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PERIODIC SOLUTIONS FOR IMPLICIT EVOLUTION INCLUSIONS

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Abstract. We consider a nonlinear implicit evolution inclusion driven by a nonlinear, nonmonotone, time-varying set-valued map and defined in the framework of an evolution triple of Hilbert spaces. Using an approximation technique and a surjectivity result for parabolic operators of monotone type, we show the existence of a periodic solution.

1. **Introduction.**

In this paper we study the following periodic implicit evolution inclusion

$$\left\{ \begin{array}{l} \dfrac{d}{dt}(Bu(t)) + A(t, u(t)) \ni 0 \text{ for almost all } t \in T = [0, b] \\ B(u(0)) = B(u(b)). \end{array} \right\} \quad (1)$$

Problem (1) is defined in the framework of an evolution triple (X, H, X^*) of Hilbert spaces (see Section 2), where $B \in \mathcal{L}(X, X^*)$ and $A : T \times X \to 2^{X^*}$ is a map measurable in $t \in T$ and such that for almost all $t \in T$, $A(t, \cdot)$ is bounded and pseudo-monotone.

Implicit evolution equations were studied by Andrews, Kuttler & Schillor [1], Barbu [2], Barbu & Favini [4], Favini & Yagi [6], Liu [11], and Showalter [15]. However, in all these works, the operator A was time-invariant and maximal monotone.

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Moreover, the aforementioned works treat the Cauchy problem. We are not aware of any work on implicit evolution equations treating the periodic problem. We mention also the works of Barbu & Favini [3] and DiBenedetto & Showalter [5], treating the case where B is nonlinear monotone. For this case the hypotheses and the techniques are different.

This paper is strongly influenced by Lions [10]. In fact, our existence result (Theorem 7) is based on a multivalued version of a surjectivity result, which was proved for the first time for single-valued maps by Lions [10, Theorem 1.2, p. 319], see Theorem 4 below. This way we can accommodate the multivalued nature of the map $A(t,x)$ in problem (1). The fact that we allow $A(t,x)$ to be set-valued broadens significantly the applicability of our work. Now we can also treat the subdifferential of continuous but not C^1-convex functionals, a situation that the single-valued formulation cannot handle. In addition, the presence of the operator B in the time derivative complicates the abstract setting. Since B can be degenerate, this adds an additional level of difficulty in the analysis of problem (1) compared to the applications studied by Lions [10, pp. 321-328]. We overcome the difficulty, using the elliptic regularization technique, also first introduced by Lions.

2. Mathematical background.

Suppose that X and Y are Banach spaces and X is continuously and densely embedded into Y. Then we know that Y^* is continuously embedded into X^* and if X is reflexive, then the embedding of Y^* into X^* is also dense.

Definition 2.1. By an "evolutions triple", we mean a triple of spaces

$$X \hookrightarrow H \hookrightarrow X^*$$

such that X is a separable reflexive Banach space, H is a separable Hilbert space identified with its dual (pivot space), and X is continuously embedded into H. We say that (X, H, X^*) is an evolution triple of Hilbert spaces, if all three spaces are Hilbert.

Evidently, $H^* = H$ is continuously and densely embedded into X^*. By $||\cdot||$ (resp $|\cdot|$, $||\cdot||_*$), we denote the norm of X (resp. of H, X^*). We have

$$|\cdot| \leq c_1||\cdot|| \text{ and } ||\cdot||_* \leq c_2|\cdot| \text{ for some } c_1, c_2 > 0.$$

We denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X) and by $(\cdot,\cdot)$ the inner product of H. We have

$$\langle\cdot,\cdot\rangle|_{H\times X} = (\cdot,\cdot).$$

Given an evolution triple (X, H, X^*) and $1 < p < \infty$, we can define the following Banach space:

$$W_p(0,b) = \{u \in L^p(T,X) : u' \in L^{p'}(T,X^*)\}.$$

In this definition, $\frac{1}{p} + \frac{1}{p'} = 1$ and the derivative u' of u is understood in the sense of vectorial distributions. A function $u \in W_p(0,b)$ viewed as a function with values in X^*, is absolutely continuous and so

$$W_p(0,b) \subseteq AC^{1,p'}(T,X^*) = W^{1,p'}((0,b),X^*).$$

Also, we know that $L^p(T,X^*)^* = L^{p'}(T,X)$. The space $W_p(0,b)$ is continuously and densely embedded into $C(T,H)$ and its elements satisfy the following integration by parts formula.

Proposition 1. *If (X, H, X^*) is an evolution triple and $u, v \in W_p(0, b)$ $(1 < p < \infty)$, then the mapping $t \mapsto (u(t), v(t))$ is absolutely continuous and*

$$\frac{d}{dt}(u(t), v(t)) = \langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle \text{ for almost all } t \in T.$$

If (X, H, X^*) is an evolution triple and X is compactly embedded into H, then $H^* = H$ is compactly embedded into X^* (Schauder's theorem) and $W_p(0, b)$ is compactly embedded into $L^p(T, H)$. For details, see Gasinski & Papageorgiou [7].

We will use the following notions from set-valued analysis (see [9]).

(a) If V, W are Hausdorff topological spaces and $G : V \to 2^W \backslash\{\emptyset\}$ is a multivalued map, then we say that $G(\cdot)$ is "upper semicontinuous" ("usc" for short), if for every closed $C \subseteq W$, the set $G^-(C) = \{v \in V : G(v) \cap C \neq \emptyset\}$ is closed.

(b) If $T = [0, b]$, Y is a separable Banach space and $G : T \to 2^Y \backslash\{\emptyset\}$ is a multivalued map, then we say that $G(\cdot)$ is "graph measurable" if

$$\mathrm{Gr}\, G = \{(t, y) \in T \times Y : y \in G(t)\} \in \mathcal{L}_T \otimes B(Y),$$

with $\mathcal{L}_T$ being the Lebesgue σ-field of T and $B(Y)$ the Borel σ-field on Y.

Given a Banach space, we will use the following notation

$$P_{f(c)}(X) = \{C \subseteq Y : C \text{ is nonempty, closed (and convex)}\}.$$

Also, if $C \subseteq Y$, then we define

$$|C| = \sup\{||c||_Y : c \in C\}.$$

Let Y be a reflexive Banach space and $A : Y \to 2^{Y^*}$ a multivalued map. We say that $A(\cdot)$ is "pseudo-monotone", if the following conditions are satisfied:

- for every $y \in Y$, $A(y)$ is nonempty, closed, and convex;
- $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets);
- if $y_n \xrightarrow{w} y$ in Y, $y_n^* \xrightarrow{w} y^*$ in Y^* with $y_n^* \in A(y_n)$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty}\langle y_n^*, y_n - y\rangle_{Y^*Y} \leq 0,$$

  then $y^* \in A(y)$ and $\langle y_n^*, y_n\rangle_{Y^*,Y} \to \langle y^*, y\rangle_{Y^*Y}$.

Any maximal monotone map $A : Y \to 2^{Y^*}\backslash\{\emptyset\}$ is pseudo-monotone (see Gasinski & Papageorgiou [7, pp. 331-332]). As in the case of maximal monotone maps, pseudo-monotone operators exhibit nice surjectivity properties. In particular, a pseudo-monotone coercive (that is, $\inf\{\langle y^*, y\rangle_{Y^*Y} : y^* \in A(y)\}/||y||_Y \to +\infty$ as $||y||_Y \to +\infty$) map is surjective (see Gasinski & Papageorgiou [7, p. 326]).

For dynamic problems (evolution equations), we have the following variant of the notion of pseudo-monotonicity.

Definition 2.2. Let Y be a reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ a linear, maximal monotone operator, and $A : Y \to 2^{Y^*}$ a multivalued map. We say that $A(\cdot)$ is "L-pseudo-monotone", if the following conditions hold:

(i) for every $y \in Y$, $A(y) \subseteq Y^*$ is nonempty, w-compact, and convex;

(ii) $A : Y \to 2^{Y^*}\backslash\{\emptyset\}$ is usc from every finite dimensional subspace of Y into Y^* furnished with the weak topology;

(iii) if $\{y_n\}_{n\geq 1} \subseteq D(L)$, $y_n \xrightarrow{w} y \in D(L)$ in Y, $L(y_n) \xrightarrow{w} L(y)$ in Y^*, $y_n^* \in A(y_n)$ for all $n \in \mathbb{N}$, $y_n^* \xrightarrow{w} y^*$ in Y^* and $\limsup_{n\to\infty}\langle y_n^*, y_n - y\rangle \leq 0$, then $y^* \in A(y)$ and $\langle y_n^*, y_n\rangle_{Y^*Y} \to \langle y^*, y\rangle_{Y^*Y}$.

These operators have nice surjectivity properties. The following result can be found in Papageorgiou, Papalini & Renzacci [12] (the single-valued version of this property is due to Lions [10]).

Theorem 2.3. *If Y is a strictly convex reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear, maximal monotone operator, and $A : Y \to 2^{Y^*}$ is bounded, L-pseudo-monotone, and coercive, then $L + A$ is surjective.*

3. **Periodic solutions.** In what follows, $T = [0, b]$ and (X, H, X^*) is an evolution triple of Hilbert spaces. We assume that X is compactly embedded into H (hence so is $H^* = H$ into X^*). The hypotheses on the data of (1) are the following:

$H(B)$: $B \in \mathcal{L}(X, X^*)$ and is symmetric and monotone.

$H(A)$: $A : T \times X \to P_{f_c}(X^*)$ is a multivalued map such that

(i) for all $x \in X$, the mapping $t \mapsto A(t, x)$ is graph measurable;
(ii) for almost all $t \in T$, the mapping $x \mapsto A(t, x)$ is pseudo-monotone;
(iii) for almost all $t \in T$ and all $x \in X$, we have

$$|A(t, x)| \leq c_1(t) + c_2||x||^{p-1}$$

with $c_1 \in L^{p'}(T), 2 \leq p < \infty$ and $c_2 > 0$;
(iv) for almost all $t \in T$ and all $x \in X$, we have

$$\inf\{\langle u^*, x\rangle : u^* \in A(t, x)\} \geq c_3||x||^p - c_4(t),$$

with $c_3 > 0$ and $c_4 \in L^1(T)$.

Let $J : X \to X^*$ be the duality (Riesz) map on the Hilbert space X. We know that $J(\cdot)$ is an isometric isomorphism (the Riesz-Fréchet theorem) which is monotone. Hence for every $\epsilon > 0$ we have $(\epsilon J + B)^{-1} \in \mathcal{L}(X^*, X)$. Then on X^* we consider the following bilinear form

$$(u, v)_* = \langle(\epsilon J + B)^{-1}u, v\rangle \text{ for all } u, v \in X^*. \tag{2}$$

Hypotheses $H(B)$ imply that $(\cdot, \cdot)_*$ is an inner product on X^*. Let $|\cdot|_*$ denote the norm corresponding to this inner product. Clearly, $|\cdot|_*$ and $||\cdot||_*$ are equivalent norms on X^*. So, if V^* denotes the space X^* equipped with the norm $|\cdot|_*$, then V^* is a Hilbert space. Using the Riesz-Fréchet theorem, we identify V^* with its dual.

Let $A_\epsilon : T \times V^* \to P_{f_c}(V^*)$ be defined by

$$A_\epsilon(t, v) = A(t, (\epsilon J + B)^{-1}v).$$

Then we introduce the multivalued Nemitsky map $\hat{A}_\epsilon : L^p(T, V^*) \to 2^{L^{p'}(T,V^*)}$ corresponding to $A_\epsilon(\cdot, \cdot)$, defined by

$$\hat{A}_\epsilon(v) = \{u \in L^{p'}(T, V^*) : u(t) \in A_\epsilon(t, v(t)) \text{ for almost all } t \in T\}.$$

Consider the function space

$$W_p^{per}((0, b), V^*) = \{u \in L^p(T, V^*) : u' \in L^{p'}(T, V^*), u(0) = u(b)\}.$$

We know that $W_p^{per}((0, b), V^*) \hookrightarrow C(T, V^*)$ and so the evaluations of u at $t = 0$ and $t = b$ make sense. Let $L : W_p^{per}((0, b), V^*) \subseteq L^p(T, V^*) \to L^{p'}(T, V^*)$ be defined by

$$L(u) = u'.$$

We know that $L(\cdot)$ is linear and maximal monotone (see Hu & Papageorgiou [9, p. 419] and Zeidler [16, p. 855]).

Proposition 2. *If hypotheses $H(B), H(A)$ hold and $\epsilon > 0$, then for every $u \in L^p(T, V^*)$, $\hat{A}_\epsilon(u) \subseteq L^{p'}(T, V^*)$ is nonempty, w-compact and convex, and the mapping $u \mapsto \hat{A}_\epsilon(u)$ is L-pseudo-monotone.*

Proof. It is clear that $\hat{A}_\epsilon(u)$ is closed, convex, and bounded, thus w-compact in $L^{p'}(T, V^*)$. We need to show that $\hat{A}_\epsilon(\cdot)$ has nonempty values. Note that hypotheses $H(A)(i), (ii)$ do not imply the graph measurability of $(t, x) \mapsto A_\epsilon(t, x)$ (see Hu & Papageorgiouo [9, p. 227]). To show the nonemptiness of $\hat{A}_\epsilon(u)$ we proceed as follows. Let $\{s_n\}_{n\geq 1} \subseteq L^p(T, V^*)$ be step functions such that

$$s_n \to u \text{ in } L^p(T, V^*), s_n(t) \to u(t) \text{ for almost all } t \in T,$$
$$|s_n(t)|_* \leq |u(t)|_* \text{ for almost all } t \in T, \text{ and for all } n \in \mathbb{N}.$$

On account of hypothesis $H(A)(i)$, for every $n \in \mathbb{N}$ the mapping

$$t \mapsto A_\epsilon(t, s_n(t)) = A(t, (\epsilon J + B)^{-1} s_n(t))$$

is graph measurable. So, we can apply the Yankov-von Neumann-Aumann selection theorem (see Hu & Papageorgiou [9, p. 158]) and obtain that $v_n : T \to V^*$ is measurable and $v_n(t) \in A_\epsilon(t, s_n(t))$ for almost all $t \in T, n \in \mathbb{N}$. Evidently, $v_n \in L^{p'}(T, V^*)$ and $\{v_n\}_{n\geq 1} \subseteq L^{p'}(T, V^*)$ is bounded. So, by passing to a suitable subsequence if necessary we may assume that

$$v_n \xrightarrow{w} v \text{ in } L^{p'}(T, V^*) \text{ as } n \to \infty. \tag{3}$$

Note that the pseudo-monotonicity of $A_\epsilon(t, \cdot)$ (see hypothesis $H(A)(ii)$) implies that $\operatorname{Gr} A_\epsilon(t, \cdot)$. is demiclosed (that is, sequentially closed in $V^* \times V^*_w$, where V^*_w denotes the Hilbert space V^* furnished with the weak topology). So, by (3) and Proposition 3.9 of Hu & Papageorgiou [9, p. 694], we have

$$v(t) \in \overline{\operatorname{conv}}\, w - \limsup_{n\to\infty} A_\epsilon(t, s_n(t)) \subseteq A_\epsilon(t, u(t)) \text{ for almost all } t \in T,$$
$$\Rightarrow v \in \hat{A}_\epsilon(u) \text{ and so } \hat{A}_\epsilon(\cdot) \text{ has nonempty values.}$$

Next, we will prove the L-pseudo-monotonicity of $\hat{A}_\epsilon$. So, let $((\cdot, \cdot))_*$ denote the duality brackets for the pair $(L^{p'}(T, V^*), L^p(T, V^*))$, that is,

$$((v, u))_* = \int_0^b (v(t), u(t))_* dt \text{ for all } u \in L^p(T, V^*), v \in L^{p'}(T, V^*). \tag{4}$$

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_p^{per}((0, b), V^*)$ such that

$$\text{“}u_n \xrightarrow{w} u \text{ in } L^p(T, V^*), u'_n \xrightarrow{w} u' \text{ in } L^{p'}(T, V^*) \text{ and } v_n \in \hat{A}_\epsilon(u_n) \text{ (for all } n \in \mathbb{N}\text{)},$$
$$\text{such that } v_n \xrightarrow{w} v \text{ in } L^{p'}(T, V^*) \text{ and } \limsup_{n\to\infty} ((v_n, u_n - u))_* \leq 0\text{”}. \tag{5}$$

We have

$$\begin{aligned} ((v_n, u_n - u))_* &= \int_0^b (v_n(t), u_n(t) - u(t))_* dt \text{ (see (4))} \\ &= \int_0^b \langle v_n(t), (\epsilon J + B)^{-1}(u_n - u)(t)\rangle dt \text{ (see (2))}. \end{aligned}$$

Let $y_n(t) = (\epsilon J + B)^{-1} u_n(t)$, $y(t) = (\epsilon J + B)^{-1} u(t)$. Then $y_n, y \in L^p(T, X)$ and we have

$$\langle v_n(t), (\epsilon J + B)^{-1}(u_n - u)(t)\rangle = \langle v_n(t), y_n(t) - y(t)\rangle$$

with $v_n(t) \in A(t, y_n(t))$ for almost all $t \in T$, all $n \in \mathbb{N}$. Evidently,

$$\{y_n\}_{n\geq 1} \subseteq L^p(T, X) \text{ is bounded (see (5))}. \tag{6}$$

Also, we have

$$\begin{aligned} & y_n' = ((\epsilon J + B)^{-1} u_n)' \\ \Rightarrow \quad & \{y_n'\}_{n\geq 1} \subseteq L^{p'}(T, X^*) \text{ is bounded (see (5)).} \end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\{y_n\}_{n\geq 1} \subseteq W_p(0,b) \text{ is bounded.}$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_p(0,b) \text{ as } n \to \infty. \tag{8}$$

Evidently, we have $y = (\epsilon J + B)^{-1} u$ and so

$$(\epsilon J + B)^{-1} u_n \xrightarrow{w} (\epsilon J + B)^{-1} u \text{ in } L^p(T, X).$$

If we denote by $((\cdot,\cdot))$ the duality brackets for the pair $(L^{p'}(T, X^*), L^p(T, X))$, that is,

$$((v, u)) = \int_0^b \langle v(t), u(t)\rangle dt \text{ for all } u \in L^p(T, X), v \in L^{p'}(T, X^*),$$

then we have

$$\limsup_{n\to\infty} ((v_n, y_n - y)) = \limsup_{n\to\infty} ((v_n, u_n - u)) \leq 0 \text{ (see (5)).}$$

Recall that $W_p(0,b)$ is continuously embedded in $C(T,H)$. So, from (8) we have

$$y_n(t) \xrightarrow{w} y(t) \text{ in } H \text{ for all } t \in T. \tag{9}$$

Let $\vartheta_n(t) = \langle v_n(t), y_n(t) - y(t)\rangle$ and let $N \subseteq T$ be the Lebesgue-null set outside of which hypotheses $H(A)(ii)$, (iii) (iv) hold. Then for $t \in T\backslash N$, we have

$$\vartheta_n(t) \geq c_3 ||y_n(t)||^p - c_4(t) - ||y(t)|| \left(c_1(t) + c_2 ||y_n(t)||^{p-1}\right) \tag{10}$$
$$\text{(see hypotheses } H(A)(iii), (iv)).$$

Let $E = \{t \in T : \liminf_{n\to\infty} \vartheta_n(t) < 0\}$. This is a Lebesgue measurable set. Suppose that $\lambda^1(E) > 0$ ($\lambda^1(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$). From (10), we see that $\{y_n(t)\}_{n\geq 1} \subseteq X$ is bounded for all $t \in E \cap (T\backslash N)$. So, on account of (9) we obtain that $y_n(t) \xrightarrow{w} y(t)$ in X. Fix $t \in E \cap (T\backslash N)$ and choose a suitable subsequence (depending on t) such that $\liminf_{n\to\infty} \vartheta_n(t) = \lim_{k\to\infty} \vartheta_{n_k}(t)$. The pseudo-monotonicity of $A(t,\cdot)$ (see hypothesis $H(A)(ii)$), implies that

$$\langle v_{n_k}(t), y_{n_k}(t) - y(t)\rangle \to 0,$$

a contradiction since $t \in E$. Therefore $\lambda^1(E) = 0$ and so we have

$$0 \leq \liminf_{n\to\infty} \vartheta_n(t) \text{ for almost all } t \in T. \tag{11}$$

Invoking Fatou's lemma, we have

$$\begin{aligned} & 0 \leq \int_0^b \liminf_{n\to\infty} \vartheta_n(t)dt \leq \liminf_{n\to\infty} \int_0^b \vartheta_n(t)dt \leq \limsup_{n\to\infty} \int_0^b \vartheta_n(t)dt \leq 0, \\ \Rightarrow \quad & \int_0^b \vartheta_n(t)dt \to \vartheta \text{ as } n \to \infty. \end{aligned} \tag{12}$$

We have $|\vartheta_n| = \vartheta_n^+ + \vartheta_n^- = \vartheta_n + 2\vartheta_n^-$ and $\vartheta_n^-(t) \to 0$ for almost all $t \in T$ (see (11)). Also, from (10) we have

$$\gamma_n(t) \leq \vartheta_n(t) \text{ for almost all } t \in T, \text{ and for all } n \in \mathbb{N},$$

and $\{\gamma_n\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable. We have

$$0 \leq \vartheta_n^-(t) \leq \gamma_n^-(t) \text{ for almost all } t \in T, \text{ and for all } n \in \mathbb{N},$$
$$\Rightarrow \quad \{\vartheta_n^-\}_{n\geq 1} \subseteq L^1(T) \text{ is uniformly integrable.}$$

Applying the extended dominated convergence theorem (see, for example, Gasinski & Papageorgiou [7, p. 901]), we have

$$\int_0^b \vartheta_n^-(t)dt \to 0,$$
$$\Rightarrow \quad \vartheta_n \to 0 \text{ in } L^1(T) \text{ (see (12))}.$$

So, by passing to a subsequence if necessary, we may assume that

$$\vartheta_n(t) \to 0 \text{ for almost all } t \in T,$$
$$\Rightarrow \quad \langle v_n(t), y_n(t) - y(t)\rangle \to 0 \text{ for almost all } t \in T.$$

Since $v_n(t) \in A(t, y_n(t))$ for almost all $t \in T$ and for all $n \in \mathbb{N}$, on account of the pseudo-monotonicity of $A(t, \cdot)$ (see hypothesis $H(A)(ii)$), we have

$$v(t) = A(t, y(t)) = A_\epsilon(t, u(t)) \text{ for almost all } t \in T$$

and $v_n(t) \xrightarrow{w} v(t)$ in X^*, $\langle v_n(t), y_n(t)\rangle \to \langle v(t), y(t)\rangle$ for almost all $t \in T$.

By the dominated convergence theorem, we have

$$v_n \xrightarrow{w} v \text{ in } L^{p'}(T, X^*),\ ((v_n, y_n)) \to ((v, y)),\ v \in \hat{A}(y),$$
$$\Rightarrow \quad v_n \xrightarrow{w} v \text{ in } L^{p'}(T, V^*),\ ((v_n, u_n)) \to ((v, u))_*,\ v \in \hat{A}_\epsilon(u).$$

Finally, using Proposition 2.23 of Hu & Papageorgiou [9, p. 43], we easily see that $\hat{A}_\epsilon(\cdot)$ is usc from finite dimensional subspaces of $L^p(T, V^*)$ into $L^{p'}(T, V^*)_w$.

Therefore we conclude that $\hat{A}_\epsilon$ is indeed L-pseudo-monotone. □

We consider the following auxiliary approximate periodic problem:

$$\left\{ \begin{array}{l} u'(t) + A_\epsilon(t, u(t)) \ni 0 \text{ for almost all } t \in T, \\ u(0) = u(b). \end{array} \right\} \tag{13}$$

Proposition 3. *If hypotheses $H(B), H(A)$ hold and $\epsilon > 0$, then problem (13) has a solution $u_\epsilon \in W_p^{per}((0, b), V^*)$.*

Proof. We rewrite (13) as the following abstract operator inclusion

$$L(u) + \hat{A}_\epsilon(u) \ni 0. \tag{14}$$

Let $v \in \hat{A}_\epsilon(u)$. We have

$$((v, u))_* = ((v, (\epsilon J + B)^{-1}u)).$$

Let $y = (\epsilon J + B)^{-1}u$. Then $v \in \hat{A}(y)$ and so, using hypothesis $H(A)(iv)$, we have

$$((v, y)) = \textstyle\int_0^b \langle v(t), y(t)\rangle dt \geq c_3 ||y||^p_{L^p(T,X)} - ||c_4||_1,$$
$$\Rightarrow \quad ((v, u))_* \geq c_5 ||u||^p_{L^p(T,V^*)} - ||c_4||_1 \text{ for some } c_5 > 0 \tag{15}$$

(recall that $|\cdot|_*$ and $||\cdot||_*$ are equivalent norms on X^*). It follows that $\hat{A}_\epsilon(\cdot)$ is coercive. Clearly, it is bounded (see hypothesis $H(A)(iii)$). Also, from Proposition 2 we know that $\hat{A}_\epsilon(\cdot)$ is L-pseudo-monotone. Since $L(\cdot)$ is maximal monotone, we can use Theorem 2.3 and find $u_\epsilon \in W_p^{per}((0, b), V^*) = D(L)$ such that it solves (14). Evidently, this is a solution of problem (13). □

Next, we will let $\epsilon \downarrow 0$ to produce a solution of problem (1).

Theorem 3.1. *If hypotheses $H(B), H(A)$ hold, then problem (1) has a solution $y \in L^p(T, X)$ which satisfies $(By)' \in L^{p'}(T, X^*)$.*

Proof. For each $\epsilon > 0$, let $u_\epsilon \in W_p^{per}((0,b), V^*)$ be a solution of the approximate problem (13) (see Proposition 3). We have

$$\left\{ \begin{array}{l} u_\epsilon'(t) + A_\epsilon(t, u_\epsilon(t)) \ni 0 \text{ for almost all } t \in T, \\ u_\epsilon(0) = u_\epsilon(b). \end{array} \right\} \tag{16}$$

We take the inner product in V^* with $u_\epsilon(t)$. Then

$$\frac{1}{2}\frac{d}{dt}|u_\epsilon'(t)|_*^2 + (v_\epsilon(t), u_\epsilon(t))_* = 0 \text{ for almost all } t \in T,$$

with $v_\epsilon \in L^{p'}(T, V^*), v_\epsilon(t) \in A_\epsilon(t, u_\epsilon(t))$ for almost all $t \in T$. Integrating on T and using (15) and the periodic conditions, we obtain

$$\begin{aligned} & c_5||u_\epsilon||_{L^p(T,V^*)} \le ||c_4||_1, \\ \Rightarrow \quad & \{u_\epsilon\}_{\epsilon>0} \subseteq L^p(T, V^*) \text{ is bounded.} \end{aligned} \tag{17}$$

We set $y_\epsilon(t) = (\epsilon J + B)^{-1} u_\epsilon(t)$. Then

$$\begin{aligned} & ||y_\epsilon(t)|| \le ||(\epsilon J + B)^{-1}||_{\mathcal{L}} ||u_\epsilon(t)||^* \\ \Rightarrow \quad & \{y_\epsilon\}_{\epsilon \in (0,1]} \subseteq L^p(T, X) \text{ is bounded (see (17)).} \end{aligned} \tag{18}$$

On account of hypothesis $H(A)(iii)$, we have

$$|A_\epsilon(t, u_\epsilon(t))| \le c_1(t) + c_2||y_\epsilon(t)||^{p-1} \text{ for almost all } t \in T. \tag{19}$$

Then it follows from (16), (18) and (19) that

$$\{u_\epsilon'\}_{\epsilon \in (0,1]} \subseteq L^{p'}(T, V^*) \text{ is bounded.}$$

This together with (17) implies that

$$\{u_\epsilon\}_{\epsilon \in (0,1]} \subseteq W^{1,p'}((0,b), V^*) \text{ is bounded (recall that } 1 < p' \le 2 \le p). \tag{20}$$

Now let $\epsilon_n = \frac{1}{n}, u_n = u_{\epsilon_n}, y_n = y_{\epsilon_n}, v_n = v_{\epsilon_n}$ for all $n \in \mathbb{N}$. Note that

$$[(n^{-1}J + B)y_n(t)]' \in L^{p'}(T, X^*).$$

We have

$$\left\{ \begin{array}{l} ((n^{-1}J + B)y_n(t))' + v_n(t) = 0 \text{ for almost all } t \in T, \\ v_n(t) \in A(t, y_n(t)) \text{ for almost all } t \in T, \\ u_n(0) = u_n(b). \end{array} \right\} \tag{21}$$

Note that

$$y_n(0) = (\epsilon J + B)^{-1} u_n(0) = (\epsilon J + B)^{-1} u_n(b) = y_n(b) \text{ for all } n \in \mathbb{N} \text{ (see (21)).} \tag{22}$$

Also, on account of (18), (20) and (21), we may assume that

$$y_n \xrightarrow{w} y \text{ in } L^p(T, X),\ u_n \xrightarrow{w} u \text{ in } W^{1,p'}((0,b), V^*),\ v_n \to v \text{ in } L^{p'}(T, X^*). \tag{23}$$

We know that $W^{1,p'}((0,b), V^*) \hookrightarrow C(T, V^*)$ continuously. Hence by (17), up to a subsequence, we have

$$u_n \xrightarrow{w} u \text{ in } C(T, V^*),$$
$$\Rightarrow \quad y_n(t) \xrightarrow{w} y(t) \text{ in } X \text{ for all } t \in T, \tag{24}$$
$$\Rightarrow \quad B(y(0)) = B(y(b)) \text{ (see (22)).} \tag{25}$$

On the first equation in (21) we act with $(y_n - y)(t)$ and then integrate over T. We obtain

$$(((\left[n^{-1}J + B\right] y_n)', y_n - y)) + ((v_n, y_n - y)) = 0 \text{ for all } n \in \mathbb{N}. \tag{26}$$

We obtain

$$\begin{aligned} &(((\left[n^{-1}J + B\right] y_p)', y_n - y)) \\ = \quad & (((\left[n^{-1}J + B\right] (y_n - y))', y_n - y)) + (((\left[n^{-1}J + B\right] y)', y_n - y)). \end{aligned} \tag{27}$$

Note that

$$(((\left[n^{-1}J + B\right] y)', y_n - y)) \to 0 \text{ as } n \to \infty \text{ (see (23))}. \tag{28}$$

Also, we have

$$\begin{aligned} &(((\left[n^{-1}J + B\right] (y_n - y))', y_n - y)) \\ &= \int_0^b \langle n^{-1}(J(y_n - y))', y_n - y\rangle dt + \int_0^b \langle (B(y_n - y))', y_n - y\rangle dt \\ &= \int_0^b \frac{1}{n}(y_n' - y', y_n - y)_X dt + \frac{1}{2}\int_0^b \frac{d}{dt}\langle B(y_n - y), y_n - y\rangle dt \\ &\text{(recall that } J(\cdot) \text{ is the Riesz map for } X \text{ and see hypothesis } H(B)) \\ &= \tfrac{1}{n}\left[||(y_p - y)(b)|| - ||(y_n - y)(0)||\right] + \tfrac{1}{2}[\langle B(y_n - y)(b), (y_n - y)(b)\rangle - \\ &\quad \langle B(y_n - y)(0), (y_n - y)(0)\rangle] \\ &= 0 \text{ for all } n \in \mathbb{N} \text{ (see (22), (24))}. \end{aligned} \tag{29}$$

So, if we return to (27) and use (28), (29) we obtain

$$\lim_{n\to\infty} (((\left[n^{-1}J + B\right] y_n)', y_n - y)) = 0. \tag{30}$$

If we use (30) in (26), we get

$$\lim_{n\to\infty} ((v_n, y_n - y)) = 0.$$

Invoking Proposition 2, we have

$$v \in \hat{A}(y) \text{ and } ((v_n, y_n)) \to ((v, y)).$$

Thus, we obtain from (21) taking the limit as $n \to \infty$

$$\left\{ \begin{array}{l} \dfrac{d}{dt}(By(t)) + A(t, y(t)) \ni 0 \text{ for almost all } t \in T, \\ B(y(0)) = B(y(b)). \end{array} \right\}$$

Therefore $y \in L^p(T, X)$ is a solution of (1) with $(By)' \in L^{p'}(T, X^*)$. □

4. **An example.** Let $T = [0, b]$ and let $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary $\partial\Omega$. We consider the following initial boundary value problem:

$$\left\{ \begin{array}{l} \dfrac{d}{dt}(m(z)u) - \operatorname{div}(a(t, z)Du) + \displaystyle\sum_{k=1}^{N}(\sin u)D_k u + \partial g(u) \ni 0 \text{ in } T \times \Omega, \\ u|_{T\times\partial\Omega} = 0, \; m(z)u(z, 0) = m(z)u(z, b) \text{ for almost all } z \in \Omega. \end{array} \right\} \tag{31}$$

We impose the following conditions on the data for problem (31):

$H(m)$: $m \in L^{N/2}(\Omega)$ if $N > 2$, $m \in L^r(\Omega)$ with $r > 1$ if $N = 2$ and $m \in L^1(\Omega)$ if $N = 1$, $m(z) \geq 0$ for almost all $z \in \Omega$, $m \not\equiv 0$.

$H(a)$: $a \in L^\infty(T \times \Omega)$ and $a(t, z) \geq a_0 > 0$ for almost all $(t, z) \in T \times \Omega$.

$H(g)$: $g : \mathbb{R} \to \mathbb{R}$ is a continuous convex function and its subdifferential $\partial g(x)$ satisfies

$$|\partial g(x)| \leq \hat{c}\,(1 + |x|^{p-1}) \text{ for all } x \in \mathbb{R}, \text{ and for some } \hat{c} > 0,\ 2 \leq p < \infty.$$

Remark 1. For any continuous convex function $g(\cdot)$, we know that $\partial g(x) \neq \emptyset$ for all $x \in \mathbb{R}$ (see Gasinski & Papageorgiou [7, p. 527]).

We introduce the following multifunction

$$N_g(u) = \{v \in L^{p'}(\Omega) : v(z) \in \partial g(u(z)) \text{ for almost all } z \in \Omega\}$$

for all $u \in H_0^1(\Omega)$. Evidently, $N_g(\cdot)$ is maximal monotone.

In this case, the evolution triple consists of the following Hilbert spaces:

$$X = H_0^1(\Omega),\ H = L^2(\Omega),\ X^* = H^{-1}(\Omega).$$

We know that $X \hookrightarrow H$ compactly (by the Sobolev embedding theorem).

Let $A_1 : T \times X \to X^*$ be the nonlinear map defined by

$$\langle A_1(t,u), h\rangle = \int_\Omega a(t,z)(Du, Dh)_{\mathbb{R}^N} dz + \int_\Omega \sin u \left(\sum_{k=1}^N D_k u\right) h dz$$
$$\text{for all } u, h \in X = H_0^1(\Omega).$$

Then the mapping $t \mapsto A_1(t,u)$ is measurable, whereas $u \mapsto A_1(t,u)$ is pseudomonotone (see, for example, Zeidler [16, p. 591]). We set

$$A(t,u) = A_1(t,u) + N_g(u).$$

Then $A(t,u)$ satisfies hypotheses $H(A)$ (see $H(a)$ and $H(g)$).

In addition, we let $B \in \mathcal{L}(X, X^*)$ be defined by

$$Bu(\cdot) = m(\cdot)u(\cdot) \text{ for all } u \in X = H_0^1(\Omega).$$

Clearly, $B(\cdot)$ satisfies $H(B)$.

We can rewrite problem (31) as the following abstract implicit evolution inclusion:

$$\left\{ \begin{array}{l} \dfrac{d}{dt}(Bu(t)) + A(t,u(t)) \ni 0 \text{ for almost all } t \in T, \\ B(u(0)) = B(u(b)). \end{array} \right\}$$

We can apply Theorem 3.1 and obtain the following result.

Proposition 4. *If hypotheses $H(m), H(a), H(g)$ hold, then problem (31) admits a solution $u \in L^p(T, H_0^1(\Omega))$ with*

$$(Bu)' \in L^{p'}(T, H^{-1}(\Omega)).$$

Remark 2. Using the methods developed in this paper one can also treat antiperiodic problems (see Gasinski & Papageorgiou [8]), problems with subdifferential terms (see Papageorgiou & Rădulescu [13]), and applications to distributed parameter control systems (see Papageorgiou, Rădulescu & Repovš [14]).

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