Dynamic Systems and Applications 22 (2013) 371-384

COMBINED EFFECTS FOR A STATIONARY PROBLEM WITH INDEFINITE NONLINEARITIES AND LACK OF COMPACTNESS

VICENŢIU D. RĂDULESCU

"Simion Stoilow" Mathematics Institute of the Romanian Academy
Bucharest, Romania
Department of Mathematics, University of Craiova, 200585 Craiova, Romania

ABSTRACT. In this paper we study the existence of entire solutions for a class of quasilinear elliptic equations on the whole space, provided that the nonlinear term has a subcritical growth. Our main results establish related nonexistence or multiplicity results.

2000 Mathematics Subject Classification: 35J60, 35J65, 35J70, 58E05

Key words: quasilinear elliptic equation, entire solution, mountain pass geometry

1. INTRODUCTION

In [4], Alama and Tarantello studied the existence and multiplicity of solutions of the equation

$$\tag{1.1} \begin{cases} -\Delta u - \lambda u = k(x)u^q - h(x)u^p, & \text{if} \quad x \in \Omega \\ u > 0, & \text{if} \quad x \in \Omega \\ u = 0, & \text{if} \quad x \in \partial\Omega\,, \end{cases}$$

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded open set with smooth boundary, the functions h, $k \in L^1(\Omega)$ are nonnegative and $1 < p < q$. For $\lambda \in \mathbb{R}$ in a neighborhood of the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$, they obtained the solvability of problem (1.1), as well as corresponding multiplicity properties, under various assumptions on h and k. More exactly, they proved existence, nonexistence and multiplicity results depending on λ and according to the integrability properties of the ratio $k(x)^{p-1}/h(x)^{q-1}$.

The work of Alama and Tarantello was carried on by Chabrowski [9] who obtained similar results for the problem

$$\tag{1.2} \begin{cases} -\Delta u + u = \lambda |u|^{q-2}u - h(x)|u|^{p-2}u, & \text{if} \quad x \in \mathbb{R}^N \\ u > 0, & \text{if} \quad x \in \mathbb{R}^N\,, \end{cases}$$

where $h > 0$ is a positive continuous function on $\mathbb{R}^N$ satisfying some integrability condition, $\lambda > 0$ is a positive parameter and $2 < q < p < 2N/(N-2)$, $N \geq 3$. More exactly, Chabrowski proved that there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$

Received November 15, 2012

equation (1.2) does not have any solution while for any $\lambda \geq \lambda_0$ equation (1.2) has at least a nontrivial solution.

Related studies with those presented above can be found in [3], [5], [7], [14], [16]. We also refer to the recent monographs [1] and [2] for related qualitative results.

In this paper, motivated by [4] and [9], we study the existence and multiplicity of solutions for the quasilinear problem

$$(1.3) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = \lambda |u|^{q-2}u - h(x)|u|^{p-2}u, & \text{if} \quad x \in \mathbb{R}^N \\ u \geq 0, & \text{if} \quad x \in \mathbb{R}^N, \end{cases}$$

where $h(x)$ is a positive continuous function on $\mathbb{R}^N$ $(N \geq 3)$ satisfying the condition

$$(1.4) \qquad \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}}\, dx < \infty,$$

$\lambda > 0$ is a positive parameter and $2 \leq m < q < p < m^\star = Nm/(N-m)$, $m < N$.

If $h = 0$, problem (1.3) is called the Lane-Emden-Fowler equation and it arises in the boundary-layer theory of viscous fluids (see [19]). This equation goes back to the paper by Lane [11] in 1869 and is originally motivated by Lane's interest in computing both the temperature and the density of mass on the surface of the sun. Problem (1.3) describes the behavior of the density of a gas sphere in hydrostatic equilibrium and the index p, which is called the polytropic index in astrophysics and is related to the ratio of the specific heats of the gas.

The main results in the present paper point out the following perturbation effects: (i) if the perturbation in the right-hand side of (1.3) is weak, then there is no solution; (ii) if the positive term in the right-hand side of (1.3) is big (this corresponding to a strong perturbation) then there are at least two different entire solutions. More precisely, we establish the non-existence of nontrivial solutions for problem (1.3) if λ is small enough (see Section 3) and the existence of at least two nontrivial solutions for problem (1.3) if λ is large enough (see Section 4).

In this paper we use standard notations and terminology. We denote by $W^{1,m}(\mathbb{R}^N)$ the Sobolev space equipped with the norm

$$\|u\|_{W^{1,m}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx \right)^{1/m}.$$

For simplicity we will often denote the above norm by $\|u\|$.

By $L_p^r(\mathbb{R}^N)$, $1 \leq p < \infty$, we denote the weighted Lebesgue space

$$L_r^p(\mathbb{R}^N) = \left\{ u;\ \int_{\mathbb{R}^N} r(x)|u|^p\, dx < \infty \right\},$$

where $r(x)$ is a positive continuous function on $\mathbb{R}^N$, equipped with the norm

$$\|u\|_{r,p}^p = \left(\int_{\mathbb{R}^N} (r(x)|u|^p)\, dx \right)^{1/p}.$$

If $r(x) \equiv 1$ on $\mathbb{R}^N$, the norm is denoted by $\|\cdot\|_p$.

2. MAIN RESULTS

In this paper we seek weak solutions for problem (1.3) in a subspace of $W^{1,m}(\mathbb{R}^N)$. Let E be the weighted Sobolev space defined by

$$E = \left\{ u \in W^{1,m}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} h(x)|u|^p\, dx < \infty \right\},$$

equipped with the norm

$$\|u\|_E^m = \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx + \left(\int_{\mathbb{R}^N} h(x)|u|^p\, dx \right)^{m/p}.$$

We define a *weak solution* for problem (1.3) as a function $u \in E$ with $u(x) \geq 0$ a.e. $x \in \mathbb{R}^N$ satisfying

$$\int_{\mathbb{R}^N} |\nabla u|^{m-2} uv\, dx + \int_{\mathbb{R}^N} |u|^{m-2}uv\, dx - \lambda \int_{\mathbb{R}^N} |u|^{q-2}uv\, dx + \int_{\mathbb{R}^N} h(x)|u|^{p-2}uv\, dx = 0,$$

for all u, $v \in E$.

The main results of this paper are the following.

Theorem 1. *There exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ problem* (1.3) *does not have a nontrivial weak solution.*

Theorem 2. *There exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ problem* (1.3) *admits at least two nontrivial weak solutions.*

Remark. In the linear case, when $m = 2$, similar results as those presented above were obtained in [4], [9], and [16], while in the general case we refer to [18].

3. PROOF OF THEOREM 1

Let $\Phi : E \to \mathbb{R}$ be the energy functional defined by

$$\Phi(u) = \frac{1}{m} \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q\, dx + \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u|^p\, dx.$$

Standard arguments assure that $\Phi \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{m-2} \nabla u \nabla v + |u|^{m-2}uv)\, dx - \lambda \int_{\mathbb{R}^N} |u|^{q-2}uv\, dx + \int_{\mathbb{R}^N} h(x)|u|^{p-2}uv\, dx$$

for any u, $v \in E$. Solutions of problem (1.3) will be found as critical points of functional Φ.

We assume by contradiction that $u \in E$ is a weak solution of problem (1.3). Then u satisfies

$$\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx + \int_{\mathbb{R}^N} h(x)|u|^p\, dx = \lambda \int_{\mathbb{R}^N} |u|^q\, dx. \tag{3.1}$$

To proceed further, we need Young's inequality

$$ab \le \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta}, \quad \forall\, a, b > 0$$

where α, $\beta > 1$ satisfy $1/\alpha + 1/\beta = 1$.

Taking $a = h(x)^{q/p}|u|^q$, $b = \lambda/[h(x)]^{q/p}$, $\alpha = p/q$ and $\beta = p/(p-q)$ we obtain that

$$h(x)^{q/p}|u|^q \frac{\lambda}{h(x)^{q/p}} \le \frac{q}{p}(h(x)^{q/p}|u|^q)^{p/q} + \frac{p-q}{p}\left(\frac{\lambda}{h(x)^{q/p}}\right)^{p/(p-q)}.$$

Integrating over $\mathbb{R}^N$ we have

$$\lambda \int_{\mathbb{R}^N} |u|^q \, dx \le \frac{q}{p}\int_{\mathbb{R}^N} h(x)|u|^p \, dx + \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}} \, dx.$$

The above inequality and relation (3.1) imply

$$\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx \le \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}} \, dx + \frac{q-p}{p}\int_{\mathbb{R}^N} h(x)|u|^p \, dx.$$

Since $q < p$ it results that $\frac{q-p}{p}\int_{\mathbb{R}^N} h(x)|u|^p \, dx < 0$ and thus

$$\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx \le \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}} \, dx. \tag{3.2}$$

Since $m < q < m^\star$ the Sobolev embedding of $W^{1,m}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ implies the existence of a positive constant C_q such that

$$C_q \left(\int_{\mathbb{R}^N} |u|^q \, dx\right)^{m/q} \le \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx.$$

We note that $\int_{\mathbb{R}^N} h(x)|u|^p \, dx \ge 0$. It follows from (3.1) that

$$\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx \le \lambda \int_{\mathbb{R}^N} |u|^q \, dx.$$

Combining the last two inequalities we obtain

$$C_q \left(\int_{\mathbb{R}^N} |u|^q \, dx\right)^{m/q} \le \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx \le \lambda \int_{\mathbb{R}^N} |u|^q \, dx. \tag{3.3}$$

Retaining the first and the last terms of (3.3) we get

$$(C_q \lambda^{-1})^{q/(q-m)} \le \int_{\mathbb{R}^N} |u|^q \, dx.$$

That inequality combined with (3.3) leads to

$$C_q[(C_q\lambda^{-1})^{q/(q-m)}]^{m/q} \le \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx$$

By relation (3.2) and the above inequality we have

$$C_q(C_q\lambda^{-1})^{m/(q-m)} \le \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx \le \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}} \, dx.$$

Retaining the first and the last term it follows that

$$\lambda > \left[C_q^{q/(q-m)} \frac{p}{p-q} \left(\int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}}\, dx \right)^{-1} \right]^{(p-q)(q-m)/(q(p-m))}.$$

Denoting the term in the right-hand side of the above inequality by $\lambda^\star$, we conclude that Theorem 1 holds true. ☐

4. PROOF OF THEOREM 2

We first establish some auxiliary results.

Lemma 1. *The functional Φ is coercive.*

Proof. To proceed to the proof of Lemma 1 we need the following inequality:

For every $k_1 > 0$, $k_2 > 0$ and $0 < s < r$ we have

$$k_1|t|^s - k_2|t|^r \le C_{rs} k_1 \left(\frac{k_1}{k_2} \right)^{s/(r-s)}, \quad \forall\, t \in \mathbb{R}, \tag{4.1}$$

where $C_{rs} > 0$ is a constant depending on r and s.

If we take in inequality (4.1) $k_1 = \frac{\lambda}{q}$, $k_2 = \frac{(m-1)h(x)}{mp}$, $s = q$ and $r = p$ ($s < r$ is verified since $q < p$) we obtain

$$\begin{aligned} \frac{\lambda}{q}|u(x)|^q - \frac{(m-1)h(x)}{mp}|u(x)|^p &\le C_{pq}\frac{\lambda}{q}\left(\frac{\lambda/q}{(m-1)h(x)/mp} \right)^{(q/(p-q))} \\ &= C_{pq}\lambda^{(p/(p-q))} \frac{1}{h(x)^{q/(p-q)}} \left(\frac{mp}{q(m-1)} \right)^{q/(p-q)} \frac{1}{q}, \\ &\quad \forall\, x \in \mathbb{R}^N, \end{aligned}$$

where $C_{pq} > 0$ is a constant depending on p and q. Relabeling $C_{pq}\left(\frac{mp}{q(m-1)}\right)^{q/(p-q)} \frac{1}{q}$ by C_{pq} and integrating the above inequality over $\mathbb{R}^N$ it follows that

$$\int_{\mathbb{R}^N} \left(\frac{\lambda}{q}|u|^q - \frac{(m-1)h(x)}{mp}|u|^p \right) dx \le C_{pq}\lambda^{(p/(p-q))} \int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}}\, dx.$$

Using the hypotheses (1.4) we deduce that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} \left(\frac{\lambda}{q}|u|^q - \frac{(m-1)h(x)}{mp}|u|^p \right) dx \le C_1.$$

Therefore

(4.2)

$$\begin{aligned} \Phi(u) &= \frac{1}{m}\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\, dx + \frac{1}{p}\int_{\mathbb{R}^N} h(x)|u|^p\, dx \\ &= \frac{1}{m}\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx - \left[\int_{\mathbb{R}^N} \left(\frac{\lambda}{q}|u|^q - \frac{(m-1)h(x)}{mp}|u|^p \right) \right] dx \\ &\quad - \int_{\mathbb{R}^N} \frac{(m-1)h(x)}{mp}|u|^p + \frac{1}{p}\int_{\mathbb{R}^N} h(x)|u|^p\, dx \\ &\ge \frac{1}{m}\int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx - C_1 + \frac{1}{mp}\int_{\mathbb{R}^N} h(x)|u|^p\, dx, \end{aligned}$$

and thus Φ is coercive. $\square$

Lemma 2. *Assume that* $\{u_n\}$ *is a sequence in* E *such that* $\Phi(u_n)$ *is bounded. Then there exists a subsequence of* $\{u_n\}$*, relabeled again* $\{u_n\}$*, which converges weakly in* E *to some* $u_0 \in E$ *and*

$$\Phi(u_0) \leq \liminf_{n\to\infty} \Phi(u_n).$$

Proof. Using inequality (4.2) we obtain

$$\Phi(u_n) \geq \frac{1}{m}\int_{\mathbb{R}^N} (|\nabla u_n|^m + |u_n|^m)\, dx + \frac{1}{mp}\int_{\mathbb{R}^N} h(x)|u_n|^p\, dx - C_1.$$

Since $\Phi(u_n)$ is bounded the above inequality implies that $\int_{\mathbb{R}^N}(|\nabla u_n|^m + |u_n|^m)\, dx$ and $\int_{\mathbb{R}^N} h(x)|u_n|^p\, dx$ are bounded. Therefore, $\{\|u_n\|_E\}$ is bounded. In fact, there exists $u_0 \in E$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \ \text{ in } \ W^{1,m}(\mathbb{R}^N) \\ u_n &\to u_0 \ \text{ in } \ L^p_h(\mathbb{R}^N) \\ u_n &\to u_0 \ \text{ in } \ L^s_{\mathrm{loc}}(\mathbb{R}^N) \ \text{ for } \ s \in [1, m^\star). \end{aligned}$$

We define

$$F(x,u) = \frac{\lambda}{q}|u|^q - h(x)\frac{|u|^p}{p}$$

and

$$f(x,u) = F_u(x,u) = \lambda|u|^{q-2}u - h(x)|u|^{p-2}u.$$

We see that

$$f_u(x,u) = \lambda(q-1)|u|^{q-2} - h(x)(p-1)|u|^{p-2}.$$

Using again inequality (4.1) for $k_1 = \lambda(q-1)$, $k_2 = h(x)(p-1)$, $s = q-2$, $r = p-2$ we obtain

$$\begin{aligned} f_u(x,u) &= \lambda(q-1)|u|^{q-2} - h(x)(p-1)|u|^{p-2} \\ &\leq C \cdot \lambda \cdot (q-1) \cdot \left(\frac{\lambda(q-1)}{h(x)(p-1)}\right)^{(q-2)/(p-q)}, \quad \forall\, x \in E, \end{aligned}$$

where C is a positive constant depending only of p and q.

This yields,

$$f_u(x,u) \leq C_{pq} \cdot \lambda \cdot \left(\frac{\lambda}{h(x)}\right)^{(q-2)/(p-q)}, \tag{4.3}$$

where C_{pq} is a positive constant depending only of p and q. According to the definition of Φ and F we obtain the following estimate for $\Phi(u_0) - \Phi(u_n)$

$$\begin{aligned} \Phi(u_0) - \Phi(u_n) &= \frac{1}{m}\int_{\mathbb{R}^N} (|\nabla u_0|^m + |u_0|^m)\, dx - \frac{1}{m}\int_{\mathbb{R}^N} (|\nabla u_n|^m + |u_n|^m)\, dx \\ &\quad + \int_{\mathbb{R}^N} [F(x,u_n) - F(x,u_0)]\, dx. \end{aligned} \tag{4.4}$$

It is clear that

$$\begin{aligned}\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt &= \frac{1}{u_n-u_0}[f(x,u_0+s(u_n-u_0))-f(x,u_0)]\\ &= \frac{1}{u_n-u_0}[F_u(x,u_0+s(u_n-u_0))-F_u(x,u_0)].\end{aligned}$$

Integrating the above relation over $[0,1]$ we obtain

$$\begin{aligned}\int_0^1\left(\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt\right)\,ds &= \frac{1}{u_n-u_0}\int_0^1[F_u(x,u_0+s(u_n-u_0))\\ &\quad -F_u(x,u_0)]\,ds\\ &= \frac{1}{(u_n-u_0)^2}[F(x,u_n)-F(x,u_0)]-\frac{f(x,u_0)}{u_n-u_0}.\end{aligned}$$

The above equality can be written in the following way

$$F(x,u_n)-F(x,u_0)=(u_n-u_0)^2\int_0^1\left(\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt\right)\,ds+(u_n-u_0)f(x,u_0). \tag{4.5}$$

Introducing relation (4.5) in relation (4.4) we get

$$\begin{aligned}\Phi(u_0)-\Phi(u_n) &= \frac{1}{m}\int_{\mathbb{R}^N}(|\nabla u_0|^m+|u_0|^m)\,dx-\frac{1}{m}\int_{\mathbb{R}^N}(|\nabla u_n|^m+|u_n|^m)\,dx\\ &\quad +\int_{\mathbb{R}^N}(u_n-u_0)f(x,u_0)\,dx+\int_{\mathbb{R}^N}(u_n-u_0)^2\int_0^1\int_0^s f_u(x,u_0\\ &\quad +t(u_n-u_0))\,dt\,ds\,dx\\ &\le \frac{1}{m}\int_{\mathbb{R}^N}(|\nabla u_0|^m+|u_0|^m)\,dx-\frac{1}{m}\int_{\mathbb{R}^N}(|\nabla u_n|^m+|u_n|^m)\,dx\\ &\quad +\int_{\mathbb{R}^N}(u_n-u_0)f(x,u_0)\,dx+C_1\int_{\mathbb{R}^N}(u_n-u_0)^2\frac{dx}{h(x)^{(q-2)/(p-q)}},\end{aligned} \tag{4.6}$$

where the last inequality follows from (4.3) and $C_1=C_{pq}\lambda^{(p-2)/(p-q)}$. It remains to show that the last two integrals converge to 0 as $n\to\infty$.

We define $J:E\to\mathbb{R}$ by

$$J(v)=\int_{\mathbb{R}^N}f(x,u_0)v\,dx.$$

Obviously, J is linear. We prove that J is also continuous. Indeed, we have

$$\begin{aligned}|J(v)| &\le \int_{\mathbb{R}^N}|f(x,u_0)|\cdot|v|\,dx=\int_{\mathbb{R}^N}|\lambda|\cdot||u_0|^{q-1}-h(x)|u_0|^{p-1}|\cdot|v|\,dx\\ &\le \lambda\int_{\mathbb{R}^N}|u_0|^{q-1}|v|\,dx+\int_{\mathbb{R}^N}h(x)|u_0|^{p-1}|v|\,dx.\end{aligned} \tag{4.7}$$

On the other hand, using Hölder's inequality, it results

$$\int_{\mathbb{R}^N}|u_0|^{q-1}|v|\,dx\le\left(\int_{\mathbb{R}^N}|u_0|^q\,dx\right)^{(q-1)/q}\left(\int_{\mathbb{R}^N}|v|^q\,dx\right)^{1/q}=\|u_0\|_q^{q-1}\|v\|_q.$$

Since $W^{1,m}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ we deduce that there exists a constant $C > 0$ such that

$$\|v\|_q \le C\|v\|_{W^{1,m}(\mathbb{R}^N)}, \quad \forall\, v \in W^{1,m}(\mathbb{R}^N).$$

Combining the last two inequalities with the fact that

$$\|v\|_{W^{1,m}(\mathbb{R}^N)} \le \|v\|_E$$

we deduce that there exists a positive constant $c_q > 0$ such that

$$\int_{\mathbb{R}^N} |u_0|^{q-1}|v|\, dx \le c_q\|v\|_E. \tag{4.8}$$

Applying again Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} h(x)|u_0|^{p-1}|v|\, dx &\le \int_{\mathbb{R}^N} (h(x)^{(p-1)/p}|u_0|^{p-1})(h(x)^{1/p}|v|)\, dx \\ &\le \left(\int_{\mathbb{R}^N} h(x)|u_0|^p\, dx\right)^{(p-1)/p} \left(\int_{\mathbb{R}^N} h(x)|v|^p\, dx\right)^{1/p} \\ &\le C_3\|v\|_{h,p} \le C_3\|v\|_E,\end{aligned} \tag{4.9}$$

where c_3 is a positive constant.

By (4.7), (4.8) and (4.9) we conclude that there exists a positive constant C_4 such that

$$|J(v)| \le C_4\|v\|_E, \quad \forall\, v \in E,$$

and thus, J is continuous. Since $\{u_n\}$ converges weakly to u_0 in E and J is linear and continuous we deduce

$$J(u_n) \to J(u_0)$$

or

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_0)(u_n-u_0)\, dx = 0. \tag{4.10}$$

In order to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{(u_n-u_0)^2}{h(x)^{(q-2)/(p-q)}}\, dx = 0$$

we take $R > 0$ sufficiently large and we observe that

(4.11)

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{(u_n-u_0)^2}{h(x)^{(q-2)/(p-q)}}\, dx &= \int_{|x|<R} \frac{(u_n-u_0)^2}{h(x)^{(q-2)/(p-q)}}\, dx + \int_{|x|\ge R} \frac{(u_n-u_0)^2}{h(x)^{(q-2)/(p-q)}}\, dx \\ &\le \left(\int_{|x|<R} \frac{dx}{h(x)^{q/(p-q)}}\, dx\right)^{(q-2)/q} \cdot \left(\int_{|x|<R} |u_n-u_0|^q\, dx\right)^{2/q} \\ &\quad + \left(\int_{|x|\ge R} \frac{dx}{h(x)^{q/(p-q)}}\right)^{(q-2)/q} \cdot \left(\int_{|x|\ge R} |u_n-u_0|^q\, dx\right)^{2/q}.\end{aligned}$$

By hypothesis (1.4) we have

$$\int_{|x|<R} \frac{dx}{h(x)^{q/(p-q)}}\, dx < \int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}}\, dx < \infty, \quad \forall\, R > 0.$$

On the other hand, for all $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$\int_{|x|\geq R_\epsilon} \frac{dx}{h(x)^{q/(p-q)}}\, dx < \epsilon.$$

Using the fact that $m < q < m^\star$ we deduce that $W^{1,m}(B_{R_\epsilon}(0))$ is compactly embedded in $L^q(B_{R_\epsilon}(0))$ and thus

$$\lim_{n\to\infty} \left(\int_{|x|<R_\epsilon} |u_n - u_0|^q \, dx \right)^{2/q} = 0.$$

Since $\{u_n - u_0\}$ is bounded in E it follows that it is bounded in $L^q(\mathbb{R}^N)$ and we find that there exists a positive constant $M > 0$ such that

$$\left(\int_{|x|\geq R_\epsilon} |u_n - u_0|^q \, dx \right)^{2/q} \leq \left(\int_{\mathbb{R}^N} |u_n - u_0|^q \, dx \right)^{2/q} < M.$$

Combining the above information with relation (4.11) we conclude that for any $\epsilon > 0$ there exists $N_\epsilon > 0$ such that for all $n \geq N_\epsilon$ we have

$$\int_{\mathbb{R}^N} \frac{(u_n - u_0)^2}{h(x)^{(q-2)/(p-q)}}\, dx \leq \epsilon + M \cdot \epsilon^{(q-2)/q}.$$

Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{(u_n - u_0)^2}{h(x)^{(q-2)/(p-q)}}\, dx = 0. \tag{4.12}$$

Since $\{u_n\}$ converges weakly to u_0 in $W^{1,m}(\mathbb{R}^N)$ Proposition III.5 in [8] implies

$$\liminf_{n\to\infty} \|u_n\|^m_{W^{1,m}(\mathbb{R}^N)} \geq \|u_0\|^m_{W^{1,m}(\mathbb{R}^N)}.$$

Passing to the limit in (4.6) and taking into account that (4.10) and (4.12) hold true we obtain

$$\Phi(u_0) \leq \liminf_{n\to\infty} \Phi(u_n).$$

Thus, Φ is weakly lower semicontinuous.

The proof of Lemma 2 is now complete. □

Proof of Theorem 2. Using Lemmas 1, 2 and Theorem 1.2 in [15] we deduce that there exists $u \in E$ a global minimizer of Φ, i.e.

$$\Phi(u) = \inf_{v\in E} \Phi(v).$$

It is obvious that u is a weak solution of problem (1.3). We prove that $u \not\equiv 0$ in E. To do that we show that $\inf_E \Phi < 0$ providing that the parameter λ is sufficiently large.

We set

$$\overline{\lambda} = \inf \left\{ \frac{q}{m} \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m)\, dx + \frac{q}{p} \int_{\mathbb{R}^N} h(x)|u|^p \, dx;\ u \in E,\ \int_{\mathbb{R}^N} |u|^q \, dx = 1 \right\}.$$

We point out that $\overline{\lambda} > 0$. Indeed, for any $u \in E$ with $\int_{\mathbb{R}^N} |u|^q \, dx = 1$ by Hölder's inequality we have

$$1 = \int_{\mathbb{R}^N} |u|^q \, dx \le \left(\int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}} \right)^{(p-q)/p} \cdot \left(\int_{\mathbb{R}^N} h(x)|u|^p \, dx \right)^{q/p}.$$

It follows that

$$\overline{\lambda} \ge \frac{q}{p} \left(\int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}} \right)^{(q-p)/p} > 0.$$

Let $\lambda > \overline{\lambda}$. Then there exists a function $u_1 \in E$ with $\int_{\mathbb{R}^N} |u_1|^q \, dx = 1$ such that

$$\lambda \int_{\mathbb{R}^N} |u_1|^q \, dx = \lambda > \frac{q}{m} \int_{\mathbb{R}^N} (|\nabla u_1|^m + |u_1|^m) \, dx + \frac{q}{p} \int_{\mathbb{R}^N} h(x)|u_1|^p \, dx.$$

This can be written as

$$\Phi(u_1) = \frac{1}{m} \int_{\mathbb{R}^N} (|\nabla u_1|^m + |u_1|^m) \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |u_1|^q \, dx + \frac{1}{p} \int_{\mathbb{R}^N} h(x)|u_1|^p \, dx < 0$$

and consequently $\inf_{u \in E} \Phi(u) < 0$. Thus, there exists $\lambda_0 = \overline{\lambda} > 0$ such that problem (1.3) has a nontrivial weak solution, $u_1 \in E$, for any $\lambda > \lambda_0$, satisfying $\Phi(u_1) < 0$. Since $\Phi(u_1) = \Phi(|u_1|)$ we may assume that $u_1 \ge 0$ a.e. in $\mathbb{R}^N$. □

In the following we are looking for a second nontrivial weak solution for problem (1.3).

Fix $\lambda \ge \lambda_0$. Set

$$g(x,t) = \begin{cases} 0, & \text{for} \quad t < 0 \\ \lambda t^{q-1} - h(x)t^{p-1}, & \text{for} \quad 0 \le t \le u_1(x) \\ \lambda u_1(x)^{q-1} - h(x)u_1(x)^{p-1}, & \text{for} \quad t > u_1(x) \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s) \, ds.$$

Define the functional $\Psi : E \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{m} \int_{\mathbb{R}^N} (|\nabla u|^m + |u|^m) \, dx - \int_{\mathbb{R}^N} G(x,u) \, dx.$$

The same arguments as those used for functional I imply that $J \in C^1(E, \mathbb{R})$ and

$$\langle \Psi^{'}(u), v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{m-2} \nabla u \nabla v + |u|^{m-2} uv) \, dx - \int_{\mathbb{R}^N} g(x,u) v \, dx,$$

for all u, $v \in E$. Moreover, it is clear that if u is a critical point of Ψ then $u \ge 0$ a.e. in $\mathbb{R}^N$.

Next, we prove

Lemma 3. *If u is a critical point of Ψ then $u \le u_1$.*

Proof. For a function v we define the positive part $v^+(x) = \max\{v(x), 0\}$. By Theorem 7.6 in [10] we deduce that if $v \in E$ then $v^+ \in E$. We have

$$\begin{aligned}
0 &= \langle \Psi'(u) - \Phi'(u_1), (u-u_1)^+ \rangle \\
&= \int_{\mathbb{R}^N} (|\nabla u|^{m-2}\nabla u - |\nabla u_1|^{m-2}\nabla u_1)\nabla(u-u_1)^+ \, dx \\
&\quad + \int_{\mathbb{R}^N} (|u|^{m-2}u - |u_1|^{m-2}u_1)(u-u_1)^+ \, dx \\
&\quad - \int_{\mathbb{R}^N} [g(x,u) - \lambda u_1^{q-1} + h(x)u_1^{p-1}](u-u_1)^+ \, dx \\
&= \int_{[u>u_1]} (|\nabla u|^{m-2}\nabla u - |\nabla u_1|^{m-2}\nabla u_1)(\nabla u - \nabla u_1) \, dx \\
&\quad + \int_{[u>u_1]} (|u|^{m-2}u - |u_1|^{m-2}u_1)(u-u_1) \, dx \\
&\geq \int_{[u>u_1]} (|\nabla u|^{m-1} - |\nabla u_1|^{m-1})(|\nabla u| - |\nabla u_1|) \, dx \\
&\quad + \int_{[u>u_1]} (|u|^{m-1} - |u_1|^{m-1})(|u| - |u_1|) \, dx \geq 0.
\end{aligned}$$

Thus, we obtain $u \leq u_1$ and the proof of Lemma 3 is complete. □

In the following we determine a critical point $u_2 \in E$ of Ψ such that $\Psi(u_2) > 0$ via the mountain pass theorem. By the above lemma we will deduce that $0 \leq u_2 \leq u_1$ in Ω. Therefore

$$g(x,u_2) = \lambda u_2^{q-1} - h(x)u_2^{p-1} \quad \text{and} \quad G(x,u_2) = \frac{\lambda}{q}u_2^q - \frac{h(x)}{p}u_2^p$$

and thus

$$\Psi(u_2) = \Phi(u_2) \quad \text{and} \quad \Psi'(u_2) = \Phi'(u_2).$$

More exactly we find

$$\Phi(u_2) > 0 = \Phi(0) > \Phi(u_1) \qquad \text{and} \qquad \Phi'(u_2) = 0\,.$$

This shows that u_2 is a weak solution of problem (1.3) such that $0 \leq u_2 \leq u_1$, $u_2 \neq 0$ and $u_2 \neq u_1$.

In order to find u_2 described above we prove

Lemma 4. *There exists $\rho \in (0, \|u_1\|)$ and $a > 0$ such that $\Psi(u) \geq a$, for all $u \in E$ with $\|u\| = \rho$.*

Proof. We have

$$\begin{aligned}\Psi(u) &= \frac{1}{m}\|u\|^m - \int_{\mathbb{R}^N} G(x,u)\,dx \\ &= \frac{1}{m}\|u\|^m - \int_{[u>u_1]} G(x,u)\,dx - \int_{[u<u_1]} G(x,u)\,dx \\ &= \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{[u>u_1]} u_1^q\,dx + \frac{1}{p}\int_{[u>u_1]} h(x)u_1^p\,dx - \frac{\lambda}{q}\int_{[u>u_1]} u^q\,dx \\ &\quad + \frac{1}{p}\int_{[u>u_1]} h(x)u^p\,dx \\ &\geq \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\,dx.\end{aligned}$$

On the other hand, the continuous Sobolev embedding of E into $L^q(\mathbb{R}^N)$ implies that there exists a positive constant $L > 0$ such that

$$|u|_q \leq L \cdot \|u\|, \quad \forall\, u \in E.$$

The above inequalities imply

$$\Psi(u) \geq \frac{1}{m}\|u\|^m - L_1\|u\|^q = \|u\|^m \left[\frac{1}{m} - L_1\|u\|^{q-m}\right],$$

where L_1 is a positive constant. Since $q > m$ it is clear that Lemma 4 holds true. □

Lemma 5. *The functional Ψ is coercive.*

Proof. For each $u \in E$ we have

$$\begin{aligned}\Psi(u) &= \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{[u>u_1]} u_1^q\,dx + \frac{1}{p}\int_{[u>u_1]} h(x)u_1^p\,dx - \frac{\lambda}{q}\int_{[u>u_1]} u^q\,dx \\ &\quad + \frac{1}{p}\int_{[u>u_1]} h(x)u^p\,dx \\ &\geq \frac{1}{m}\|u\|^m - \frac{\lambda}{q}\int_{\mathbb{R}^N} u_1^q\,dx \\ &= \frac{1}{m}\|u\|^m - L_2\,,\end{aligned}$$

where L_2 is a positive constant. The above inequality implies that $\Psi(u) \to \infty$ as $\|u\| \to \infty$, that is, Ψ is coercive. The proof of Lemma 5 is complete. □

Proof of Theorem 2 completed. Using Lemma 4 and the mountain pass theorem (see [6] with the variant given by Theorem 1.15 in [17]) we deduce that there exists a sequence $(u_n) \subset E$ such that

$$\Psi(u_n) \to c > 0 \quad \text{and} \quad \Psi'(u_n) \to 0 \tag{4.13}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

By relation (4.13) and Lemma 5 we obtain that (u_n) is bounded and thus passing eventually to a subsequence, still denoted by (u_n), we may assume that there exists $u_2 \in E$ such that u_n converges weakly to u_2. Standard arguments based on the Sobolev embeddings will show that

$$\lim_{n\to\infty} \langle \Psi^{'}(u_n), v\rangle = \langle \Psi^{'}(u_2), v\rangle\,,$$

for any $v \in C_0^\infty(\mathbb{R}^N)$. Taking into account that $E \subset W^{1,m}(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,m}(\mathbb{R}^N)$ the above information implies that u_2 is a weak solution of problem (1.3).

We conclude that problem (1.3) has two nontrivial weak solutions. The proof of Theorem 2 is complete. □

We point out that the proof of Theorem 2 is similar with those of Theorems 2.1 and 2.2 in [4]. However, our method in finding the second solution is somewhat different since we use the mountain pass theorem while in [4] the authors appeals to sub and super-solutions method. Our idea is frequently used when we deal with quasilinear problems see, e.g., Perera [13] or Mihăilescu and Rădulescu [12].

On the other hand, we point out that equation (1.3) can be studied also in the case when p is supercritical using similar arguments, since the $|u|^p$ term in the energy will be coercive. In that cases standard regularity results will lead to stronger results in what concerns the smoothness of solutions since in that case $W^{1,m}$ is embedded in C^1.

Acknowledgements. The author acknowledges the support through Grant CNCS PCE-47/2011.

REFERENCES

[1] R. P. Agarwal and D. O'Regan, *Ordinary and Partial Differential Equations. With Special Functions, Fourier Series, and Boundary Value Problems*, Universitext, Springer, New York, 2009.

[2] R. P. Agarwal, K. Perera, and D. O'Regan, *Morse Theoretic Aspects of p-Laplacian Type Operators*, Mathematical Surveys and Monographs, vol. 161, American Mathematical Society, Providence, RI, 2010.

[3] S. Alama and G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, *Calc. Var. Partial Differential Equations* **1** (1993), 439–479.

[4] S. Alama and G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, *J. Funct. Anal.* **141** (1996), 159–215.

[5] A. Ambrosetti, H. Brezis, and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519–543.

[6] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349–381.

[7] T. Bartsch and M. Willem, On elliptic equations with concave and convex nonlinearities, *Proc. Amer. Math. Soc.* **123** (1995), 3555–3561.

[8] H. Brezis, *Analyse Fonctionnelle: Thorie et Applications*, Masson, Paris, 1992.

[9] J. Chabrowski, Elliptic variational problems with indefinite nonlinearities, *Topological Methods in Nonlinear Analysis* **9** (1997), 221–231.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.

[11] H. Lane, On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment, *Am. J. Sci.* **50** (1869), 57–74.

[12] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **462** (2006), 2625–2641.

[13] K. Perera, Multiple positive solutions for a class of quasilinear elliptic boundary-value problems, *Electronic Journal of Differential Equations* **7** (2003), 1–5.

[14] V. Rădulescu and D. Repovš, Combined effects in nonlinear problems arising in the study of anisotropic continuous media, *Nonlinear Analysis: Theory, Methods and Applications* **75** (2012), 1523–1529.

[15] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

[16] S. B. Tshinanga, On multiple solutions of semilinear elliptic equations on unbounded domains with concave and convex nonlinearities, *Nonlinear Analysis: Theory, Methods and Applications* **28** (1997), 809–814.

[17] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[18] J. Yang and X. Zhu, On the existence of nontrivial solutions of a quasilinear elliptic boundary value problem for unbounded domains. (I) Positive mass case, *Acta Math. Sci.* **7** (1987), 341–359.

[19] J.S.W. Wong, On the generalized Emden-Fowler equation, *SIAM Rev.* **17** (1975), 339-360.