ weakly in } H^1(\mathbb{R}^{N+1}_+)$$

and $\widetilde{W}$ satisfies

$$\begin{cases} -\Delta \widetilde{W} + m^2 \widetilde{W} = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial \widetilde{W}}{\partial y} = (|x|^{-\mu} * F(\gamma(\widetilde{W})))f(\gamma(\widetilde{W})) - V(x')\gamma(\widetilde{W}) & \text{on } \mathbb{R}^N. \end{cases} \tag{3.15}$$

With the same arguments as in the proof of Step 1, we can get a contradiction. So, $\gamma(W_\varepsilon) \to \gamma(W)$ strongly in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*)$, which implies

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(W_\varepsilon)))F(\gamma(W_\varepsilon))dx = \int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(W)))F(\gamma(W))dx. \tag{3.16}$$

Then we deduce that

$$\begin{aligned} \varlimsup_{\varepsilon\to 0} \Gamma_\varepsilon(v^1_{1,\varepsilon}) &\geq \varlimsup_{\varepsilon\to 0} P_\varepsilon(v^1_{1,\varepsilon}) \\ &= \varlimsup_{\varepsilon\to 0} \Big(\frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla W_\varepsilon|^2 + m^2|W_\varepsilon|^2)dxdy + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(\varepsilon x + x_\varepsilon)|\gamma(W_\varepsilon)|^2 dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(W_\varepsilon)))F(\gamma(W_\varepsilon))dx\Big) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|\nabla W|^2 + m^2|W|^2)dxdy + \frac{1}{2}\int_{\mathbb{R}^N} V(x_0)|\gamma(W)|^2 dx \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
& -\frac{1}{2}\int\limits_{\mathbb{R}^N}(|x|^{-\mu}*F(\gamma(W)))F(\gamma(W))dx \\
& = J_{V(x_0)}(W)\geq E_{V_0}.
\end{aligned}
$$

Now, by the estimate (3.3), we get

$$
\overline{\lim_{\varepsilon\to 0}}\Big(\Gamma_\varepsilon(v_{2,\varepsilon})+\Gamma_\varepsilon(v^1_{1,\varepsilon})\Big)=\overline{\lim_{\varepsilon\to 0}}\big(\Gamma_\varepsilon(v_{2,\varepsilon})+\Gamma_\varepsilon(v_{1,\varepsilon})\big)\leq \overline{\lim_{\varepsilon\to 0}}\,\Gamma_\varepsilon(v_\varepsilon)\leq E_{V_0}. \tag{3.18}
$$

Therefore, by (3.10) and (3.17), by choosing $d>0$ small enough,

$$
\lim_{\varepsilon\to 0}\Gamma_\varepsilon(v_{1,\varepsilon})=\lim_{\varepsilon\to 0}\Gamma_\varepsilon(v^1_{1,\varepsilon})=E_{V_0}. \tag{3.19}
$$

By (3.10), $\|v_{2,\varepsilon}\|\to 0$ as $\varepsilon\to 0$. Recalling that E_a is strictly increasing for $a>0$, we obtain $x_0\in\mathcal{M}$ and $W(x,y)=U(x-\overline{x}',y)$ for some $U\in S_{V_0}$ and $\overline{x}'\in\mathbb{R}^N$. Moreover, by (3.16), (3.17) and (3.19), we have

$$
\begin{aligned}
& \lim_{\varepsilon\to 0}\Big(\int\limits_{\mathbb{R}^{N+1}_+}(|\nabla W_\varepsilon|^2+m^2|W_\varepsilon|^2)dxdy+\int\limits_{\mathbb{R}^N}V_\varepsilon(\varepsilon x+x_\varepsilon)|\gamma(W_\varepsilon)|^2dx\Big) \\
& \to \int\limits_{\mathbb{R}^{N+1}_+}(|\nabla W|^2+m^2|W|^2)dxdy+\int\limits_{\mathbb{R}^N}V(x_0)|\gamma(W)|^2dx.
\end{aligned}
$$

Then $W_\varepsilon\to W$ strongly in $H^1(\mathbb{R}^{N+1}_+)$. Let $\overline{x}'_\varepsilon=\overline{x}'+x_\varepsilon/\varepsilon$, then $\varepsilon\overline{x}'_\varepsilon\to x_0\in\mathcal{M}$ and $v_{1,\varepsilon}(x,y)\to\varphi_\varepsilon(x-\overline{x}'_\varepsilon,y)U(x-\overline{x}'_\varepsilon,y)$ in $H^1(\mathbb{R}^{N+1}_+)$ as $\varepsilon\to 0$, which implies that

$$
v_{1,\varepsilon}=\varphi_\varepsilon(x-\overline{x}'_\varepsilon,y)U(x-\overline{x}'_\varepsilon,y)+o_\varepsilon(1)\text{ in }H^1(\mathbb{R}^{N+1}_+).
$$

The proof is now complete. □

Let

$$
D_\varepsilon=\max_{s\in[0,T]}\Gamma_\varepsilon(\iota_\varepsilon(s)).
$$

Then, we can see easily that the following properties hold.

Proposition 3.4.
(i) *We have* $\lim\limits_{\varepsilon\to 0}C_\varepsilon=\lim\limits_{\varepsilon\to 0}D_\varepsilon=E_{V_0}$.
(ii) *For any* $d>0$, *there exists* $\alpha_0>0$ *such that for* $\varepsilon>0$ *small,*

$$
\Gamma_\varepsilon(\iota_\varepsilon(s))\geq D_\varepsilon-\alpha_0 \textit{ implies that } \iota_\varepsilon(s)\in X^{d/2}_\varepsilon.
$$

As a consequence of Proposition 3.3, we have

Proposition 3.5. *For sufficiently small $d > 0$, there exist constants $\varrho > 0$ and $\varepsilon_0 > 0$, such that $|\Gamma'_\varepsilon(v)| \geq \varrho$ for $v \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{\frac{d}{2}})$ and $\varepsilon \in (0, \varepsilon_0)$.*

Proof. By contradiction, we can assume that there exist $\{\varepsilon_j\}$ with $\lim\limits_{j\to\infty} \varepsilon_j = 0$ and $\{v_{\varepsilon_j}\}$ with $v_{\varepsilon_j} \in X_{\varepsilon_j}^d \setminus X_{\varepsilon_j}^{\frac{d}{2}}$, such that

$$\lim_{j\to\infty} \Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \leq E_{V_0} \quad \text{and} \quad \lim_{j\to\infty} \Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0.$$

Thus, by Proposition 3.3, there exist $\{x_{\varepsilon_j}\} \subset \mathbb{R}^N$, $x_0 \in \mathcal{M}$ and $U \in S_{V_0}$ such that $\lim\limits_{\varepsilon_j\to 0} |\varepsilon_j x_{\varepsilon_j} - x_0| = 0$ and

$$\lim_{\varepsilon_j\to 0} \left\| v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot - x_{\varepsilon_j}, \cdot) U(\cdot - x_{\varepsilon_j}, \cdot) \right\| = 0.$$

By the definition of X_{ε_j}, we see that $\lim\limits_{\varepsilon_j\to 0} \operatorname{dist}(v_{\varepsilon_j}, X_{\varepsilon_j}) = 0$. This contradicts $v_{\varepsilon_j} \notin X_{\varepsilon_j}^{\frac{d}{2}}$. Thus, we complete the proof. $\square$

Now, we fix $d > 0$ such that Proposition 3.5 holds. Choose $R_0 > 0$ large enough such that $O \subset (\mathbb{R}^N \times \{0\}) \cap B(0, R_0)$ and $\iota_\varepsilon(s) \in H_0^1(B(0, \frac{R}{\varepsilon}))$ for any $s \in [0, T]$, $R > R_0$, $\varepsilon > 0$ small enough.

Proposition 3.6. *Given $\varepsilon > 0$ sufficiently small, then there exists a sequence $\{v_n^R\} \subset X_\varepsilon^{\frac{d}{2}} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, \frac{R}{\varepsilon}))$, such that $\lim\limits_{n\to\infty} \|\Gamma'_\varepsilon(v_n^R)\| = 0$ in $H_0^1(B(0, \frac{R}{\varepsilon}))$.*

Proof. The proof uses some ideas found in [5]. To the contrary, for $\varepsilon > 0$ small enough, there exists $a_R(\varepsilon) > 0$ such that $\|\Gamma'_\varepsilon(v)\| \geq a_R(\varepsilon)$ for any $v \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, \frac{R}{\varepsilon}))$. By Proposition 3.4, we know that there exists $\alpha_0 \in (0, E_{V_0})$ such that if $\varepsilon > 0$ small enough and $\Gamma_\varepsilon(\iota_\varepsilon(s)) \geq D_\varepsilon - \alpha_0$, then $\iota_\varepsilon(s) \in X_\varepsilon^{\frac{d}{2}} \cap H_0^1(B(0, \frac{R}{\varepsilon}))$. Thus, by a deformation argument in $H_0^1\left(B(0, \frac{R}{\varepsilon})\right)$, there exist a $\kappa_0 \in (0, \alpha_0)$ and a path $\iota \in \mathcal{C}\left([0, T], H^1(\mathbb{R}_+^{N+1})\right)$ such that

$$\iota(s) \begin{cases} = \iota_\varepsilon(s) & \text{if } \iota_\varepsilon(s) \in \Gamma_\varepsilon^{D_\varepsilon - \alpha_0} \\ \in X_\varepsilon^d & \text{if } \iota_\varepsilon(s) \notin \Gamma_\varepsilon^{D_\varepsilon - \alpha_0}, \end{cases}$$

and

$$\Gamma_\varepsilon(\iota(s)) < D_\varepsilon - \kappa_0, s \in [0, T]. \tag{3.20}$$

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}_+^{N+1})$ be such that $\psi(x, y) = 1$ for $x \in O^\delta$ and $0 < y < \delta$, $\psi(x, y) = 0$ for $x \notin O^{2\delta}$ and $y \geq 2\delta$, $\psi(x, y) \in [0, 1]$ and $|\nabla\psi| \leq \frac{2}{\delta}$. For $\iota(s) \in X_\varepsilon^d$, we define $\iota_1(s) = \psi_\varepsilon \iota(s)$, $\iota_2(s) = (1 - \psi_\varepsilon)\iota(s)$, where $\psi_\varepsilon = \psi(\varepsilon x, \varepsilon y)$. Then

$$
\begin{aligned}
Q_\varepsilon(\iota(s)) &= \left(\int_{\mathbb{R}^N} \chi_\varepsilon |\iota_1(s)|^2 + \int_{\mathbb{R}^N} \chi_\varepsilon |\iota_2(s)|^2 - 1 \right)_+^2 \\
&\geq \left(\int_{\mathbb{R}^N} \chi_\varepsilon |\iota_1(s)|^2 - 1 \right)_+^2 + \left(\int_{\mathbb{R}^N} \chi_\varepsilon |\iota_2(s)|^2 - 1 \right)_+^2 \\
&= Q_\varepsilon\left(\iota_1(s)\right) + Q_\varepsilon(\iota_2(s)).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\Gamma_\varepsilon\left(\iota(s)\right) = {} & \Gamma_\varepsilon\left(\iota_1(s)\right) + \Gamma_\varepsilon\left(\iota_2(s)\right) + Q_\varepsilon(\iota(s)) - Q_\varepsilon(\iota_1(s)) - Q_\varepsilon(\iota_2(s)) \\
& + \int_{\mathbb{R}_+^{N+1}} \left(\psi_\varepsilon(1-\psi_\varepsilon)|\nabla \iota(s)|^2 + m^2 \psi_\varepsilon(1-\psi_\varepsilon)|\iota(s)|^2 \right) dxdy \\
& + \int_{\mathbb{R}^N} V_\varepsilon \psi_\varepsilon(1-\psi_\varepsilon)|\gamma(\iota(s))|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota(s))))F(\gamma(\iota(s)))dx \\
& + \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota_1(s))))F(\gamma(\iota_1(s)))dx \qquad (3.21) \\
& + \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota_2(s))))F(\gamma(\iota_2(s)))dx + o_\varepsilon(1) \\
\geq {} & \Gamma_\varepsilon\left(\iota_1(s)\right) + \Gamma_\varepsilon\left(\iota_2(s)\right) + \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota_2(s))))F(\gamma(\iota_2(s)))dx \\
& - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota(s))))F(\gamma(\iota(s)))dx \\
& + \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota_1(s))))F(\gamma(\iota_1(s)))dx + o_\varepsilon(1).
\end{aligned}
$$

By [40, Proposition 7], we know

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(\iota_2(s))))F(\gamma(\iota_2(s)))dx = 0
$$

and

$$
\int_{\mathbb{R}^N} [|x|^{-\mu} * F(\gamma(\iota(s)))]F(\gamma(\iota(s)))dx = \int_{\mathbb{R}^N} [|x|^{-\mu} * F(\gamma(\iota_1(s)))]F(\gamma(\iota_1(s)))dx + o_\varepsilon(1).
$$

Then, by (3.21), we can obtain

$$\Gamma_\varepsilon(\iota(s)) \geq \Gamma_\varepsilon(\iota_1(s)) + o_\varepsilon(1). \tag{3.22}$$

Let

$$\iota_1^1(s)(x) = \begin{cases} \iota_1(s)(x), & \text{for } x \in O_\varepsilon^{2\delta}, \\ 0, & \text{for } x \notin O_\varepsilon^{2\delta}, \end{cases}$$

then

$$\Gamma_\varepsilon(\iota_1(s)) \geq \Gamma_\varepsilon(\iota_1^1(s)). \tag{3.23}$$

Since $0 < \alpha_0 < E_{V_0}$, we immediately see that $\iota_1^1(s) \in \Phi_\varepsilon$. Thus, thanks to [16, Proposition 3.4] and (3.23), we deduce that

$$\max_{s\in T} \Gamma_\varepsilon(\iota(s)) \geq E_{V_0} + o_\varepsilon(1).$$

Combining with (3.20), we get $E_{V_0} \leq D_\varepsilon - \kappa_0$, which is a contradiction. □

Proposition 3.7. *Given $\varepsilon, d > 0$ sufficiently small, Γ_ε has a nontrivial critical point $v \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$.*

Proof. Let $\{v_n^R\}$ be a Palais-Smale sequence of Γ_ε obtained above, then due to $v_n^R \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$, $\{v_n^R\}$ is uniformly bounded in $H_0^1\left(B(0, \frac{R}{\varepsilon})\right)$ for n. Up to a subsequence, as $n \to \infty$, $v_n^R \to v_\varepsilon^R$ strongly in $H_0^1\left(B(0, \frac{R}{\varepsilon})\right)$ and v_ε^R is a critical point of Γ_ε on $H_0^1\left(B(0, \frac{R}{\varepsilon})\right)$ and satisfies

$$\begin{cases} -\Delta v_\varepsilon^R + m^2 v_\varepsilon^R = 0 & \text{in } B(0, \dfrac{R}{\varepsilon}), \\ -\dfrac{\partial v_\varepsilon^R}{\partial y}(x,0) = (|x|^{-\mu} * F(v_\varepsilon^R(x,0))) f((v_\varepsilon^R(x,0)) - \widetilde{V}_\varepsilon v_\varepsilon^R(x,0) & x \in \mathbb{R}^N \text{ with } |x| = \dfrac{R}{\varepsilon}, \end{cases} \tag{3.24}$$

where

$$\widetilde{V}_\varepsilon^R = V_\varepsilon + 4\left(\int_{\mathbb{R}^N} \chi_\varepsilon |\gamma(v_\varepsilon^R)|^2 dx - 1\right)_+ \chi_\varepsilon.$$

Since $v_\varepsilon^R \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1\left(B(0, \frac{R}{\varepsilon})\right)$, we deduce that both $\{\|v_\varepsilon^R\|\}_R$ and $\{\Gamma_\varepsilon(v_\varepsilon^R)\}_R$ are uniformly bounded for $\varepsilon > 0$ sufficiently small. Hence also $\{Q_\varepsilon(v_\varepsilon^R)\}_R$ is uniformly bounded for $\varepsilon > 0$ sufficiently small. Now a Moser iteration scheme like Proposition 2.2 yields that $\{v_\varepsilon^R\}_R$ is bounded in $L^\infty(\mathbb{R}_+^{N+1})$ uniformly for $\varepsilon > 0$ sufficiently small. Taking into account that $\{Q_\varepsilon(v_\varepsilon^R)\}_R$ is uniformly bounded in $L^\infty(\mathbb{R}_+^{N+1})$. Similar as in Lemma 2.9, we know

$$|v_\varepsilon^R(x,y)| \leq Ce^{-m(\sqrt{|x|^2+y^2}-2R_0)}, \; x \in \mathbb{R}^N, y \in \mathbb{R}_+.$$

We assume, without loss of generality, that $\{v_\varepsilon^R\}_R$ weakly converges to some v_ε in $H^1(\mathbb{R}_+^{N+1})$ as $R \to +\infty$ and $v_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ is a nontrivial critical point of Γ_ε, that is,

$$\begin{cases} -\Delta v_\varepsilon + m^2 v_\varepsilon = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\dfrac{\partial v_\varepsilon}{\partial y}(x,0) = (|x|^{-\mu} * F(v_\varepsilon(x,0)))f(v_\varepsilon(x,0)) - \widetilde{V}_\varepsilon v_\varepsilon(x,0) & \text{on } \mathbb{R}^N, \end{cases} \tag{3.25}$$

where

$$\widetilde{V}_\varepsilon = V_\varepsilon + 4\left(\int_{\mathbb{R}^N} \chi_\varepsilon |\gamma(v_\varepsilon)|^2 dx - 1\right)_+ \chi_\varepsilon.$$

Obviously, $0 \notin X_\varepsilon^d$ if $d > 0$ small enough. So $v_\varepsilon \not\equiv 0$ if $d > 0$ small. □

Proof of Theorem 1.2. By Proposition 3.7, there exist $d > 0$ and $\varepsilon_0 > 0$, such that Γ_ε has a nontrivial critical point $v_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ for $\varepsilon \in (0, \varepsilon_0)$. As in the proof of Proposition 2.2, we have $\gamma(v_\varepsilon) \in L^q(\mathbb{R}^N)$ for all $q \in [2, +\infty]$ and $\{v_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R}_+^{N+1})$. By Proposition 3.3, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}_+^{N+1} \setminus ((\mathcal{M}^{2\beta})_\varepsilon \times [0,+\infty))} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2) dxdy = 0.$$

It follows that

$$\lim_{\varepsilon \to 0} \sup_{(x,y) \in \mathbb{R}_+^{N+1} \setminus ((\mathcal{M}^{2\beta})_\varepsilon \times [0,+\infty))} |v_\varepsilon(x,y)| = 0$$

and as in Proposition 3.7 we deduce an exponential decay of the trace $\gamma(v_\varepsilon)$ away from $\mathbb{R}^N \setminus (\mathcal{M}^{2\beta})_\varepsilon$:

$$0 < \gamma(v_\varepsilon)(x,y) \le C_1 e^{-C_2 \operatorname{dist}(x, (\mathcal{M}^{2\beta})_\varepsilon)},$$

which yields that $Q_\varepsilon(v_\varepsilon) = 0$ for small $\varepsilon > 0$. Therefore, v_ε is a critical point of P_ε. This completes the proof. □

4. Multi-peak solution case

In this section, by using the method of penalization argument [6], we construct a positive solution having multiple concentration regions that concentrate at the minimum points of the potential V, as $\varepsilon \to 0$.

Fixing an arbitrary $\alpha > 0$ and $v \in H^1(\mathbb{R}_+^{N+1})$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in O^\varepsilon, \\ \varepsilon^{-\alpha}, & \text{if } x \in \mathbb{R}^N \setminus O^\varepsilon, \end{cases}$$

$$\chi_\varepsilon^i(x) = \begin{cases} 0, & \text{if } x \in O_\varepsilon^i, \\ \varepsilon^{-\alpha}, & \text{if } x \in \mathbb{R}^N \setminus O_\varepsilon^i, \end{cases}$$

and

$$Q_\varepsilon(v) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v)^2 dx - 1 \right)_+^2, \quad Q_\varepsilon^i(v) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon^i \gamma(v)^2 dx - 1 \right)_+^2.$$

Let $\Gamma_\varepsilon, \Gamma_\varepsilon^i (i = 1, 2, \cdots, k) : H^1(\mathbb{R}_+^{N+1}) \to \mathbb{R}$ be given by

$$\Gamma_\varepsilon(v) = P_\varepsilon(v) + Q_\varepsilon(v), \quad \Gamma_\varepsilon^i(v) = P_\varepsilon(v) + Q_\varepsilon^i(v),$$

where P_ε is given in Section 3. It is standard to check that $\Gamma_\varepsilon, \Gamma_\varepsilon^i \in \mathcal{C}^1(H^1(\mathbb{R}_+^{N+1}))$.

Now, we construct a set of approximate solutions of (1.2). Let

$$\delta = \frac{1}{10} \min \left\{ \operatorname{dist}(\mathcal{M}', \mathbb{R}^N \setminus O'), \min_{i \neq j} \operatorname{dist}(O^i, O^j) \right\}.$$

For some $x_i \in (\mathcal{M}^i)^\beta$, $1 \le i \le k$, and $U_i \in S_{m_i}$, we define

$$U_\varepsilon^{x_1, x_2, \cdots, x_k}(x, y) = \sum_{i=1}^k \varphi_\varepsilon \left(x - \frac{x_i}{\varepsilon}, y \right) U_i \left(x - \frac{x_i}{\varepsilon}, y \right),$$

where S_{m_i} is the set of least energy solutions of (1.9) with $a = m_i$ and φ is given in Section 3. As in [40], we will find a solution near the set

$$\overline{X}_\varepsilon = \{ U_\varepsilon^{x_1, x_2, \cdots, x_k} \mid x_i \in (\mathcal{M}^i)^\beta, U_i \in S_{m_i}, i = 1, 2, \cdots, k \}$$

for sufficiently small $\varepsilon > 0$.

For each $1 \le i \le k$, choosing some $U_i \in S_{m_i}$ and $x_i \in \mathcal{M}^i$ but fixed, define

$$W_{\varepsilon,t}^i(x, y) \equiv (\varphi_\varepsilon U_{i,t})(x - \frac{x_i}{\varepsilon}, y), \text{ where } U_{i,t}(x, y) = U_i(\frac{x}{t}, y).$$

Then, as in Lemma 3.1, we have the following property.

Lemma 4.1. *There exist $T_i > 0, i = 1, 2, \cdots, k$, such that for $\varepsilon > 0$ small enough, $\Gamma_\varepsilon(W_{\varepsilon,T_i}^i) < -2, i = 1, 2, \cdots, k$.*

For any $1 \le i \le k$, let $\iota_\varepsilon^i(t)(x, y) = W_{\varepsilon,t}^i(x, y)$ for $t > 0$. Due to $\lim_{t \to 0} W_{\varepsilon,t}^i = 0$, let $\iota_\varepsilon^i(0) = 0$. Next, we define the min-max value C_ε^i:

$$C_\varepsilon^i = \inf_{\iota \in \Phi_\varepsilon^i} \max_{s_i \in [0, T_i]} \Gamma_\varepsilon^i(\iota(s_i)),$$

where $\Phi_\varepsilon^i = \{\iota \in \mathcal{C}([0, T_i], H^1(\mathbb{R}_+^{N+1})) : \iota(0) = 0, \iota(T_i) = \iota_\varepsilon^i(T_i)\}$. Similar to Propositions 2 and 3 in [5], we have for any $1 \le i \le k$,

$$\lim_{\varepsilon\to 0} C_\varepsilon^i = E_{m_i}.$$

Let

$$\iota'_\varepsilon(s) = \sum_{i=1}^k \iota_\varepsilon^i(s_i),\ s = (s_1, s_2, \cdots, s_k)$$

and

$$D'_\varepsilon = \max_{s\in\mathbb{T}} \Gamma_\varepsilon(\iota'_\varepsilon(s)),$$

where $\mathbb{T} \equiv [0, T_1] \times \cdot \times [0, T_k]$. Since $\operatorname{supp}(\gamma_\varepsilon(s)) \subset (\mathcal{M}')_\varepsilon^\beta$ for each $s \in \mathbb{T}$, it follows that

$$\Gamma_\varepsilon(\iota'_\varepsilon(s)) = P_\varepsilon(\iota'_\varepsilon(s)) = \sum_{i=1}^k P_\varepsilon(\iota_\varepsilon^i(s)).$$

By the Pohozǎev identity, for any $1 \le i \le k$, we have

$$\begin{aligned} J_{m_i}(U_{i,t}) = &\left(\frac{t^{N-2}}{2} - \frac{N-1}{2N-\mu}\frac{t^{2N-\mu}}{2}\right)\int_{\mathbb{R}_+^{N+1}} |\nabla U_i|^2 dxdy \\ &+ m^2\left(\frac{t^N}{2} - \frac{N+1}{2N-\mu}\frac{t^{2N-\mu}}{2}\right)\int_{\mathbb{R}_+^{N+1}} |U_i|^2 dxdy \\ &+ \left(\frac{t^N}{2} - \frac{N}{2N-\mu}\frac{t^{2N-\mu}}{2}\right) m_i \int_{\mathbb{R}^N} |\gamma(U_i)|^2 dx. \end{aligned}$$

Let

$$g_1(t) = \frac{t^{N-2}}{2} - \frac{N-1}{2N-\mu}\frac{t^{2N-\mu}}{2},\ \ g_2(t) = \frac{t^N}{2} - \frac{N+1}{2N-\mu}\frac{t^{2N-\mu}}{2},\ \ g_3(t) = \frac{t^N}{2} - \frac{N}{2N-\mu}\frac{t^{2N-\mu}}{2}.$$

By straightforward computation we deduce that $g'_j(t) > 0$ for $t \in (0, 1)$ and $g'_j(t) < 0$ for $t > 1$, $j = 1, 2, 3$. Thus, for any $1 \le i \le k$, the function $J_{m_i}(U_{i,t})$ achieves a unique maximum point at $t = 1$ for $t > 0$, that is,

$$\max_{t>0} J_{m_i}(U_{i,t}) = J_{m_i}(U_i) = E_{m_i},$$

which leads to the following conclusion.

Proposition 4.2.

(i) $\lim_{\varepsilon\to 0} D'_\varepsilon = \sum_{i=1}^k E_{m_i} := E$;

(ii) $\limsup_{\varepsilon\to 0} \max_{s\in\mathbb{T}} \Gamma_\varepsilon(\iota'_\varepsilon(s)) \le \widetilde{E}$, *where* $\widetilde{E} = \max_{1\le j\le k}(\sum_{i\ne j} E_{m_i})$;

(iii) *for any* $d>0$, *there exists* $\alpha_0>0$ *such that for* $\varepsilon>0$ *small,*

$$\Gamma_\varepsilon(\iota'_\varepsilon(s)) \ge D'_\varepsilon - \alpha_0 \text{ implies that } \iota'_\varepsilon(s) \in \overline{X}_\varepsilon^{d/2}.$$

Proposition 4.3. *Let* $\{\varepsilon_j\}$ *with* $\lim\limits_{j\to\infty}\varepsilon_j = 0$, $\{v_{\varepsilon_j}\}\subset \overline{X}_{\varepsilon_j}^d$ *be such that*

$$\lim_{j\to\infty}\Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \le E \text{ and } \lim_{j\to\infty}\Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0.$$

Then for sufficiently small $d>0$, *there exist, up to a subsequence,* $\{x_j^i\}\subset\mathbb{R}^N$, $i=1,2,\cdots,k$, *points* $x^i\in\mathcal{M}^i$, $U_i\in S_{m_i}$ *such that*

$$\lim_{j\to\infty}\left|\varepsilon_j x_j^i - x^i\right| = 0,$$

and

$$\lim_{j\to\infty}\left\| v_{\varepsilon_j} - \sum_{i=1}^k \varphi_{\varepsilon_j}(\cdot - x_j^i,\cdot)U_i(\cdot - x_j^i,\cdot)\right\| = 0.$$

Proof. Without confusion, we write ε for ε_j. Since S_{m_i} is compact, there exist $Z_i\in S_{m_i}$, $x_\varepsilon^i\in(\mathcal{M}^i)^\beta$, $x^i\in(\mathcal{M}^i)^\beta$, $i=1,2,\cdots,k$, $\lim\limits_{\varepsilon\to 0}x_\varepsilon^i = x^i$, such that up to a subsequence, denoted still by $\{v_\varepsilon\}$, we have for small $\varepsilon>0$,

$$\left\| v_\varepsilon - \sum_{i=1}^k \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon^i}{\varepsilon},\cdot\right) Z_i\left(\cdot - \frac{x_\varepsilon^i}{\varepsilon},\cdot\right)\right\| \le 2d. \tag{4.1}$$

Set $v_{1,\varepsilon}(x,y) = \sum\limits_{i=1}^k \varphi_\varepsilon\left(x - \frac{x_\varepsilon^i}{\varepsilon}, y\right) v_\varepsilon$, $v_{2,\varepsilon}(x,y) = v_\varepsilon(x,y) - v_{1,\varepsilon}(x,y)$.

Similar to (3.8), we can get

$$\lim_{\varepsilon\to 0}\int\limits_{\bigcup_{i=1}^k B\left(\frac{x_\varepsilon^i}{\varepsilon},\frac{2\beta}{\varepsilon}\right)\setminus B\left(\frac{x_\varepsilon^i}{\varepsilon},\frac{\beta}{\varepsilon}\right)} |\gamma(v_\varepsilon)|^q dx = 0, \quad \text{for any } 2<q<2^*.$$

Then, by the proof of Step 1 and Step 2 in Proposition 3.3, we deduce that

$$\Gamma_\varepsilon(v_\varepsilon) \ge \Gamma_\varepsilon(v_{1,\varepsilon}) + \Gamma_\varepsilon(v_{2,\varepsilon}) + O(\varepsilon) \tag{4.2}$$

and for $d,\varepsilon>0$ small enough,

$$\Gamma_\varepsilon(v_{2,\varepsilon}) \geq c_0\|v_{2,\varepsilon}\|^2 \tag{4.3}$$

for some $c_0 > 0$.

For each $i = 1, 2, \cdots, k$, we define

$$v^i_{1,\varepsilon}(x, y) = \begin{cases} v_{1,\varepsilon}(x, y), & x \in O^i_\varepsilon, \\ 0, & x \notin O^i_\varepsilon, \end{cases}$$

and set $W^i_\varepsilon(x, y) = v^i_{1,\varepsilon}\left(x + \frac{x^i_\varepsilon}{\varepsilon}, y\right)$. Then for fixed $i \in \{1, 2, \cdots, k\}$, we can assume, up to a subsequence that as $\varepsilon \to 0$,

$$W^i_\varepsilon \rightharpoonup W^i \text{ weakly in } H^1(\mathbb{R}^{N+1}_+),$$

and W^i is a solution of

$$\begin{cases} -\Delta W^i + m^2 W^i = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial W^i}{\partial y} = (|x|^{-\mu} * F(\gamma(W^i)))f(\gamma(W^i)) - V(x^i)\gamma(W^i) & \text{on } \mathbb{R}^N. \end{cases}$$

Similar to (3.17), for any $i = 1, 2, \cdots, k$, we deduce that

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}}\, \Gamma_\varepsilon(v^i_{1,\varepsilon}) \geq \overline{\lim_{\varepsilon\to 0}}\, P_\varepsilon(v^i_{1,\varepsilon}) \\ =&\overline{\lim_{\varepsilon\to 0}} \Big(\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+} |\nabla W^i_\varepsilon|^2 + m^2|W^i_\varepsilon|^2 dxdy + \frac{1}{2}\int\limits_{\mathbb{R}^N} V_\varepsilon(x + x^i_\varepsilon/\varepsilon)|\gamma(W^i_\varepsilon)|^2 dx \\ &- \frac{1}{2}\int\limits_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(W^i_\varepsilon)))F(\gamma(W^i_\varepsilon))dx\Big) \\ \geq&\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+} |\nabla W^i|^2 + m^2|W^i|^2 dxdy + \frac{1}{2}\int\limits_{\mathbb{R}^N} V(x^i)|\gamma(W^i)|^2 dx \\ &- \frac{1}{2}\int\limits_{\mathbb{R}^N} (|x|^{-\mu} * F(\gamma(W^i)))F(\gamma(W^i))dx \\ =&J_{V(x^i)}(W^i) \geq E_{m_i}. \end{aligned} \tag{4.4}$$

Now, by the estimate (4.2), we get

$$\overline{\lim_{\varepsilon\to 0}} \left(\Gamma_\varepsilon(v_{2,\varepsilon}) + \sum_{i=1}^k \Gamma_\varepsilon(v^i_{1,\varepsilon})\right) \leq \overline{\lim_{\varepsilon\to 0}}\, \Gamma_\varepsilon(v_\varepsilon) \leq E = \sum_{i=1}^k E_{m_i}. \tag{4.5}$$

On the other hand, by (4.3) and (4.4), by choosing $d > 0$ small enough,

$$\overline{\lim_{\varepsilon\to 0}}\left(\Gamma_\varepsilon(v_{2,\varepsilon})+\sum_{i=1}^{k}\Gamma_\varepsilon(v_{1,\varepsilon}^i)\right)\geq\sum_{i=1}^{k}E_{m_i}. \tag{4.6}$$

Therefore, (4.5) and (4.6) imply that by choosing $d>0$ small enough, for any $i=1,2,\cdots,k$

$$\overline{\lim_{\varepsilon\to 0}}\,\Gamma_\varepsilon(v_{2,\varepsilon})\to 0 \ \text{ and } \ \lim_{\varepsilon\to 0}\Gamma_\varepsilon(v_{1,\varepsilon}^i)=E_{m_i}. \tag{4.7}$$

By (4.3), $\|v_{2,\varepsilon}\|\to 0$ as $\varepsilon\to 0$. By (4.4), we have $J_{V(x^i)}(W^i)=E_{m_i}$. Recalling that E_a is strictly increasing for $a>0$, we obtain $x^i\in\mathcal{M}^i$ and $W^i(\cdot,y)=U_i(\cdot-x_i,y)$ for some $U_i\in S_{m_i}$ and $x_i\in\mathbb{R}^N$. Moreover, by (4.4) and (4.7), we have

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+}|\nabla W_\varepsilon^i|^2+m^2|W_\varepsilon^i|^2dxdy+\frac{1}{2}\int\limits_{\mathbb{R}^N}V_\varepsilon(x+x_\varepsilon^i/\varepsilon)|\gamma(W_\varepsilon^i)|^2dx$$
$$\to\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+}|\nabla W^i|^2+m^2|W^i|^2dxdy+\frac{1}{2}\int\limits_{\mathbb{R}^N}V(x^i)|\gamma(W_\varepsilon^i)|^2dx.$$

It follows that $W_\varepsilon^i\to W^i$ strongly in $H^1(\mathbb{R}^{N+1}_+)$. Letting $\overline{x}_\varepsilon^i=x_i+x_\varepsilon^i/\varepsilon$, then $\varepsilon\overline{x}_\varepsilon^i\to x^i\in\mathcal{M}^i$ and $v_{1,\varepsilon}^i\to\varphi_\varepsilon(\cdot-\overline{x}_\varepsilon^i,y)U_i(\cdot-\overline{x}_\varepsilon^i,y)$ in $H^1(\mathbb{R}^{N+1}_+)$ as $\varepsilon\to 0$, which implies that

$$v_{1,\varepsilon}=\sum_{i=1}^{k}v_{1,\varepsilon}^i=\sum_{i=1}^{k}\varphi_\varepsilon(x-\overline{x}_\varepsilon^i,y)U_i(x-\overline{x}_\varepsilon^i,y)+o_\varepsilon(1)\text{ in }H^1(\mathbb{R}^{N+1}_+).$$

This completes the proof. □

Immediately, as a consequence of Proposition 4.3, we have the following property.

Proposition 4.4. *For sufficiently small $d>0$, there exist constants $\varrho>0$ and $\varepsilon_0>0$, such that $|\Gamma_\varepsilon'(v)|\geq\varrho$ for $v\in\Gamma_\varepsilon^{D_\varepsilon'}\cap(\overline{X}_\varepsilon^d\setminus\overline{X}_\varepsilon^{\frac{d}{2}})$ and $\varepsilon\in(0,\varepsilon_0)$.*

We fix $d>0$ such that Proposition 4.4 holds. Choose $R_0>0$ large enough such that $O'\subset(\mathbb{R}^N\times\{0\})\cap B(0,R_0)$ and $\iota_\varepsilon'(s)\in H_0^1(B(0,\frac{R}{\varepsilon}))$ for any $s\in\mathbb{T}$, $R>R_0$.

Proposition 4.5. *Given $\varepsilon>0$ sufficiently small, then there exists a sequence $\{v_n^R\}\subset\overline{X}_\varepsilon^{\frac{d}{2}}\cap\Gamma_\varepsilon^{D_\varepsilon'}\cap H_0^1(B(0,\frac{R}{\varepsilon}))$, such that $\lim\limits_{n\to\infty}\|\Gamma_\varepsilon'(u_n^R)\|=0$ in $H_0^1(B(0,\frac{R}{\varepsilon}))$.*

Proof. Arguing by contradiction, for $\varepsilon>0$ small enough, there exists $a_R(\varepsilon)>0$ such that $\|\Gamma_\varepsilon'(v)\|\geq a_R(\varepsilon)$ for any $v\in\overline{X}_\varepsilon^d\cap\Gamma_\varepsilon^{D_\varepsilon'}\cap H_0^1(B(0,\frac{R}{\varepsilon}))$. By Proposition 4.2, we know that there exists $\alpha_0\in(0,E-\widetilde{E})$ such that if $\varepsilon>0$ small enough and $\Gamma_\varepsilon(\iota_\varepsilon'(s))\geq D_\varepsilon'-\alpha_0$, then $\iota_\varepsilon'(s)\in\overline{X}_\varepsilon^{\frac{d}{2}}\cap H_0^1(B(0,\frac{R}{\varepsilon}))$. Thus, by a deformation argument in $H_0^1\left(B(0,\frac{R}{\varepsilon})\right)$, there exist a $\kappa_0\in(0,\alpha_0)$ and a path $\iota\in\mathcal{C}\left(\mathbb{T},H^1(\mathbb{R}^{N+1}_+)\right)$ such that

$$\iota(s)\begin{cases} =\iota'_\varepsilon(s) & \text{if } \iota'_\varepsilon(s)\in\Gamma_\varepsilon^{D'_\varepsilon-\alpha_0} \\ \in\overline{X}^d_\varepsilon & \text{if } \iota'_\varepsilon(s)\notin\Gamma_\varepsilon^{D'_\varepsilon-\alpha_0}, \end{cases}$$

and

$$\Gamma_\varepsilon(\iota(s)) < D'_\varepsilon - \kappa_0, s\in\mathbb{T}. \tag{4.8}$$

Let $\psi\in C_0^\infty(\mathbb{R}^{N+1}_+)$ be such that $\psi(x,y)=1$ for $x\in(O')^\delta$ and $0<y<\delta$, $\psi(x,y)=0$ for $x\notin(O')^{2\delta}$ and $y\geq 2\delta$, $\psi(x,y)\in[0,1]$ and $|\nabla\psi|\leq\frac{2}{\delta}$. For $\iota(s)\in\overline{X}^d_\varepsilon$, we define $\iota_1(s)=\psi_\varepsilon\iota(s)$, $\iota_2(s)=(1-\psi_\varepsilon)\iota(s)$, where $\psi_\varepsilon=\psi(\varepsilon x,\varepsilon y)$. Then, by (3.22), we can obtain

$$\Gamma_\varepsilon(\iota(s))\geq\Gamma_\varepsilon(\iota_1(s))+o_\varepsilon(1). \tag{4.9}$$

For $i=1,2,\cdots,k$, let

$$\iota_1^i(s)(x)=\begin{cases}\iota_1(s)(x), & \text{for } x\in(O^i)^{2\delta}_\varepsilon,\\ 0, & \text{for } x\notin(O^i)^{2\delta}_\varepsilon,\end{cases}$$

then

$$\Gamma_\varepsilon(\iota_1(s))\geq\sum_{i=1}^k\Gamma_\varepsilon(\iota_1^i(s))=\sum_{i=1}^k\Gamma_\varepsilon^i(\iota_1^i(s)). \tag{4.10}$$

Since $0<\alpha_0<E-\tilde{E}$, by Proposition 4.2, for all $i\in\{1,2,\cdots,k\}$, $\iota_1^i(s)\in\Phi^i_\varepsilon$. Thus, thanks to [16, Proposition 3.4] and (4.10), we deduce that

$$\max_{s\in\mathbb{T}}\Gamma_\varepsilon(\iota(s))\geq E+o_\varepsilon(1).$$

Combining with (4.8), we get $E\leq D'_\varepsilon-\kappa_0$, which is a contradiction. □

Proof of Theorem 1.3. Similar to Proposition 3.7, we have that there exist $d>0$ and $\varepsilon_0>0$, such that Γ_ε has a nontrivial critical point $v_\varepsilon\in\overline{X}^d_\varepsilon\cap\Gamma_\varepsilon^{D'_\varepsilon}$ for $\varepsilon\in(0,\varepsilon_0)$. By the proof of Theorem 1.2, v_ε is a critical point of P_ε and

$$0<\gamma(v_\varepsilon)(x,y)\leq C_1e^{-C_2\operatorname{dist}(x,(\mathcal{M}')^{2\beta}_\varepsilon)}.$$

This completes the proof. □

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