Research Article

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Concentrating solutions for singularly perturbed fractional (N/s)-Laplacian equations with nonlocal reaction

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Abstract: This paper is concerned with the following fractional (N/s)-Laplacian Choquard equation:

$$\varepsilon^N(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = \varepsilon^\mu\Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big)f(u), \quad x \in \mathbb{R}^N,$$

where $(-\Delta)^s_{N/s}$ denotes the (N/s)-Laplacian operator, $0 < \mu < N$, and V and f are continuous real functions satisfying some mild assumptions. Applying the weak growth conditions on the exponential critical nonlinearity f and without using the strictly monotone condition, we use some refined analysis and develop the arguments in the existing results to establish the existence of the ground state solution of the fractional (N/s)-Laplacian Choquard equation. Moreover, we also study the concentration phenomenon of the ground state solutions. As far as we know, our results seem to be new concerning the fractional (N/s)-Laplacian equation with the Choquard reaction.

Keywords: Fractional Choquard equation, semiclassical states, critical exponential growth, Trudinger–Moser inequality

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1 Introduction and main results

In this paper, we are concerned with the following nonlinear fractional (N/s)-Laplacian equation with the nonlocal Choquard reaction:

$$\varepsilon^N(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = \varepsilon^\mu\Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big)f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where ε is a small positive parameter, $0 < s < 1$, $0 < \mu < N = ps$ with $p \geq 2$, $*$ represents the convolution between two functions. Here $(-\Delta)^s_{N/s}$ denotes the fractional p-Laplacian operator, which, up to normalization

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factors, can be defined by

$$(-\Delta)^s_{N/s}u(x) = C(N,s)\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{N/s-2}(u(x)-u(y))}{|x-y|^{2N}}\,\mathrm{d}y$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. Throughout this paper, we omit the normalizing constant to simplify the expressions. Further, V is the absorption potential, and the nonlinear function F is the primitive function of f. In what follows, we introduce some relevant results about the fractional Sobolev space. For each $s \in (0,1)$ and $p > 2$, we consider the Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty\}.$$

Here, $[u]_{W^{s,p}(\mathbb{R}^N)}$ is the Gagliardo seminorm

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}.$$

It is well known that the space $(W^{s,p}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}(\mathbb{R}^N)})$, where

$$\|\cdot\|^p_{W^{s,p}(\mathbb{R}^N)} = [\,\cdot\,]^p_{s,p} + \|\cdot\|^p_p \quad \text{and} \quad \|u\|_p = \Big(\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x\Big)^{\frac{1}{p}},$$

is a uniformly convex Banach space, particularly reflexive, and separable. We also recall that $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$; see [3, Theorem 7.38].

There are many applications for fractional p-Laplacian and nonlocal operators of elliptic type, including optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, and water waves; for more information, see [21, 23, 28] and the references therein. Several academics, like Pucci, Xiang and Zhang [43], Xiang, Zhang and Rădulescu [49, 50], among others, concentrated on the investigation of such fractional p-Laplacian problems. For a detailed analysis of nonlocal fractional problems, we also refer to the work of Molica Bisci, Rădulescu and Servadei [34].

The Choquard reaction

$$\Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big)f(u),$$

which appears in many intriguing physical conditions in quantum theory and is important in explicating the finite-range many-body interactions, is another intriguing phenomenon in our work. Pekar [42] described the quantum mechanics of a polaron at rest by proposing the nonlocal Choquard problem for the first time. In an attempt to approximate the Hartree–Fock theory of one-component plasma, Lieb [27] noted that Choquard sketched out the phenomena of an electron trapped in its hole using such an equation. Studying elliptic problems with the nonlocal Choquard reaction is becoming more and more popular due to the nonlocal characteristic. It is important to note that the majority of study on the Choquard equation is based on the crucial inequality listed below, which will also be crucial throughout this paper.

Lemma 1.1 (Hardy–Littlewood–Sobolev inequality). *Let* $1 < r, t < \infty$ *and* $0 < \mu < N$ *with*

$$\frac{1}{r} + \frac{1}{t} + \frac{N-\mu}{N} = 2.$$

If $f \in L^r(\mathbb{R}^N)$ *and* $h \in L^t(\mathbb{R}^N)$, *then there exists a sharp constant* $\mathcal{C} = C(r,t,\mu) > 0$, *independent of* f *and* h, *such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y \le \mathcal{C}\|f\|_{L^r(\mathbb{R}^N)}\|h\|_{L^t(\mathbb{R}^N)}.$$

Under the help of the above Hardy–Littlewood–Sobolev inequality and the Sobolev embedding

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N) \quad \text{for all } \frac{N}{s} \le t \le \Big(\frac{N}{s}\Big)^* = \frac{Np}{N-ps},$$

we know that the power range of k is

$$\frac{(N+\mu)p}{2N} \leq k \leq \frac{p(N+\mu)}{2(N-ps)},$$

when dealing with the equation (1.1) with pure power nonlinearity $f(u) = |u|^k$ variationally in the case that $sp < N$. Numerous studies have been conducted in this field using variation methods; for example, [1, 29, 30, 35–37, 47], among others, for the case where $s = 1$ and $sp < N$, and [16, 33], among others, for the case where $0 < s < 1$ and $sp < N$.

The existence and asymptotic behavior of the solutions to problem (1.1) as $\varepsilon \to 0$, also referred to as the *semiclassical* problem, are of significant importance in studies of standing waves to the nonlinear Choquard equation. It provides important physical insights and is used to explain how quantum physics and classical mechanics interact. The existence and concentration of ground state solutions under the scenario $sp < N$ are extensively discussed in the literature with regard to the relative progress of the fractional semiclassical Choquard problem. For further information, see [8, 22, 45, 52] and any related references. For the cases of $s = 1$ and $sp < N$, we additionally cite [7, 14] and the references therein.

We study the existence and concentration of the ground state solutions to problem (1.1) for the situation $sp = N$, in contrast to the studies listed. The Sobolev embedding, as was previously mentioned, is continuous but lacks a sense of critical growth. Now, we remind the readers of the critical growth in the space $W^{s,p}(\mathbb{R}^N)$, where $sp = N$ is specified by the Trudinger–Moser inequality. In the Sobolev–Slobodeckij spaces, there are a number of results on the Trudinger–Moser inequality [2, 10, 31, 39–41]. Based on the finding [41] and using a slightly altered version of the Trudinger–Moser sequence, Parini and Ruf [40] established the local fractional Trudinger–Moser inequality for the N-dimensional fractional p-Laplacian equation. Following that, Zhang [53] generalized the local fractional Trudinger–Moser inequality [40] to the entire space.

Lemma 1.2. *Let $s \in (0,1)$ and $sp = N$. Then for every $0 \leq \alpha < \alpha_*$, the inequality*

$$\sup_{u \in W^{s,p}(\mathbb{R}^N),\, \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{\frac{N}{N-s}})\,\mathrm{d}x < +\infty \tag{1.2}$$

holds, where

$$\Phi_{N,s}(t) = e^t - \sum_{i=0}^{j_p-2} \frac{t^j}{j!} \quad \textit{and} \quad j_p := \min\{j \in \mathbb{N} : j \geq p\}.$$

Moreover, for $\alpha \geq \alpha^*_{s,N}$,

$$\sup_{u \in W^{s,p}(\mathbb{R}^N),\, \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{\frac{N}{N-s}})\,\mathrm{d}x = +\infty.$$

As explored by Zhang [53, Remark 1.2], $\alpha^*_{s,N}$ is just an upper bound of α_*; they did not give the precise value of α_*. Motivated by the inequality explored in Lemma 1.2, it is natural to say that

(F1) a continuous nonlinearity f has critical exponential growth if there exists $\alpha_0 > 0$ such that

$$\lim_{t\to+\infty} f(t)\exp(-\alpha t^{\frac{N}{N-s}}) = 0 \quad \text{for all } \alpha > \alpha_0$$

and

$$\lim_{t\to+\infty} f(t)\exp(-\alpha t^{\frac{N}{N-s}}) = +\infty \quad \text{for all } \alpha < \alpha_0.$$

Several authors addressed the existence, multiplicity, and concentration of semiclassical states for the local Schrödinger equations with critical exponential growth based on the Trudinger–Moser inequalities. We refer to [6, 17–19, 54] and their references for the most significant developments in this field. Only a few works in this field deal with semiclassical states in the context of the semiclassical problem with nonlocal reactions. The presence and concentration of semiclassical ground state solutions of the equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\alpha-2}\Big[\frac{1}{|x|^\alpha} * F(u)\Big] f(u) \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

were established by Alves, Cassani, Tarsi, and Yang [4] by studying the existence of a nontrivial solution for the critical nonlocal equation with periodic potential. On V, they made the following assumptions:

(V1) $V(x) \geq V_0$ in $\mathbb{R}^2$ for some $V_0 > 0$.

(V2) $0 < \inf_{x\in\mathbb{R}^2} V(x) = V_0 < V_\infty = \liminf_{|x|\to\infty} V(x) < \infty$.

Condition (V2) is introduced by Rabinowitz [44]. They also assumed that the nonlinearity satisfies the following assertions:

(f1) $f(s) = 0$ for all $s \leq 0$ and $0 \leq f(s) \leq Ce^{4\pi s^2}$, $s \geq 0$.

(f2) There exist $s_0 > 0$, $M_0 > 0$ and $q \in (0, 1]$ such that $0 < s^q F(s) \leq M_0 f(s)$ for all $|s| \geq s_0$.

(f3) There exist $p > \frac{2-\alpha}{2}$ and $C_p > 0$ such that $f(s) \sim C_p s^p$, as $s \to 0$.

(f4) There exists $K > 1$ such that $f(s)s > KF(s)$ for all $s > 0$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

(f5) It holds

$$\lim_{s\to+\infty} \frac{sf(s)F(s)}{e^{8\pi s^2}} \geq \beta$$

with

$$\beta > \inf_{\rho>0} \frac{e^{\frac{4-\mu}{4}} V_0 \rho^2}{16\pi^2 \rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}.$$

(f6) $s \to f(s)$ is strictly increasing on $(0, +\infty)$.

Applying the mountain pass lemma and checking that the mountain pass level shall be less than the critical level provided that the potential V is periodic, they could obtain the existence of a nontrivial solution for the periodic Choquard problem. To be specific, they need to give the upper bound of the mountain pass level, that is, to show that for some $n \in \mathbb{N}$,

$$g(t) := \frac{1}{2} t^2 (1 + \delta_n) - \frac{1}{2} \int_{B_\rho} [I_\mu * F(tw_n)] F(tw_n)\,\mathrm{d}x < \frac{2+\mu}{8}, \quad t \geq 0, \tag{1.4}$$

where B_ρ denotes the open ball centered at the origin with radius $\rho > 0$ in $\mathbb{R}^2$, w_n is the Moser-type function, and

$$\delta_n = \sup_{|x|\leq\rho} V(x)\Big(\frac{1}{4\log n} - \frac{1}{4n^2 \log n} - \frac{1}{2n^2}\Big) > 0.$$

Such an estimation contributes to excluding the vanishing case of the Cerami sequence $\{u_n\}$. Another heuristic proof in their work is that they considered a sequence of measures that has uniformly bounded total variation and by using the Radon–Nikodym theorem to show that if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \Big[\frac{1}{|x|^{2-\mu}} * F(u_n)\Big] f(u_n)\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^2} \Big[\frac{1}{|x|^{2-\mu}} * F(u)\Big] f(u)\varphi\,\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

Here, we refer to [15] as an example of a work that used the proof of this property of the Choquard reaction to study the nonlocal Choquard problem in the exponential critical condition. Yang [51] carried out additional research on the nonlocal Choquard-type equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\alpha-2}\Big[\frac{1}{|x|^\alpha} * (P(x)F(u))\Big] P(x)f(u) \quad \text{in } \mathbb{R}^2. \tag{1.5}$$

Also, they found the existence and concentration of solutions to equation (1.5), and it is clear that equation (1.5) is more challenging to solve than equation (1.3). We refer the readers to [51] for further information.

Let us also have a quick look at the relative progress of semiclassical problems with the fractional p-Laplacian. For the case $s = \frac{1}{2}$ and $N = 1$, Alves, do Ó and Miyagaki [5] studied the concentration phenomenon of solutions to the problem

$$\varepsilon(-\Delta)^{1/2} u + V(x)u = f(u) \quad \text{in } \mathbb{R}.$$

They gave a hypothesis on the growth of f at the infinity as follows:

(f7) There exist a constant $p > 2$ and a suitable constant $C_p > 0$ such that

$$f(t) \geq C_p t^{p-1} \quad \text{for all } t > 0.$$

They made use of this supposition to derive directly that the critical level is below the mountain pass level of the associated energy functional. In the general case, we notice that there is only a paper dealing with the semiclassical problem with fractional (N/s)-Laplacian. Thin [46] is concerned with the following fractional (N/s)-Laplacian equation:

$$\varepsilon^N(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

By Nehari manifold, variational method, concentration compactness principle and Ljusternik–Schnirelman theory, they obtained the existence, multiplicity and concentration of solutions to equation (1.6) under the following assumptions:

(F_1) The nonlinearity $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(t) = 0$ for all $t \in (-\infty, 0]$, $f(t) > 0$ for all $t > 0$, and there exist constants $\alpha_0 \in (0, \alpha^*)$, $b_1, b_2 > 0$ such that for any $t \in \mathbb{R}$,

$$|f'(t)| \le b_1|t|^{p-2} + b_2|t|^{p-2}\Phi_{N,s}(\alpha_0|t|^{\frac{N}{N-s}}),$$

where

$$\Phi_{N,s}(y) = e^y - \sum_{i=0}^{j_p-2}\frac{y^j}{j!}, \quad j_p = \min\{j \in \mathbb{N} : j \ge p\}$$

and α_* is given in Lemma 1.2.

(F_2) There exists $\mu > N/s$ such that $f(t)t - \mu F(t) \ge 0$ for all $t \in \mathbb{R}$, where $F(t) = \int_0^\tau f(\tau)\,d\tau$.

(F_3) It holds

$$\lim_{t\to 0^+}\frac{f'(t)}{t^{N/s-2}} = 0.$$

(F_4) There exists $\gamma_1 > 0$ large enough such that $F(t) \ge \gamma_1|t|^\mu$ for all $t \ge 0$.

(F_5) $f(t)/t^{p-1}$ is a strictly increasing function of $t \ge 0$.

It is worth mentioning that they also used the kind of condition (F_4). Both conditions like (f5), (f7) and condition like (F_4) define the behaviors of the nonlinearity at infinity, but the later two cases also establish the growth condition at the origin as a global assumption. It is important to note that these kinds of conditions make estimation easier because C_p and γ_1 can have as large values as possible. This uncertainty cannot actually be verified, which has some influence on potential follow-up uses. Hypothesis (f5) has the advantage of allowing the use of classical Moser-type functions as test functions to control the energy level. As a result, it is natural to expect that (f5) could be weakened appropriately for the search for ground state solutions to (1.1).

Inspired by the preceding works, we will establish the existence and concentration of the ground state solutions to equation (1.1), while also confirming the preceding expectation. Before we present this result, we make the following assumption about V:

(V) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and it satisfies

$$1 < \inf_{x\in\mathbb{R}^N} V(x) = V_0 < V_\infty = \liminf_{|x|\to\infty} V(x) < \infty.$$

The lower bound is intended to make it easier for readers to see the estimation process between different equations by removing some of the influence caused by the switch of the potential term and linear potential with fixed constants. In addition to (V) and (F1), we present the following hypotheses:

(F2) It holds

$$f(t) = o(t^{\frac{N+\mu}{2s}-1}) \quad \text{as } t \to 0,$$

$f(t) = 0$ for all $t \in (-\infty, 0]$, and $f(t) > 0$ for all $t > 0$.

(F3) There exists $\bar{\mu} > (N-s)/s$ such that $f(t)t \ge \bar{\mu}F(t) \ge 0$ for all $t > 0$.

(F4) There exist M_0 and $t_0 > 0$ such that, for any $t \ge t_0$, we have $F(t) \le M_0|f(t)|$.

(F5) $t \mapsto f(t)$ is nondecreasing on $(0, +\infty)$.

(F6) It holds

$$\liminf_{t\to\infty}\frac{f(t)}{\exp(\alpha_0 t^{N/(N-s)})} = \kappa,$$

and κ satisfies

$$\begin{aligned}\Big\{1 + Am\rho^N\delta_n + \frac{(N-s)\ln[C_1\Lambda(\kappa-\varepsilon)^{-2}(1+\varepsilon)(1+m\rho^N A\delta_n)](1+m\rho^N A\delta_n)}{s(N+\mu)\ln n} \\ - \frac{C_3(1+\varepsilon)\alpha^*_{s,N}(N-s)}{2\alpha_0 Ns\ln n}\Big[\frac{2\alpha_0 N}{(N+\mu)\alpha^*_{s,N}}\Big]^{\frac{N-s}{s}}\Big\} \\ < \Big(\frac{\alpha_{N,s}}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}},\end{aligned}$$

where $\varepsilon > 0$ is small enough and the definitions of A, Λ, ρ, δ_n, $\alpha_{N,s}$ will be introduced in the subsequent Section 3.

Our result reads as follows.

Theorem 1.3. *Assume that V and f satisfy* (V) *and* (F1)–(F6). *Then, for any $\varepsilon > 0$ small, problem* (1.1) *has at least one positive ground state solution. Moreover, if we replace* (F2) *with*

(F2') $f(t) = o(t^{N/s-1})$ *as $t \to 0$,*

and let u_ε denote one of these positive solutions with $\eta_\varepsilon \in \mathbb{R}$ being its global maximum, then we have

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

Let us now sketch the strategies and highlights that will be used to prove Theorem 1.3. Our arguments are based on variational methods and some refined analysis, and we will make use of the Nehari method to deal with the problem. In particular, by establishing the ground state solutions of the autonomous problem (3.1) introduced in Section 3, and by analyzing the energy level of the energy functionals of the autonomous problem (3.1) and the original problem (1.1), we can further establish the existence result of Theorem 1.3. We notice that the strict monotonicity conditions (f6) and (F_5) play an important role in confirming the uniqueness of the projection from the space $H^1(\mathbb{R}^N)$ or $W^{s,p}(\mathbb{R}^N)$ to the corresponding Nehari manifold, respectively. They both require the uniqueness of the projection when analyzing the asymptotic behavior of the ground state energy level of the singular perturbation equation (1.1) and excluding the vanishing case of a Palais–Smale sequence. It is reasonable to expect that the strict monotonicity condition could be relaxed in order to search for the ground state solutions of the singular perturbation equation (1.1), and we will answer affirmatively under this condition (F5).

It is well known that we meet several difficulties due to the exponential critical growth of nonlinearity. First, as previously stated, it is difficult to demonstrate that the Fréchet derivative of the Choquard term is weakly sequentially continuous. Unlike the heuristic proof discussed in [4, 15], we will develop some arguments to directly verify this property. Another challenge is the lack of compactness, and we must ensure that the energy functional meets certain compactness requirements at some minimax level. This type of estimate is more delicate in the fractional p-Laplacian equation (1.1) with the Choquard term, because we proposed a weaker hypothesis (F6) and the appearance of the fractional p-Laplacian. Based on the arguments presented in [11–13], we will further develop it into our current workspace, which is not trivial. As far as we know, our results in the fields of the fractional p-Laplacian equation are new.

The structure of this paper is the following. In Section 2, we introduce the variational setting of problem (1.1) and present some preliminary results. Section 3 is devoted to demonstrating the existence of a positive ground state solution of the autonomous equation while also providing proofs of refined delicate energy estimation and the convergence of the Fréchet derivative of the Choquard term. Sections 4 and 5 will discuss the existence of ground state solutions to the singular perturbation equation (1.1), as well as the concentration phenomenon of the ground state solutions.

2 Some preliminaries and mountain pass geometry

First, we introduce some notations that will clarify what follows.

- C, c, C_i, c_i ($i = 1, 2, \dots$) denote positive constants which may vary from line to line.
- For any exponent $p > 1$, p' denotes the conjugate of p and is given as $p' = p/(p-1)$.

- $B_r(x)$ denotes the ball of radius r centered at $x \in \mathbb{R}^N$.
- The arrows $\rightharpoonup$ and $\to$ denote the weak convergence and strong convergence, respectively.
- $L^s(\mathbb{R}^N)$ $(1 \le s < +\infty)$ denotes the Lebesgue space with the norm

$$\|u\|_s = \Big(\int_{\mathbb{R}^N} |u|^s \,dx\Big)^{\frac{1}{s}}.$$

- To make the notation concise, we set, for $\alpha > 0$ and $t \in \mathbb{R}$,

$$\mathcal{H}(\alpha, t) = \exp(\alpha |t|^{\frac{N}{N-s}}) - S_{k_p-2}(\alpha, t) = \sum_{k=k_p-1}^{+\infty} \frac{\alpha^k}{k!} |t|^{\frac{N}{N-s}k},$$

where

$$S_{k_p-2}(\alpha, t) = \sum_{k=0}^{k_p-2} \frac{\alpha^k}{k!} |t|^{\frac{N}{N-s}k}$$

and $k_p = \min\{k \in \mathbb{N} : k \ge p\}$.

Observe that, making the change of variable $x \to \varepsilon x$, problem (1.1) is equivalent to the following problem:

$$(-\Delta)^s_{N/s} u + V(\varepsilon x)|u|^{\frac{N}{s}-2}u = \Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big) f(u) \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

Evidently, if u is a solution of equation (2.1), then $v(x) := u(x/\varepsilon)$ is a solution of equation (1.1). To study the original equation (1.1), we only need to look at the equivalent equation (2.1).

For any fixed $\varepsilon > 0$, we define the working space

$$E_\varepsilon = \Big\{u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,dx < \infty\Big\} \tag{2.2}$$

endowed with the norm

$$\|u\|_\varepsilon = ([u]^p_{s,p} + \|u\|^p_{V_\varepsilon,p})^{\frac{1}{p}},$$

where

$$\|u\|_{V_\varepsilon,p} = \Big(\int_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,dx\Big)^{\frac{1}{p}}.$$

From condition (V), we can see that $\|\cdot\|_\varepsilon$ and the norm of $W^{s,p}(\mathbb{R}^N)$ are equivalent.

To deal with the nonlocal-type problem (2.1), besides the classical Hardy–Littlewood–Sobolev inequality (see Lemma 1.1) which will be frequently used throughout this paper, we also use the following inequality.

Lemma 2.1 (Cauchy–Schwarz-type inequality [32, Section 5]). *For $f, h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, there holds*

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^{N-\mu}} * |f|\Big)|h| \,dx \le \Big[\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^{N-\mu}} * |f|\Big)|f| \,dx \int_{\mathbb{R}^N} \Big(\frac{1}{|x|^{N-\mu}} * |h|\Big)|h| \,dx\Big]^{\frac{1}{2}}. \tag{2.3}$$

We define the energy functional associated with problem (2.1):

$$\mathcal{J}_\varepsilon(u) = \frac{s}{N}\|u\|^{N/s}_\varepsilon - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|x-y|^{N-\mu}} \,dx\,dy. \tag{2.4}$$

Using Lemma 1.1 and some standard arguments, we can easily check that $\mathcal{J}_\varepsilon$ is well defined on E_ε and belongs to $\mathcal{C}^1$ with its derivative given by

$$\begin{aligned}\langle \mathcal{J}'_\varepsilon(u), v\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{N/s-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}} \,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{N/s-2}uv \,dx \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))f(u(x))v(x)}{|x-y|^{N-\mu}} \,dx\,dy \quad \text{for all } u, v \in E_\varepsilon.\end{aligned} \tag{2.5}$$

Hence, it is obvious that the solutions of equation (2.1) correspond to critical points of $\mathcal{J}_\varepsilon$. To obtain the positive ground state solutions of equation (2.1), we need to define the Nehari manifold and the ground state energy related to $\mathcal{J}_\varepsilon$:

$$\mathcal{N}_\varepsilon := \{u \in E_\varepsilon \setminus \{0\} : \langle \mathcal{J}_\varepsilon'(u), u\rangle = 0\} \quad \text{and} \quad c_\varepsilon := \inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon. \tag{2.6}$$

Obviously, $\mathcal{N}_\varepsilon$ contains all nontrivial critical points of $\mathcal{J}_\varepsilon$, and if c_ε is achieved by $u_\varepsilon \in \mathcal{N}_\varepsilon$, then u_ε is called a ground state solution of equation (2.1).

Next, we verify some properties for the Nehari manifold $\mathcal{N}_\varepsilon$. Before that, we give several preliminary results which will be used in this part.

Lemma 2.2. *If $u \in W^{s,p}(\mathbb{R}^N)$, then for any $\alpha > 0$ it holds*

$$\int_{\mathbb{R}^N} \mathcal{H}(\alpha, u)\, dx < +\infty.$$

For the detailed proof of Lemma 2.2, we refer to [53, Corollary 2.4] and [40, Proposition 3.2]. Following that, we give a property of $\mathcal{H}(\alpha, t)$, which can be seen as the corollary of [24, Lemma A.2].

Lemma 2.3. *It holds*

$$\mathcal{H}(\alpha, t)^b \le \mathcal{H}(b\alpha, t)$$

for all $\alpha > 0$, $t > 0$ and $b \ge 1$, where

$$\mathcal{H}(\alpha, t) = \exp(\alpha |t|^{\frac{N}{N-s}}) - S_{k_p-2}(\alpha, t), \quad S_{k_p-2}(\alpha, t) = \sum_{k=0}^{k_p-2} \frac{\alpha^k}{k!} |t|^{\frac{N}{N-s}k}, \quad k_p = \min\{k \in \mathbb{N} : k \ge p\}.$$

Lemma 2.4. *Assume that* (F1)–(F3) *hold. Then there exists $\lambda > 0$, independent of ε, such that*

$$\|u\|_\varepsilon \ge \lambda \quad \text{for all } u \in \mathcal{N}_\varepsilon.$$

Proof. For any $u \in \mathcal{N}_\varepsilon$, we have

$$\|u\|_\varepsilon^{N/s} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))f(u(x))u(x)}{|x-y|^{N-\mu}}\, dx\, dy.$$

Recall that, by (F1) and (F2), for each $\alpha > \alpha_0$ and close to α_0 and $q > N/s$, there exist constants $\delta > 0$ and $C_{q,\delta} > 0$ such that

$$f(t)t \le \delta |t|^{\frac{N+\mu}{2s}} + C_{q,\delta}|t|^q \mathcal{H}(\alpha, t)$$

for all $t > 0$. This together with (F3) implies that, for all $t > 0$, one has

$$F(t) \le \frac{1}{\bar{\mu}} f(t)t \le \frac{\delta}{\bar{\mu}}|t|^{\frac{N+\mu}{2s}} + \frac{C_{q,\delta}}{\bar{\mu}}|t|^q \mathcal{H}(\alpha, t).$$

Then by Lemmas 2.2 and 2.3, it is easy to verify

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))f(u(x))u(x)}{|x-y|^{N-\mu}}\, dx\, dy \\
&\quad \le \mathcal{C}_0 \|F(u)\|_{\frac{2N}{N+\mu}} \|f(u)u\|_{\frac{2N}{N+\mu}} \\
&\quad \le \mathcal{C}_0' \Big[\Big(\frac{\delta}{\bar{\mu}}\Big)^{\frac{2N}{N+\mu}} \int_{\mathbb{R}^N} |u|^{N/s}\, dx + \Big(\frac{C_{q,\delta}}{\bar{\mu}}\Big)^{\frac{2N}{N+\mu}} \int_{\mathbb{R}^N} |u|^{\frac{2Nq}{N+\mu}} \mathcal{H}(\alpha, u)^{\frac{2N}{N+\mu}}\, dx\Big]^{\frac{N+\mu}{N}} \\
&\quad \le \mathcal{C}_0'' \Big(\frac{C_{q,\delta}}{\bar{\mu}}\Big)^2 \Big(\int_{\mathbb{R}^N} |u|^{\frac{4Nq}{N+\mu}}\, dx\Big)^{\frac{N+\mu}{2N}} \Big(\int_{\mathbb{R}^N} \mathcal{H}\Big(\frac{4N\alpha}{N+\mu}, u\Big)\, dx\Big)^{\frac{N+\mu}{2N}} + \mathcal{C}_0'' \Big(\frac{\delta}{\bar{\mu}}\Big)^2 \|u\|_{N/s}^{(N+\mu)/s} \\
&\quad \le C_0''' (\|u\|_{N/s}^{(N+\mu)/s} + \|u\|_{4Nq/(N+\mu)}^{2q}).
\end{aligned}$$

Considering that $(N+\mu)/s > p$ and $2q > p$, we can see there exists $\lambda > 0$ such that $\|u\|_\varepsilon \geq \lambda$. This completes the proof. □

The following result demonstrates that the energy functional $\mathcal{J}_\varepsilon$ satisfies the geometric structure of the mountain pass theorem.

Lemma 2.5 (Mountain pass geometry). *Assume that* (F1)–(F3) *are satisfied. Then the following conclusions hold:*
(i) *There exist $\tau > 0$ and $\rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \tau$, provided that $\|u\|_\varepsilon = \rho$.*
(ii) *There exists $v \in E_\varepsilon$ with $\|v\|_\varepsilon > \rho$ such that $\mathcal{J}_\varepsilon(v) < 0$.*

Proof. (i) By a standard argument as in the proof of Lemma 2.4, one can easily obtain the conclusion that there exist $\tau > 0$ and $\rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \tau$, provided that $\|u\|_\varepsilon = \rho$.

(ii) For all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $\|u\|_\varepsilon = 1$, from (F6) and all $t > 0$, we obtain

$$\begin{aligned}\mathcal{J}_\varepsilon(tu) &= \frac{st^{N/s}}{N}\|u\|_\varepsilon^{N/s} - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(tu(y))F(tu(x))}{|x-y|^{N-\mu}}\,dx\,dy \\ &\leq \frac{st^{N/s}}{N}\|u\|_\varepsilon^{N/s} - \frac{\kappa^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{e^{\alpha t^{N/(N-s)}u(y)^{N/(N-s)}}e^{\alpha t^{N/(N-s)}u(x)^{N/(N-s)}}}{|x-y|^{N-\mu}}\,dx\,dy.\end{aligned}$$

Since $\|u\|_\varepsilon = 1$, we can find a bounded Ω with a positive measure in $\mathbb{R}^N$ such that $|u(x)| \geq \zeta > 0$ when $x \in \Omega$. Without loss of generality, we assume that $0 \in \Omega$. So, as Ω is bounded, there exists $r \in \mathbb{R}$ such that $B_r(0) \in \Omega$. By the fact that

$$\int_{B_r(0)}\int_{B_r(0)}\frac{1}{|x-y|^{N-\mu}}\,dx\,dy \geq \mathcal{C}_{N,\mu}r^{N+\mu},$$

where $\mathcal{C}_{N,\mu}$ is a positive constant, we have

$$\mathcal{J}_\varepsilon(tu) \leq \frac{st^{N/s}}{N} - \kappa^2\mathcal{C}_{N,\mu}r^{N+\mu}e^{\alpha t^{N/(N-s)}\zeta^{N/(N-s)}}e^{\alpha t^{N/(N-s)}\zeta^{N/(N-s)}}.$$

Since the $t^{N/s}$ has growth smaller than $e^{t^{N/(N-s)}}$ as $t \to +\infty$, we have $\mathcal{J}_\varepsilon(tu) \to -\infty$. Taking $v = \rho_1 u$, where $\rho_1 > \rho > 0$ large enough, we can see that conclusion (ii) holds. □

According to Lemma 2.5, we can use a version of the mountain pass theorem without the Palais–Smale condition [48] to deduce the existence of a Palais–Smale sequence $\{u_n\}$ at level $\tilde{c}_\varepsilon$, namely

$$\mathcal{J}_\varepsilon(u_n) \to \tilde{c}_\varepsilon \quad \text{and} \quad \mathcal{J}'_\varepsilon(u_n) \to 0,$$

where $\tilde{c}_\varepsilon$ is the mountain pass level $\mathcal{J}_\varepsilon$ defined by

$$\tilde{c}_\varepsilon = \inf_{l\in\Gamma}\max_{t\in(0,1)}\mathcal{J}_\varepsilon(l(t)),$$

and

$$\Gamma = \{l \in \mathcal{C}([0,1],E_\varepsilon) : l(0) = 0,\ \mathcal{J}_\varepsilon(l(1)) < 0\}.$$

Lemma 2.6. *Suppose that* (F1)–(F3) *and* (F5) *are satisfied. Then, for all $u \in \mathcal{N}_\varepsilon$, one has*

$$\mathcal{J}_\varepsilon(u) \geq \max_{t\geq 0}\mathcal{J}_\varepsilon(tu).$$

Proof. For any fixed $u \in E_\varepsilon$ and any $t \geq 0$, we define $\mathbb{J}(t)$ as follows:

$$\begin{aligned}\mathbb{J}(t) := &-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))F(u(y))}{|x-y|^{N-\mu}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(tu(x))F(tu(y))}{|x-y|^{N-\mu}}\,dx\,dy \\ &+ \frac{s-st^{N/s}}{N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))f(u(y))u(y)}{|x-y|^{N-\mu}}\,dx\,dy.\end{aligned}$$

From (F3) and (F5), one has

$$\begin{aligned}\mathcal{I}'(t) &= \int_{\mathbb{R}^{2N}} \frac{F(tu(x))f(tu(y))u(y)}{|x-y|^{N-\mu}}\,dx\,dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x)))f(u(y))t^{(N-s)/s}u(y)}{|x-y|^{N-\mu}}\,dx\,dy \\ &= \int_{\mathbb{R}^{2N}} \frac{t^{(N-s)/s}[u(x)]^{(N-s)/s}u(y)}{|x-y|^{N-\mu}}\Big[\frac{F(tu(x))}{t^{(N-s)/s}[u(x)]^{(N-s)/s}}f(tu(y)) - \frac{F(u(x))}{[u(x)]^{(N-s)/s}}f(u(y))\Big]\,dx\,dy \\ &\begin{cases}\geq 0, & t \geq 1,\\ \leq 0, & 0 < t < 1,\end{cases}\end{aligned}$$

which implies that $\mathcal{I}(t) \geq \mathcal{I}(1) = 0$ immediately. With the help of this, we can easily deduce that

$$\begin{aligned}\mathcal{I}_\varepsilon(u) - \mathcal{I}_\varepsilon(tu) &= \frac{s - st^{N/s}}{N}\|u\|_\varepsilon - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{N-\mu}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(tu(x))F(tu(y))}{|x-y|^{N-\mu}}\,dx\,dy \\ &= \frac{s - st^{N/s}}{N}\langle \mathcal{J}'_\varepsilon(u), u\rangle - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{N-\mu}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(tu(x))F(tu(y))}{|x-y|^{N-\mu}}\,dx\,dy \\ &\quad + \frac{s - st^{N/s}}{N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))f(u(x))u(x)}{|x-y|^{N-\mu}}\,dx\,dy \\ &\geq \frac{s - st^{N/s}}{N}\langle \mathcal{J}'_\varepsilon(u), u\rangle + \mathcal{I}(t) \\ &\geq \frac{s - st^{N/s}}{N}\langle \mathcal{J}'_\varepsilon(u), u\rangle \quad \text{for all } u \in E_\varepsilon,\ t \geq 0.\end{aligned}$$

Together with the definition of the Nehari manifold $\mathcal{N}_\varepsilon$, this completes the proof. □

Lemma 2.7. *Let $u \in E_\varepsilon \setminus \{0\}$. Then there exists $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$.*

Proof. Let $u \in E_\varepsilon \setminus \{0\}$. We define the function $\psi(t) = \mathcal{J}_\varepsilon(tu)$ for $t > 0$. From Lemma 2.5, we know that $\psi(0) = 0$, $\psi(t) > 0$ for t sufficiently small, and $\psi(t) < 0$ for t sufficiently large. Therefore, there exists $t = t_u$ such that $\max_{t>0}\psi(t)$ is achieved at t_u, so $\psi'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. □

Applying Lemmas 2.6 and 2.7, we can see that the ground state energy value c_ε has a minimax characterization given by

$$c_\varepsilon = \tilde{c}_\varepsilon = \inf_{u\in E_\varepsilon\setminus\{0\}} \max_{t\geq 0} \mathcal{J}_\varepsilon(tu). \tag{2.7}$$

Moreover, there is a constant $c > 0$ independent of ε such that $c_\varepsilon > c > 0$.

Lemma 2.8. *Suppose that* (F1)–(F3) *are satisfied. Let $\{u_n\}$ be a Palais–Smale sequence at level $c > 0$ for $\mathcal{J}_\varepsilon$. Then $\{u_n\}$ is bounded in E_ε and $\|u_n^-\|_\varepsilon = o(1)$.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence at level $c > 0$ for $\mathcal{J}_\varepsilon$. We have

$$\begin{aligned}c + 1 + \|u_n\|_\varepsilon &\geq \mathcal{J}_\varepsilon(u_n) - \frac{s}{2(N-s)}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle \\ &= \frac{s(N-2s)}{2N(N-s)}\|u_n\|_\varepsilon^{N/s} + \frac{s}{2(N-s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(y))[f(u_n(x))u_n(x) - [(N-s)/s]F(u_n(x))]}{|x-y|^{N-\mu}}\,dx\,dy.\end{aligned}$$

Thus, by (F3), we can conclude that $\{u_n\}$ is bounded. Since $\{u_n\}$ is bounded, we have

$$\langle \mathcal{J}'_\varepsilon(u_n), u_n^-\rangle = o_n(1).$$

Recalling that $f(t) = 0$ for $t < 0$, and by the inequality

$$|a-b|^{s-2}(a-b)(a^- - b^-) \geq |a^- - b^-|^s \quad \text{for all } s > 1, \tag{2.8}$$

we get

$$\begin{aligned}\|u_n^-\|_\varepsilon^{N/s} &\leq \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{N/s-2}(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{N/s-2}u_n u_n^-\,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u_n(y))f(u_n(x))u_n^-(x)}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y = o(1),\end{aligned}$$

which implies that $\|u_n^-\|_\varepsilon \to 0$ in E_ε. Consequently, we may assume that $u_n \geq 0$ for any $n \in \mathbb{N}$. The proof is now complete. □

3 The autonomous problem

For our scope, we shall also investigate the limit problem associated with problem (2.1). To this end, we first discuss in this section the existence of the positive ground state solutions to the autonomous problem.

Let $m > \inf_{x\in\mathbb{R}^N} V(x) = V_0$. We consider the following autonomous problem:

$$(-\Delta)^s_{N/s}u + m|u|^{\frac{N}{s}-2}u = \Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big)f(u) \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

The corresponding energy functional of problem (3.1) is defined by

$$\mathcal{J}_m(u) = \frac{s}{N}\|u\|_m^{N/s} - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.2}$$

where

$$\|u\|_m = ([u]^{N/s}_{s,N/s} + m\|u\|^{N/s}_{N/s})^{\frac{s}{N}}.$$

By a standard argument explored in Section 2, we can easily see that $\mathcal{J}_m \in \mathcal{C}^1(W^{s,N/s}(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{J}_m'(u), v\rangle &= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{N/s-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}^N} m|u|^{\frac{N}{s}-2}uv\,\mathrm{d}x \\ &\quad - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(x))f(u(y))v(y)}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for any $u, v \in W^{s,N/s}(\mathbb{R}^N)$. Accordingly, we use $\mathcal{N}_m$ and c_m to denote the corresponding Nehari manifold and the ground state energy of $\mathcal{J}_m$, where

$$\mathcal{N}_m := \{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{J}_m'(u), u\rangle = 0\} \quad \text{and} \quad c_m = \inf_{\mathcal{N}_m} \mathcal{J}_m.$$

Inspired by the arguments explored in Section 2, we can easily establish the mountain pass geometry of the energy functional $\mathcal{J}_m$. Then we know that there exists the $(\mathrm{PS})_{c_m}$ sequence $\{u_n\} \subset W^{s,p}(\mathbb{R}^N)$, i.e.

$$\mathcal{J}_m'(u_n) \to 0, \quad \mathcal{J}_m(u_n) \to c_m := \inf_{\gamma_m\in\Gamma_m}\max_{t\in[0,1]} \mathcal{J}_m(\gamma_m(t)), \tag{3.3}$$

where

$$\Gamma_m := \{\gamma_m \in \mathcal{C}^1([0,1], W^{s,p}(\mathbb{R}^N)) : \gamma_m(0) = 0,\ \mathcal{J}_m(\gamma_m(1)) < 0\}.$$

For each $u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$, there exists $t = t(u)$ such that

$$\mathcal{J}_m(t(u)u) = \max_{s\geq 0} \mathcal{J}_m(su) \quad \text{and} \quad t(u)u \in \mathcal{N}_m.$$

Following the standard arguments in Section 3, we also have

$$c_m = \inf_{\mathcal{N}_m} \mathcal{J}_m(u) = \inf_{u\in W^{s,p}(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} \mathcal{J}_m(tu).$$

We now state the main result for the autonomous equation (3.1).

Lemma 3.1. *Assume that* (F5) *holds. Then equation* (3.1) *has at least one positive ground state solution u such that $\mathfrak{J}_m(u) = c_m$.*

Proof. Let $\{u_n\}$ be the (PS) sequence obtained in (3.3), that is,

$$\frac{s}{N}\|u_n\|_m^{N/s} - \frac{1}{2}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(u_n)\Big)F(u_n) \to c_m \quad \text{as } n \to \infty.$$

For any $v \in W^{s,N/s}(\mathbb{R}^N)$, we have

$$|\langle \mathfrak{J}_m'(u_n), v\rangle| \le o_n(1)\|v\|_m.$$

From Lemma 2.8, we can deduce that $\|u_n\|_m \le \mathfrak{C}$, where $\mathfrak{C}$ is a positive constant. Therefore, from (F3) we have

$$\bar{\mu}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(u_n)\Big)F(u_n)\,\mathrm{d}x \le \int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(u_n)\Big)f(u_n)u_n\,\mathrm{d}x = \|u_n\|_m \le \mathfrak{C}. \tag{3.4}$$

Passing to a subsequence if necessary, we have that $u_n \rightharpoonup u$ in E_m, $u_n \to u$ in $L^{N/s}_{\mathrm{loc}}(\mathbb{R}^N)$, and $u_n \to u$ a.e. on $\mathbb{R}^N$. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^{N/s}dx = 0,$$

together with Lions's concentration compactness principle [48, Lemma 1.21], we know that $u_n \to 0$ in $L^q(\mathbb{R}^N)$, $q \in (N/s, +\infty)$. For small enough $\varepsilon > 0$, we choose $M_\varepsilon > (M_0\mathfrak{C})/\varepsilon > t_0$ in assumption (F4). Then it follows from (F3), (F4) and (3.4) that

$$\begin{aligned}\int_{|u_n|\ge M_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]F(u_n)\,\mathrm{d}x &\le M_0\int_{|u_n|\ge M_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]|f(u_n)|\,\mathrm{d}x\\ &\le \frac{M_0}{M_\varepsilon}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]f(u_n)u_n\,\mathrm{d}x\\ &< \varepsilon.\end{aligned}$$

Using (F2), (F3), Lemma 1.1, (2.3), (3.4), and the Sobolev embedding, we can choose $N_\varepsilon \in (0,1)$ such that

$$\begin{aligned}&\int_{|u_n|\le N_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]F(u_n)\,\mathrm{d}x\\ &\quad\le \frac{1}{\bar{\mu}}\int_{|u_n|\le N_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]f(u_n)u_n\,\mathrm{d}x\\ &\quad\le \frac{\varepsilon}{\bar{\mu}}\int_{|u_n|\le N_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]|u_n|^{(N+\mu)/2s}\,\mathrm{d}x\\ &\quad\le \frac{\varepsilon}{\bar{\mu}}\Big(\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]F(u_n)\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{|u_n|\le N_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * |u_n|^{(N+\mu)/2s}\Big]|u_n|^{(N+\mu)/2s}\,\mathrm{d}x\Big)^{\frac{1}{2}}\\ &\quad\le \frac{\varepsilon\mathfrak{C}_0^{1/2}\mathfrak{C}^{1/2}}{\bar{\mu}^{3/2}}\|u_n\|_{N/s}^{(N+\mu)/2s}\\ &\quad\le C_1\|u_n\|_m^{(N+\mu)/2s}\\ &\quad\le C_1\mathfrak{C}^{(N+\mu)/2s}.\end{aligned}$$

Similarly, we have

$$\int_{N_\varepsilon\le|u_n|\le M_\varepsilon}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]F(u_n) \le \varepsilon.$$

Due to the arbitrariness of $\varepsilon > 0$, we obtain

$$\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]F(u_n)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Hence, according to the fact that $\{u_n\}$ is a (PS) sequence, we have the following proposition.

Proposition 3.2. *It holds*

$$\lim_{n\to\infty}\|u_n\|_m^{N/s}=\frac{Nc_m}{s}\le\Big[\frac{\alpha_{N,s}(N+\mu)}{2N\alpha_0}\Big]^{\frac{N-s}{s}}.$$

Proof. In the previous work [25], Kozono, Sato and Wadade proved that there exist positive constants $\alpha_{N,s}$ and $C_{N,s}$, depending on N and s, such that

$$\int_{\mathbb{R}^N}\Big(e^{\alpha|t|^{\frac{N}{N-s}}}-\sum_{j=0}^{j_p-2}\frac{\alpha^j|t|^{jN/(N-s)}}{j!}\Big)\,\mathrm{d}x\le C_{N,s}\quad\text{for all }\alpha\in(0,\alpha_{N,s}),\tag{3.5}$$

for all

$$u\in W^{s,N/s}(\mathbb{R}^N)\quad\text{with }\|u\|_{W^{s,N/s}(\mathbb{R}^N)}\le 1,$$

and where $j_p=\min\{j\in\mathbb{N}:j\ge p\}$. Inequality (3.5) is better for us to consider the behavior of a sequence compared to Lemma 1.2, and it is obvious that $\alpha_{N,s}\le\alpha_*\le\alpha_{s,N}^*$. To prove Proposition 3.2, it is enough to prove that there exists $\bar{n}\in\mathbb{N}$ such that

$$\max_{t\ge0}\mathcal{J}_m(tw_{\bar{n}})\le\frac{s}{N}\Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}},\tag{3.6}$$

where the $w_n(x)$ are the Moser-type functions supported in $B_\rho(0)$ as follows:

$$w_n(x)=\Big(\frac{N}{\alpha_{s,N}^*}\Big)^{\frac{N-s}{N}}\begin{cases}|\ln n|^{\frac{N-s}{N}} & \text{if }|x|\le\frac{\rho}{n},\\ \frac{|\ln\frac{\rho}{|x|}|}{|\ln n|^{s/N}} & \text{if }\frac{\rho}{n}\le|x|\le\rho,\\ 0 & \text{if }|x|\ge\rho,\end{cases}$$

which belong to $W_0^{s,p}(\mathbb{R}^N)$. For $s\in(0,1)$, as explored in [40], we cannot expect that $[w_n]_{W^{s,p}(\mathbb{R}^N)}$ is constant. Following the estimation in [40] and after some basic calculations, we know that

$$[w_n]_{W^{s,N/s}(\mathbb{R}^N)}^{N/s}\le 1+\mathcal{O}\Big(\frac{1}{\log n}\Big)$$

and

$$\int_{\mathbb{R}^N}m|w_n(x)|^p\,\mathrm{d}x\le Cmw_N\rho^N\Big(\frac{N}{\alpha_{s,N}^*}\Big)^{\frac{N-s}{s}}\mathcal{O}\Big(\frac{1}{\log n}\Big).$$

This implies

$$\|w_n\|_m^{N/s}\le 1+\Big[Cmw_N\rho^N\Big(\frac{N}{\alpha_{s,N}^*}\Big)^{\frac{N-s}{s}}+1\Big]\mathcal{O}\Big(\frac{1}{\log n}\Big)=:1+Cmw_N\rho^N\Big(\frac{N}{\alpha_{s,N}^*}\Big)^{\frac{N-s}{s}}\delta_n,\tag{3.7}$$

where

$$\delta_n=\Big(1+\frac{(\alpha_{s,N}^*)^{(N-s)/s}}{cmw_N\rho^N N^{(N-s)/s}}\Big)\mathcal{O}\Big(\frac{1}{\log n}\Big).$$

Further, we know

$$\int_{B_{\rho/n}(0)}\int_{B_{\rho/n}(0)}\frac{1}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y\ge C(\mu,N)\Big(\frac{\rho}{n}\Big)^{N+\mu},$$

where $C(\mu,N)$ is a positive constant.

Let us argue by contradiction and suppose that (3.6) does not hold. So, for all n, let $t_n>0$ be such that

$$\mathcal{J}_m(t_nw_n)=\max_{t\ge0}\mathcal{J}_m(tw_n)\ge\frac{s}{N}\Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}},\tag{3.8}$$

where t_n satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_m(tw_n)\Big|_{t=t_n}=0.$$

Together with estimate (3.7), we have

$$t_n^{N/s}\Big(1 + Cmw_N\rho^N\Big(\frac{N}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}\delta_n\Big) \geq \Big[\frac{\alpha_{N,s}(N+\mu)}{2N\alpha_0}\Big]^{\frac{N-s}{s}}. \tag{3.9}$$

By (F6), we know that there exists $t_\varepsilon > 0$ such that, for any $t \geq t_\varepsilon$, we have

$$f(t) \geq (\kappa - \varepsilon)e^{\alpha_0 t^{N/(N-s)}} \quad \text{and} \quad F(t) \geq \frac{(N-s)(\kappa-\varepsilon)}{N\alpha_0} t^{\frac{-s}{N-s}} e^{\alpha_0 t^{N/(N-s)}}.$$

From now on, all inequalities hold for large $n \in \mathbb{N}$, and it is obvious that $t_n w_n \geq t_\varepsilon$ under this condition. From (F6) and (3.8), we have

$$\begin{aligned}
&t_n^p\Big(1 + Cmw_N\rho^N\Big(\frac{N}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}\delta_n\Big)\\
&\quad\geq t_n^p\|w_n\|_m^{\frac{N}{s}}\\
&\quad\geq \int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{F(t_n w_n(y))}{|x-y|^{N-\mu}}\,\mathrm{d}y\Big)f(t_n w_n(x))t_n w_n(x)\,\mathrm{d}x\\
&\quad\geq \int_{B_{\rho/n}(0)}\Big(\int_{B_{\rho/n}(0)}\frac{F(t_n w_n(y))}{|x-y|^{N-\mu}}\,\mathrm{d}y\Big)f(t_n w_n(x))t_n w_n(x)\,\mathrm{d}x\\
&\quad\geq (\kappa-\varepsilon)^2 t_n^{(N-2s)/(N-s)}(\ln n)^{\frac{N-2s}{N}}\frac{C_{\mu,\beta,N}(N-s)\rho^{N+\mu}}{N^{2s/N}\alpha_0(\alpha^*_{s,N})^{(N-2s)/s}}\exp\Big[\frac{2\alpha_0 t_n^{N/(N-s)}N\ln n}{\alpha^*_{s,N}} - (N+\mu)\ln n\Big].
\end{aligned}$$

This implies that there exists a constant $C_1 > 0$ such that

$$\Big[\frac{2\alpha_0 t_n^{N/(N-s)}N}{\alpha^*_{s,N}} - (N+\mu)\Big]\ln n \leq C_1,$$

that is,

$$t_n^{N/s} \leq \Big[\frac{(N+\mu)\alpha^*_{s,N}}{2N\alpha_0}\Big]^{\frac{N-s}{s}}\Big(1 + \frac{C_2}{\log n}\Big). \tag{3.10}$$

Combining (3.9) and (3.10), we obtain that, for any small $\varepsilon > 0$,

$$\Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}}(1-\varepsilon) \leq t_n^p \leq \Big[\frac{(N+\mu)\alpha^*_{s,N}}{2N\alpha_0}\Big]^{\frac{N-s}{s}}(1+\varepsilon).$$

Taking this range into consideration, we have

$$\begin{aligned}
I(t_n w_n) &= \frac{s}{N}t_n^{N/s}\|w_n\|_m^{N/s} - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(t_n w_n(y))F(t_n w_n(x))}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \frac{s}{N}t_n^{N/s}\Big(1 + Cmw_N\rho^N\Big(\frac{N}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}\delta_n\Big) - \frac{1}{2}\int_{B_{\rho/n}(0)}\int_{B_{\rho/n}(0)}\frac{F(t_n w_n(y))F(t_n w_n(x))}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \frac{s}{N}t_n^{N/s}\Big(1 + Cmw_N\rho^N\Big(\frac{N}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}\delta_n\Big)\\
&\qquad - \frac{(N-s)^2 C(\mu,N)}{2N^2\alpha_0^2}\Big(\frac{\rho}{n}\Big)^{N+\mu}\frac{(k-\varepsilon)^2 e^{2\alpha_0 t_n^{N/(N-s)}N(\alpha^*_{s,N})^{-1}\ln n}}{t_n^{2s/(N-s)}(\frac{N}{\alpha^*_{s,N}})^{2s/N}(\ln n)^{2s/N}}\\
&\leq \frac{s}{N}t_n^{N/s}\Big(1 + Cmw_N\rho^N\Big(\frac{N}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}\delta_n\Big)\\
&\qquad - \frac{(N-s)^2 C(\mu,N)}{2^{1-2s/N}N^2\alpha_0^{2-2s/N}(N+\mu)^{2s/N}(1+\varepsilon)}\Big(\frac{\rho}{n}\Big)^{N+\mu}\frac{(k-\varepsilon)^2 e^{2\alpha_0 t_n^{N/(N-s)}N(\alpha^*_{s,N})^{-1}\ln n}}{(\ln n)^{2s/N}}\\
&=: \varphi(t_n),
\end{aligned}$$

Now consider the following notations:

$$A := Cw_N\Big(\frac{N}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}$$

and

$$B := \frac{(N-s)^2\rho^{N+\mu}C(\mu,N)}{2^{1-2s/N}N^2\alpha_0^{2-2s/N}(N+\mu)^{2s/N}}.$$

Thus,

$$\varphi(t_n) = \frac{s}{N}t_n^{N/s}(1+Am\rho^N\delta_n) - B\frac{(k-\varepsilon)^2e^{2\alpha_0 t_n^{N/(N-s)}N(\alpha^*_{s,N})^{-1}\ln n}}{(1+\varepsilon)(\ln n)^{2s/N}n^{N+\mu}},$$

and there exists $\hat{t}_n$ such that $\varphi'(\hat{t}_n) = 0$. Thus we have

$$\hat{t}_n^{(N-s)/s}(1+Am\rho^N\delta_n) = B\frac{(k-\varepsilon)^2e^{2\alpha_0\hat{t}_n^{N/(N-s)}N(\alpha^*_{s,N})^{-1}\ln n}}{(1+\varepsilon)(\ln n)^{2s/N}n^{N+\mu}}\,\frac{2\alpha_0N^2\ln n\,\hat{t}_n^{s/(N-s)}}{\alpha^*_{s,N}(N-s)}.$$

This implies that

$$\begin{aligned}
e^{2\alpha_0\hat{t}_n^{N/(N-s)}N(\alpha^*_{s,N})^{-1}\ln n} &= \frac{\hat{t}_n^{(N-s)/s-s/(N-s)}(1+\varepsilon)(1+Am\rho^N\delta_n)(\ln n)^{2s/N}n^{N+\mu}\alpha^*_{s,N}(N-s)}{B(k-\varepsilon)^2 2\alpha_0N^2\ln n}\\
&\le \frac{C_1(1+\varepsilon)(1+Am\rho^N\delta_n)n^{N+\mu}\alpha^*_{s,N}(N-s)}{B(k-\varepsilon)^2 2\alpha_0N^2}\\
&= \frac{C_1\Lambda(1+\varepsilon)(1+Am\rho^N\delta_n)n^{N+\mu}}{(k-\varepsilon)^2},
\end{aligned}$$

where

$$\Lambda = \frac{[\alpha^*_{s,N}(N-s)]}{2B\alpha_0N^2}.$$

Thus, one has

$$\hat{t}_n^{N/s} \le \Big[\frac{(N+\mu)\alpha^*_{s,N}}{2\alpha_0N}\Big]^{\frac{N-s}{s}}\Big(1+\frac{(N-s)\ln[C_1\Lambda(\kappa-\varepsilon)^{-2}(1+\varepsilon)(1+m\rho^NA\delta_n)]}{s(N+\mu)\ln n}\Big),$$

where

$$\begin{aligned}
\varphi(\hat{t}_n) &= \frac{s}{N}\hat{t}_n^{N/s}(1+Am\rho^N\delta_n) - \frac{\hat{t}_n^{(N-s)/s-s/(N-s)}(1+\varepsilon)(1+Am\rho^N\delta_n)\alpha^*_{s,N}(N-s)}{2\alpha_0N^2\ln n}\\
&\le \frac{s}{N}\hat{t}_n^{N/s}(1+Am\rho^N\delta_n) - (1+\varepsilon)(1+Am\rho^N\delta_n)\frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0N^2\ln n}\\
&= \frac{s}{N}(1+Am\rho^N\delta_n)\Big[\hat{t}_n^{N/s} - \frac{(1+\varepsilon)C_3\alpha^*_{s,N}(N-s)}{2\alpha_0Ns\ln n}\Big]\\
&\le \frac{s}{N}(1+Am\rho^N\delta_n)\Big[\Big[\frac{(N+\mu)\alpha^*_{s,N}}{2\alpha_0N}\Big]^{\frac{N-s}{s}}\\
&\qquad\times\Big(1+\frac{(N-s)\ln[C_1\Lambda(\kappa-\varepsilon)^{-2}(1+\varepsilon)(1+m\rho^NA\delta_n)]}{s(N+\mu)\ln n}\Big) - \frac{(1+\varepsilon)C_3\alpha^*_{s,N}(N-s)}{2\alpha_0Ns\ln n}\Big]\\
&= \frac{s}{N}\Big[\frac{(N+\mu)\alpha^*_{s,N}}{2\alpha_0N}\Big]^{\frac{N-s}{s}}\Big[1+Am\rho^N\delta_n + \frac{(N-s)\ln[C_1\Lambda(\kappa-\varepsilon)^{-2}(1+\varepsilon)(1+m\rho^NA\delta_n)](1+m\rho^NA\delta_n)}{s(N+\mu)\ln n}\\
&\qquad - \frac{C_3(1+\varepsilon)\alpha^*_{s,N}(N-s)}{2\alpha_0Ns\ln n}\Big[\frac{2\alpha_0N}{(N+\mu)\alpha^*_{s,N}}\Big]^{\frac{N-s}{s}}\Big] + \mathcal{O}\Big(\frac{1}{\log^2 n}\Big).
\end{aligned}$$

Recall that

$$\delta_n = \Big(1+\frac{(\alpha^*_{s,N})^{(N-s)/s}}{cmw_N\rho^NN^{(N-s)/s}}\Big)\mathcal{O}\Big(\frac{1}{\ln n}\Big).$$

From (F6) and by taking a suitable ρ, we know that there exists ε small enough such that

$$\begin{aligned}\Big[1 + Am\rho^N\delta_n + &\frac{(N-s)\ln[C_1\Lambda(\kappa-\varepsilon)^{-2}(1+\varepsilon)(1+m\rho^N A\delta_n)](1+m\rho^N A\delta_n)}{s(N+\mu)\ln n} \\ &- \frac{C_3(1+\varepsilon)\alpha^*_{s,N}(N-s)}{2\alpha_0 Ns\ln n}\Big[\frac{2\alpha_0 N}{(N+\mu)\alpha^*_{s,N}}\Big]^{\frac{N-s}{s}}\Big] \\ &< \Big(\frac{\alpha_{N,s}}{\alpha^*_{s,N}}\Big)^{\frac{N-s}{s}}.\end{aligned}$$

Then we have

$$\mathcal{J}_m(t_n w_n) \le \varphi(t_n) < \frac{s}{N}\Big[\frac{(N+\mu)\alpha_{N,s}}{2\alpha_0 N}\Big]^{\frac{N-s}{s}}.$$

which is a contradiction to (3.8). This concludes the proof. □

Thus, there exist $\delta > 0$ small and $n_0 \in \mathbb{N}$ large such that

$$\|u_n\|_m^{N/s} \le \Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}}(1-\delta) \quad \text{for all } n \ge n_0. \tag{3.11}$$

In light of the Hardy–Littlewood–Sobolev inequality, we have

$$\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big]f(u_n)u_n\,\mathrm{d}x \le \mathcal{C}_0\|F(u_n)\|_{2N/(N+\mu)}\|f(u_n)u_n\|_{2N/(N+\mu)}.$$

For any $\varepsilon > 0$ and $q > N/s$, there is $\vartheta > 0$ close to 0 such that

$$\begin{aligned}\|F(u_n)\|_{2N/(N+\mu)} &\le \|f(u_n)u_n\|_{2N/(N+\mu)} \\ &\le \varepsilon\|u_n\|_{N/s}^{(N+\mu)/2N} + C_{\varepsilon,q}\Big[\int_{\mathbb{R}^N}\mathcal{H}((1+\vartheta)\alpha_0,|u_n|)^{\frac{2N}{N+\mu}}|u_n|^{\frac{2Nq}{N+\mu}}\,\mathrm{d}x\Big]^{\frac{N+\mu}{2N}} \\ &\le \varepsilon\|u_n\|_{N/s}^{(N+\mu)/2N} + C_{\varepsilon,q}\|u_n\|^q_{2Nqr'/(N+\mu)}\Big(\int_{\mathbb{R}^N}\mathcal{H}\Big(\frac{2Nr(1+\vartheta)\alpha_0}{N+\mu},|u_n|\Big)\,\mathrm{d}x\Big)^{\frac{N+\mu}{2Nr}},\end{aligned}$$

where $r, r' > 1$ satisfy that $1/r + 1/r' = 1$. By choosing $r > 1$ sufficient close to 1 such that $1 < r(1+\vartheta) < (1-\delta)^{-1}$, we have

$$\frac{2Nr\alpha_0(1+\vartheta)\|u_n\|_m^{N/(N-s)}}{N+\mu} < \alpha_{N,s} \quad \text{for all } n \ge n_0.$$

Thus for any $n \ge n_0$, we have

$$\int_{\mathbb{R}^N}\mathcal{H}\Big(\frac{2Nr(1+\vartheta)\alpha_0}{N+\mu},|u_n|\Big)\,\mathrm{d}x = \int_{\mathbb{R}^N}\mathcal{H}\Big(\frac{2Nr(1+\vartheta)\alpha_0\|u_n\|_m^{N/(N-s)}}{N+\mu},\frac{|u_n|}{\|u_n\|_m}\Big)\,\mathrm{d}x \le C_{N,s}.$$

Therefore, we can conclude that

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(u_n)\Big)f(u_n)u_n\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Then we have that

$$0 < c_m = \frac{s}{N}\|u_n\|_m^{N/s} + o(1) \quad \text{and} \quad o(1) = \|u_n\|_m^{N/s},$$

which is not possible. Hence, there exist $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}$ such that

$$\lim_{n\to\infty}\int_{B_1(y_n)}|u_n|^{\frac{N}{s}}\,\mathrm{d}x \ge \delta.$$

Letting $\tilde{u}_n(x) = u_n(x + y_n)$, we have

$$\lim_{n\to\infty} \int_{B_1(0)} |\tilde{u}_n|^{\frac{N}{s}}\,dx \geq \delta.$$

Since $\|\tilde{u}_n\|_m = \|u_n\|_m$, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ we have $\langle \mathcal{J}_m'(\tilde{u}_n), \varphi\rangle = o_n(1)$. Consequently, $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in X, $\tilde{u}_n \to \tilde{u}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $N/s \leq q < \infty$, and $\tilde{u}_n \to \tilde{u}$ a.e. on $\mathbb{R}^N$. In virtue of the boundedness of the sequence $\{\tilde{u}_n\}$, we have

$$\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(\tilde{u}_n)\Big] f(\tilde{u}_n)\tilde{u}_n\,dx \leq C.$$

Now, we need to show the following proposition.

Proposition 3.3. *For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have*

$$\lim_{n\to\infty} \langle \mathcal{J}_m'(\tilde{u}_n), \varphi\rangle = \langle \mathcal{J}_m'(\tilde{u}), \varphi\rangle = 0.$$

Proof. By Fatou's Lemma, we know

$$\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(\tilde{u})\Big] f(\tilde{u})\tilde{u}\,dx \leq K_0. \tag{3.12}$$

Take $\Omega = \operatorname{supp}\varphi$. For any given $\varepsilon > 0$, let $M_\varepsilon := K_0\|\varphi\|_\infty \varepsilon^{-1}$. Then, for n large enough, we have

$$\int_{\{|u_n|\geq M_\varepsilon\}\cup\{|\tilde{u}|=M_\varepsilon\}} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] |f(u_n)\varphi|\,dx \leq \frac{2\varepsilon}{K_0} \int_{|u_n|\geq \frac{M_\varepsilon}{2}} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] f(u_n)u_n\,dx \leq 2\varepsilon \tag{3.13}$$

and

$$\int_{|\tilde{u}|\geq M_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * F(\tilde{u})\Big] |f(\tilde{u})\varphi|\,dx \leq \frac{\varepsilon}{K_0} \int_{|\tilde{u}|\geq M_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * F(\tilde{u}))\Big] f(\tilde{u})\tilde{u}\,dx < \varepsilon. \tag{3.14}$$

Since

$$|f(u_n)|\chi_{|u_n|\leq M_\varepsilon} \to |f(\tilde{u})|\chi_{|\tilde{u}|\leq M_\varepsilon} \quad \text{a.e. in } \Omega \setminus D_\varepsilon,$$

where $D_\varepsilon = \{x \in \Omega : |\tilde{u}(x)| = M_\varepsilon\}$, and

$$|f(u_n)|\chi_{|u_n|\leq M_\varepsilon} \leq \max_{|t|\leq M_\varepsilon} |f(t)| < \infty \quad \text{for all } x \in \Omega,$$

the Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty} \int_{\{\Omega\setminus D_\varepsilon\}\cup\{|u_n|\leq M_\varepsilon\}} |f(u_n)|^{\frac{2}{1+\mu}}\,dx = \int_{(\Omega\setminus D_\varepsilon)\cup\{|\tilde{u}|\leq M_\varepsilon\}} |f(\tilde{u})|^{\frac{2}{1+\mu}}\,dx. \tag{3.15}$$

Here, we choose $K_\varepsilon > t_0$ such that

$$\|\varphi\|_\infty \Big(\frac{M_0K_0}{K_\varepsilon}\Big)^{\frac{1}{2}} \Big[2\mathcal{C}_0 \int_\Omega |f(\tilde{u})|^{\frac{2N}{N+\mu}}\,dx\Big]^{\frac{N+\mu}{2N}} < \varepsilon \tag{3.16}$$

and

$$\int_{|\tilde{u}|\leq M_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * (F(\tilde{u})\chi_{|\tilde{u}|\geq K_\varepsilon})\Big] |f(\tilde{u})\varphi|\,dx < \varepsilon.$$

From (F4), Lemma 1.1, (2.3), (3.15), and (3.16), we have

$$\begin{aligned}
&\int_{|u_n|\geq M_\varepsilon\cap\{|\tilde{u}|\neq M_\varepsilon\}} \Big[\frac{1}{|x|^{N-\mu}} * (F(u_n)\chi_{|u_n|\geq K_\varepsilon})\Big] |f(u_n)\varphi|\,dx \\
&\leq \|\varphi\|_\infty \Big[\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * (F(u_n)\chi_{|u_n|\geq K_\varepsilon})\Big] F(u_n)\chi_{|u_n|\geq K_\varepsilon}\,dx\Big]^{\frac{1}{2}} \\
&\quad\times \Big[\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * |f(u_n)|\chi_{\{\Omega\setminus D_\varepsilon\}\cap\{|u_n|\leq M_\varepsilon\}}\Big] |f(u_n)|\chi_{\{\Omega\setminus D_\varepsilon\}\cap\{|u_n|\leq M_\varepsilon\}}\,dx\Big]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le \|\varphi\|_\infty \Big[\int\limits_{|u_n|\ge K_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] F(u_n)\,\mathrm{d}x \Big]^{\frac12} \\
&\quad \times \Big[\int\limits_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * |f(u_n)|\chi_{\{\Omega\setminus D_\varepsilon\}\cap\{|u_n|\le M_\varepsilon\}}\Big] |f(u_n)|\chi_{\{\Omega\setminus D_\varepsilon\}\cap\{|u_n|\le M_\varepsilon\}}\,\mathrm{d}x \Big]^{\frac12} \\
&\le \|\varphi\|_\infty \Big[\int\limits_{|u_n|\ge K_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] F(u_n)\,\mathrm{d}x \Big]^{\frac12} \times \Big[\mathcal{C}_0 \int\limits_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\le M_\varepsilon\}} |f(u_n)|^{\frac{2N}{N+\mu}}\,\mathrm{d}x \Big]^{\frac{N+\mu}{2N}} \\
&\le \|\varphi\|_\infty \Big[\frac{M_0}{K_\varepsilon} \int\limits_{|u_n|\ge K_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] f(u_n)u_n\,\mathrm{d}x \Big]^{\frac12} \times \Big[2\mathcal{C}_0 \int\limits_{\Omega} |f(\bar u)|^{\frac{2N}{N+\mu}}\,\mathrm{d}x + o(1) \Big]^{\frac{N+\mu}{2N}} \\
&\le \|\varphi\|_\infty \Big(\frac{M_0K_0}{K_\varepsilon} \Big)^{\frac12} \Big[2\mathcal{C}_0 \int\limits_{\Omega} |f(\bar u)|^{\frac{2N}{N+\mu}}\,\mathrm{d}x \Big]^{\frac{N+\mu}{2N}} + o(1) \\
&< \varepsilon + o(1).
\end{aligned}
$$

For any $x \in \mathbb{R}$, define $\zeta_n(x)$ and $\bar\zeta$ as follows:

$$
\zeta_n(x) := \frac{1}{A_\mu}\Big(\frac{1}{|x|^{N-\mu}} * (|F(u_n)|\chi_{|u_n|\le K_\varepsilon}) \Big) = \int\limits_{\mathbb{R}^N} \frac{|F(u_n)|\chi_{|u_n|\le K_\varepsilon}}{|x-y|^{N-\mu}}\,\mathrm{d}y
$$

and

$$
\bar\zeta(x) := \frac{1}{A_\mu}\Big(\frac{1}{|x|^{N-\mu}} * (|F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}) \Big) = \int\limits_{\mathbb{R}^N} \frac{|F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}}{|x-y|^{N-\mu}}\,\mathrm{d}y.
$$

Then we have

$$
\begin{aligned}
|\zeta_n(x) - \bar\zeta(x)| &\le \int\limits_{\mathbb{R}^N} \frac{||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}|}{|x-y|^{N-\mu}}\,\mathrm{d}y \\
&\le \Big[\int\limits_{|x-y|\le R} ||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}|^{\frac{2N-\mu}{\mu}}\,\mathrm{d}y \Big]^{\frac{\mu}{2N-\mu}} \times \Big(\int\limits_{|x-y|\le R} \frac{1}{|x-y|^{(2N-\mu)/2}}\,\mathrm{d}y \Big)^{\frac{2N-2\mu}{2N-\mu}} \\
&\quad + \Big[\int\limits_{|x-y|>R} ||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}|^{\frac{N+\mu}{2\mu}}\,\mathrm{d}y \Big]^{\frac{2\mu}{N+\mu}} \times \Big(\int\limits_{|x-y|>R} \frac{1}{|x-y|^{N+\mu}}\,\mathrm{d}y \Big)^{\frac{N-\mu}{N+\mu}} \\
&\le \Big(\frac{2Nw_N}{\mu} \cdot R^{\frac{\mu}{2}} \Big)^{\frac{2N-2\mu}{2N-\mu}} \Big[\int\limits_{|x-y|\le R} ||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}|^{\frac{2N-\mu}{\mu}}\,\mathrm{d}y \Big]^{\frac{\mu}{2N-\mu}} \\
&\quad + \Big(\frac{Nw_N}{\mu R^\mu} \Big)^{\frac{N-\mu}{N+\mu}} \Big(\int\limits_{|x-y|>R} ||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}|^{\frac{N+\mu}{2\mu}}\,\mathrm{d}y \Big)^{\frac{2\mu}{N+\mu}} \\
&\le \Big(\frac{2Nw_N}{\mu} \cdot R^{\frac{\mu}{2}} \Big)^{\frac{2N-2\mu}{2N-\mu}} \Big[\int\limits_{|x-y|\le R} ||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar u)|\chi_{|\bar u|\le K_\varepsilon}|^{\frac{2N-\mu}{\mu}}\,\mathrm{d}y \Big]^{\frac{\mu}{2N-\mu}} \\
&\quad + C_\varepsilon \Big(\frac{Nw_N}{\mu R^\mu} \Big)^{\frac{N-\mu}{N+\mu}} \big[\|u_n\|_{(N+\mu)^2/4\mu s}^{(N+\mu)/s} + \|\bar u\|_{(N+\mu)^2/4\mu s}^{(N+\mu)/s} \big] \\
&\le \Big(\frac{2Nw_N}{\mu} \cdot R^{\frac{\mu}{2}} \Big)^{\frac{2N-2\mu}{2N-\mu}} o_n(1) + \tilde C_\varepsilon \Big(\frac{Nw_N}{\mu R^\mu} \Big)^{\frac{N-\mu}{N+\mu}},
\end{aligned}
$$

which implies that, for any $x \in \mathbb{R}^N$, we have $\zeta_n(x) \to \bar\zeta(x)$. Similarly, we know, for any $x \in \mathbb{R}^N$, that $|\zeta_n(x)| \le \mathcal{M}$. It follows that

$$
|\zeta_n(x) f(u_n(x))\chi_{|u_n|\le M_\varepsilon}(x)\varphi(x)| \le \mathcal{M}\|\varphi\|_\infty \max_{|t|\le M_\varepsilon} |f(t)| \quad \text{for all } x \in \Omega.
$$

Therefore, together with $\zeta_n(x) \to \bar{\zeta}(x)$ and the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int\limits_{\{|u_n|\le M_\varepsilon\}\cap\{|\bar{u}|\ne M_\varepsilon\}} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\chi_{|u_n|\le K_\varepsilon}\Big]|f(u_n)\varphi|\,\mathrm{d}x \to \int\limits_{|\bar{u}|<M_\varepsilon} \Big[\frac{1}{|x|^{N-\mu}} * F(\bar{u})\chi_{|\bar{u}|\le K_\varepsilon}\Big]|f(\bar{u})\varphi|\,\mathrm{d}x.$$

This concludes the proof. □

Proposition 3.3 implies that $\tilde{u}$ is a nontrivial solution of (3.1), and it is easy to see that $\mathcal{J}_m(\tilde{u}) \ge c_m$. By Fatou's lemma, we know

$$\begin{aligned}
c_m &= \lim_{n\to\infty}\Big[\mathcal{J}_m(\tilde{u}_n) - \frac{s}{N}\langle \mathcal{J}'_m(\tilde{u}_n), \tilde{u}_n\rangle\Big]\\
&= \lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}} * F(\tilde{u}_n)\Big)\Big(\frac{s}{N}f(\tilde{u}_n)\tilde{u}_n - \frac{1}{2}F(\tilde{u}_n)\Big)\,\mathrm{d}x\\
&\ge \int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}} * F(\tilde{u})\Big)\Big(\frac{s}{N}f(\tilde{u})\tilde{u} - \frac{1}{2}F(\tilde{u})\Big)\,\mathrm{d}x\\
&= \mathcal{J}_m(\tilde{u}) - \frac{s}{N}\langle \mathcal{J}'_m(\tilde{u}), \tilde{u}\rangle\\
&= \mathcal{J}_m(\tilde{u}).
\end{aligned}$$

Therefore $\mathcal{J}_m(\tilde{u}) = c_m$. Combining with $\mathcal{J}'_m(\tilde{u}) = 0$, we complete the proof. □

4 Existence of positive solutions

In this section, we are going to prove the existence of positive ground state solutions to equation (1.1). Consider the following equation:

$$(-\Delta)^s_{N/s}u + V_0|u|^{\frac{N}{s}-2}u = \Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big)f(u) \quad \text{in } \mathbb{R}^N, \tag{4.1}$$

where V_0 is given in (V). In view of Lemma 3.1, we know that (4.1) possesses a positive ground state solution u_0 satisfying

$$\mathcal{J}_{V_0}(u_0) = c_{V_0} = \inf_{\mathcal{N}_{V_0}} \mathcal{J}_{V_0},$$

where

$$\mathcal{J}_{V_0}(u) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{N/s}}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}^N} V_0|u|^{N/s}\,\mathrm{d}x - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\mathcal{N}_{V_0} = \{u \in E_{V_0} \setminus \{0\} : \langle \mathcal{J}'_{V_0}(u), u\rangle = 0\}.$$

The definitions of E_{V_0} and $\langle \mathcal{J}'_{V_0}(u), u\rangle$ are similar to (2.2) and (2.5).

Consider another equation:

$$(-\Delta)^s_{N/s}u + V_\infty|u|^{\frac{N}{s}-2}u = \Big(\frac{1}{|x|^{N-\mu}} * F(u)\Big)f(u) \quad \text{in } \mathbb{R}^N, \tag{4.2}$$

where V_∞ is given in (V). In view of Lemma 3.1, we know that (4.2) possesses a positive ground state solution u_0 satisfying

$$\mathcal{J}_{V_\infty}(u_0) = c_{V_\infty} = \inf_{\mathcal{N}_{V_\infty}} \mathcal{J}_{V_\infty},$$

where the definitions of $\mathcal{J}_{V_\infty}$ and $\mathcal{N}_{V_\infty}$ are similar to $\mathcal{J}_{V_0}$ and $\mathcal{N}_{V_0}$.

We begin this section by analyzing the comparison relationship of the ground state energy level between problem (2.1) and problem (4.1), which is very important in our arguments.

Lemma 4.1. *Assume that* (V) *and* (F1)–(F6) *hold. Let* c_ε *be the minimax value defined by* (2.6). *Then the following assertions hold:*

(i) $\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}$.

(ii) $\lim_{\varepsilon\to 0} c_\varepsilon < c_{V_\infty}$.

Proof. (i) Given $\delta > 0$, fix $w_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ satisfying

$$w_\delta \in \mathcal{N}_{V_0}, \quad w_\delta \to w \text{ in } E_{V_0}, \quad \mathcal{J}_{V_0}(w_\delta) < c_{V_0} + \delta.$$

Now, let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta = 1$ on $B_1(0)$, and $\eta = 0$ on $\mathbb{R}^N \setminus B_2(0)$, and define $v_n(x) = \eta(\varepsilon_n x) w_\delta(x)$, where $\varepsilon_n \to 0$ as $n \to +\infty$. Clearly,

$$v_n \to w_\delta \quad \text{in } E_{V_0}, \text{ as } n \to +\infty.$$

By Lemma 2.7, there exists $t_n > 0$ such that $t_n v_n \in \mathcal{N}_{\varepsilon_n}$. We claim that t_n is bounded; otherwise, $|t_n| \to +\infty$. Consequently,

$$\begin{aligned} c_{\varepsilon_n} \le \mathcal{I}_{\varepsilon_n}(t_n v_n) = \frac{s t_n^{N/s}}{N} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v_n(x) - v_n(y)]^{N/s}}{|x-y|^{2N}}\,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon_n x)|v_n|^{\frac{N}{s}}\,dx \\ - \frac{1}{2}\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(t_n v_n)\Big] F(t_n v_n)\,dx. \end{aligned}$$

Together with $\langle \mathcal{I}'_{\varepsilon_n}(t_n v_n), t_n v_n\rangle = 0$, we obtain

$$\begin{aligned} t_n^{N/s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v_n(x) - v_n(y)]^{N/s}}{|x-y|^{2N}}\,dx\,dy + t_n^{N/s}\int_{\mathbb{R}^N} V(\varepsilon_n x)|v_n|^{\frac{N}{s}}\,dx &= \int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\mu}} * F(t_n v_n)\Big] f(t_n v_n) t_n v_n\,dx \\ &\ge C t_n^{2\bar{\mu}} \int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{N-\mu}} * |v_n|^{\bar{\mu}}\Big]|v_n|^{\bar{\mu}}\,dx, \end{aligned}$$

which means that $\{t_n\}$ is bounded and, up to a subsequence, we have $t_n \to T_0 \ge 0$. Notice that there exists a constant $\bar{c} > 0$, independent of ε, such that $c_{\varepsilon_n} > \bar{c} > 0$, which implies that $T_0 > 0$. Then, by the characteristic of c_{ε_n} and w_δ, we have

$$\begin{aligned} c_{\varepsilon_n} \le \mathcal{I}_{\varepsilon_n}(t_n v_n) &= \mathcal{J}_{V_0}(t_n v_n) + \frac{s t_n^{N/s}}{N}\int_{\mathbb{R}^N} [V(\varepsilon_n x) - V_0] v_n^{N/s}\,dx \\ &= \mathcal{J}_{V_0}(T_0 w_\delta) + o_n(1) \\ &\le \mathcal{J}_{V_0}(w_\delta) + o_n(1) \\ &\le c_{V_0} + \delta. \end{aligned}$$

Since δ, ε_n are arbitrary, it follows that $\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V_0}$. On the other hand, we already know that, for any $\varepsilon > 0$, we have $c_\varepsilon \ge c_{V_0}$, which implies that $\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}$.

(ii) Since $V_0 < V_\infty$, by a standard argument, we know that $c_{V_0} < c_{V_\infty}$, which together with (i) implies that $\lim_{\varepsilon\to 0} c_\varepsilon < c_{V_\infty}$. The proof is now complete. □

Hence, by Lemma 4.1, there exists a $\varepsilon_0 > 0$ such that

$$c_\varepsilon < \frac{s}{N}\Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}} \quad \text{for all } \varepsilon \in [0, \varepsilon_0).$$

Now we prove the existence result of the positive ground state solution of problem (2.1).

Lemma 4.2. *Assume that* (V) *and* (F1)–(F6) *hold. There exists* $\varepsilon_0 > 0$ *such that equation* (2.1) *has a positive ground state solution* u_ε *for all* $\varepsilon < \varepsilon_0$.

Proof. Since $\mathcal{I}_\varepsilon$ satisfies the mountain pass geometry, there exists a Palais–Smale sequence $\{u_n\}$ at level c_ε, namely

$$\mathcal{I}_\varepsilon(u_n) \to c_\varepsilon < \frac{s}{N}\Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}}, \quad \mathcal{I}'_\varepsilon(u_n) \to 0, \qquad n \to \infty.$$

Inspired by Lemma 2.8, we obtain that $\|u_n\|_\varepsilon$ is bounded in E_ε, and $\{u_n\}$ is a nonnegative Palais–Smale sequence, without loss of generality. Thus, we have $u_n \rightharpoonup u_\varepsilon$ in E_ε.

Next, we claim that $\mathcal{J}'_\varepsilon(u_\varepsilon) = 0$. In fact, if $u_\varepsilon = 0$, then the claim is completed. If $u_\varepsilon \neq 0$, by $\mathcal{J}'_\varepsilon(u_n) \to 0$, there exists a constant C such that

$$\int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] f(u_n) u_n \,\mathrm{d}x = \|u_n\|_\varepsilon^{N/s} \leq C. \tag{4.3}$$

This together with Proposition 3.3 implies that, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big] f(u_n)\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(u_\varepsilon)\Big] f(u_\varepsilon)\varphi \,\mathrm{d}x.$$

Inspired by the arguments in [53], for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{N/s-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{N/s-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{2N}} \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^{N/s-2}u_n\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} V(x)|u|^{N/s-2}u\varphi \,\mathrm{d}x,$$

which implies that

$$\langle \mathcal{J}'_\varepsilon(u_\varepsilon), \varphi\rangle = \lim_{n\to\infty} \langle \mathcal{J}'_\varepsilon(u_n), \varphi\rangle \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

By the characterization of u_ε and Fatou's lemma, we have

$$\begin{aligned}
c_\varepsilon \leq \mathcal{J}_\varepsilon(u_\varepsilon) &= \mathcal{J}_\varepsilon(u_\varepsilon) - \frac{s}{N}\langle \mathcal{J}'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_\varepsilon(y))[(\frac{s}{N})f(u_\varepsilon(x))u_\varepsilon(x) - (\frac{1}{2})F(u_\varepsilon(x))]}{|x-y|^{N-\mu}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_n(y))[\frac{s}{N}f(u_n(x))u_n(x) - (\frac{1}{2})F(u_n(x))]}{|x-y|^{N-\mu}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \liminf_{n\to\infty} \Big[\mathcal{J}_\varepsilon(u_n) - \frac{s}{N}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle\Big] \\
&\leq c_\varepsilon.
\end{aligned}$$

Together with $\mathcal{J}'_\varepsilon(u_\varepsilon) = 0$, we have $\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon$. Then, by (F3), we have

$$\begin{aligned}
c_\varepsilon &= \mathcal{J}_\varepsilon(u_\varepsilon) - \frac{1}{2\bar{\mu}}\langle \mathcal{J}'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle \\
&\leq \Big(\frac{s}{N} - \frac{1}{2\bar{\mu}}\Big)\|u_\varepsilon\|_\varepsilon^{N/s} + \frac{1}{2\bar{\mu}} \int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(u_\varepsilon)\Big](f(u_\varepsilon)u_\varepsilon - \bar{\mu}F(u_\varepsilon)) \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \liminf_{n\to\infty}\Big(\Big(\frac{s}{N} - \frac{1}{2\bar{\mu}}\Big)\|u_n\|_\varepsilon^{N/s} + \frac{1}{2\bar{\mu}} \int_{\mathbb{R}^N} \Big[\frac{1}{|x|^{N-\mu}} * F(u_n)\Big](f(u_n)u_n - \bar{\mu}F(u_n)) \,\mathrm{d}x\,\mathrm{d}y\Big) \\
&\leq \liminf_{n\to\infty}\Big(\mathcal{J}_\varepsilon(u_n) - \frac{1}{2\bar{\mu}}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle\Big) \\
&\leq c_\varepsilon.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|u_n\|_\varepsilon = \|u_\varepsilon\|_\varepsilon.$$

So, $u_n \to u_\varepsilon \geq 0$ in E_ε, showing that $\mathfrak{I}_\varepsilon$ verifies the $(PS)_{c_\varepsilon}$ condition. There it is only needed to show that $u_\varepsilon \neq 0$ for small $\varepsilon > 0$. Next, arguing by contradiction, assume that $u_\varepsilon = 0$. By the arguments explored in the proof of Lemma 3.1, we can deduce that there exists $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}$ such that

$$\lim_{n\to\infty} \int_{B_1(y_n)} |u_n|^{\frac{N}{s}} \, dx \geq \delta. \tag{4.4}$$

Defining $\tilde{u}_n(x) = u_n(x + y_n)$, we have

$$\int_{B_1(0)} |\tilde{u}_n|^2 \, dx \geq \delta.$$

It is easy to see that $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in E_ε, $\tilde{u}_n \to \tilde{u}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $q \in [N/s, +\infty)$, $\tilde{u}_n \to \tilde{u}$ a.e. on $\mathbb{R}^N$. By using the fact that $\tilde{u}_n \geq 0$ for all $n \in \mathbb{N}$, there exist $\zeta > 0$ and a subset $\Omega \subset \mathbb{R}^N$ with a positive measure such that $\tilde{u}(x) > \zeta$ for all $x \in \Omega$.

Let $\{t_n\}$ be the sequence such that $\{t_n u_n\} \subset \mathcal{N}_{V_\infty}$. Hence,

$$\mathcal{J}_{V_\infty}(t_n u_n) \geq c_{V_\infty} \quad \text{and} \quad \langle \mathcal{J}'_{V_\infty}(t_n u_n), t_n u_n\rangle = 0. \tag{4.5}$$

If $t_n \to +\infty$ as $n \to +\infty$, by (4.5) and for n large enough, we have

$$\begin{aligned}
0 &= t_n^{-N/s} \langle \mathcal{J}'_{V_\infty}(t_n u_n), t_n u_n\rangle \\
&= t_n^{-N/s} \langle \mathcal{J}'_{V_\infty}(t_n \tilde{u}_n), t_n \tilde{u}_n\rangle \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{N/s}}{|x-y|^{2N}} \, dx\, dy + \int_{\mathbb{R}^N} V_\infty |\tilde{u}_n|^{N/s} \, dx - t_n^{-N/s} \int_\Omega\int_\Omega \frac{F(t_n\tilde{u}_n(y))}{|x-y|^{N-\mu}} f(t_n\tilde{u}_n(x)) t_n \tilde{u}_n(x) \, dx\, dy.
\end{aligned}$$

By (F5), we know that

$$\liminf_{n\to+\infty} t_n^{-N/s} \int_\Omega\int_\Omega \frac{F(t_n\tilde{u}_n(y))}{|x-y|^{N-\mu}} f(t_n\tilde{u}_n(x)) t_n \tilde{u}_n(x) \, dx\, dy = +\infty.$$

Together with the boundedness of $\|\tilde{u}_n\|_\varepsilon$, we get a contradiction, which implies that $\{t_n\}$ is bounded. Without loss of generality, we may assume that $0 \leq t_n \leq T_1$. Given $\varsigma > 0$, by condition (V), there exists $R = R(\varsigma) > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \varsigma \quad \text{for any } x \in \mathbb{R}^N \setminus B_R(0). \tag{4.6}$$

Then, by the arbitrariness of ς, we have

$$\begin{aligned}
c_\varepsilon + o_n(1) &= \mathfrak{I}_\varepsilon(u_n) \\
&\geq \mathfrak{I}_\varepsilon(t_n u_n) + \frac{s - st^{N/s}}{N} \langle \mathfrak{I}'_\varepsilon(u_n), u_n\rangle \\
&= \mathfrak{I}_\varepsilon(t_n u_n) + o_n(1) \\
&= \mathfrak{I}_{V_\infty}(t_n u_n) + \frac{st_n^{N/s}}{N} \int_{\mathbb{R}} [V(\varepsilon x) - V_\infty]|u_n|^{N/s} \, dx + o_n(1) \\
&\geq c_{V_\infty} + \frac{st_n^{N/s}}{N} \int_{B_R(0)} [V(\varepsilon x) - V_\infty]|u_n|^{N/s} \, dx + \frac{st_n^{N/s}}{N} \int_{\mathbb{R}^N\setminus B_R(0)} [V(\varepsilon x) - V_\infty]|u_n|^{N/s} \, dx + o_n(1) \\
&\geq c_{V_\infty} - \frac{s(V_\infty - V_0)T_1^{N/S}}{N} \int_{B_R(0)} |u_n|^{N/s} \, dx - \frac{sT_1^{N/s}}{N} \sup_{\mathbb{R}^N\setminus B_R(0)} [V_\infty - V(\varepsilon x)]\|u_n\|^{N/s}_{N/S} + o_n(1) \\
&\geq c_\varepsilon + \frac{st_\infty^{N/s}}{N} \int_{\mathbb{R}^N} [V_\infty - V(\varepsilon x)]|u_\infty|^{N/s} \, dx - \frac{sT_1^{N/s}}{N} \sup_{\mathbb{R}^N\setminus B_R(0)} [V_\infty - V(\varepsilon x)]\|u_n\|^{N/s}_{N/s} + o_n(1) \\
&\geq c_\varepsilon + \frac{st_\infty^{N/s}}{2N} \int_{\mathbb{R}} [V_\infty - V(\varepsilon x)]|u_\infty|^{N/s} \, dx + o_n(1) \\
&> c_\varepsilon,
\end{aligned}$$

which is a contradiction due to $u_\varepsilon = 0$. Thus, we have completed the proof. □

5 Concentration of positive solutions

In this section, we are going to show the concentration behavior of positive ground state solutions.

Lemma 5.1. *Let $\varepsilon_n \to 0$ and let $\{u_n\}$ be the sequence of solutions obtained in Lemma 4.2. Then there exists a sequence $\{y_n\} \subset \mathbb{R}$ such that $v_n = u_n(x + y_n)$ has a convergent subsequence in E. Moreover, up to a subsequence, $y_n \to y \in M$.*

Proof. Let $\{u_n\}$ be the sequence of solutions obtained in Lemma 4.2. It is easy to see that $c_{\varepsilon_n} = \mathcal{I}_{\varepsilon_n}(u_n) \to c_{V_0}$, $\{u_n\}$ is bounded in E_{V_0}, and

$$0 < c_{V_0} = \limsup_{n\to\infty} c_{\varepsilon_n} < \frac{s}{N} \cdot \Big[\frac{(N+\mu)\alpha_{N,s}}{2N\alpha_0}\Big]^{\frac{N-s}{s}}.$$

Following the arguments explored in the proof of Theorem 3.1, there exist $r, \delta > 0$ and $\tilde{y}_n \in \mathbb{R}$ such that

$$\liminf_{n\to\infty} \int_{B_r(\tilde{y}_n)} |u_n|^{\frac{N}{s}}\,dx \geq \delta.$$

Setting $v_n(x) = u_n(x + \tilde{y}_n)$, up to a subsequence if necessary, we may assume $v_n \rightharpoonup v \not\equiv 0$ in E_{V_0}. Let $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$. Then

$$c_{V_0} \leq \mathcal{J}_{V_0}(\tilde{v}_n) = \mathcal{J}_{V_0}(t_n u_n) \leq \mathcal{I}_{\varepsilon_n}(t_n u_n) \leq \mathcal{I}_{\varepsilon_n}(u_n) \to c_{V_0},$$

which implies that $\mathcal{J}_{V_0}(\tilde{v}_n) \to c_{V_0}$ as $n \to \infty$. Then the sequence $\{\tilde{v}_n\}$ is a minimizing sequence, and, by the Ekeland variational principle [20], we may also assume that it is a bounded (PS) sequence at c_{V_0}. Thus, for some subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in E_{V_0} with $\tilde{v} \neq 0$ and $\mathcal{J}'_{V_0}(\tilde{v}) = 0$. Applying the same arguments as the ones used in the proof of Lemma 3.1, we have that $\tilde{v}_n \to \tilde{v}$ in E_{V_0}. Since $\{t_n\}$ is bounded, we can assume that, for some subsequence, $t_n \to T_2 > 0$, and so $v_n \to v$ in E_{V_0}, where $v = \tilde{v}/T_2$.

Next, we will show that $\{y_n\} = \{\varepsilon_n \tilde{y}_n\}$ has a subsequence satisfying $y_n \to y \in M$. We point out that $\{y_n\}$ is bounded in $\mathbb{R}^N$. Indeed, if not, there would exist a subsequence, which we still denote by $\{y_n\}$, such that $|y_n| \to \infty$. Since $\tilde{v}_n \to \tilde{v}$ in E_{V_0} and $V_0 < V_\infty$, we have

$$\begin{aligned}
c_{V_0} &= \frac{s}{N}\|\tilde{v}\|_{V_0}^{N/s} - \frac{1}{2}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(\tilde{v})\Big)F(\tilde{v})\,dx \\
&< \frac{s}{N}\|\tilde{v}\|_{V_\infty}^{N/s} - \frac{1}{2}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(\tilde{v})\Big)F(\tilde{v})\,dx \\
&\leq \liminf_{n\to\infty}\Big[\frac{s}{N}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\tilde{v}_n(x)-\tilde{v}_n(y))^{N/s}}{|x-y|^{2N}}\,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\tilde{v}_n^{N/s}\,dx\Big) - \frac{1}{2}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(\tilde{v})\Big)F(\tilde{v})\,dx\Big] \\
&= \liminf_{n\to\infty}\Big[\frac{st_n^{N/s}}{N}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_n(x)-u_n(y))^{N/s}}{|x-y|^{2N}}\,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon_n x)u_n^{N/s}\,dx\Big) \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{N-\mu}} * F(t_n u_n)\Big)F(t_n u_n)\,dx\Big] \\
&= \liminf_{n\to\infty}\mathcal{I}_{\varepsilon_n}(t_n u_n) \\
&\leq \liminf_{n\to\infty}\mathcal{I}_{\varepsilon_n}(u_n) \\
&= c_{V_0},
\end{aligned}$$

and hence the absurd shows that $\{y_n\}$ stays bounded and, up to a subsequence, $y_n \to y \in \mathbb{R}^N$. Then necessarily $y \in M$; otherwise, we would again get a contradiction as above. □

Let $\varepsilon_n \to 0$ as $n \to \infty$, and let u_n be the ground state solution of

$$(-\Delta)^s_{N/s}u + V(\varepsilon_n x)|u|^{\frac{N}{s}-2}u = \Big[\frac{1}{|x|^{N-\mu}} * F(u)\Big]f(u) \quad \text{in } \mathbb{R}^N.$$

From Lemma 4.1 we know that

$$\lim_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) \to c_{V_0}.$$

Then there exists a subsequence $\tilde{y}_n \in \mathbb{R}$ such that $v_n = u_n(x + \tilde{y}_n) > 0$ is a solution of

$$(-\Delta)^s_{N/s} u + V_n(x)|u|^{\frac{N}{s}-2}u = \Big[\frac{1}{|x|^{N-\mu}} * F(u)\Big] f(u) \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$. Moreover, $\{v_n\}$ has a convergent subsequence in E_{V_0} and $y_n \to y \in M$, up to a subsequence, where $y_n = \varepsilon_n \tilde{y}_n$. Hence, there exists $h \in W^{s,p}(\mathbb{R}^N)$ such that

$$|v_n(x)| \le h(x) \quad \text{a.e. in } \mathbb{R}^N \text{ for all } n \in \mathbb{N}.$$

Lemma 5.2. *Assume that* (V), (F1), (F3)–(F6) *and*

$$\lim_{t\to 0} \frac{f(t)}{|t|^{N/s-1}} = 0 \tag{5.2}$$

hold. Then there exists $C > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Proof. Using similar arguments to the ones explored by [4], we can obtain that

$$\Big|\frac{1}{|x|^{N-\mu}} * F(h)\Big| \le C.$$

Since F is a nondecreasing function, we know that

$$0 \le \Big[\frac{1}{|x|^{N-\mu}} * F(v_n)\Big] \le \Big[\frac{1}{|x|^{N-\mu}} * F(h)\Big].$$

For any $R > 0$, $0 < r \le R/2$, let $\eta \in \mathcal{C}^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$, with $\eta(x) = 1$ when $|x| \ge R$, $\eta(x) = 0$ when $|x| \le R - r$, and $|\nabla\eta| \le 2/r$. Following the technique explored in [38], for $L > 0$, we set $v_{L,n} = \min\{v_n, L\}$ and

$$\gamma(v_n) = \gamma_{L,\beta}(v_n) = \eta^p v_{L,n}^{p(\beta-1)} v_n,$$

with $\beta > 1$ to be determined later. Set

$$\Lambda(t) = \frac{|t|^p}{p} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(\tau))^{\frac{1}{p}}\, d\tau.$$

Thus we have the conclusion as in [9]:

$$\Lambda'(a-b)(\gamma(a)-\gamma(b)) \ge |\Gamma(a)-\Gamma(b)|^p \quad \text{for any } a, b \in \mathbb{R},$$

from which we have

$$|\Gamma(v_n(x)) - \Gamma(v_n(y))|^p \le |v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((\eta^p v_{L,n}^{p(\beta-1)} v_n)(x) - (\eta^p v_{L,n}^{p(\beta-1)} v_n)(y)). \tag{5.3}$$

Using $\gamma(v_n) = \eta^p v_{L,n}^{p(\beta-1)} v_n$ as a test function in (5.1), in view of (5.3) and the Cauchy inequality, we have

$$\begin{aligned}
[\Gamma(v_n)]^p_{s,p} &+ \int_{\mathbb{R}^N} V_n(x)\eta^p |v_n|^p v_{L,n}^{p(\beta-1)}\, dx \\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{2N}} [(\eta^p v_{L,n}^{p(\beta-1)} v_n)(x) - (\eta^p v_{L,n}^{p(\beta-1)} v_n)(y)]\, dx\, dy \\
&\qquad + \int_{\mathbb{R}^N} V_n(x)\eta^p |v_n|^p v_{L,n}^{p(\beta-1)}\, dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(v_n(y)) f(v_n(x))\eta^p v_{L,n}^{p(\beta-1)} v_n}{|x-y|^{N-\mu}}\, dx\, dy \\
&\le C \int_{\mathbb{R}^N} f(v_n(x))\eta^p v_{L,n}^{p(\beta-1)} v_n\, dx.
\end{aligned}$$

Since $\Gamma(v_n) \geq \frac{1}{\beta}\eta v_n v_{L,n}^{\beta-1}$ and the embedding

$$W^{s,N/s}(\mathbb{R}^N) \to L^\theta(\mathbb{R}^N), \quad \theta \geq \frac{N}{s},$$

is continuous, there exists a suitable constant $S_* > 0$ such that

$$\|\Gamma(v_n)\|_{V_0/2}^{N/s} \geq S_*\|\Gamma(v_n)\|_\theta^{N/s} \geq \frac{1}{\beta^p}S_*\|\eta v_n v_{L,n}^{\beta-1}\|_\theta^{N/s},$$

where the norm $\|\cdot\|_{V_0/2}$ is defined by

$$\|u\|_{V_0/2} = \big([u]_{s,p}^p + \|u\|_{V_0/2,p}^p\big)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{V_0/2,p} = \Big(\int_{\mathbb{R}^N}(V_0|u|^p)/2\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

Recall that for each $\varepsilon > 0$, by the hypothesis (5.2), there exist ε and $C_\varepsilon > 0$ such that, for any $u \in W^{s,N/s}(\mathbb{R}^N)$,

$$f(u)u \leq \varepsilon|u|^{N/s} + C_{q,\varepsilon}|u|^{N/s}\mathcal{H}(\beta\alpha_0, u)$$

holds. Then we obtain

$$\begin{aligned}\frac{1}{\beta^p}S_*\|\eta v_n v_{L,n}^{\beta-1}\|_\theta^{N/s} &+ \int_{\mathbb{R}^N}V_n(x)\eta^p|v_n|^p v_{L,n}^{p(\beta-1)}\,\mathrm{d}x\\ &\leq \varepsilon\int_{\mathbb{R}^N}|\eta v_n v_{L,n}^{\beta-1}|^{N/s}\,\mathrm{d}x + C_{q,\varepsilon}\int_{\mathbb{R}^N}\eta^p|v_n|^{N/s}v_{L,n}^{p(\beta-1)}\mathcal{H}(\beta\alpha_0, v_n)\,\mathrm{d}x.\end{aligned}$$

Choose $0 < \varepsilon < V_0/2$. Then we have

$$\frac{1}{\beta^p}S_*\|\eta v_n v_{L,n}^{\beta-1}\|_\theta^{N/s} \leq C_{q,\varepsilon}\Big(\int_{\mathbb{R}^N\setminus B_{R-r}(0)}|v_n v_{L,n}^{\beta-1}|^{\frac{Nt}{s(t-1)}}\,\mathrm{d}x\Big)^{\frac{t-1}{t}}\Big(\int_{\mathbb{R}^N}\mathcal{H}(t\beta\alpha_0, v_n)\,\mathrm{d}x\Big)^{\frac{1}{t}}.$$

Take $w_{L,n} = \eta v_n v_{L,n}^{\beta-1}$. By using the Trudinger–Moser inequality in $W^{s,N/s}(\mathbb{R}^N)$ with $t > 1$ and t near 1 and $\frac{t}{t-1} \gg \frac{N}{s}$ such that $\frac{pt}{t-1} < \theta$, there exists a constant $D > 0$ such that

$$\|w_{L,n}\|_\theta^p \leq D\beta^p\|v_n v_{L,n}^{\beta-1}\|_{(pt/(t-1))(|x|\geq R-r)}^p.$$

Note that

$$\begin{aligned}|v_{L,n}|_{\theta\beta(|x|\geq R)}^{p\beta} &= \Big(\int_{\mathbb{R}^N\setminus B_R(0)}|v_{L,n}|^{\theta\beta}\Big)^{\frac{N}{s\theta}}\\ &\leq \Big(\int_{\mathbb{R}^N}|\eta v_n v_{L,n}^{\beta-1}|^\theta\,\mathrm{d}x\Big)^{\frac{N}{s\theta}}\\ &= \|w_{L,n}\|_\theta^p\\ &\leq D\beta^p\|v_n v_{L,n}^{\beta-1}\|_{(pt/(t-1))(|x|\geq R-r)}^p\\ &\leq D\beta^p\|v_n\|_{(p\beta t/(t-1))(|x|\geq R-r)}^{p\beta}.\end{aligned}$$

By applying Fatou's lemma, we deduce

$$\|v_n\|_{\theta\beta(|x|\geq R)}^{p\beta} \leq D\beta^p\|v_n\|_{(p\beta t/(t-1))(|x|\geq R-r)}^{p\beta}.$$

Now, we set $\beta = \frac{\theta(t-1)}{pt} > 1$. Then we have

$$\begin{aligned}\|v_n\|_{\theta\beta^2(|x|\geq R)} &\leq D^{\frac{1}{p\beta^2}}\beta^{\frac{1}{\beta^2}}\|v_n\|_{(p\beta^2t/(t-1))(|x|\geq R-r)}\\ &= D^{\frac{1}{p\beta^2}}\beta^{\frac{1}{\beta^2}}\|v_n\|_{\theta\beta(|x|\geq R-r)}\\ &\leq D^{\frac{1}{p}(\frac{1}{\beta}+\frac{2}{\beta^2})}\beta^{\frac{1}{\beta}+\frac{2}{\beta^2}}\|v_n\|_{(p\beta t/(t-1))(|x|\geq R-2r)}.\end{aligned}$$

Following the arguments explored by [26], we set $r = 2^{-(m+1)}R$. Iterating the above process, we can infer that

$$\|v_n\|_{\theta\beta^m(|x|\geq R)} \leq D^{\sum_{j=1}^{m}\frac{1}{p\beta^j}}\beta^{\sum_{j=1}^{m}\frac{j}{\beta^j}}\|v_n\|_{(p\beta t/(t-1))(|x|\geq R/2)}, \quad \text{where } m \text{ is a positive integer.} \tag{5.4}$$

Taking the limit in (5.4) as $m \to \infty$, we get that, for all n,

$$\|v_n\|_{\infty(|x|\geq R)} \leq C\|v_n\|_{(p\beta t/(t-1))(|x|\geq R/2)}, \tag{5.5}$$

where

$$C = D^{\sum_{j=1}^{\infty}\frac{1}{p\beta^j}}\beta^{\sum_{j=1}^{\infty}\frac{j}{\beta^j}} < +\infty.$$

Set $\tilde{y}(v_n) = v_n v_{L,n}^{p(\beta-1)}$. Repeating the above process and after a minor modification, we have

$$\|v_n\|_\infty \leq C,$$

where

$$C = D^{\sum_{j=1}^{\infty}\frac{1}{p\beta^j}}\beta^{\sum_{j=1}^{\infty}\frac{j}{\beta^j}}\|v_n\|_{(p\beta t/(t-1))} < +\infty.$$

Then, using the fact that $v_n \to v$ in E_{V_0} on the right-hand side of (5.5), for any $n \in \mathbb{N}$ and for each $\delta > 0$ fixed, there exists $R > 0$ such that $\|v_n\|_{\infty(|x|\geq R)} < \delta$. Thus,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in } n \in \mathbb{N},$$

and the proof is complete. □

5.1 Proof of Theorem 1.3 completed

Let b_n denote a maximum point of v_n, and recall that

$$\delta \leq \int_{B_r(\tilde{y}_n)} |u_n|^{N/s}\,dx = \int_{B_r(0)} |v_n|^{N/s}\,dx \leq w_N r^N |v_n|_\infty^{N/s} \leq C.$$

Then the sequence $\{v_n\}$ is a bounded sequence in $\mathbb{R}^N$. Thus, there exists $R > 0$ such that $\{b_n\} \subset B_R(0)$, and the global maximum of u_{ε_n} is attained at $z_n = b_n + \tilde{y}_n$ and

$$\varepsilon_n z_n = \varepsilon_n b_n + \varepsilon_n \tilde{y}_n = \varepsilon_n b_n + y_n.$$

From the boundedness of $\{b_n\}$ we have

$$\lim_{n\to\infty} z_n = y,$$

which together with (V_2) yields

$$\lim_{n\to\infty} V(\varepsilon z_n) = V_0.$$

If u_ε is a positive solution of (2.1), the function $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of (1.1). Thus, the maxima points η_ε and z_ε of w_ε and u_ε, respectively, satisfy the equality $\eta_\varepsilon = \varepsilon z_\varepsilon$, and in turn

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

The proof is now complete.

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