Differential and Integral Equations Volume 27, Numbers 1-2 (2014) , 35–44

POSITIVE HOMOCLINIC SOLUTIONS FOR THE DISCRETE p-LAPLACIAN WITH A COERCIVE WEIGHT FUNCTION

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(Submitted by: Roger Temam)

Abstract. We study a p-Laplacian difference equation on the set of integers, involving a coercive weight function and a reaction term satisfying the Ambrosetti–Rabinowitz condition. By means of critical-point theory and a discrete maximum principle, we prove the existence of a positive homoclinic solution.

1. Introduction

The main purpose of the present paper is to extend a classical result of Ambrosetti and Rabinowitz [3] to the framework of difference equations on infinite sets. The pioneering application of the mountain-pass theorem concerns the nonlinear elliptic problem

$$\begin{cases} -\Delta u + a(x)u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary and $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a smooth function such that the following hypotheses are fulfilled:

(i) there exists $C > 0$ such that

$$|f(x,u)| \le C\,(1 + |u|^p) \text{ for all } x \in \Omega \text{ and for all } u \ge 0,$$

with $1 < p < (N+2)/(N-2)$ if $N \ge 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$;

(ii) $f(x,0) = f_u(x,0) = 0$;

Accepted for publication: July 2013.
AMS Subject Classifications: 39A10, 47J30.

(*iii*) there exists $\mu > 2$ such that

$$0 < \mu F(x,u) \leq u f(x,u) \text{ for all } u \text{ large enough,}$$

where $F(x,u) = \int_0^u f(x,t)\,dt$;

(*iv*) $a \in L^\infty(\Omega)$ and the operator $-\Delta + a(x)I$ is coercive in $H_0^1(\Omega)$; that is, there exists $C > 0$ such that for all $u \in H_0^1(\Omega)$,

$$\int_\Omega \left(|\nabla u|^2 + a(x)u^2\right) dx \geq C\,\|u\|^2_{H_0^1(\Omega)}.$$

Under these assumptions, Ambrosetti and Rabinowitz [3] proved that problem (1.1) has at least one nonzero solution. Moreover, the same result holds true if the above subcritical condition (*i*) is replaced with the weaker assumption

$$f(x,u) = o(|u|^{(N+2)/(N-2)}) \text{ as } |u| \to \infty, \text{ uniformly in } x \in \overline{\Omega}.$$

In dealing with problems on *unbounded* domains, due to the lack of compact embeddings in Sobolev spaces, weight functions are often assumed to be coercive: this approach was first used by Omana and Willem [15] for finding homoclinic orbits of a Hamiltonian system on $\mathbb{R}$. We refer to the recent book by Ciarlet [10] for several related examples and applications.

In the present paper we deal with the following nonlinear second-order difference equation:

$$\begin{cases} -\Delta\phi_p(\Delta u(k-1)) + a(k)\phi_p(u(k)) = f(k,u(k)) & \text{for all } k \in \mathbb{Z} \\ u(k) \to 0 & \text{as } |k| \to \infty. \end{cases} \tag{1.2}$$

Here $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, $a : \mathbb{Z} \to \mathbb{R}$ is a positive and coercive weight function, while $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, the forward difference operator is defined as

$$\Delta u(k-1) = u(k) - u(k-1) \text{ for all } k \in \mathbb{Z}.$$

Difference equations represent the discrete counterpart of ordinary differential equations, and are usually studied in connection with numerical analysis. Existence of a solution for a nonlinear difference equation can be proved via fixed-point theory or by means of nonlinear operator theory (for an exhaustive description of the subject, we refer the reader to the monograph of Agarwal [1]).

Variational methods for difference equations, which allow one to achieve multiplicity results, were introduced by Agarwal, Perera, and O'Regan [2]. Later on, such methods received considerable attention. We mention here the works of Cabada, Iannizzotto, and Tersian [6]; Candito and Giovannelli [8]; Candito and Molica Bisci [9]; and Mihăilescu, Rădulescu, and Tersian

[13] (for the anisotropic case). In all these papers, variational methods are applied to boundary-value problems on *bounded* discrete intervals (that is, sets of the type $\{0, \dots n\}$). Most results combine minimization and versions of the minimax principle, which usually do not require the Palais–Smale condition as the energy functional is defined on a finite-dimensional Banach space.

When dealing with difference equations on *unbounded* discrete intervals (typically, on the whole set of integers $\mathbb{Z}$), with asymptotic conditions of homoclinic or heteroclinic type, the finite-dimensional variational framework cannot be employed: namely, solutions are sought in sequence spaces of ℓ^p-type. The lack of compactness of Palais–Smale sequences in such spaces represents a severe difficulty in such cases. Thus, many authors have developed mixed methods to deal with such problems. For instance, Cabada and Iannizzotto [5] first study a Dirichlet problem on the bounded interval $\{-n, \dots, n\}$ and then, letting $n \to \infty$, use a compactness argument to prove the existence of a homoclinic solution on $\mathbb{Z}$. We also recall the work of Cabada, Li, and Tersian [7], where a problem with periodic coefficients is proved to have a non-zero homoclinic solution. A similar approach was extended by Mihăilescu, Rădulescu, and Tersian [14] to the anisotropic case.

Ma and Guo [12] introduced coercive weight functions for a semilinear difference equation on $\mathbb{Z}$ ($p = 2$). So, the energy functional turns out to be defined on a subspace of ℓ^2 which is still infinite-dimensional but compactly embedded into ℓ^2: such a compact embedding is a key tool to prove the Palais–Smale condition. The same approach was recently extended by Iannizzotto and Tersian [11] to fully nonlinear equations of the type (1.2), with $p > 1$, employing techniques of functional analysis, which led to some multiplicity results for problem (1.2) under convenient hypotheses of the reaction term f (namely, $f(k, \cdot)$ is assumed to be $(p-1)$-superlinear at 0 and $(p-1)$-sublinear at infinity).

Here we consider a more general class of reaction terms than in [11], including $(p-1)$-superlinear mappings both at 0 and at infinity, subject to a version of the Ambrosetti–Rabinowitz condition. Let $a : \mathbb{Z} \to \mathbb{R}$ and the continuous mapping $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfy the following hypotheses:

- (A) $a(k) \geq a_0 > 0$ for all $k \in \mathbb{Z}$, $a(k) \to +\infty$ as $|k| \to \infty$;
- (F_1) $\lim_{t \to 0^+} \frac{f(k,t)}{t^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;
- (F_2) $0 < \mu F(k,t) \leq f(k,t)t$ for all $k \in \mathbb{Z}$, $t > 0$ ($\mu > p$),

where $F : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(k,t) = \int_0^t f(k,\tau)\, d\tau \text{ for all } k \in \mathbb{Z},\ t \in \mathbb{R}.$$

Here is our main result:

Theorem 1. *If* (A), (F_1), *and* (F_2) *are satisfied, then problem* (1.2) *admits at least a positive solution.*

The paper has the following structure: in Section 2 we collect some preliminary results, and in Section 3 we prove a maximum principle for problem (1.2) and Theorem 1.

2. Preliminaries

In this section we will recall some technical results which will be used later. We begin by defining some Banach spaces. For all $1 \le p < +\infty$, we denote by ℓ^p the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_p^p = \sum_{k \in \mathbb{Z}} |u(k)|^p < +\infty.$$

Moreover, we denote by ℓ^∞ the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_\infty = \sup_{k \in \mathbb{Z}} |u(k)| < +\infty$$

(we are slightly distorting notation, as the symbols $\ell^{(\cdot)}$ usually denote spaces of functions defined in $\mathbb{N}$, but the main properties still hold in our case). By classical results of functional analysis we know that, for all $1 < p < +\infty$, $(\ell^p, \|\cdot\|_p)$ is a uniformly convex (hence, reflexive) Banach space with dual $(\ell^q, \|\cdot\|_q)$ $(1/p + 1/q = 1)$. Moreover, $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space. For all $1 \le p \le r \le +\infty$, the embedding $\ell^p \hookrightarrow \ell^r$ is continuous. We recall some classical inequalities: the *Hölder inequality*

$$\Big| \sum_{k \in \mathbb{Z}} u(k) v(k) \Big| \le \|u\|_p \|v\|_q \text{ for all } u \in \ell^p,\ v \in \ell^q, \tag{2.1}$$

and the *Minkowski inequality*

$$\Big(\sum_{k \in \mathbb{Z}} |u(k) + v(k)|^p \Big)^{\frac{1}{p}} \le \|u\|_p + \|v\|_p \text{ for all } u, v \in \ell^p. \tag{2.2}$$

Moreover, for all $1 < p < +\infty$ there exists $c > 0$ such that either

$$(\phi_p(x) - \phi_p(y))\,(x - y) \ge c|x - y|^p \text{ for all } x, y \in \mathbb{R}, \text{ if } p \ge 2, \tag{2.3}$$

or

$$(\phi_p(x) - \phi_p(y))\,(x - y) \ge c\,(|x| + |y|)^{p-2}\,|x - y|^2 \text{ for all } x, y \in \mathbb{R}, \text{ if } 1 < p < 2. \tag{2.4}$$

In the sequel, we will need the following technical result:

Lemma 2. [11, Lemma 4] *If S is a compact subset of ℓ^p, then for all $\varepsilon > 0$ there exists $h \in \mathbb{N}$ such that*

$$\Big(\sum_{|k|\geq h} |u(k)|^p\Big)^{\frac{1}{p}} < \varepsilon \text{ for all } u \in S.$$

We set

$$X = \Big\{u : \mathbb{Z} \to \mathbb{R} \;:\; \sum_{k\in\mathbb{Z}} a(k)|u(k)|^p < \infty\Big\},\ \|u\| = \Big[\sum_{k\in\mathbb{Z}} a(k)|u(k)|^p\Big]^{\frac{1}{p}}.$$

Clearly we have

$$\|u\|_\infty \leq \|u\|_p \leq a_0^{-\frac{1}{p}}\|u\| \text{ for all } u \in X. \tag{2.5}$$

Proposition 3. [11, Proposition 3] *$(X, \|\cdot\|)$ is a reflexive Banach space, and the embedding $X \hookrightarrow \ell^p$ is compact.*

We denote by $(X^*, \|\cdot\|_*)$ the topological dual of $(X, \|\cdot\|)$.

We recall that a functional $J \in C^1(X)$ is said to satisfy the *Palais–Smale condition* ((PS) for short) if every sequence (u_n) in X such that $(J(u_n))$ is bounded and $J'(u_n) \to 0$ in X^* admits a convergent subsequence. Such a condition is an essential hypothesis in the following mountain-pass theorem, due to Pucci and Serrin [16]:

Theorem 4. [16, Theorem 1] *If X is a reflexive Banach space, $J \in C^1(X)$ satisfies (PS), and there exist $\check{u} \in X$ and real numbers $0 < \rho_1 < \rho_2$ such that $\|\check{u}\| > \rho_2$ and*

$$\inf_{\rho_1\leq\|u\|\leq\rho_2} J(u) = \alpha \geq \max\{J(0), J(\check{u})\},$$

then there exists a critical point $\hat{u} \in X$ of J such that $J(\hat{u}) \geq \alpha$.

3. Proof of Theorem 1

For the convenience of the reader, we split the proof of our main result into several steps. First, we provide problem (1.2) with a variational formulation. We denote $t^\pm = \max\{\pm t, 0\}$ and set

$$f_+(k,t) = f(k,t^+),\ F_+(k,t) = \int_0^t f_+(k,\tau)\,d\tau \text{ for all } k \in \mathbb{Z},\ t \in \mathbb{R}.$$

Note that $f_+ : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is zero for all $t \leq 0$ as $f(k,0) = 0$ for all $k \in \mathbb{Z}$ (by (F_1)). We define an energy functional for (1.2) by setting for all $u \in X$

$$J(u) = \frac{1}{p}\sum_{k\in\mathbb{Z}} [|\Delta u(k-1)|^p + a(k)|u(k)|^p] - \sum_{k\in\mathbb{Z}} F_+(k,u(k)).$$

Lemma 5. *If* (A) *and* (F_1) *are satisfied, then* $J \in C^1(X)$*. Moreover, if* $u \in X$ *is a critical point of* J*, then* $u(k) \geq 0$ *for all* $k \in \mathbb{Z}$ *and* u *is a solution of* (1.2)*.*

Proof. By [11, Propositions 5, 6, 7], $J \in C^1(X)$ and for all $u, v \in X$ we have

$$\langle J'(u), v\rangle = \sum_{k\in\mathbb{Z}} [\phi_p(\Delta u(k-1))\Delta v(k-1) + a(k)\phi_p(u(k))v(k)] - \sum_{k\in\mathbb{Z}} f_+(k, u(k))v(k). \tag{3.1}$$

Now assume $u \in X$ and $J'(u) = 0$. For all $k \in \mathbb{Z}$ we have

$$\phi_p(\Delta u(k-1))\Delta u^-(k-1) \leq 0, \;\; f_+(k, u(k))u^-(k) = 0.$$

From (3.1) with $v = u^-$, we obtain

$$\begin{aligned} 0 &= \langle J'(u), -u^-\rangle \\ &= \sum_{k\in\mathbb{Z}} \left[\phi_p(\Delta u(k-1))\Delta u^-(k-1) + a(k)\phi_p(u(k))u^-(k)\right] \leq -\|u^-\|^p, \end{aligned}$$

which implies $u^- = 0$, that is, $u(k) \geq 0$ for all $k \in \mathbb{Z}$. Now fix $h \in \mathbb{Z}$ and define $e_h \in X$ by setting $e_h(k) = \delta_{h,k}$ for all $k \in \mathbb{Z}$. From (3.1) with $v = e_h$ we have

$$-\Delta\phi_p(\Delta u(h-1)) + a(h)\phi_p(u(h)) = f(h, u(h)).$$

Moreover, clearly $u(k) \to 0$ as $|k| \to +\infty$, so u is in fact a solution of (1.2). □

Next, we prove a maximum principle for problem (1.2).

Lemma 6. *If* (A) *and* (F_1) *are satisfied, and* $u \in X$ *is a solution of* (1.2) *such that* $u(k) \geq 0$ *for all* $k \in \mathbb{Z}$ *and* $u \neq 0$*, then* $u(k) > 0$ *for all* $k \in \mathbb{Z}$*.*

Proof. Arguing by contradiction, assume that $u(h) = 0$ for some $h \in \mathbb{Z}$. By (1.2) we have

$$\phi_p(\Delta u(h)) = \phi_p(\Delta u(h-1)),$$

which implies $u(h+1) = -u(h-1)$, so (recall that u takes non-negative values)

$$u(h-1) = u(h) = u(h+1) = 0.$$

An easy inductive argument shows now that $u(k) = 0$ for all $k \in \mathbb{Z}$, a contradiction. □

The crucial step of our argument is the following lemma:

Lemma 7. *If (A), (F_1), and (F_2) are satisfied, then J satisfies (PS).*

Proof. We follow Ma and Guo [12]. Let (u_n) be a sequence in X such that $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ in X^*. For $n \in \mathbb{N}$ big enough we have $\|J'(u_n)\|_* < \mu$. There exists $c_1 > 0$ such that

$$\begin{aligned} c_1 + \|u_n\| &\geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\ &\geq \Big(\frac{1}{p} - \frac{1}{\mu}\Big)\sum_{k\in\mathbb{Z}}\Big[|\Delta u_n(k-1)|^p + a(k)|u_n(k)|^p\Big] \\ &\quad + \sum_{k\in\mathbb{Z}}\Big[\frac{1}{\mu}f_+(k,u_n(k))u_n(k) - F_+(k,u_n(k))\Big] \geq \Big(\frac{1}{p} - \frac{1}{\mu}\Big)\|u_n\|^p, \end{aligned}$$

which (as $p > 1$) implies that (u_n) is bounded. By Proposition 3, passing if necessary to a subsequence, we may assume $u \rightharpoonup u$ in X and $u_n \to u$ in ℓ^p for some $u \in X$. We assume $p \geq 2$ and choose $c_2 > 0$ such that

$$\|u_n\|_p < c_2 \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Fix $\varepsilon > 0$. By (F_1), there exists $\delta > 0$ such that

$$|f_+(k,t)| \leq |t|^{p-1} \text{ for all } k \in \mathbb{Z} \text{ and } |t| < \delta. \tag{3.3}$$

By Lemma 2, there exists $h \in \mathbb{N}$ such that

$$\begin{aligned} &\Big(\sum_{|k|>h}|u_n(k)|^p\Big)^{\frac{1}{p}} < \frac{\varepsilon}{2^{p+3}3c_2^{p-1}} \text{ for all } n \in \mathbb{N} \text{ and} \\ &\Big(\sum_{|k|>h}|u(k)|^p\Big)^{\frac{1}{p}} < \frac{\varepsilon}{2^{p+3}3c_2^{p-1}}. \end{aligned} \tag{3.4}$$

Moreover, choosing h even bigger if necessary, we have $|u_n(k)| < \delta$ for all $n \in \mathbb{N}$ and $|k| > h$, and $|u(k)| < \delta$ for all $|k| > h$. Due to the continuity of the finite sum, we have for $n \in \mathbb{N}$ big enough

$$\sum_{|k|\leq h}|\phi_p(\Delta u_n(k-1)) - \phi_p(\Delta u(k))|\,|\Delta u_n(k-1) - \Delta u(k-1)| < \frac{\varepsilon}{6} \tag{3.5}$$

and

$$\sum_{|k|\leq h}|f_+(k,u_n(k)) - f_+(k,u(k))|\,|u_n(k) - u(k)| < \frac{\varepsilon}{6}. \tag{3.6}$$

Since $J'(u_n) \to 0$ in X^*, for $n \in \mathbb{N}$ big enough we have

$$\big|\langle J'(u_n), u_n - u\rangle\big| < \frac{\varepsilon}{6}. \tag{3.7}$$

Besides, $u_n \rightharpoonup u$ in X yields for $n \in \mathbb{N}$ big enough

$$\left|\langle J'(u), u_n - u\rangle\right| < \frac{\varepsilon}{6}. \tag{3.8}$$

Now, for $n \in \mathbb{N}$ big enough we have

$$\begin{aligned}
c\|u_n-u\|^p &\le \sum_{k\in\mathbb{Z}} a(k)\left(\phi_p(u_n(k))-\phi_p(u(k))\right)(u_n(k)-u(k)) \text{ (see (2.3))}\\
&= \langle J'(u_n)-J'(u), u_n-u\rangle\\
&\quad - \sum_{k\in\mathbb{Z}}\left(\phi_p(\Delta u_n(k-1))-\phi_p(\Delta u(k-1))\right)(\Delta u_n(k-1)-\Delta u(k-1))\\
&\quad + \sum_{k\in\mathbb{Z}}\left(f_+(k,u_n(k))-f_+(k,u(k))\right)(u_n(k)-u(k)) \text{ (see (3.1))}\\
&< \frac{2\varepsilon}{3} - \sum_{|k|>h}\left(\phi_p(\Delta u_n(k-1))-\phi_p(\Delta u(k-1))\right)(\Delta u_n(k-1)-\Delta u(k-1))\\
&\quad + \sum_{k\in\mathbb{Z}}\left(f_+(k,u_n(k))-f_+(k,u(k))\right)(u_n(k)-u(k)) \text{ (see (3.5)–(3.8))}\\
&\le \frac{2\varepsilon}{3} + \Big[\sum_{|k|>h}|\phi_p(\Delta u_n(k-1))-\phi_p(\Delta u(k-1))|^q\Big]^{\frac{1}{q}}\\
&\quad \times \Big[\sum_{|k|>h}|\Delta u_n(k-1)-\Delta u(k-1)|^p\Big]^{\frac{1}{p}}\\
&\quad + \Big[\sum_{|k|>h}|f_+(k,u_n(k))-f_+(k,u(k))|^q\Big]^{\frac{1}{q}}\Big[\sum_{|k|>h}|u_n(k)-u(k)|^p\Big]^{\frac{1}{p}} \text{(see (2.1))}\\
&\le \frac{2\varepsilon}{3} + \Big[2^{p-1}\|u_n\|_p^{p-1}+2^{p-1}\|u\|_p^{p-1}\Big]\\
&\quad \times \Big[\Big(\sum_{|k|>h}|\Delta u_n(k-1)|^p\Big)^{\frac{1}{p}} + \Big(\sum_{|k|>h}|\Delta u(k-1)|^p\Big)^{\frac{1}{p}}\Big]\\
&\quad + \Big[\|u_n\|_p^{p-1}+\|u\|_p^{p-1}\Big]\Big[\Big(\sum_{|k|>h}|u_n(k)|^p\Big)^{\frac{1}{p}}+\Big(\sum_{|k|>h}|u(k)|^p\Big)^{\frac{1}{p}}\Big] \text{(see (2.2), (3.3))}\\
&\le \frac{2\varepsilon}{3} + 2^p c_2^{p-1}\frac{2\varepsilon}{2^{p+3}3c_2^{p-1}} + 2c_2^{p-1}\frac{2\varepsilon}{2^{p+3}3c_2^{p-1}} < \varepsilon \text{ (see (3.2), (3.4))}.
\end{aligned}$$

This finally implies $u_n \to u$ in X. If $1 < p < 2$ we argue in an analogous way using (2.4) instead of (2.3). Thus, J satisfies (PS). □

Now we are in a suitable position to conclude the proof of our main result:

Proof of Theorem 1—Conclusion. We are going to apply Theorem 4. Clearly $J(0) = 0$; besides, J satisfies (PS) (see Lemma 7). Fix $0 < \varepsilon < a_0/2^p$. By (F_1), there exists $\delta > 0$ such that

$$F_+(k,t) \le \frac{\varepsilon}{p}|t|^p \text{ for all } k \in \mathbb{Z},\ |t| \le \delta.$$

Set $\rho_2 = a_0^{\frac{1}{p}}\delta$ and $\rho_1 = \rho_2/2$. For all $u \in X$ such that $\rho_1 \le \|u\| \le \rho_2$, by (2.5) we have $\|u\|_\infty \le \delta$; hence,

$$J(u) \ge \frac{\|u\|^p}{p} - \sum_{k\in\mathbb{Z}} F_+(k,u(k)) \ge \frac{\rho_1^p}{p} - \frac{\varepsilon}{p}\|u\|_p^p \ge \left(\frac{1}{2^p} - \frac{\varepsilon}{a_0}\right)\frac{\rho_2^p}{p},$$

so

$$\alpha \ge \left(\frac{1}{2^p} - \frac{\varepsilon}{a_0}\right)\frac{\rho_2^p}{p} > 0.$$

By standard integration, (F_2) implies that there exists a mapping $b : \mathbb{Z} \to (0, +\infty)$ such that

$$F(k,t) \ge b(k)t^\mu \text{ for all } k \in \mathbb{Z},\ t \ge 1. \tag{3.9}$$

Fix $h \in \mathbb{Z}$. For all $\sigma \ge 1$, by (3.9) we have

$$J(\sigma e_h) = \frac{1}{p}(2 + a(h))\sigma^p - F(h,\sigma) \le \frac{1}{p}(2 + a(h))\sigma^p - b(h)\sigma^\mu,$$

which goes to $-\infty$ as $\sigma \to +\infty$ (recall that $\mu > p$). So, we can choose $\sigma \ge 1$ big enough and set $\check{u} = \sigma e_h$, so that $\|\check{u}\| > \rho_2$ and $J(\check{u}) \le 0$. By Theorem 4 there exists $\hat{u} \in X$ such that $J'(\hat{u}) = 0$ and $J(\hat{u}) \ge \alpha$. From Lemma 5 we know that $\hat{u}(k) \ge 0$ for all $k \in \mathbb{Z}$ and that $\hat{u}$ solves (1.2). Moreover, since $\alpha > 0$ we also have $\hat{u} \ne 0$, which, by Lemma 6, implies $\hat{u}(k) > 0$ for all $k \in \mathbb{Z}$. □

Finally, we present two simple examples:

Example 8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = \phi_r(t)$ for all $t \in \mathbb{R}$ ($p < r < +\infty$). Then, it is easily seen that f satisfies hypotheses (F_1) and (F_2). So, given a weight function a satisfying (A), problem (1.2) has at least a positive solution $\hat{u}$ (in this case, as f is odd, $-\hat{u}$ is a negative solution of (1.2)).

Example 9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \dfrac{t}{t+1} + r\ t^{r-1}\ln(t+1) & \text{if } t > 0. \end{cases}$$

Then, it is easily seen that f satisfies hypotheses (F_1) and (F_2). So, given a weight function a satisfying (A), problem (1.2) has at least a positive solution $\hat{u}$.

Acknowledgments. V. Rădulescu acknowledges the support through Grant CNCS PCE-47/2011. This work was partially written while V. Rădulescu was visiting the University of Catania with the support of the GNAMPA Visiting Professors Programme 2012/2013.

References

[1] R.P. Agarwal, "Difference Equations and Inequalities," Marcel Dekker Inc., 2000.
[2] R.P. Agarwal, K. Perera, and D. O'Regan, *Multiple positive solutions of singular and nonsingular discrete problems via variational methods*, Nonlinear Anal., 58 (2004), 69–73.
[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis, 14 (1973), 349–381.
[4] A. Cabada and J.A. Cid, *Solvability of some Φ-Laplacian singular difference equations defined on the integers*, ASJE - Mathematics, 34 (2009), 75–81.
[5] A. Cabada and A. Iannizzotto, *Existence of homoclinic constant sign solutions for a difference equation on the integers* (preprint).
[6] A. Cabada, A. Iannizzotto, and S. Tersian, *Multiple solutions for discrete boundary value problems*, J. Math. Anal. Appl., 356 (2009), 418–428.
[7] A. Cabada, C. Li, and S. Tersian, *On homoclinic solutions of a semilinear p-Laplacian difference equation with periodic coefficients*, Adv. Difference Equ., 2010 (2010), 17 pp.
[8] P. Candito and N. Giovannelli, *Multiple solutions for a discrete boundary value problem involving the p-Laplacian*, Comput. Math. Appl., 56 (2008), 959–964.
[9] P. Candito and G. Molica Bisci, *Existence of two solutions for a second-order discrete boundary value problem*, Adv. Nonlinear Studies, 11 (2011), 443–453.
[10] P.G. Ciarlet, "Linear and Nonlinear Functional Analysis with Applications," Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.
[11] A. Iannizzotto and S. Tersian, *Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory*, J. Math. Anal. Appl., 403 (2013), 173–182.
[12] M. Ma and Z. Guo, *Homoclinic orbits for second order self-adjoint difference equations*, J. Math. Anal. Appl., 323 (2005), 513–521.
[13] M. Mihăilescu, V. Rădulescu, and S. Tersian, *Eigenvalue problems for anisotropic discrete boundary value problems*, J. Difference Equ. Appl., 15 (2009), 557–567.
[14] M. Mihăilescu, V. Rădulescu, and S. Tersian, *Homoclinic solutions of difference equations with variable exponents*, Topol. Methods Nonlinear Anal., 38 (2011), 277–289.
[15] W. Omana and M. Willem, *Homoclinic orbits for a class of Hamiltonian systems*, Diff. Int. Equations, 5 (1992), 1115–1120.
[16] P. Pucci and J. Serrin, *A mountain pass theorem*, J. Differential Equations, 60 (1985), 142–149.