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INVERSE PROBLEMS FOR ANISOTROPIC OBSTACLE PROBLEMS WITH MULTIVALUED CONVECTION AND UNBALANCED GROWTH

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Abstract. The prime goal of this paper is to introduce and study a highly nonlinear inverse problem of identification discontinuous parameters (in the domain) and boundary data in a nonlinear variable exponent elliptic obstacle problem involving a nonhomogeneous, nonlinear partial differential operator, which is formulated the sum of a weighted anisotropic p-Laplacian and a weighted anisotropic q-Laplacian (called the weighted anisotropic (p,q)-Laplacian), a multivalued reaction term depending on the gradient, two multivalued boundary conditions and an obstacle constraint. We, first, employ the theory of nonsmooth analysis and a surjectivity theorem for pseudomonotone operators to prove the existence of a nontrivial solution of the anisotropic elliptic obstacle problem, which relies on the first eigenvalue of the Steklov eigenvalue problem for the p_--Laplacian. Then, we introduce the parameter-to-solution map for the anisotropic elliptic obstacle problem, and establish a critical convergence result of the Kuratowski type to parameter-to-solution map. Finally, a general framework is proposed to examine the solvability of the nonlinear inverse problem.

1. **Introduction.** Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and its boundary $\Gamma := \partial\Omega$ be Lipschitz continuous such that Γ is decomposed into four mutually disjoint parts Γ_a, Γ_b, Γ_c, and Γ_d with Γ_a having positive Lebesgue measure. Also, let $\beta > 0$ and $p, q, \theta \in C_+(\overline{\Omega})$ (see Section 2, below) satisfy $1 < q(x) < p(x) < \theta(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where p^* is the critical Sobolev variable exponent to p in the domain Ω (given in (5) for $s = p$). Given two multivalued mappings $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ and $U\colon \Gamma_d \times \mathbb{R} \to 2^{\mathbb{R}}$, three functions $a\colon \Omega \to (0, +\infty)$, $b\colon \Omega \to (0, +\infty)$ and $h\colon \Gamma_b \to \mathbb{R}$, a convex function $\psi\colon \Gamma_c \times \mathbb{R} \to \mathbb{R}$ and an obstacle function $\Phi\colon \Omega \to \mathbb{R}$, in the present paper, we are interesting in the study of the following anisotropic elliptic obstacle inclusion problem with the weighted anisotropic (p,q)-Laplacian, a multivalued convection term, and two multivalued boundary conditions:

$$\begin{aligned}
-\Delta^a_{p(x)}u - \Delta^b_{q(x)}u + \beta|u|^{\theta(x)-2}u &\in f(x,u,\nabla u) && \text{in} \quad \Omega,\\
u &= 0 && \text{on} \quad \Gamma_a,\\
\frac{\partial u}{\partial \nu_{a,b}} &= h(x) && \text{on} \quad \Gamma_b,\\
-\frac{\partial u}{\partial \nu_{a,b}} &\in \partial_c\psi(x,u) && \text{on} \quad \Gamma_c,\\
\frac{\partial u}{\partial \nu_{a,b}} &\in U(x,u) && \text{on} \quad \Gamma_d,\\
u(x) &\leq \Phi(x) && \text{in} \quad \Omega.
\end{aligned} \tag{1}$$

Here

$$\frac{\partial u}{\partial \nu_{a,b}} := \left(a(x)|\nabla u|^{p(x)-2}\nabla u + b(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nu,$$

where ν is the outward unit normal vector on Γ, $\Delta^a_{p(x)}$ stands for the weighted anisotropic $p(x)$-Laplace differential operator with respect to the weight $a \in L^\infty(\Omega)$ defined by

$$\Delta^a_{p(x)}u := \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) \text{ for all } u \in W^{1,p(\cdot)}(\Omega),$$

and $W^{1,p(\cdot)}(\Omega)$ is the variable exponent Sobolev space.

In conclusion, the novelty of the present paper is the fact that problem (1) models numerous interesting and challenge phenomena. We emphasize that the differential operator involved in problem (1) is a nonhomogeneous and nonlinear partial differential operator with different anisotropic growth, called weighted anisotropic (p,q)-Laplacian, which is the sum of a weighted anisotropic p-Laplace operator $(\Delta^a_{p(\cdot)})$ and a weighted anisotropic q-Laplace operator $(\Delta^b_{p(\cdot)})$, where $p,q\in C_+(\Omega)$ are such that $1<q(x)<p(x)$ for a. e. $x\in\Omega$. Especially, we have

- if $a\equiv b\equiv 1$, then the weighted anisotropic (p,q)-Laplacian reduces to the anisotropic (p,q)-Laplacian

$$\Delta_{p(x)}u+\Delta_{q(x)}u:=\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u+|\nabla u|^{q(x)-2}\nabla u\right) \text{ for all } u\in W^{1,p(\cdot)}(\Omega),$$

- when p,q are constants such that $1<q<p$, then the weighted anisotropic (p,q)-Laplacian becomes to the weighted (p,q)-Laplacian

$$\Delta^a_p u+\Delta^b_q u:=\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u+b(x)|\nabla u|^{q-2}\nabla u\right) \text{ for all } u\in W^{1,p}(\Omega).$$

In fact, the main motivation to study the weighted anisotropic (p,q)-Laplacian is that this partial differential operator has two important properties

- the presence of unbalanced growth,
- variable exponents structure and nonuniform parameters (or weights),

which could explain and describe exactly various complicated problems and natural phenomena in the Mechanics, Physics and Engineering Sciences. For example, assume that a body (or the material) occupying the domain Ω is anisotropic and heterogeneous. If u is the temperature field (resp. electric potential or magnetic potential), then the first inclusion in (1) stands for a generalized anisotropic version of the Fourier constitutive law of heat conduction (resp. nonlinear constitutive relation for electric potential or nonlinear constitutive law for magnetic fluids), where the thermal conductivity a (resp. the dielectric coefficient and magnetic permeability) effectively depends on the space variable x. In consideration of the advantage of weighted anisotropic (p,q)-Laplacian, it permits us to apply model (1) as a powerful mathematical tool for solving the problems arising in electrostatics, magnetostatics, and stationary heat transfer in anisotropic and heterogeneous materials.

We notice some impressive results concerning the anisotropic p-Laplacian, the weighted (p,q)-Laplacian, and anisotropic (p,q)-Laplacian. Bai-Papageorgiou-Zeng [1] have combined variational tools combined with suitable truncations and comparison techniques to study a parametric nonlinear, nonhomogeneous Dirichlet problem driven by the (p,q)-Laplacian with a reaction involving a singular term plus a superlinear reaction which does not satisfy the Ambrosetti-Rabinowitz condition, and established a bifurcation-type theorem describing in a precise way the dependence of the set of positive solutions on a parameter. Ciraolo-Figalli-Roncoroni [8] characterized the solutions to the critical p-Laplacian equation induced by a smooth norm inside any convex cone, and applied optimal transport method to prove a general class of (weighted) anisotropic Sobolev inequalities inside arbitrary convex cones. By variational method based on critical point theory and Morse theory (critical groups), Gasiński-Papageorgiou [21] studied a nonlinear Neumann problem driven by the anisotropic p-Laplacian differential operator and with a superlinear reaction which does not satisfy the usual in such cases Ambrosetti-Rabinowitz condition, and proved that the nonlinear Neumann problem has at least three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative). Mercuri-Riey-Sciunzi [42] considered the weak solutions to a class of Dirichlet boundary value problems involving the p-Laplace operator, and proved that the second weak derivatives are in L^q with q as large as it is desirable, provided p is sufficiently close to $p_0=2$.

Some related developments can be found in Hannukainen-Hyvonen-Mustonen [29], Gasiński-Papageorgiou [20], Baroni-Colombo-Mingione al. [2, 3], Colombo-Mingione [9], Papageorgiou-Rădulescu-Repovš [53, 54], Zeng-Bai-Gasiński-Winkert [61], Marcellini [40, 41], and Liu-Motreanu-Zeng [38].

The second interesting feature of the paper consists in the presence of the multivalued convection term. It is well-known that in a single or a multiphase fluid flow, the convection effect may appear spontaneously because of the combined effects of material heterogeneity and the influence of body forces on a fluid (commonly density and gravity). Whereas, the reaction terms which depend on the gradient of unknown functions can precisely model the convection effect for various fluids flow. From the point of view of methodology, the multivalued convection term appeared in the problem (1) causes tremendous difficulty from two perspectives. On the one hand, the multivalued convection phenomenon has a nonvariational character, so we cannot apply the standard variational tools for the corresponding energy functionals. On the other hand, the discontinuity of the multivalued convection term renders many regularity theorems inapplicable. These two issues motivate us to develop a new pattern and techniques to handle with such kinds of problems. This is, actually, another motivation of the present paper.

We also refer to the recent works involving convection terms or multivalued terms. Applying the Kakutani-Ky Fan fixed point theorem for multivalued operators along with the theory of nonsmooth analysis and variational methods for pseudomonotone operators, Zeng-Rădulecu-Winkert [64] examined the existence of solutions to a mixed boundary value problem with a nonhomogeneous, nonlinear differential operator (called double phase operator), a nonlinear convection term (a reaction term depending on the gradient), three multivalued terms and an implicit obstacle constraint. Ghergu-Rădulescu [23] established some bifurcation results for a singular Lane-Emden-Fowler equation with a convection term, in the meanwhile, the authors utilized the sub- and supersolutions method together with various techniques related to the maximum principles to obtain the asymptotic behaviour of the solution around the bifurcation point. Via employing nonlinear Trudinger-Moser inequality and Galerkin approximation approach, de Araujo-Faria [10] verified the existence of positive solutions to a new class of quasilinear elliptic equations with exponential nonlinearity combined with convection term. For more details with respect to the direction of problems having convection terms or multivalued terms, we refer to El Manouni-Marino-Winkert [15], Figueiredo-Madeira [17], Gasiński-Papageorgiou [19], Papageorgiou-Rădulescu-Repovš [52], Marano-Winkert [39] and Gasiński-Winkert [22].

Another novelty of the paper is the multivalued boundary conditions, which have been widely applied to various problems arising in contact mechanics, diffusion of fluids through a semipermeable membrane, optimal transport, heat conductivity and so on. It is worthy to point out that, in the present paper, the multivalued boundary conditions involved in problem (1) are a multivalued monotone boundary condition, which is formulated by convex subdifferential of a convex functional, and a generalized multivalued boundary condition, which is nonmonotone in general. A classical example for multivalued monotone boundary conditions is the Coulomb law of dry friction, which is formulated by

$$\left\{\begin{array}{ll} \|\sigma_\tau\| \le \mu & \text{if } \mathbf{v}_\tau = 0, \\ \sigma_\tau = -\dfrac{\mu \mathbf{v}_\tau}{\|\mathbf{v}_\tau\|} & \text{if } \mathbf{v}_\tau \neq 0, \end{array}\right. \quad \text{on } \Gamma_c,$$

where σ_τ stands for the friction, μ is the coefficient of dry friction, $\mathbf{v}_\tau$ is the tangential velocity on the contact boundary Γ_c. This constitutive law is equivalent to the following inclusion form

$$-\sigma_\tau \in \mu\partial_c\|\mathbf{v}_\tau\|,$$

where $\partial_c\|\cdot\|$ is the convex subdifferential operator of $\mathbf{v}_\tau \mapsto \|\mathbf{v}_\tau\|$. On the other hand, when U is specialized by the generalized Clakre subdiffrential of a locally Lipschitz function (which is not convex in general), i.e., $U(x,s) = \partial j(x,s)$ for a.e. $x \in \Gamma_d$ and all $s \in \mathbb{R}$, then the relation $\frac{\partial u}{\partial \nu_{a,b}} \in U(x,u)$ reduces to the following multivalued nonmonotone boundary condition

$$\frac{\partial u}{\partial \nu_{a,b}} \in \partial j(x,u) \text{ on } \Gamma_d, \tag{2}$$

where $j\colon \Gamma_d \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with respect to the second variable, and $\partial j(x,s)$ stands for the Clarke subdifferential of $s \mapsto j(x,s)$. In fact, the multivalued nonmonotone boundary condition (2) has been used commonly in numerous nonsmooth mechanics problems and semipermeability problems, for instance, Liu et al. [36] used the theory of nonsmooth analysis and Lagrange multipliers method to establish the remarkable existence and convergence results for an elastic frictional contact problem with nonmonotone subdifferential boundary conditions. Concerning the research of problems with multivalued boundary conditions, we refer to the recent contribution of Migórski-Ochal [46], Liu [37], Naniewicz-Panagiotopoulos [49], Panagiotopoulos [50, 51], Migórski-Pączka [48], Li-Liu [32], Han [27], Liu el al. [35, 34], Zeng-Migórski-Khan [63] and etc.

The fourth feature of the paper is the presence of obstacle effect. The study of obstacle problems goes back to the pioneering contributions of J.-L. Lions [33]. Various classes of obstacle problems arise naturally when describing phenomena in real-world problems. Many of these models, such as the fluid filtration through a porous medium, osmosis, optimal stopping, heat control, etc., are described in monographs by Duvaut-Lions [14] and Rodrigues [57]. Recently, obstacle effects arising in dynamic vehicle routing problems, contact problems in mechanics, fluids flow models, the penetration phenomenon of the magnetic field, etc., have been studied in Brezis-Kinderlehrer-Lewy [5], Wang-Han-Cheng [59], Han-Sofonea [28] and the cited references therein.

Parameter identification is an inverse problem taking place in material model development, which raises much interest in recent years, for example, Cakoni-Moskow-Pangburn [7] considered the two scale asymptotic expansion for a transmission problem modeling scattering by a bounded inhomogeneity with a periodic coefficient in the lower order term of the Helmholtz equation, and shown a new convergence estimate for the second order boundary corrector on a square, and Guzina-Cakoni-Bellis [26] investigated the possibility of multi-frequency reconstruction of sound-soft and penetrable obstacles via the linear sampling method involving either far-field or near-field observations of the scattered field. Finally, we note that the inverse problem under investigation is motivated by the problem of identification of a discontinuous coefficient in an elliptic variational inequality, see Gutman [25], Zeev-Cakoni [60], Migórski-Ochal [45], Cakoni-Haddar-Lechleiter [6], Zeng-Bai-Winkert-Yao [62] and Migórski-Khan-Zeng [44, 43]. Since our direct problem is governed by the weighted anisotropic (p,q)-Laplacian, the identification problem for (1) seeks to determine the coefficients a and b (e.g., the permeabilities of the medium), and a function h (representing a flux of heat, of fluid, or electricity, depending on a model) on a part of the boundary in such a way that the solution u (that can be either a temperature, a pressure, or an electric potential) matches the observed or measured data z. More precisely, since the problem (1) is not uniquely solvable, in general, which leads to that the inverse problem is inevitable a double minimization one, see Problem 4. The compactness of the set (28) of admissible parameters represents the crucial consequence.

The main contribution of the paper is twofold. The first contribution of the paper is to examine the existence of a weak nontrivial solution to the anisotropic elliptic obstacle inclusion problem, problem (1), by employing the theory of nonsmooth analysis and a surjectivity theorem for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping. However, the second goal of the paper is to develop the new sufficient conditions for determining the solvability of the nonlinear inverse problem under consideration. To the best of our knowledge, this is the first work that combines the weighted anisotropic (p,q)-Laplacian along with an obstacle constraint, a multivalued convection term (a reaction term depending on the gradient), and multivalued mixed boundary conditions.

The paper is organized as follows. Section 2 recalls a preliminary material including p-Laplacian eigenvalue problem with the Steklov boundary condition, the necessary results in Lebesgue and Sobolev spaces with variable exponents, and a surjectivity result for multivalued pseudomonotone operators. In Section 3, we first impose the assumptions on the data of problem (1) and then examine the nonemptiness and compactness of the solution set to this problem. In Section 4, we present a new existence result to the nonlinear inverse problem under consideration.

2. **Mathematical prerequisites.**

In this section, we collect some the basic definitions and tools that will be needed in the sequel to derive the main results of the paper.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ such that Γ is decomposed into four mutually disjoint parts Γ_a, Γ_b, Γ_c and Γ_d with Γ_a having positive Lebesgue measure, and let $1 \leq \delta < \infty$. Let $M(\Omega)$ be the space of all measurable functions $u\colon \Omega \to \mathbb{R}$, and we always identify two functions which differ on a Lebesgue-null set. Let D be a nonempty subset of $\overline{\Omega}$. In what follows, we denote by $L^\delta(D) := L^\delta(D;\mathbb{R})$ and $L^\delta(D;\mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_{\delta,D}$, that is,

$$\|u\|_{\delta,D} := \left(\int_D |u|^\delta \,\mathrm{d}x\right)^{\frac{1}{\delta}} \quad \text{for all } u \in L^\delta(D).$$

We set $L^\delta(D)_+ := \{u \in L^\delta(D) \mid u(x) \geq 0 \text{ for a.e. } x \in D\}$. Moreover, $W^{1,\delta}(\Omega)$ stands for the Sobolev space endowed with the norm $\|\cdot\|_{1,\delta,\Omega}$, namely,

$$\|u\|_{1,\delta,\Omega} := \|u\|_{\delta,\Omega} + \|\nabla u\|_{\delta,\Omega} \quad \text{for all } u \in W^{1,\delta}(\Omega).$$

Now we review the r-Laplacian eigenvalue problem with Steklov boundary condition given by

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u && \text{in } \Omega,\\ |u|^{r-2}u\cdot\nu &= \lambda|u|^{r-2}u && \text{on } \Gamma. \end{aligned} \tag{3}$$

It is well-known that problem (3) has a smallest eigenvalue $\lambda_{1,r}^S > 0$ that is isolated and simple (see, for instance [30]). Also, it is not difficult to see that $\lambda_{1,r}^S > 0$ can be characterized by

$$\lambda_{1,r}^S = \inf_{u\in W^{1,r}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^r\,dx + \int_\Omega |u|^r\,dx}{\displaystyle\int_\Gamma |u|^r\,d\Gamma}. \tag{4}$$

For the sake of convenience, in what follows, we denote by $u_{1,r}^S$ the first eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}^S$ which, indeed, belongs to $\operatorname{int}\big(C^1(\overline{\Omega})_+\big)$, where $\operatorname{int}\big(C^1(\overline{\Omega})_+\big)$ stands for the interior of $C^1(\overline{\Omega})_+ := \{u \in C^1(\overline{\Omega}) \mid u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\}$. Without any loss of generality, we suppose that $\|u_{1,r}^S\|_{r,\Gamma} = 1$.

We introduce a subset $C_+(\overline{\Omega})$ of $C(\overline{\Omega})$ defined by

$$C_+(\overline{\Omega}) := \{a \in C(\overline{\Omega}) \mid 1 < a(x) \text{ for all } x \in \overline{\Omega}\}.$$

For the sake of convenience, in the sequel, for any $r \in C_+(\overline{\Omega})$, we define

$$r_- := \min_{x\in\overline{\Omega}} r(x) \text{ and } r_+ := \max_{x\in\overline{\Omega}} r(x).$$

Let $p \in C_+(\overline{\Omega})$. In what follows, we denote by $p' \in C_+(\overline{\Omega})$ the conjugate variable exponent to p, namely,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \text{ for all } x \in \overline{\Omega}.$$

Also, we denote by s^* and s_* the critical Sobolev variable exponents to $s \in C_+(\overline{\Omega})$ in the domain and on the boundary, respectively, given by

$$s^*(x) = \begin{cases} \frac{Ns(x)}{N-s(x)} & \text{if } s(x) < N, \\ +\infty & \text{if } s(x) \geq N, \end{cases} \text{ for all } x \in \overline{\Omega}, \tag{5}$$

and

$$s_* = \begin{cases} \frac{(N-1)s(x)}{N-s(x)} & \text{if } s(x) < N, \\ +\infty & \text{if } s(x) \geq N \end{cases} \text{ for all } x \in \overline{\Omega}, \tag{6}$$

respectively.

Let $r \in C_+(\overline{\Omega})$, let us recall the variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ defined by

$$L^{r(\cdot)}(\Omega) := \left\{ u \in M(\Omega) \mid \int_\Omega |u|^{r(x)}\,\mathrm{d}x < +\infty \right\}.$$

It is well-known that $L^{r(\cdot)}(\Omega)$ is equipped with the Luxemburg norm given by

$$\|u\|_{r(\cdot),\Omega} := \inf\left\{ \lambda > 0 \mid \int_\Omega \left(\frac{|u|}{\lambda}\right)^{r(x)} \mathrm{d}x \leq 1 \right\},$$

to be a separable and reflexive Banach space, the dual space of $L^{r(\cdot)}(\Omega)$ is $L^{r(\cdot)'}(\Omega)$ (i.e., $L^{r(\cdot)}(\Omega)^* = L^{r(\cdot)'}(\Omega)$), and the following Hölder inequality holds:

$$\int_\Omega |uv|\,\mathrm{d}x \leq \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(\cdot),\Omega}\|v\|_{r(\cdot)',\Omega} \leq 2\|u\|_{r(\cdot),\Omega}\|v\|_{r(\cdot)',\Omega}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r(\cdot)'}(\Omega)$.

Remark 2.1. It is not difficult to see that if $r_1, r_2 \in C_+(\overline{\Omega})$ are such that $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega).$$

For any $r \in C_+(\overline{\Omega})$, we consider the modular function $\varrho_{r(\cdot),\Omega}\colon L^{r(\cdot)}(\Omega) \to \mathbb{R}_+ := [0, +\infty)$ given by

$$\varrho_{r(\cdot),\Omega}(u) := \int_\Omega |u|^{r(x)}\,\mathrm{d}x \text{ for all } u \in L^{r(\cdot)}(\Omega). \tag{7}$$

The following proposition delivers some important relations between the norm of variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ and the modular function $\varrho_{r(\cdot),\Omega}$ (defined in (7)).

Proposition 2.2. *If $r \in C_+(\overline{\Omega})$ and $u \in L^{r(\cdot)}(\Omega)$, then we have the following assertions:*

(i) $\|u\|_{r(\cdot),\Omega} = \lambda \Longleftrightarrow \varrho_{r(\cdot),\Omega}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$*;*
(ii) $\|u\|_{r(\cdot),\Omega} < 1$ *(resp.* $= 1, > 1$*)* $\Longleftrightarrow \varrho_{r(\cdot),\Omega}(u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) $\|u\|_{r(\cdot),\Omega} < 1 \Longrightarrow \|u\|^{r_+}_{r(\cdot),\Omega} \leq \varrho_{r(\cdot),\Omega}(u) \leq \|u\|^{r_-}_{r(\cdot),\Omega}$*;*
(iv) $\|u\|_{r(\cdot),\Omega} > 1 \Longrightarrow \|u\|^{r_-}_{r(\cdot),\Omega} \leq \varrho_{r(\cdot),\Omega}(u) \leq \|u\|^{r_+}_{r(\cdot),\Omega}$*;*
(v) $\|u\|_{r(\cdot),\Omega} \to 0 \Longleftrightarrow \varrho_{r(\cdot),\Omega}(u) \to 0$*;*
(vi) $\|u\|_{r(\cdot),\Omega} \to +\infty \Longleftrightarrow \varrho_{r(\cdot),\Omega}(u) \to +\infty$*.*

Let D be a nonempty subset of $\overline{\Omega}$. In what follows, we denote by $\|\cdot\|_{r(\cdot),D}$ and by the norm of variable exponent Lebesgue space $L^{r(\cdot)}(D)$. Set $\varrho_{r(\cdot),D}(u) = \int_D |u|^{r(x)}\,\mathrm{d}x$ for $u \in L^{r(\cdot)}(D)$.

On the other hand, let us recall the corresponding variable exponent Sobolev spaces, which could be formulated by the same way by applying the variable exponent Lebesgue spaces. Let $r \in C_+(\overline{\Omega})$, we denote by $W^{1,r(\cdot)}(\Omega)$ the variable exponent Sobolev space given in

$$W^{1,r(\cdot)}(\Omega) := \left\{ u \in L^{r(\cdot)}(\Omega) \mid |\nabla u| \in L^{r(\cdot)}(\Omega) \right\}.$$

It can prove that variable exponent Sobolev space $W^{1,r(\cdot)}(\Omega)$ is equipped with the norm

$$\|u\|_{1,r(\cdot),\Omega} := \|u\|_{r(\cdot),\Omega} + \|\nabla u\|_{r(\cdot),\Omega} \text{ for all } u \in W^{1,r(\cdot)}(\Omega)$$

to be a separable and reflexive Banach space, where $\|\nabla u\|_{r(\cdot),\Omega} := \||\nabla u|\|_{r(\cdot),\Omega}$. We also consider a subspace $W_0^{1,r(\cdot)}(\Omega)$ of $W^{1,r(\cdot)}(\Omega)$ defined by $W_0^{1,r(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(\cdot),\Omega}}$. For space $W_0^{1,r(\cdot)}(\Omega)$, it is well-known that the Poincaré inequality is valid

$$\|u\|_{r(\cdot),\Omega} \leq c_0 \|\nabla u\|_{r(\cdot),\Omega} \text{ for all } u \in W_0^{1,r(\cdot)}(\Omega)$$

for some $c_0 > 0$. So, in what follows, we adopt the equivalent norm $\|\cdot\|_{1,r(\cdot),0,\Omega}$ to $W_0^{1,r(\cdot)}(\Omega)$

$$\|u\|_{1,r(\cdot),0,\Omega} = \|\nabla u\|_{r(\cdot),\Omega} \text{ for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

Moreover, we introduce a subset V of $W^{1,p(\cdot)}(\Omega)$ given by

$$V := \left\{ u \in W^{1,p(\cdot)}(\Omega) \mid u = 0 \text{ for a.e. } x \in \Gamma_a \right\}.$$

Since Γ_a has a positive measure, it follows from Gossez [24, Corollary 5.8] that the norm $\|\cdot\|_{1,r(\cdot),\Omega}$ on V is equivalent to the one, $\|\cdot\|_{1,r(\cdot),0,\Omega}$ and V endowed the norm

$$\|u\|_V := \|u\|_{1,r(\cdot),0,\Omega} \text{ for all } v \in V,$$

becomes a reflexive Banach space.

In the sequel, we denote by $C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ the set of all functions $r\colon \overline{\Omega} \to \mathbb{R}$ that are log-Hölder continuous, namely, there is a constant $C > 0$ satisfying

$$|r(x) - r(y)| \le \frac{C}{|\log|x-y||} \text{ for all } x, y \in \overline{\Omega} \text{ with } |x-y| < \tfrac{1}{2}.$$

The following proposition gives several important embeddings results, its detailed proof can be founded in Diening-Harjulehto-Hästö-Ružička [13, Corollary 8.3.2] and Fan [16, Proposition 2.1].

Proposition 2.3. *The following statements hold*

(i) *if* $r \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ *and* $s \in C(\overline{\Omega})$ *is such that*

$$1 \le s(x) \le r^*(x) \text{ for all } x \in \overline{\Omega},$$

then the embedding is continuous

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega).$$

(ii) *if* $s \in C_+(\overline{\Omega})$ *is such that*

$$1 \le s(x) < r^*(x) \text{ for all } x \in \overline{\Omega},$$

then the embedding is compact

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega).$$

Proposition 2.4. *The following statements hold*

(i) *if* $r \in C_+(\overline{\Omega}) \cap W^{1,\varsigma}(\Omega)$ *for some* $\varsigma > N$ *and* $s \in C(\overline{\Omega})$ *is such that*

$$1 \le s(x) \le r_*(x) \text{ for all } x \in \overline{\Omega},$$

then the embedding is continuous

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\partial\Omega).$$

(ii) *if* $s \in C_+(\overline{\Omega})$ *is such that*

$$1 \le s(x) < r_*(x) \text{ for all } x \in \overline{\Omega},$$

then the embedding is compact

$$W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\partial\Omega).$$

Remark 2.5. The embeddings in Propositions 2.3 and 2.4 remain valid, if we replace the space $W^{1,r(\cdot)}(\Omega)$ by V.

Throughout the paper the symbols " $\xrightarrow{w}$ " and "$\to$" stand for the weak and the strong convergence, respectively, in various spaces. For any $a \in L^\infty(\Omega)$ with $\inf_{x\in\Omega} a(x) > 0$, we introduce the nonlinear operator $\widetilde{A}\colon V \to V^*$ given by

$$\langle \widetilde{A}(u), v\rangle := \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x, \tag{8}$$

for $u, v \in V$ with $\langle\cdot,\cdot\rangle$ being the duality pairing between V and its dual space V^*. Arguing as in the proof of Proposition 2.5 of Gasiński-Parpagerogiou [18] or Rădulescu-Repovš [56] (p.40), we have the following result which states main properties of $\widetilde{A}\colon V \to V^*$.

Proposition 2.6. *The operator $\widetilde{A}$ defined by (8) is bounded, continuous, monotone (hence maximal monotone) and of type* (S_+)*, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } V \quad \text{and} \quad \limsup_{n\to\infty} \langle \widetilde{A}u_n, u_n - u\rangle \le 0,$$

imply $u_n \to u$ in V.

We now recall some notions and results concerning nonlinear and nonsmooth analysis as well as multivalued analysis. We review the notions of pseudomonotonicity and generalized pseudomonotonicity in the sense of Brezis for multivalued operators (see e.g., Migórski et al. [47, Definition 3.57]) which will be useful in the sequel.

Definition 2.7. Let X be a reflexive real Banach space. The operator $A\colon X \to 2^{X^*}$ is called

(a) pseudomonotone (in the sense of Brezis) if the following conditions hold:
  (i) the set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$.
  (ii) A is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*.
  (iii) if $\{u_n\} \subset X$ with $u_n \xrightarrow{w} u$ in X and $u_n^* \in A(u_n)$ are such that
  $$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$
  then to each element $v \in X$, there exists $u^*(v) \in A(u)$ with
  $$\langle u^*(v), u - v\rangle_{X^*\times X} \le \liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

(b) generalized pseudomonotone (in the sense of Brezis) if the following holds: Let $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with $u_n^* \in A(u_n)$. If $u_n \xrightarrow{w} u$ in X and $u_n^* \xrightarrow{w} u^*$ in X^* and
$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$
then the element u^* lies in $A(u)$ and
$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}.$$

It is not difficult to see that every pseudomonotone operator is generalized pseudomonotone, see e.g. Migórski-Ochal-Sofonea [47, Proposition 3.58] or Denkowski et al. [12, Proposition 1.3.65]. Also, under an additional assumption of boundedness, we obtain the converse statement, see e.g. Migórski-Ochal-Sofonea [47, Proposition 3.58] or Denkowski et al. [12, Proposition 1.3.66].

Proposition 2.8. *Let X be a reflexive real Banach space and assume that $A\colon X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^*.*
(ii) *$A\colon X \to 2^{X^*}$ is bounded.*
(iii) *if $u_n \xrightarrow{w} u$ in X and $u_n^* \xrightarrow{w} u^*$ in X^* with $u_n^* \in A(u_n)$ and if*
$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$
then $u^* \in A(u)$ and
$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}.$$

Then the operator $A\colon X \to 2^{X^*}$ is pseudomonotone.

Additionally, we recall the following definition, see, for example, Papageorgiou-Winkert [55, Definition 6.7.4].

Definition 2.9. Let (X,τ) be a Hausdorff topological space and let $\{A_n\}\subset 2^X$ be a sequence of sets. We define the τ-Kuratowski lower limit of the sets A_n by

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x\in X \mid x=\tau\text{-}\lim_{n\to\infty} x_n,\, x_n\in A_n \text{ for all } \ n\geq 1 \right\},$$

and the τ-Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n\to\infty} A_n := \left\{ x\in X \mid x=\tau\text{-}\lim_{k\to\infty} x_{n_k},\, x_{n_k}\in A_{n_k},\, n_1<n_2<\ldots<n_k<\ldots \right\}.$$

If

$$A=\tau\text{-}\liminf_{n\to\infty} A_n=\tau\text{-}\limsup_{n\to\infty} A_n,$$

then A is called τ-Kuratowski limit of the sets A_n.

We conclude this section by recalling the following surjectivity theorem for multivalued mappings which is formulated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping. The following theorem was proved in Le [31, Theorem 2.2]. We use the notation $B_R(0):=\{u\in X \mid \|u\|_X<R\}$.

Theorem 2.10. *Let X be a real reflexive Banach space, let $G\colon D(G)\subset X\to 2^{X^*}$ be a maximal monotone operator, let $F\colon D(F)=X\to 2^{X^*}$ be a bounded multivalued pseudomonotone operator and let $L\in X^*$. Assume that there exist $u_0\in X$ and $R\geq \|u_0\|_X$ such that $D(G)\cap B_R(0)\neq\emptyset$ and*

$$\langle \xi+\eta-L,u-u_0\rangle_{X^*\times X}>0 \tag{9}$$

for all $u\in D(G)$ with $\|u\|_X=R$, for all $\xi\in G(u)$ and for all $\eta\in F(u)$. Then the inclusion

$$F(u)+G(u)\ni L$$

has a solution in $D(G)$.

Remark 2.11. Indeed, it is obvious that if we can prove the following result

$$\lim_{\|u\|_X\to+\infty,u\in D(G)} \frac{\langle \xi+\eta,u-u_0\rangle_{X^*\times X}}{\|u\|_X}=+\infty, \tag{10}$$

then the estimate condition (9) holds automatically for some R large enough.

3. Existence of solutions for anisotropic obstacle inclusion problems.

The section is concerned with the investigation of solvability of anisotropic obstacle inclusion problem, problem (1), with a multivalued reaction term which depends on the gradient of unknown function, and complicated multivalued boundary conditions which contain a multivalued monotone boundary condition and a generalized multivalued boundary condition that is nonmonotone in general.

In order to obtain the existence of a nontrivial (weak) solution to problem (1), we make the following assumptions on the data of problem (1).

H(f): The multivalued convection mapping $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N\to 2^{\mathbb{R}}$ has nonempty, bounded, closed and convex values such that $0\notin f(x,0,0)$ for a. e. $x\in\Omega$ and the following conditions are satisfied

(i) the multivalued mapping $x \mapsto f(x,s,\xi)$ is measurable in Ω for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$;
(ii) the multivalued mapping $(s,\xi) \mapsto f(x,s,\xi)$ is upper semicontinuous for a. e. $x \in \Omega$;
(iii) there exist $\alpha_f \in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)_+$ and $a_f, b_f \geq 0$ such that
$$|\eta| \leq a_f|\xi|^{\frac{p(x)(r(x)-1)}{r(x)}} + b_f|s|^{r(x)-1} + \alpha_f(x)$$
for all $\eta \in f(x,s,\xi)$, all $s \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$ and a. e. $x \in \Omega$, where $r \in C_+(\overline{\Omega})$ is such that
$$r(x) < p^*(x) \text{ for all } x \in \overline{\Omega},$$
with the critical Sobolev variable exponent p^* in the domain Ω given in (5) for $s = p$;
(iv) there exist $\beta_f \in L^1(\Omega)_+$ and constants c_f, $d_f \geq 0$ satisfying
$$\eta s \leq c_f|\xi|^{p(x)} + d_f|s|^{p_-} + \beta_f(x)$$
for all $\eta \in f(x,s,\xi)$, all $s \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$ and a. e. $x \in \Omega$.

H(Φ): The function $\Phi\colon \Omega \to [0,\infty)$ is such that $\Phi \in M(\Omega)$.

H(ψ): The function $\psi\colon \Gamma_c \times \mathbb{R} \to \mathbb{R}$ is such that
(i) for all $s \in \mathbb{R}$, $x \mapsto \psi(x,s)$ is measurable on Γ_c;
(ii) for a. e. $x \in \Gamma_c$, $s \mapsto \psi(x,s)$ is convex and lower semicontinous;
(iii) for each $u \in L^{p_-}(\Gamma_c)$, the function $x \mapsto \psi(x,u(x))$ belongs to $L^1(\Gamma_c)$, i.e., $\int_\Omega \psi(x,u(x))\,\mathrm{d}\Gamma < +\infty$ for all $u \in L^{p_-}(\Gamma_c)$.

H(U): $U\colon \Gamma_d \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies the following conditions:
(i) $U(x,s)$ is a nonempty, bounded, closed and convex set in $\mathbb{R}$ for a. e. $x \in \Gamma_d$ and all $s \in \mathbb{R}$;
(ii) $x \mapsto U(x,s)$ is measurable on Γ_d for all $s \in \mathbb{R}$;
(iii) $s \mapsto U(x,s)$ is u.s.c. for a. e. $x \in \Gamma_d$;
(iv) there exist $\alpha_U \in L^{\frac{\delta(\cdot)}{\delta(\cdot)-1}}(\Gamma_d)_+$ and $a_U \geq 0$ such that
$$|U(x,s)| \leq \alpha_U(x) + a_U|s|^{\delta(x)-1}$$
for a. e. $x \in \Gamma_d$ and all $s \in \mathbb{R}$, where $\delta \in C_+(\overline{\Gamma}_d)$ is such that
$$\delta(x) < p_*(x) \text{ for all } x \in \overline{\Gamma}_d$$
with the critical Sobolev variable exponent p_* on the boundary Γ given in (6);
(v) there exist $\beta_U \in L^1(\Gamma_d)_+$ and $b_U \geq 0$ satisfying
$$\xi s \leq b_U|s|^{p_-} + \beta_U(x)$$
for all $\xi \in U(x,s)$, all $s \in \mathbb{R}$ and a. e. $x \in \Gamma_d$.

H(0): a, $b \in L^\infty(\Omega)$ are such that $\inf_{x\in\Omega} a(x) \geq c_\Lambda > 0$ and $b(x) \geq 0$ for a. e. $x \in \Omega$, and $h \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Gamma_b)$.

H(1): The inequality holds
$$c_\Lambda - c_f - b_U\big(\lambda^S_{1,p_-}\big)^{-1} > 0,$$
where λ^S_{1,p_-} is the first eigenvalue of the p_--Laplacian with the Steklov boundary condition (see (3) and (4)).

Remark 3.1. A concrete example for function ψ is given as follows

$$\psi(x,s)=\pi(x)|s| \text{ for all } s\in\mathbb{R} \text{ and a. e. } x\in\Gamma_c,$$

where $\pi\in L^{p'_-}(\Gamma_c)_+$. In this case, the convex subdifferential of ψ is formulated by

$$\partial_c\psi(x,s)=\begin{cases}\pi(x) & \text{if } s>0,\\ \pi(x)[-1,1] & \text{if } s=0,\\ -\pi(x) & \text{if } s<0,\end{cases} \quad \text{for a. e. } x\in\Gamma_c.$$

Let $j\colon\Gamma_d\times\mathbb{R}\to\mathbb{R}$ be such that $x\mapsto j(x,s)$ is measurable on Γ_d for all $s\in\mathbb{R}$ and $s\mapsto j(x,s)$ is locally Lipschitz continuous for a. e. $x\in\Gamma_d$. If the generalized Clarke subdifferential $s\mapsto\partial j(x,s)$ of j fulfills the following conditions:

H(j)(i) there exist $\alpha_j\in L^{\frac{\delta(\cdot)}{\delta(\cdot)-1}}(\Gamma_d)_+$ and $a_j\geq 0$ such that

$$|\xi|\leq\alpha_j(x)+a_j|s|^{\delta(x)-1}$$

for all $\xi\in\partial j(x,s)$, a. e. $x\in\Gamma_d$ and all $s\in\mathbb{R}$, where $\delta\in C_+(\overline{\Gamma}_d)$ is such that

$$\delta(x)<p_*(x) \text{ for all } x\in\overline{\Gamma}_d,$$

H(j)(ii) there exist $\beta_j\in L^1(\Gamma_d)_+$ and $b_j\geq 0$ satisfying

$$\xi s\leq b_j|s|^{p_-}+\beta_j(x)$$

for all $\xi\in\partial j(x,s)$, all $s\in\mathbb{R}$ and a. e. $x\in\Gamma_d$,

then hypotheses H(U) hold automatically.

Moreover, it should be pointed out that if hypotheses H(f)(iv) and (U)(v) are replaced by the following conditions, respectively:

H(f)(iv)' there exist $\beta_f\in L^1(\Omega)_+$ and constants c_f, $d_f\geq 0$ satisfying

$$\eta s\leq c_f|\xi|^{\vartheta_1(x)}+d_f|s|^{p_-}+\beta_f(x)$$

for all $\eta\in f(x,s,\xi)$, all $s\in\mathbb{R}$, all $\xi\in\mathbb{R}^N$ and a. e. $x\in\Omega$, where $\vartheta_1\in C_+(\overline{\Omega})$ is such that $\vartheta_1(x)<p(x)$ for all $x\in\overline{\Omega}$,

H(U)(v)' there exist $\beta_U\in L^1(\Gamma_d)_+$ and $b_U\geq 0$ satisfying

$$\xi s\leq b_U|s|^{\vartheta_2(x)}+\beta_U(x)$$

for all $\xi\in U(x,s)$, for all $s\in\mathbb{R}$ and for a. a. $x\in\Gamma_d$, where $\vartheta_2\in C_+(\overline{\Gamma}_d)$ is such that $\vartheta_2(x)<p(x)$ for all $x\in\overline{\Gamma}_d$,

then hypothesis H(1) can be removed. Let $\varepsilon>0$ be arbitrary. In fact, it follows from the Young inequality that there exist constants $c_1(\varepsilon),c_2(\varepsilon)>0$ satisfying

$$\begin{aligned}&\eta s\leq c_f|\xi|^{\vartheta_1(y)}+d_f|s|^{p_-}+\beta_f(y)\leq\varepsilon|\xi|^{p(y)}+c_1(\varepsilon)+d_f|s|^{p_-}+\beta_f(y),\\ &\zeta s\leq b_U|s|^{\vartheta_2(x)}+\beta_U(x)\leq\varepsilon|s|^{p(x)}+c_2(\varepsilon)+\beta_U(x)\end{aligned}$$

for all $\eta\in f(y,s,\xi)$, all $\zeta\in U(x,s)$, all $s\in\mathbb{R}$, all $\xi\in\mathbb{R}^N$, a. e. $y\in\Omega$ and a. e. $x\in\Gamma_d$. Observe that if we take $\varepsilon\in\left(0,\frac{c_\Lambda}{1+\left(\lambda^S_{1,p_-}\right)^{-1}}\right)$, then the inequality in H(1) is satisfied automatically.

Let us consider a subset K of V defined by

$$K:=\{v\in V\mid v\leq\Phi \text{ in } \Omega\}. \tag{11}$$

Remark 3.2. Under the hypothesis H(Φ), we can see that the set K is a nonempty, closed and convex subset of V. In fact, from H(Φ), we can see that $\Phi(x)\ge 0$ for a. e. $x\in\Omega$, so, it holds $0\in K$, that is, $K\neq\emptyset$. The convexity of K is obvious. Let $\{u_n\}\subset K$ be a sequence such that $u_n\to u$ in V as $n\to\infty$ for some $u\in V$. Keeping in mind that the embedding of V to $L^{p_-}(\Omega)$ is continuous, hence it has $u_n\to u$ in $L^{p_-}(\Omega)$ as $n\to\infty$. Passing to a subsequence if necessary, we have $u_n(x)\to u(x)$ as $n\to\infty$ for a. e. $x\in\Omega$. Therefore, we can see that $\Phi(x)\ge\lim_{n\to\infty}u_n(x)=u(x)$ for a. e. $x\in\Omega$, thus, $u\in K$. This means that K is closed.

Next, we give the definition of weak solutions to problem (1).

Definition 3.3. A function $u\in K$ is said to be a weak solution of problem (1), if there exist functions $\eta\in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)$ and $\xi\in L^{\frac{\delta(\cdot)}{\delta(\cdot)-1}}(\Gamma_d)$ with $\eta(x)\in f(x,u(x),\nabla u(x))$ for a. e. $x\in\Omega$, $\xi(x)\in U(x,u(x))$ for a. e. $x\in\Gamma_d$ such that the following inequality

$$\begin{aligned}&\int_\Omega\Big(a(x)|\nabla u|^{p(x)-2}\nabla u+b(x)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(v-u)\,\mathrm{d}x\\&+\beta\int_\Omega|u|^{\theta(x)-2}u(v-u)\,\mathrm{d}x+\int_{\Gamma_c}\psi(x,v)\,\mathrm{d}\Gamma-\int_{\Gamma_c}\psi(x,u)\,\mathrm{d}\Gamma\\&\ge\int_\Omega\eta(x)(v-u)\,\mathrm{d}x+\int_{\Gamma_b}h(x)(v-u)\,\mathrm{d}\Gamma+\int_{\Gamma_d}\xi(x)(v-u)\,\mathrm{d}\Gamma\end{aligned}\tag{12}$$

is satisfied for all $v\in K$, where the set K is defined by (11).

We are now in a position to deliver the main result in the section by the following theorem which reveals that for each triple of functions $(a,b,h)\in L^\infty(\Omega)_+\times L^\infty(\Omega)_+\times L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Gamma_b)$ with $a(x)>0$ and $b(x)\ge 0$ for a. e. $x\in\Omega$, the solution set to problem (1), denoted by $S(a,b,h)$, is nonempty, bounded, and weakly closed.

Theorem 3.4. *Assume that* H(f), H(0), H(1), H(ψ), H(U) *and* H(Φ) *are satisfied. Then, the solution set of problem* (1) *is nonempty, bounded, closed and weakly closed (hence, weakly compact).*

Proof. **Existence.** Assume that functions $a,b\in L^\infty(\Omega)$ and $h\in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Gamma_b)$ satisfy condition H(0). For any $u\in V$ fixed, by virtue of hypotheses H(f)(i) and (ii), we can apply Yankov-von Neumann-Aumann selection theorem (see e.g. Denkowski et al. [11, Theorem 4.3.7]) to conclude that there exists a measurable selection $\eta\colon\Omega\to\mathbb{R}$ satisfying $\eta(x)\in f(x,u(x),\nabla u(x))$ for a. e. $x\in\Omega$. On the other hand, from hypothesis H(f)(iii) and the elementary inequality $(|r_1|+|r_2|)^s\le 2^{s-1}(|r_1|^s+|r_2|^s)$ for all r_1, $r_2\in\mathbb{R}$ and $s\ge 1$, implies

$$\begin{aligned}&\int_\Omega|\eta(x)|^{r(x)'}\,\mathrm{d}x\le\int_\Omega\Big(a_f|\nabla u|^{\frac{p(x)(r(x)-1)}{r(x)}}+b_f|u|^{r(x)-1}+\alpha_f(x)\Big)^{r(x)'}\,\mathrm{d}x\\&\le\int_\Omega\Big(2^{2r(x)'-2}|\nabla u|^{p(x)}+2^{2r(x)'-2}|u|^{r(x)}+2^{r(x)'-1}\alpha_f(x)^{r(x)'}\Big)\,\mathrm{d}x\\&\le M_1\int_\Omega\Big(|\nabla u|^{p(x)}+|u|^{r(x)}+\alpha_f(x)^{r(x)'}\Big)\,\mathrm{d}x\\&\le M_1\left(\varrho_{p(\cdot),\Omega}(|\nabla u|)+\varrho_{r(\cdot),\Omega}(u)+\varrho_{r'(\cdot),\Omega}(\alpha_f)\right)\\&\le M_1\Big(\max\left\{\|u\|_V^{p_-},\|u\|_V^{p_+}\right\}+\max\left\{\|u\|_{r(\cdot),\Omega}^{r_-},\|u\|_{r(\cdot),\Omega}^{r_+}\right\}+\varrho_{r'(\cdot),\Omega}(\alpha_f)\Big),\end{aligned}$$

where $M_1 := \max_{x\in\overline{\Omega}} 2^{2r(x)'-2}$. For any $k \in C_+(\overline{\Omega})$ with $k(x) < p^*(x)$ for all $x \in \overline{\Omega}$, in the sequel, let $C_k > 0$ be such that

$$\|u\|_{k(\cdot),\Omega} \le C_k \|u\|_V \text{ for all } u \in V.$$

Keeping in mind that the embedding of V to $L^{r(\cdot)}(\Omega)$ is continuous (see hypothesis H(f)(iii)), we have

$$\begin{aligned}&\int_\Omega |\eta(x)|^{r(x)'}\,\mathrm{d}x \\ &\le M_1\left(\max\left\{\|u\|_V^{p_-}, \|u\|_V^{p_+}\right\} + \max\left\{C_r^{r_-}\|u\|_V^{r_-}, C_r^{r_+}\|u\|_V^{r_+}\right\} + \varrho_{r'(\cdot),\Omega}(\alpha_f)\right).\end{aligned} \tag{13}$$

This turns out that η belongs to $L^{r(\cdot)'}(\Omega)$. Under the above analysis, we are now in a position to introduce the Nemytskij operator $N_f\colon V \subset L^{r(\cdot)}(\Omega) \to 2^{L^{r(\cdot)'}(\Omega)}$ corresponding to the multivalued mapping f given by

$$N_f(u) := \left\{\eta \in L^{r(\cdot)'}(\Omega) \mid \eta(x) \in f(x, u(x), \nabla u(x)) \text{ for a. e. } x \in \Omega\right\}$$

for all $u \in V$.

Analogously, using hypotheses H(U)(i), (ii) and (iii) and Yankov-von Neumann-Aumann selection theorem, for each $u \in L^{\delta(\cdot)}(\Gamma_d)$, we are able to find a measurable function $\xi\colon \Gamma_d \to \mathbb{R}$ satisfying $\xi(x) \in U(x, u(x))$ for a. e. $x \in \Gamma_d$ and

$$\begin{aligned}\varrho_{\delta(\cdot)',\Gamma_d}(\xi) &= \int_{\Gamma_d} |\xi(x)|^{\delta(x)'}\,\mathrm{d}\Gamma \le \int_{\Gamma_d}\left(\alpha_U(x) + a_U|u|^{\delta(x)-1}\right)^{\delta(x)'}\mathrm{d}\Gamma \\ &\le M_2\int_{\Gamma_d}\left(\alpha_U(x)^{\delta(x)'} + |u|^{\delta(x)}\right)\mathrm{d}\Gamma = M_2\left(\varrho_{\delta(\cdot)',\Gamma_d}(\alpha_U) + \varrho_{\delta(\cdot),\Gamma_d}(u)\right) \\ &\le M_2\left(\varrho_{\delta(\cdot)',\Gamma_d}(\alpha_U) + \max\left\{\|u\|_{\delta(\cdot),\Gamma_d}^{\delta_-}, \|u\|_{\delta(\cdot),\Gamma_d}^{\delta_+}\right\}\right)\end{aligned} \tag{14}$$

for some $M_2 > 0$. So, in what follows, we denote by $N_U\colon L^{\delta(\cdot)}(\Gamma_d) \to 2^{L^{\delta(\cdot)'}(\Gamma_d)}$ the Nemytskij operator associated with the multivalued mapping U given by

$$N_U(u) := \left\{\eta \in L^{\delta(\cdot)'}(\Gamma_d) \mid \eta(x) \in U(x, u(x)) \text{ for a. e. } x \in \Gamma_d\right\}$$

for all $u \in L^{\delta(\cdot)}(\Gamma_d)$. Denote by $\iota\colon V \to L^{r(\cdot)}(\Omega)$ and $\omega\colon V \to L^{\theta(\cdot)}(\Omega)$ the embedding operators of V to $L^{r(\cdot)}(\Omega)$ and of V to $L^{\theta(\cdot)}(\Omega)$ with the adjoint operators $\iota^*\colon L^{r(\cdot)'}(\Omega) \to V^*$ and $\omega^*\colon L^{\theta(\cdot)'}(\Omega) \to V^*$. Also, let $\gamma\colon V \to L^{\delta(\cdot)}(\Gamma_d)$ be the trace operator of V into $L^{\delta(\cdot)}(\Gamma_d)$ with its adjoint operator $\gamma^*\colon L^{\delta(\cdot)'}(\Gamma_d) \to V^*$.

On the other hand, let us consider the indicator function $I_K\colon V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ of K given by

$$I_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise,}\end{cases} \quad \text{for all } u \in V.$$

It is not difficult to observe that $u \in K$ is a solution of problem (1), if and only if it solves the following problem

$$\begin{aligned}&\int_\Omega\left(a(x)|\nabla u|^{p(x)-2}\nabla u + b(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x \\ &+ \beta\int_\Omega |u|^{\theta(x)-2}u(v-u)\,\mathrm{d}x + \int_{\Gamma_c}\psi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_c}\psi(x,u)\,\mathrm{d}\Gamma + I_K(v) - I_K(u) \\ &\ge \int_\Omega \eta(x)(v-u)\,\mathrm{d}x + \int_{\Gamma_b} h(x)(v-u)\,\mathrm{d}\Gamma + \int_{\Gamma_d}\xi(x)(v-u)\,\mathrm{d}\Gamma\end{aligned} \tag{15}$$

for all $v \in V$ for some $\eta \in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)$ and $\xi \in L^{\frac{\delta(\cdot)}{\delta(\cdot)-1}}(\Gamma_d)$ with $\eta(x) \in f(x, u(x), \nabla u(x))$ for a. e. $x \in \Omega$ and $\xi(x) \in U(x, u(x))$ for a. e. $x \in \Gamma_d$. In addition, we consider the function $\widetilde{I_K}\colon V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ defined by

$$\widetilde{I_K}(u) := I_K(u) + \int_{\Gamma_c} \psi(x, u)\,\mathrm{d}\Gamma$$

for all $u \in V$. We assert that $\widetilde{I_K}$ is a proper, convex, and lower semicontinuous function. Because V is a subset of the effective domain of function $V \ni u \mapsto \int_{\Gamma_c} \psi(x, u)\,\mathrm{d}\Gamma \in \overline{\mathbb{R}}$ (see hypothesis H(ψ)(iii)). So, it is sufficient to show that $V \ni u \mapsto \int_{\Gamma_c} \psi(x, u)\,\mathrm{d}\Gamma \in \overline{\mathbb{R}}$ is a proper, convex, and lower semicontinuous function. The convexity of $V \ni u \mapsto \int_{\Gamma_c} \psi(x, u)\,\mathrm{d}\Gamma \in \overline{\mathbb{R}}$ is the direct consequence of hypothesis H(ψ)(ii). Let sequence $\{u_n\} \subset V$ be such that $u_n \to u$ in V as $n \to \infty$ for some $u \in V$. So, $u_n \to u$ in $L^1(\Gamma_c)$. Without any loss of generality, we may assume that $u_n(x) \to u(x)$ as $n \to \infty$ for a. e. $x \in \Gamma_c$. Employing Fatou lemma, we have

$$\liminf_{n\to\infty} \int_{\Gamma_c} \psi(x, u_n(x))\,\mathrm{d}\Gamma \geq \int_{\Gamma_c} \inf_{n\to\infty} \psi(x, u_n)\,\mathrm{d}\Gamma \geq \int_{\Gamma_c} \psi(x, u)\,\mathrm{d}\Gamma.$$

This reveals that $V \ni u \mapsto \int_{\Gamma_c} \psi(x, u)\,\mathrm{d}\Gamma \in \overline{\mathbb{R}}$ is l.s.c.

Whereas, we can use a standard argument to find that $u \in K$ is a solution of the inequality (15), if and only if, it solves the following inclusion problem

$$Au + \omega^* Bu - \iota^* N_f(u) - \gamma^* N_f(u) + \partial_c \widetilde{I_K}(u) \ni h \qquad \text{in } V^*, \tag{16}$$

where the nonlinear functions $A\colon V \to V^*$ and $B\colon L^{\theta(\cdot)}(\Omega) \to L^{\theta(\cdot)'}(\Omega)$ are given by

$$\langle Au, v\rangle := \int_\Omega \Big(a(x)|\nabla u|^{p(x)-2}\nabla u + b(x)|\nabla u|^{q(x)-2}\nabla u\Big) \cdot \nabla v\,\mathrm{d}x \text{ for all } u, v \in V,$$

$$\langle Bw, z\rangle_{L^{\theta(\cdot)'}(\Omega)\times L^{\theta(\cdot)}(\Omega)} := \beta \int_\Omega |w|^{\theta(x)-2} wz\,\mathrm{d}x \text{ for all } w, z \in L^{\theta(\cdot)}(\Omega),$$

respectively.

Next, we are going to invoke Theorem 2.10 for examining the existence of a nontrivial solution to problem (16). By the definition of A and Proposition 2.6, we can see that A is a bounded map. From the definition of B, it is not difficult to prove that B is also a bounded operator. Therefore, taking account into (13) and (14), we conclude that $F\colon V \to 2^{V^*}$, $F(u) = Au + \omega^* Bu - \iota^* N_f(u) - \gamma^* N_f(u)$ for all $u \in V$, is a bounded map thanks to the boundedness of ω, ι and γ. Using the convexity of f and U, we can verify that for each $u \in V$ the sets $\iota^* N_f(u)$ and $\gamma^* N_f(u)$ are both convex in V^*. Indeed, we also can prove that $F(u)$ is closed in V^* for each $u \in V$. Moreover, we shall show that F is generalized pseudomonotone. Let sequences $\{u_n\} \subset V$, $\{\zeta_n\} \subset V^*$ and $(u, \zeta) \in V \times V^*$ be such that $\zeta_n \in F(u_n)$ for each $n \in \mathbb{N}$,

$$\zeta_n \xrightarrow{w} \zeta,\ u_n \xrightarrow{w} u \text{ in } V, \text{ and } \limsup_{n\to\infty} \langle \zeta_n, u_n - u\rangle \leq 0. \tag{17}$$

Then, for every $n \in \mathbb{N}$, there are $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ such that

$$\zeta_n = Au_n + \omega^* Bu_n - \iota^* \eta_n - \gamma^* \xi_n \text{ for all } n \in \mathbb{N}.$$

From (13) and (14), we can observe that sequences $\{\eta_n\} \subset L^{r(\cdot)'}(\Omega)$ and $\{\xi_n\} \subset L^{\delta(\cdot)'}(\Gamma_d)$ are both bounded independently of n. Passing to a subsequence if necessary, we may assume that

$$\eta_n \xrightarrow{w} \eta \text{ in } L^{r(\cdot)'}(\Omega), \text{ and } \xi_n \xrightarrow{w} \xi \text{ in } L^{\delta(\cdot)'}(\Gamma_d) \text{ as } n \to \infty$$

for some $(\eta, \xi) \in L^{r(\cdot)'}(\Omega) \times L^{\delta(\cdot)'}(\Gamma_d)$. Keeping in mind that the embeddings of V to $L^{\theta(\cdot)}(\Omega)$ and V to $L^{r(\cdot)}(\Omega)$, and the trace operator $\gamma\colon V \to L^{\delta(\cdot)}(\Gamma_d)$ are compact (see Propositions 2.3 and 2.4), then we have

$$\begin{aligned}\lim_{n\to\infty}\langle \omega^* B u_n, u_n - u\rangle &= \lim_{n\to\infty}\langle B u_n, u_n - u\rangle_{L^{\theta'(\cdot)}(\Omega)\times L^{\theta(\cdot)}(\Omega)} = 0,\\ \lim_{n\to\infty}\langle \iota^*\eta_n, u_n - u\rangle &= \lim_{n\to\infty}\langle \eta_n, \iota(u_n - u)\rangle_{L^{r(\cdot)'}(\Omega)\times L^{r(\cdot)}(\Omega)} = 0,\\ \lim_{n\to\infty}\langle \gamma^*\xi_n, u_n - u\rangle &= \lim_{n\to\infty}\langle \xi_n, \gamma(u_n - u)\rangle_{L^{\delta(\cdot)'}(\Gamma_d)\times L^{\delta(\cdot)}(\Gamma_d)} = 0.\end{aligned} \tag{18}$$

Using (17) and (18), one has

$$\begin{aligned}0 \geq& \limsup_{n\to\infty}\langle \zeta_n, u_n - u\rangle\\ \geq& \limsup_{n\to\infty}\langle A u_n, u_n - u\rangle + \liminf_{n\to\infty}\langle \omega^* B u_n, u_n - u\rangle + \liminf_{n\to\infty}\langle \iota^*\eta_n, u - u_n\rangle\\ &+ \liminf_{n\to\infty}\langle \gamma^*\xi_n, u - u_n\rangle\\ \geq& \limsup_{n\to\infty}\langle A u_n, u_n - u\rangle.\end{aligned}$$

The latter together with the monotonicity of $s \mapsto b(x)|s|^{q(x)-2}s$ deduces

$$0 \geq \limsup_{n\to\infty}\langle \widetilde{A} u_n, u_n - u\rangle.$$

Taking into account the inequality above and the fact that $\widetilde{A}$ is of type (S_+) (see Proposition 2.6), we obtain

$$u_n \to u \text{ in } V \text{ as } n \to \infty.$$

Passing to a subsequence if necessary, we may suppose that

$$u_n(x) \to u(x) \text{ and } \nabla u_n(x) \to \nabla u(x) \text{ as } n \to \infty \text{ for a.e. } x \in \Omega, \tag{19}$$

due to the continuity of the embedding of V to $W^{1,p_-}(\Omega)$. Employing Mazur's theorem, we are able to find a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ of convex combinations of $\{\eta_n\}_{n\in\mathbb{N}}$ satisfying

$$\chi_n \to \eta \text{ in } L^{r(\cdot)'}(\Omega) \text{ as } n \to \infty.$$

This allows one to suppose that $\chi_n(x) \to \eta(x)$ for a.e. $x \in \Omega$ (owing to the continuity of the embedding of $L^{r(\cdot)'}(\Omega)$ to $L^{r'_-}(\Omega)$). From the convexity of f, it finds

$$\chi_n(x) \in f(x, u_n(x), \nabla u_n(x)) \text{ for a.e. } x \in \Omega.$$

Because f is u.s.c. and has nonempty, bounded, closed and convex values (see hypotheses H(f)(i) and (ii)). This allows us to invoke Denkowski et al. [11, Proposition 4.1.9] to admit that the graph of $(s,\xi) \mapsto f(x,s,\xi)$ is closed for a.e. $x \in \Omega$. Besides, we use the convergences (19) and $\chi_n(x) \to \eta(x)$ for a.e. $x \in \Omega$ to confess

$$\eta(x) \in f(x, u(x), \nabla u(x)) \text{ for a.e. } x \in \Omega.$$

This means that $\eta \in N_f(u)$. Analogously, we could apply the same arguments to infer that $\xi \in N_U(u)$. However, it follows from the continuity of A, B and the convergence (17) that

$$\zeta_n = Au_n + \omega^* Bu_n - \iota^*\eta_n - \gamma^*\xi_n \xrightarrow{w} Au + \omega^* Bu - \iota^*\eta - \gamma^*\xi = \zeta \text{ in } V^*. \quad (20)$$

This means that $\zeta \in F(u)$. Note that

$$\begin{aligned}
&\lim_{n\to\infty} \langle \zeta_n, u_n\rangle \\
&= \lim_{n\to\infty} \langle Au_n + \omega^* Bu_n - \iota^*\eta_n - \gamma^*\xi_n, u_n\rangle \\
&= \lim_{n\to\infty} \langle Au_n + \omega^* Bu_n, u_n\rangle - \lim_{n\to\infty} \langle \eta_n, \iota u_n\rangle_{L^{r(\cdot)'}(\Omega)\times L^{r(\cdot)}(\Omega)} \\
&\quad - \lim_{n\to\infty} \langle \xi_n, \gamma u_n\rangle_{L^{\delta(\cdot)'}(\Gamma_d)\times L^{\delta(\cdot)}(\Gamma_d)} \\
&= \langle Au + \omega^* Bu - \iota^*\eta - \gamma^*\xi, u\rangle \\
&= \langle \zeta, u\rangle,
\end{aligned}$$

we use the equality above and (20) to profess that F is a generalized pseudomonotone operator. We are now in a position to employ Proposition 2.8 to admit that F is pseudomonotone.

Furthermore, we show that F is coercive. For any $u \in V$ and $\zeta \in F(u)$, we have

$$\begin{aligned}
&\langle \zeta, u\rangle \quad (21)\\
&= \langle Au, u\rangle + \langle Bu, u\rangle_{L^{\theta(\cdot)'}(\Omega)\times L^{\theta(\cdot)}(\Omega)} - \langle \eta, u\rangle_{L^{r(\cdot)'}(\Omega)\times L^{r(\cdot)}(\Omega)} \\
&\quad - \langle \xi, u\rangle_{L^{\delta(\cdot)'}(\Gamma_d)\times L^{\delta(\cdot)}(\Gamma_d)} \\
&\geq \int_\Omega c_\Lambda |\nabla u|^{p(x)} + b(x)|\nabla u|^{q(x)}\,\mathrm{d}x + \beta \int_\Omega |u|^{\theta(x)}\,\mathrm{d}x - \int_\Omega \eta(x)u(x)\,\mathrm{d}x \\
&\quad - \int_{\Gamma_d} \xi(x)u(x)\,\mathrm{d}\Gamma \\
&\geq c_\Lambda \varrho_{p(\cdot),\Omega}(|\nabla u|) + \beta\varrho_{\theta(\cdot),\Omega}(u) - \int_\Omega c_f|\nabla u|^{p(x)} + d_f|u|^{p_-} + \beta_f(x)\,\mathrm{d}x \\
&\quad - \int_{\Gamma_d} b_U|u|^{p_-} + \beta_U(x)\,\mathrm{d}\Gamma \\
&\geq (c_\Lambda - c_f)\,\varrho_{p(\cdot),\Omega}(|\nabla u|) + \beta\varrho_{\theta(\cdot),\Omega}(u) - d_f\|u\|_{p_-,\Omega}^{p_-} - \|\beta_f\|_{1,\Omega} \\
&\quad - b_U\|u\|_{p_-,\Gamma_d}^{p_-} - \|\beta_U\|_{1,\Gamma_d},
\end{aligned}$$

where $\eta \in N_f(u)$ and $\xi \in N_U(u)$ are such that $\zeta = Au + \omega^* Bu - \iota^*\eta - \gamma^*\xi$. Employing the variational characteristic of the smallest eigenvalue $\lambda_{1,p_-}^S > 0$ to Steklov eigenvalue problem for the p_--Laplacian (see (4)), we have

$$b_U\|u\|_{p_-,\Gamma_d}^{p_-} \leq b_U\left(\lambda_{1,p_-}^S\right)^{-1}\left(\|\nabla u\|_{p_-,\Omega}^{p_-} + \|u\|_{p_-,\Omega}^{p_-}\right). \quad (22)$$

Inserting (22) into (21), it yields

$$\begin{aligned}
\langle \zeta, u\rangle &\geq (c_\Lambda - c_f)\,\varrho_{p(\cdot),\Omega}(|\nabla u|) + \beta\varrho_{\theta(\cdot),\Omega}(u) - d_f\|u\|_{p_-,\Omega}^{p_-} - \|\beta_f\|_{1,\Omega} \\
&\quad - b_U\left(\lambda_{1,p_-}^S\right)^{-1}\left(\|\nabla u\|_{p_-,\Omega}^{p_-} + \|u\|_{p_-,\Omega}^{p_-}\right) - \|\beta_U\|_{1,\Gamma_d} \\
&= (c_\Lambda - c_f)\,\varrho_{p(\cdot),\Omega}(|\nabla u|) + \beta\varrho_{\theta(\cdot),\Omega}(u) - \left(d_f + b_U\left(\lambda_{1,p_-}^S\right)^{-1}\right)\|u\|_{p_-,\Omega}^{p_-}
\end{aligned}$$

$$
\begin{aligned}
&- \|\beta_f\|_{1,\Omega} - b_U \left(\lambda^S_{1,p_-}\right)^{-1} \|\nabla u\|^{p_-}_{p_-,\Omega} - \|\beta_U\|_{1,\Gamma_d} \\
\geq &\left(c_\Lambda - c_f - b_U \left(\lambda^S_{1,p_-}\right)^{-1}\right) \varrho_{p(\cdot),\Omega}(|\nabla u|) + \beta \varrho_{\theta(\cdot),\Omega}(u) - \|\beta_f\|_{1,\Omega} - M_3 \\
&- \|\beta_U\|_{1,\Gamma_d} - \left(d_f + b_U \left(\lambda^S_{1,p_-}\right)^{-1}\right) \|u\|^{p_-}_{p_-,\Omega}
\end{aligned}
$$

for some M_3 >0, where the last inequality is obtained via using Young inequality, namely,

$$
\begin{aligned}
\|\nabla u\|^{p_-}_{p_-,\Omega} &= \int_\Omega |\nabla u|^{p_-} \, \mathrm{d}x \\
&= \int_{\{x\in\Omega \,|\, p(x)=p_-\}} |\nabla u|^{p(x)} \, \mathrm{d}x + \int_{\{x\in\Omega \,|\, p(x)>p_-\}} |\nabla u|^{p_-} \, \mathrm{d}x \\
&\leq \int_{\{x\in\Omega \,|\, p(x)=p_-\}} |\nabla u|^{p(x)} \, \mathrm{d}x + \int_{\{x\in\Omega \,|\, p(x)>p_-\}} |\nabla u|^{p(x)} \, \mathrm{d}x + M_4 \\
&= \int_\Omega |\nabla u|^{p(x)} \, \mathrm{d}x + M_4 = \varrho_{p(\cdot),\Omega}(|\nabla u|) + M_4
\end{aligned}
$$

for some $M_4 > 0$. Let $\varepsilon = \frac{\beta}{2\left(\left(\lambda^S_{1,p_-}\right)^{-1} b_U + d_f\right)}$. Employing Young inequality again, we get

$$
\|u\|^{p_-}_{p_-,\Omega} = \int_\Omega |u|^{p_-} \, \mathrm{d}x \leq \varepsilon \int_\Omega |u|^{\theta(x)} \, \mathrm{d}x + M_5 = \varepsilon \varrho_{\theta(\cdot),\Omega}(u) + M_5
$$

for some $M_5 > 0$. From the last three inequalities, we have

$$
\begin{aligned}
&\langle \zeta, u \rangle \\
\geq &\left(c_\Lambda - c_f - b_U \left(\lambda^S_{1,p_-}\right)^{-1}\right) \varrho_{p(\cdot),\Omega}(|\nabla u|) + \frac{\beta}{2} \varrho_{\theta(\cdot),\Omega}(u) - \|\beta_f\|_{1,\Omega} \\
&- M_6 - \|\beta_U\|_{1,\Gamma_d} \\
\geq &\left(c_\Lambda - c_f - b_U \left(\lambda^S_{1,p_-}\right)^{-1}\right) \min\left\{\|u\|^{p_-}_V, \|u\|^{p_+}_V\right\} - \|\beta_f\|_{1,\Omega} - M_6 \\
&- \|\beta_U\|_{1,\Gamma_d} + \frac{\beta}{2} \min\left\{\|u\|^{\theta_-}_{\theta(\cdot),\Omega}, \|u\|^{\theta_+}_{\theta(\cdot),\Omega}\right\}
\end{aligned} \tag{23}
$$

with some $M_6 > 0$. This means that F is coercive. Recall that $\widetilde{I_K}$ is a proper, convex, and lower semicontinuous function, then $\widetilde{I_K}$ is bounded from below by an affine function (see, e.g., [4, Proposition 1.10]), thus, there exist constants $\alpha_K, \beta_K \geq 0$ such that

$$
\widetilde{I_K}(v) \geq -\alpha_K \|v\|_V - \beta_K \text{ for all } v \in K. \tag{24}
$$

Notice that $0 \in K$, by virtue of definition of convex subgradient, for any $\varsigma \in \partial_c \widetilde{I_K}(u)$ we have

$$
\begin{aligned}
\langle \varsigma, u \rangle &= - \langle \varsigma, -u \rangle \geq \widetilde{I_K}(u) - \widetilde{I_K}(0) = \widetilde{I_K}(u) - \int_{\Gamma_c} \psi(x,0) \, \mathrm{d}\Gamma \\
&\geq - \alpha_K \|u\|_V - \beta_K - \int_{\Gamma_c} \psi(x,0) \, \mathrm{d}\Gamma.
\end{aligned} \tag{25}
$$

Combining (23) and (25), it has

$$\lim_{\|u\|_V\to\infty, u\in K} \frac{\langle \zeta+\varsigma, u\rangle}{\|u\|_V} = +\infty.$$

This indicates that the result (10) is valid with $u_0 = 0$.

Therefore, all conditions of Theorem 2.10 are verified. Using this theorem, we conclude that there exists a function $u \in K$ such that inclusion (16) holds true, which is also a solution of problem (1). However, the condition $0 \notin f(x,0,0)$ for a. e. $x \in \Omega$ points out that $u \in K$ is a nontrivial solution of problem (1), thus, $S(a,b,h) \neq \emptyset$.

Boundedness. Arguing by contradiction, we suppose that the solution set $S(a,b,h)$ is unbounded. Therefore, we are able to find a sequence $\{u_n\} \subset S(a,b,h)$ such that

$$\|u_n\|_V \to +\infty \text{ as } n \to \infty.$$

Employing the same arguments as in the proof of the first part, we take $v = 0$ into (12) with $u = u_n$ and get the following estimates

$$\begin{aligned}
&\left(c_\Lambda - c_f - b_U\left(\lambda_{1,p_-}^S\right)^{-1}\right)\min\{\|u_n\|_V^{p_-}, \|u_n\|_V^{p_+}\} + \frac{\beta}{2}\min\{\|u_n\|_{\theta(\cdot),\Omega}^{\theta_-}, \|u\|_{\theta(\cdot),\Omega}^{\theta_+}\}\\
&\le \|\beta_f\|_{1,\Omega} + M_7 + \|\beta_U\|_{1,\Gamma_d} + \int_{\Gamma_c}\psi(x,0)\,\mathrm{d}\Gamma + M_8\|h\|_{p(\cdot)',\Gamma_b}\|u_n\|_V + \alpha_K\|u_n\|_V\\
&\quad + \beta_K
\end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$, for some $M_7, M_8 > 0$. This, obviously, leads to a contradiction. Therefore, the solution set $S(a,b,h)$ is bounded in V.

Closedness. Let a sequence $\{u_n\} \subset S(a,b,h)$ be such that

$$u_n \xrightarrow{w} u \text{ in } V \text{ as } n \to \infty$$

for some $u \in K$. Hence, we could take $\eta_n \in N_f(u_n)$ and $\xi_n \in N_U(u_n)$ such that

$$\begin{aligned}
&\int_\Omega \left(a(x)|\nabla u_n|^{p(x)-2}\nabla u_n + b(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)\,\mathrm{d}x\\
&+\beta\int_\Omega |u_n|^{\theta(x)-2}u_n(v-u_n)\,\mathrm{d}x + \int_{\Gamma_c}\psi(x,v)\,\mathrm{d}\Gamma - \int_{\Gamma_c}\psi(x,u_n)\,\mathrm{d}\Gamma\\
&\ge \int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x + \int_{\Gamma_b} h(x)(v-u_n)\,\mathrm{d}\Gamma + \int_{\Gamma_d}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma
\end{aligned} \tag{27}$$

for all $v \in K$. The boundedness of operators N_f and N_U (see (13) and (14)) allows one to suppose that there are functions $\eta \in L^{r(\cdot)'}(\Omega)$ and $\xi \in L^{\delta(\cdot)'}(\Gamma_d)$ satisfying

$$\eta_n \xrightarrow{w} \eta \text{ in } L^{r(\cdot)'}(\Omega) \text{ and } \xi_n \xrightarrow{w} \xi \text{ in } L^{\delta(\cdot)'}(\Gamma_d) \text{ as } n \to \infty.$$

Taking $v = u$ in (27) and passing to the upper limit as $n \to \infty$ in the resulting inequality, we get

$$\begin{aligned}
&\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle\\
&\le \limsup_{n\to\infty}\langle Bu_n, u-u_n\rangle_{L^{\theta(\cdot)'}(\Omega)\times L^{\theta(\cdot)}(\Omega)} + \int_{\Gamma_c}\psi(x,u\,\mathrm{d}\Gamma - \liminf_{n\to\infty}\int_{\Gamma_c}\psi(x,u_n)\,\mathrm{d}\Gamma\\
&\quad - \lim_{n\to\infty}\int_\Omega \eta_n(u-u_n)\,\mathrm{d}x - \lim_{n\to\infty}\int_{\Gamma_b} h(u-u_n)\,\mathrm{d}\Gamma - \lim_{n\to\infty}\int_{\Gamma_d}\xi_n(u-u_n)\,\mathrm{d}\Gamma\\
&\le 0.
\end{aligned}$$

We apply Proposition 2.6 to obtain that $u_n \to u$ in V as $n \to \infty$. Using the upper semicontinuity of f and U, we obtain $\eta \in N_f(u)$ and $\xi \in N_U(u)$. Passing to the upper limit as $n \to \infty$ in inequality (27), we have $u \in S(a,b,h)$. This means that $S(a,b,h)$ is weakly closed. This completes the proof. □

4. **Inverse problems for anisotropic obstacle inclusion systems.** The current section is devoted to consider a nonlinear inverse problem of identification of parameters (in the domain) and boundary data in the problem (1), and to develop a general framework for solving the nonlinear inverse problem. More precisely, we shall introduce a highly nonlinear regularized optimization problem, Problem 4 below, to identify two discontinuous parameters (a,b), which control the weighted anisotropic (p,q)-Laplacian, and a discontinuous boundary data h on the part Γ_d.

For any $g \in L^1(\Omega)$ fixed, we denote by $TV(g)$ the total variation of function g given by

$$TV(g) := \sup_{\varphi \in C^1(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega g(x) \operatorname{div} \varphi(x)\,\mathrm{d}x \mid |\varphi(x)| \le 1 \text{ for all } x \in \Omega \right\}.$$

In what follows, we denote by $BV(\Omega)$ the function space of all integrable functions with bounded variation, that is,

$$BV(\Omega) := \{g \in L^1(\Omega) \mid TV(g) < +\infty\}.$$

It is well-known that the function space $BV(\Omega)$ endowed with the norm

$$\|g\|_{BV(\Omega)} := \|g\|_{1,\Omega} + TV(g) \text{ for all } g \in BV(\Omega)$$

becomes a Banach space. Let H be a nonempty, closed and convex subset in $L^{p(\cdot)'}(\Gamma_b)$, and consider the set of admissible parameters Λ for weighted anisotropic (p,q)-Laplacian defined by

$$\Lambda := \{a \in L^\infty(\Omega) \cap BV(\Omega) \mid 0 < c_\Lambda \le a(x) \le d_\Lambda \text{ for a. e. } x \in \Omega\}, \tag{28}$$

where c_Λ and d_λ are positive constants. It is obvious to see that Λ is a closed, and convex subset of $BV(\Omega)$ and $L^\infty(\Omega)$.

Let κ, τ and $\mu > 0$ be given regularization coefficients and $z \in L^{p(\cdot)}(\Omega;\mathbb{R}^N)$ the known observed or measured data. We consider the following inverse problem which is modelled by a highly nonlinear regularized optimal control system:

Problem. *Find a^*, $b^* \in \Lambda$ and $h^* \in H$ such that*

$$\inf_{a,b\in\Lambda,\, h\in H} L(a,b,h) = L(a^*,b^*,h^*), \tag{29}$$

where the cost functional $L\colon \Lambda^2 \times H \to \mathbb{R}$ is formulated by

$$\begin{aligned} L(a,b,h) := \min_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x + \kappa\, TV(a) + \tau\, TV(b) \\ + \mu \int_{\Gamma_b} |h|^{p(x)'}\,\mathrm{d}\Gamma, \end{aligned} \tag{30}$$

and $S(a,b,h)$ is the solution set of problem (1) corresponding a, $b \in L^\infty(\Omega)$ and $h \in L^{p(\cdot)'}(\Gamma_b)$.

The main result of the section is provided by the following theorem, which presents the sufficient conditions for determining the existence of an optimal solution to nonlinear regularized optimization problem, Problem 4.

Theorem 4.1. *If the hypotheses of Theorem 3.4 are fulfilled, then the solution set of Problem 4 is nonempty and weakly compact.*

Proof. The proof of the theorem is divided into four steps.

Step 1. For each $(a, b, h) \in \Lambda^2 \times H$ fixed, the cost functional L defined by (30) is well-defined.

Let $(a, b, h) \in \Lambda^2 \times H$ be fixed. To this end, we only prove that the minimizer of $\inf_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x$ is reachable. Assume that $\{u_n\} \subset S(a, b, h)$ is a minimizing sequence of the problem $\inf_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x$. Then,

$$\inf_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x = \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x.$$

From Theorem 3.4, we can see that the set $S(a, b, h)$ is bounded in V, so does $\{u_n\}$. Therefore, there exists a subsequence of $\{u_n\}$, denoted still in the same way, such that $u_n \xrightarrow{w} u^*$ in V as $n \to \infty$ for some $u^* \in V$. This together with the weak closedness of $S(a, b, h)$ implies that $u^* \in S(a, b, h)$. Note that the function $L^{p(\cdot)}(\Omega) \ni u \mapsto \int_\Omega |u|^{p(x)}\,\mathrm{d}x \in \mathbb{R}$ is convex and continuous, so, it is weakly semicontinuous. Hence, one has

$$\begin{aligned}\inf_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x &= \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x\\ &\ge \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x \ge \inf_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x.\end{aligned}$$

This indicates that for every $(a, b, h) \in \Lambda^2 \times H$ we are able to find a function $u^* \in S(a, b, h)$ satisfying

$$\inf_{u\in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x = \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x,$$

and hence, $L(a, b, h)$ is well defined.

Let $(a, b, h) \in \Lambda^2 \times H$ and $u \in S(a, b, h)$ be arbitrary. A simple calculation (see e.g. (26)) gives

$$\begin{aligned}&\left(c_\Lambda - c_f - b_U\left(\lambda^S_{1,p_-}\right)^{-1}\right)\min\left\{\|u\|_V^{p_-}, \|u\|_V^{p_+}\right\} + \frac{\beta}{2}\min\left\{\|u\|^{\theta_-}_{\theta(\cdot),\Omega}, \|u\|^{\theta_+}_{\theta(\cdot),\Omega}\right\}\\ &\le \|\beta_f\|_{1,\Omega} + M_9 + \|\beta_U\|_{1,\Gamma_d} + \int_{\Gamma_c}\psi(x,0)\,\mathrm{d}\Gamma + \alpha_K\|u\|_V + M_{10}\|h\|_{p(\cdot)',\Gamma_b}\|u\|_V\\ &\quad + \beta_K\end{aligned}$$

for some $M_9, M_{10} > 0$. We infer that S maps bounded sets of $BV(\Omega)\times BV(\Omega)\times H$ into bounded sets of K.

Step 2. If sequence $\{(a_n, b_n, h_n)\} \subset \Lambda^2 \times H$ is such that $\{a_n\}$, $\{b_n\}$ are bounded in $BV(\Omega)$, and $(a_n, b_n) \to (a, b)$ in $L^1(\Omega) \times L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in H for some $(a, b, h) \in L^1(\Omega)^2 \times H$, then $(a, b) \in \Lambda^2$ and one has

$$\emptyset \ne w\text{–}\limsup_{n\to\infty} S(a_n, b_n, h_n) \subset S(a, b, h). \tag{31}$$

Let sequence $\{(a_n, b_n, h_n)\} \subset \Lambda^2 \times H$ be such that $(a_n, b_n) \to (a, b)$ in $L^1(\Omega)^2$ and $h_n \xrightarrow{w} h$ in H for some $(a, b, h) \in L^1(\Omega)^2 \times H$. Hence, by the properties of Λ (that is, Λ is nonempty, closed and convex in $BV(\Omega)$ and $L^1(\Omega)$), one has $(a, b, h) \in \Lambda^2 \times H$. By virtue of boundedness of $\{(a_n, b_n)\} \subset BV(\Omega) \times BV(\Omega)$ and the map S, we obtain that $\cup_{n\ge1} S(a_n, b_n, h_n)$ is bounded in K. The latter together with the reflexivity of V concludes that the set $w\text{–}\limsup_{n\to\infty} S(a_n,b_n,h_n)$ is nonempty.

Let $u\in w\text{–}\limsup_{n\to\infty} S(a_n,b_n,h_n)$ be arbitrary. Passing to a subsequence if necessary, there exists a sequence $\{u_n\}\subset K$ satisfying

$$u_n\in S(a_n,b_n,h_n) \text{ and } u_n \xrightarrow{w} u \text{ in } V \text{ as } n\to\infty.$$

Hence, for each $n\in\mathbb{N}$, we are able to find $\eta_n\in N_f(u_n)$ and $\xi_n\in N_U(u_n)$ such that

$$\begin{aligned}&\int_\Omega \Big(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n+b_n(x)|\nabla u_n|^{q(x)-2}\nabla u_n\Big)\cdot\nabla(v-u_n)\,\mathrm{d}x\\&+\beta\int_\Omega |u_n|^{\theta(x)-2}u_n(v-u_n)\,\mathrm{d}x+\int_{\Gamma_c}\psi(x,v)\,\mathrm{d}\Gamma-\int_{\Gamma_c}\psi(x,u_n)\,\mathrm{d}\Gamma\\&\ge\int_\Omega \eta_n(x)(v-u_n)\,\mathrm{d}x+\int_{\Gamma_b}h_n(x)(v-u_n)\,\mathrm{d}\Gamma+\int_{\Gamma_d}\xi_n(x)(v-u_n)\,\mathrm{d}\Gamma \qquad (32)\end{aligned}$$

for all $v\in K$. Letting $v=u$ for (32), we obtain

$$\begin{aligned}&\int_\Omega \Big(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n+b_n(x)|\nabla u_n|^{q(x)-2}\nabla u_n\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\&\le\beta\int_\Omega |u_n|^{\theta(x)-2}u_n(u-u_n)\,\mathrm{d}x+\int_{\Gamma_c}\psi(x,u)\,\mathrm{d}\Gamma-\int_{\Gamma_c}\psi(x,u_n)\,\mathrm{d}\Gamma\\&\quad-\int_\Omega \eta_n(x)(u-u_n)\,\mathrm{d}x-\int_{\Gamma_b}h_n(x)(u-u_n)\,\mathrm{d}\Gamma-\int_{\Gamma_d}\xi_n(x)(u-u_n)\,\mathrm{d}\Gamma. \qquad (33)\end{aligned}$$

By hypotheses H(f)(iii) and H(U)(iv), we can see that sequences $\{\eta_n\}$ and $\{\xi_n\}$ are bounded in $L^{r(\cdot)'}(\Omega)$ and $L^{\delta(\cdot)'}(\Gamma_d)$, respectively, due to (13) and (14). By the compactness of the embeddings V to $L^{\theta(\cdot)}(\Omega)$, $L^{r(\cdot)}(\Omega)$ and of the trace of V to $L^{\delta(\cdot)}(\Gamma_d)$ (see Propositions 2.3 and 2.4), it finds

$$\left\{\begin{aligned}&\lim_{n\to\infty}\beta\int_\Omega |u_n|^{\theta(x)-2}u_n(u_n-u)\,\mathrm{d}x=0,\quad \lim_{n\to\infty}\int_\Omega \eta_n(x)(u-u_n)\,\mathrm{d}x=0,\\&\lim_{n\to\infty}\int_{\Gamma_c}h_n(x)(u-u_n)\,\mathrm{d}\Gamma=0,\quad \lim_{n\to\infty}\int_{\Gamma_d}\xi_n(x)(u-u_n)\,\mathrm{d}\Gamma=0.\end{aligned}\right. \qquad (34)$$

However, the weak lower semicontinuity of $V\ni u\mapsto\int_{\Gamma_c}\psi(x,u)\,\mathrm{d}\Gamma\in\mathbb{R}$ deduces

$$\int_{\Gamma_c}\psi(x,u)\,\mathrm{d}\Gamma-\liminf_{n\to\infty}\int_{\Gamma_c}\psi(x,u_n)\,\mathrm{d}\Gamma\le 0. \qquad (35)$$

Using the Hölder inequality, we have

$$\begin{aligned}&\int_\Omega\Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\&\ge-\int_\Omega |a_n(x)-a(x)||\nabla u|^{p(x)-1}|\nabla(u_n-u)|\,\mathrm{d}x\\&\ge-\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\||a_n(\cdot)-a(\cdot)||\nabla u|\|_{\frac{p(\cdot)}{p(\cdot)-1},\Omega}\|u_n-u\|_V\\&\ge-\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\|u_n-u\|_V\times\min\Bigg\{\left(\int_\Omega |a_n(x)-a(x)|^{\frac{p(x)}{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\left(\frac{p}{p-1}\right)_-},\\&\quad\left(\int_\Omega |a_n(x)-a(x)|^{\frac{p(x)}{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\left(\frac{p}{p-1}\right)_+}\Bigg\},\end{aligned}$$

where the last inequality is obtained by using Proposition 2.2. Since $a_n \to a$ in $L^1(\Omega)$, so, we may assume that $a_n(x) \to a(x)$ for a. e. $x \in \Omega$. Note $\{u_n\} \subset V$ and $\{a_n\} \subset L^\infty(\Omega)$ are bounded. This allows us to invoke Lebesgue dominated convergence theorem to get

$$\lim_{n\to\infty}\left(\int_\Omega |a_n(x)-a(x)|^{\frac{p(x)}{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\left(\frac{p}{p-1}\right)_\pm}\times\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\|u_n-u\|_V=0.$$

Hence, we have

$$\lim_{n\to\infty}\int_\Omega\Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\geq 0. \tag{36}$$

Keeping in mind that $u_n \xrightarrow{w} u$ in V as $n\to\infty$, so, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega\Big(a(x)|\nabla u|^{p(x)-2}\nabla u+b(x)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(u-u_n)\,\mathrm{d}x\\&=\lim_{n\to\infty}\langle Au,u_n-u\rangle=0.\end{aligned} \tag{37}$$

We use the monotonicity of $s\mapsto|s|^{q(x)-2}s$ to find

$$\begin{aligned}&\int_\Omega\Big(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n+b_n(x)|\nabla u_n|^{q(x)-2}\nabla u_n\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&=\int_\Omega a_n(x)\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(a(x)|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(b_n(x)|\nabla u_n|^{q(x)-2}\nabla u_n\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\geq\int_\Omega a_n(x)\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(a(x)|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(b_n(x)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\geq\int_\Omega a_n(x)\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big((a_n(x)-a(x))|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(a(x)|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big((b_n(x)-b(x))|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(b(x)|\nabla u|^{q(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)\,\mathrm{d}x.\end{aligned} \tag{38}$$

Passing to the upper limit as $n \to \infty$ for (33) and using inequalities (34), (35), (36), (37) and (38), we conclude that

$$\limsup_{n\to\infty} \int_\Omega a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x \le 0.$$

The latter combined with the nonnegativity of $(|s|^{p(x)-2}s - |t|^{p(x)-2}t)(s-t)$ for all $s, t \in \mathbb{R}^N$ implies

$$\lim_{n\to\infty} \int_\Omega a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x = 0. \tag{39}$$

On the other hand, it follows from Simom [58, formula (2.2)], that the following inequalities hold

$$c_t|\xi-\eta|^t \le \left(|\xi|^{t-2}\xi - |\eta|^{t-2}\eta\right)\cdot(\xi-\eta), \qquad \text{if } \ t \ge 2, \tag{40}$$

$$C_t|\xi-\eta|^2 \le \left(|\xi|^{t-2}\xi - |\eta|^{t-2}\eta\right)\cdot(\xi-\eta)\left(|\xi|^t + |\eta|^t\right)^{\frac{2-t}{t}}, \qquad \text{if } \ 1 \le t < 2, \tag{41}$$

for all $\xi,\ \eta \in \mathbb{R}^N$, where the constants $c_t,\ C_t > 0$ are independent of $\xi,\ \eta \in \mathbb{R}^N$ given by

$$c_t = 5^{\frac{2-t}{2}} \text{ and } C_t = (t-1)2^{\frac{(t-1)(t-2)}{t}}.$$

Set $c_p := \min_{x\in\overline{\Omega}} 5^{\frac{2-p(x)}{2}}$ and $C_p := \min_{x\in\overline{\Omega}}(p(x)-1)2^{\frac{(p(x)-1)(p(x)-2)}{p(x)}}$. For $p \in C_+(\overline{\Omega})$, it is obvious that the domain Ω could be decomposed into two mutually disjoint parts $\Omega_{p\ge2}$ and $\Omega_{p<2}$, i.e., $\Omega = \Omega_{p\ge2}\cup\Omega_{p<2}$ and $\Omega_{p\ge2}\cap\Omega_{p<2} = \emptyset$, where $\Omega_{p\ge2}$ and $\Omega_{p<2}$ are given by

$$\Omega_{p\ge2} := \{x\in\Omega \mid p(x)\ge 2\} \text{ and } \Omega_{p<2} := \{x\in\Omega \mid p(x) < 2\}.$$

In the part $\Omega_{p\ge2}$, we can use (40) to get

$$\begin{aligned}
&\int_{\Omega_{p\ge2}} a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x \\
&\ge \int_{\Omega_{p\ge2}} a_n(x) c_{p(x)} |\nabla u_n - \nabla u|^{p(x)}\,\mathrm{d}x \\
&\ge c_\Lambda c_p \varrho_{p(\cdot),\Omega_{p\ge2}}(|\nabla u_n - \nabla u|).
\end{aligned} \tag{42}$$

Set $\Omega_n = \{x\in\Omega \mid \nabla u_n \ne 0\} \cup \{x\in\Omega \mid \nabla u \ne 0\}$ and $\Sigma_n = \{x\in\Omega \mid \nabla u = \nabla u_n = 0\}$. So, it is valid that $\Omega = \Omega_n\cup\Sigma_n$ and $\Omega_n\cap\Sigma_n = \emptyset$. Using the absolute continuity of the Lebesgue integral, it gives

$$\int_{\Sigma_n} a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x = 0.$$

Hence,

$$\begin{aligned}
&\int_\Omega a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x \\
&= \int_{\Omega_n} a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x \\
&\quad + \int_{\Sigma_n} a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x \\
&= \int_{\Omega_n} a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u \right) \cdot \nabla(u_n - u)\,\mathrm{d}x.
\end{aligned}$$

Regarding the part $\Omega_{p<2}$, it follows from (41) that

$$\begin{aligned}
&\int_{\Omega_{p<2}} a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\,\mathrm{d}x\\
&=\int_{\Omega_n\cap\Omega_{p<2}} a_n(x)\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(u_n-u)\times\\
&\quad\frac{\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{2-p(x)}{p(x)}}}{\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{2-p(x)}{p(x)}}}\,\mathrm{d}x\\
&\geq\int_{\Omega_n\cap\Omega_{p<2}} C_{p(x)}a_n(x)\,|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x\\
&\geq C_p\int_{\Omega_n\cap\Omega_{p<2}} a_n(x)\,|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x\\
&\geq c_\Lambda C_p\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\,\mathrm{d}x.
\end{aligned}$$

Since $1<p(x)<2$, we have $\frac{2}{p(x)}>1$. By the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n-\nabla u|^{p(x)}\,\mathrm{d}x=\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n-\nabla u|^{2\times\frac{p(x)}{2}}\,\mathrm{d}x\\
&=\int_{\Omega_n\cap\Omega_{p<2}}\left(|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\right)^{\frac{p(x)}{2}}\times\\
&\quad\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)^{\frac{2-p(x)}{2}}\,\mathrm{d}x\\
&\leq\left[\frac{1}{\left(\frac{2}{p}\right)_-}+\frac{1}{\left(\frac{2}{2-p}\right)_-}\right]\|l_1\|_{\frac{2}{p(\cdot)},\Omega_n\cap\Omega_{p<2}}\|l_2\|_{\frac{2}{2-p(\cdot)},\Omega_n\cap\Omega_{p<2}},
\end{aligned}$$

where functions l_1,l_2 are given by

$$\begin{aligned}
l_1(x)&=\left(|\nabla u_n-\nabla u|^2\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}}\right)^{\frac{p(x)}{2}},\\
l_2(x)&=\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\right)^{\frac{2-p(x)}{2}}.
\end{aligned}$$

Comparing between the norm and the modular, see Proposition 2.2, it yields

$$\begin{aligned}
&\|l_1\|_{\frac{2}{p(\cdot)},\Omega_n\cap\Omega_{p<2}}\|l_2\|_{\frac{2}{2-p(\cdot)},\Omega_n\cap\Omega_{p<2}}\\
&\leq\max\left\{\left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\,\mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_-}},\left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\,\mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_+}}\right\}\times\\
&\quad\max\left\{\left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\,\mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}},\left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\,\mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}}\right\}.
\end{aligned}$$

From the last two inequalities, it has

$$\begin{aligned}
&\int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n - \nabla u|^{p(x)}\, \mathrm{d}x \\
&\leq \left[\frac{1}{\left(\frac{2}{p}\right)_-} + \frac{1}{\left(\frac{2}{2-p}\right)_-}\right] \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \right. \\
&\quad \left. \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_+}} \right\} \times \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \right. \\
&\quad \left. \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\},
\end{aligned}$$

that is,

$$\begin{aligned}
&\left[\frac{1}{\left(\frac{2}{p}\right)_-} + \frac{1}{\left(\frac{2}{2-p}\right)_-}\right]^{-1} \int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n - \nabla u|^{p(x)}\, \mathrm{d}x \times \\
&\quad \left(\max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\}\right)^{-1} \\
&\leq \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_+}} \right\}.
\end{aligned}$$

Let $\overline{M}_0 > 0$ be such that

$$\begin{aligned}
&\overline{M}_0 \\
&\geq \max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_2(x)^{\frac{2}{2-p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{2-p}\right)_+}} \right\} \\
&\quad \times \left[\frac{1}{\left(\frac{2}{p}\right)_-} + \frac{1}{\left(\frac{2}{2-p}\right)_-}\right]
\end{aligned}$$

for all $n \in \mathbb{N}$, thanks to the boundedness of $\{u_n\}$. It follows from (39) that the limit superior of $\int_\Omega l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x$ is strictly smaller than one. Therefore, we have

$$\begin{aligned}
&\max\left\{ \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_-}}, \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{1}{\left(\frac{2}{p}\right)_+}} \right\} \\
&= \left(\int_{\Omega_n\cap\Omega_{p<2}} l_1(x)^{\frac{2}{p(x)}}\, \mathrm{d}x\right)^{\frac{p_-}{2}}.
\end{aligned}$$

Hence, it has

$$\int_{\Omega_{p<2}} a_n(x) \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right) \cdot \nabla(u_n - u)\, \mathrm{d}x$$

$$\geq c_\Lambda C_p \int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n - \nabla u|^2 \left(|\nabla u_n|^{p(x)} + |\nabla u_n|^{p(x)}\right)^{\frac{p(x)-2}{p(x)}} \,\mathrm{d}x$$
$$\geq c_\Lambda C_p \left(\overline{M}_0^{-1} \int_{\Omega_n\cap\Omega_{p<2}} |\nabla u_n - \nabla u|^{p(x)} \,\mathrm{d}x\right)^{\frac{p_-}{2}}$$
$$= c_\Lambda C_p \left(\overline{M}_0^{-1} \int_{\Omega_{p<2}} |\nabla u_n - \nabla u|^{p(x)} \,\mathrm{d}x\right)^{\frac{p_-}{2}}.$$

Inserting the inequality above and (42) into (39), it gives

$$\varrho_{p(\cdot),\Omega}(|\nabla u_n - \nabla u|) = \int_\Omega |\nabla u_n - \nabla u|^{p(x)} \,\mathrm{d}x \to 0.$$

The latter combined with Proposition 2.2 implies

$$\|u_n - u\|_V \to 0, \text{ i.e., } u_n \to u \text{ in } V \text{ as } n \to \infty.$$

Additionally, the boundedness of $\{\eta_n\}$ and $\{\xi_n\}$, and the reflexivity of $L^{r(\cdot)'}(\Omega)$ and $L^{\delta(\cdot)'}(\Gamma_d)$ point out that there exist subsequences of $\{\eta_n\}$ and $\{\xi_n\}$, denoted still in the same way, and functions $\eta \in L^{r(\cdot)'}(\Omega)$ and $\xi \in L^{\delta(\cdot)'}(\Gamma_d)$ satisfying

$$\eta_n \xrightarrow{w} \eta \text{ in } L^{r(\cdot)'}(\Omega) \text{ and } \xi_n \xrightarrow{w} \xi \text{ in } L^{\delta(\cdot)'}(\Gamma_d) \text{ as } n \to \infty.$$

Since $u_n \to u$ in V as $n \to \infty$. Without any loss of generality, we may assume that $\nabla u_n(x) \to \nabla u(x)$ and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. Using the same arguments as before we did (see the proof of Theorem 3.4), we conclude that $\eta \in N_f(u)$ and $\xi \in N_U(u)$. Exploiting the Lebesgue dominated convergence theorem, it yields

$$\lim_{n\to\infty} \int_\Omega \left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n + b_n(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right) \cdot \nabla(v - u_n) \,\mathrm{d}x$$
$$= \int_\Omega \lim_{n\to\infty} \left(a_n(x)|\nabla u_n|^{p(x)-2}\nabla u_n + b_n(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right) \cdot \nabla(v - u_n) \,\mathrm{d}x$$
$$= \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u + b(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla(v - u) \,\mathrm{d}x,$$

because of the boundedness of $\{a_n\}$, $\{b_n\} \subset L^\infty(\Omega)$ and $\{u_n\} \subset V$. Letting $n \to \infty$ in inequality (32) and using the convergence results above we deduce

$$\int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u_n + b(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla(v - u) \,\mathrm{d}x$$
$$+ \beta \int_\Omega |u|^{\theta(x)-2}u(v - u) \,\mathrm{d}x + \int_{\Gamma_c} \psi(x, v) \,\mathrm{d}\Gamma - \int_{\Gamma_c} \psi(x, u) \,\mathrm{d}\Gamma$$
$$\geq \int_\Omega \eta(x)(v - u) \,\mathrm{d}x + \int_{\Gamma_b} h(x)(v - u) \,\mathrm{d}\Gamma + \int_{\Gamma_d} \xi(x)(v - u) \,\mathrm{d}\Gamma$$

for all $v \in K$. This implies that $u \in K$ is a solution of problem (1) corresponding to $(a, b, h) \in \Lambda^2 \times H$, namely, $u \in S(a, b, h)$. So, we get $\emptyset \neq w\text{–}\limsup_{n\to\infty} S(a_n, b_n, h_n) \subset S(a, b, h)$. Hence, (31) is valid.

Step 3. If sequence $\{(a_n, b_n, h_n)\} \subset \Lambda^2 \times H$ is such that $\{a_n\}$, $\{b_n\}$ are bounded in $BV(\Omega)$, and $(a_n, b_n) \to (a, b)$ in $L^1(\Omega) \times L^1(\Omega)$ and $h_n \xrightarrow{w} h$ in H for some $(a, b, h) \in L^1(\Omega)^2 \times H$, then the inequality is valid

$$L(a, b, h) \leq \liminf_{n\to\infty} L(a_n, b_n, h_n). \tag{43}$$

Let $\{(a_n, b_n, h_n)\} \subset \Lambda^2 \times H$ be such that $\{a_n\}$, $\{b_n\}$ are bounded in $BV(\Omega)$, and $(a_n, b_n) \to (a, b)$ in $L^1(\Omega)^2$ and $h_n \xrightarrow{w} h$ in $L^{p(\cdot)'}(\Gamma_b)$ as $n \to \infty$ for some $(a, b, h) \in L^1(\Omega)^2 \times H$. By virtue of Step 2, we know that $(a, b) \in \Lambda \times \Lambda$. Suppose that sequence $\{u_n\} \in K$ is such that

$$\begin{cases} u_n \in S(a_n, b_n, h_n) \\ \inf_{u \in S(a_n,b_n,h_n)} \int_\Omega |\nabla u - z|^{p(x)} \, \mathrm{d}x = \int_\Omega |\nabla u_n - z|^{p(x)} \, \mathrm{d}x \end{cases} \tag{44}$$

for each $n \in \mathbb{N}$. Remembering that $\cup_{n\geq 1} S(a_n, b_n, h_n)$ is bounded, passing to a subsequence if necessary, we may suppose that $u_n \xrightarrow{w} u^*$ in V as $n \to \infty$ for some $u^* \in K$, i.e., $u^* \in w\text{–}\limsup_{n\to\infty} S(a_n, b_n, h_n)$. Applying Step 2, it yields $u^* \in S(a, b, h)$. Therefore, it follows from the lower semicontinuity of the function $L^1(\Omega) \ni a \mapsto TV(a) \in \mathbb{R}$, and the weak lower semicontinuity of $V \ni u \mapsto \int_\Omega |\nabla u - z|^{p(x)} \, \mathrm{d}x \in \mathbb{R}$ and $L^{p(\cdot)'}(\Gamma_b) \ni h \mapsto \int_{\Gamma_b} |h(z)|^{p'(x)} \, \mathrm{d}\Gamma \in \mathbb{R}$ that

$$\begin{aligned}
&\liminf_{n\to\infty} L(a_n, b_n, h_n) \\
&= \liminf_{n\to\infty} \left[\int_\Omega |\nabla u_n - z|^{p(x)} \, \mathrm{d}x + \kappa TV(a_n) + \tau TV(b_n) + \mu \int_{\Gamma_b} |h_n(z)|^{p'(x)} \, \mathrm{d}\Gamma \right] \\
&\geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)} \, \mathrm{d}x + \liminf_{n\to\infty} \kappa TV(a_n) + \liminf_{n\to\infty} \tau TV(b_n) \\
&\quad + \liminf_{n\to\infty} \mu \int_{\Gamma_b} |h_n(z)|^{p'(x)} \, \mathrm{d}\Gamma \\
&\geq \int_\Omega |\nabla u^* - z|^{p(x)} \, \mathrm{d}x + \kappa TV(a) + \tau TV(b) + \mu \int_{\Gamma_b} |h(z)|^{p'(x)} \, \mathrm{d}\Gamma \\
&\geq \inf_{u \in S(a,b,h)} \int_\Omega |\nabla u - z|^{p(x)} \, \mathrm{d}x + \kappa TV(a) + \tau TV(b) + \mu \int_{\Gamma_b} |h(z)|^{p'(x)} \, \mathrm{d}\Gamma \\
&= L(a, b, h).
\end{aligned}$$

Hence (43) follows.

Step 4. The solution set of Problem 4 is nonempty and weakly compact.

It follows from the formulation of the cost functional L that L is nonnegative. Let $\{(a_n, b_n, h_n)\} \subset \Lambda^2 \times H$ be a minimizing sequence of problem (29), that is,

$$\inf_{a,b\in\Lambda,\, h\in H} L(a, b, h) = \liminf_{n\to\infty} L(a_n, b_n, h_n). \tag{45}$$

By virtue of definitions of L and Λ, we can see that $\{a_n\} \subset \Lambda$, $\{b_n\} \subset \Lambda$ are bounded in $BV(\Omega) \cap L^\infty(\Omega)$, and $\{h_n\}$ is bounded in $L^{p(\cdot)'}(\Gamma_b)$. Passing to a subsequence if necessary, we have

$$a_n \to a^*,\ b_n \to b^* \text{ in } L^1(\Omega) \text{ and } h_n \xrightarrow{w} h^* \text{ in } L^{p(\cdot)'}(\Gamma_b) \tag{46}$$

for some $(a^*, b^*, h^*) \in \Lambda^2 \times L^{p(\cdot)'}(\Gamma_b)$, where we have used the closedness of Λ in $L^1(\Omega)$ and the compactness of the embedding of $BV(\Omega)$ to $L^1(\Omega)$. Let $\{u_n\} \subset K$ be a sequence such that (44) holds. From the convergence (46) and boundedness of S, we conclude that $\{u_n\}$ is bounded in V. So, we are able to select a subsequence of $\{u_n\}$, denoted still in the same way, such that $u_n \xrightarrow{w} u^*$ in V as $n \to \infty$ for some $u^* \in K$. It is clear from Step 2 that $u^* \in S(a^*, b^*, h^*)$. Therefore, we have

$$\liminf_{n\to\infty} L(a_n, b_n, h_n)$$

$$
\begin{aligned}
&= \liminf_{n\to\infty} \left[\int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x + \kappa TV(a_n) + \tau TV(b_n) + \mu \int_{\Gamma_b} |h_n(z)|^{p'(x)}\,\mathrm{d}\Gamma \right] \\
&\geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n - z|^{p(x)}\,\mathrm{d}x + \liminf_{n\to\infty} \kappa TV(a_n) + \liminf_{n\to\infty} \tau TV(b_n) \\
&\quad + \liminf_{n\to\infty} \mu \int_{\Gamma_b} |h_n(z)|^{p'(x)}\,\mathrm{d}\Gamma \\
&\geq \int_\Omega |\nabla u^* - z|^{p(x)}\,\mathrm{d}x + \kappa TV(a^*) + \tau TV(b^*) + \mu \int_{\Gamma_b} |h^*(z)|^{p'(x)}\,\mathrm{d}\Gamma \\
&\geq \inf_{u\in S(a^*,b^*,h^*)} \int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x + \kappa TV(a^*) + \tau TV(b^*) + \mu \int_{\Gamma_b} |h^*(z)|^{p'(x)}\,\mathrm{d}\Gamma \\
&= L(a^*,b^*,h^*) \\
&\geq \inf_{a,b\in\Lambda,\, h\in H} L(a,b,h).
\end{aligned} \tag{47}
$$

The latter combined with (45) implies that $(a^*,b^*,h^*) \in \Lambda^2 \times H$ is a solution to Problem 4.

Finally, we prove the weak compactness of solution set to Problem 4. Let $\{(a_n,b_n,h_n)\}$ be a sequence of solutions to Problem 4. It is obvious that $\{a_n\} \subset \Lambda$, $\{b_n\} \subset \Lambda$ are bounded in $BV(\Omega) \cap L^\infty(\Omega)$, and $\{h_n\}$ is bounded in $L^{p(\cdot)'}(\Gamma_b)$. Using the same arguments, we may assume that (46) holds with some $(a^*,b^*,h^*) \in \Lambda^2 \times L^{p(\cdot)'}(\Gamma_b)$. Likewise, there exists a sequence $\{u_n\}$ such that (44) is fulfilled and $u_n \xrightarrow{w} u^*$ in V as $n \to \infty$ for some $u^* \in S(a^*,b^*,h^*)$. As we did before, we prove the validity of (47). This means that $(a^*,b^*,h^*) \in \Lambda^2 \times H$ is a solution to Problem 4. Consequently, the solution set of Problem 4 is weakly compact. This completes the proof. □

Remark 4.2. The results of this section remain valid if the functional (30) is replaced by the following regularized cost functional

$$
\begin{aligned}
J(a,b,h) = \min_{u\in S(a,b,h)} &\left(\int_\Omega |\nabla u - z|^{p(x)}\,\mathrm{d}x \right)^{\frac{1}{p_-}} + \kappa\, TV(a) + \tau\, TV(b) \\
&+ \mu \left(\int_{\Gamma_b} |h|^{p(x)'}\,\mathrm{d}\Gamma \right)^{\frac{1}{p'_-}}.
\end{aligned}
$$

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