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LINEAR ELLIPTIC SYSTEMS INVOLVING FINITE RADON MEASURES

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1. Statement of the main result

The study of elliptic boundary value problems with L^1 or Radon measure data has been initiated in the last few decades by the pioneering works of Stampacchia [12], Brezis-Strauss [7], Brezis [5], [6].

Let Ω be a smooth bounded domain in $\mathbf{R}^N$. Consider the problem

$$\begin{cases} -\text{div } (a_i(x)\nabla u_i) + \displaystyle\sum_{j=1}^{d} b_{ij}(x)u_j = f_i, \quad \text{in } \Omega, \text{ for } i = 1, \cdots, d \\ u_i = 0, \quad \text{on } \Gamma_{\mathcal{D}}, \text{ for } i = 1, \cdots, d \\ \dfrac{\partial u_i}{\partial \nu} = g_i, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d\,. \end{cases} \tag{1.1}$$

Here, ν denotes the unit normal outward vector, $d \geq 1$ is an integer, and $a_i, b_{ij} \in L^\infty(\Omega)$, for $1 \leq i, j \leq d$. We point out that we make no symmetry assumption on the coefficients b_{ij}. We assume that $\{\Gamma_{\mathcal{D}}, \Gamma_{\mathcal{N}}\}$ realize an open partition of the boundary $\partial\Omega$, i.e., $\Gamma_{\mathcal{D}} \cap \Gamma_{\mathcal{N}} = \emptyset$ and $\overline{\Gamma_{\mathcal{D}}} \cup \overline{\Gamma_{\mathcal{N}}} = \partial\Omega$. Moreover, we suppose that $\Gamma_{\mathcal{D}}$ has nonzero $(N-1)$-Lebesgue measure, namely, $\text{m}eas_{N-1}(\Gamma_{\mathcal{D}}) > 0$. We also assume that the elliptic operator is not degenerate, i.e., there exists $\alpha > 0$ such that

$$a_i(x) \geq \alpha \qquad \text{for a.e. } x \in \Omega \text{ and any } i = 1, \cdots, d. \tag{1.2}$$

Set $E^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega); u = 0 \text{ on } \Gamma_{\mathcal{D}}\}$ and $E := \bigcap_{1 \leq p < \frac{N}{N-1}} (E^{1,p}(\Omega))^d$. We denote throughout by $\|\cdot\|_p$ (resp. $\|\cdot\|_{p,d}$) the norm in the space $L^p(\Omega)$

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(resp. $(L^p(\Omega))^d$). We also denote by $|\cdot|_p$ (resp. $|\cdot|_{p,d}$) the norm in the space $E^{1,p}(\Omega)$ (resp. $(E^{1,p}(\Omega))^d$).

We suppose that the associated bilinear form is coercive, namely there exists $\beta > 0$ such that, for every $u = (u_1, \cdots, u_d) \in (E^{1,2}(\Omega))^d$,

$$\int_\Omega \Big(\sum_{i=1}^d a_i |\nabla u_i|^2 + \sum_{i,j} b_{ij} u_i u_j \Big) dx \geq \beta \, |u|_{2,d}^2 . \tag{1.3}$$

We assume that f_i and g_i are bounded measures (finite Radon measures) on Ω, respectively $\Gamma_{\mathcal{N}}$, that is, $f_i \in \mathcal{M}(\Omega)$ and $g_i \in \mathcal{M}(\Gamma_{\mathcal{N}})$, for any $i = 1, \cdots, d$,.

If $\Gamma_{\mathcal{N}} = \emptyset$ and $f \in (\mathcal{M}(\Omega))^d$ Stampacchia introduced in [12] a duality method combined with a $C^{0,\alpha}$-regularity argument. The purpose of this paper is to study the general elliptic system (1.1) which involves mixed boundary conditions. As in Stampacchia's framework, our arguments are restricted to a linear setting. The proof relies on the crucial observation (see Lemma 1) that L^1 boundedness implies the boundedness in the space E. As we shall observe in Lemma 1, this becomes true because the L^{p^*}-boundedness implies $E^{1,p}$-boundedness, for any $p < \frac{N}{N-1}$.

Definition 1. A function $u = (u_1, \cdots, u_d) \in E$ is said to be a solution of the problem (1.1) provided that

$$\int_\Omega a_i \nabla u_i \cdot \nabla \varphi + \int_\Omega \Big(\sum_{j=1}^d b_{ij} u_j \Big) \varphi = \int_\Omega f_i \varphi + \int_{\Gamma_{\mathcal{N}}} a_i g_i \varphi,$$

for any $i = 1, \cdots, d$ and for every $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on $\Gamma_{\mathcal{D}}$.

Theorem 1. *Assume that hypotheses* (1.2) *and* (1.3) *are fulfilled. Then, for any bounded measures* $f \in (\mathcal{M}(\Omega))^d$ *and* $g \in (\mathcal{M}(\Gamma_{\mathcal{N}}))^d$ *the problem* (1.1) *has at least one solution.*

We point out that the celebrated non-uniqueness example constructed in Serrin [11] shows that Problem (1) may have several solutions (see also Prignet [10], p. 329).

2. PROOF OF THEOREM 1

Let $f^n = (f_i^n)_{1 \leq i \leq d} \in (L^2(\Omega))^d$ and $g^n = (g_i^n)_{1 \leq i \leq d} \in (L^2(\Gamma_{\mathcal{N}}))^d$ be such that

$$f^n \rightharpoonup f \qquad \text{weakly in the sense of measures in } (\mathcal{M}(\Omega))^d \tag{2.1}$$

$$g^n \rightharpoonup g \qquad \text{weakly in the sense of measures in } (\mathcal{M}(\Gamma_{\mathcal{N}}))^d \tag{2.2}$$

$$\|f^n\|_{1,d} \leq \|f\|_{(\mathcal{M}(\Omega))^d} \tag{2.3}$$

$$\|g^n\|_{1,d} \leq \|g\|_{(\mathcal{M}(\Gamma_{\mathcal{N}}))^d} . \tag{2.4}$$

Consider the problem

$$\begin{cases} -\text{div } (a_i \nabla u_i^n) + \displaystyle\sum_{j=1}^{d} b_{ij} u_j^n = f_i^n, \quad \text{in } \Omega, \text{ for } i = 1, \cdots, d \\ u_i^n = 0, \quad \text{on } \Gamma_{\mathcal{D}}, \text{ for } i = 1, \cdots, d \\ \dfrac{\partial u_i^n}{\partial \nu} = g_i^n, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d . \end{cases} \tag{2.5}$$

Using the coercivity condition (1.3) and applying the Lax-Milgram Lemma we find that problem (2.5) has a unique solution $u^n \in (E^{1,2}(\Omega))^d$.

Proposition 1. *The sequence $(u^n)_n$ is bounded in $(L^1(\Omega))^d$.*

Proof of Proposition 1. We argue by contradiction and assume that $\|u^n\|_{1,d} \to \infty$. Set $v_i^n = \frac{u_i^n}{\|u^n\|_{1,d}}$, for every $1 \leq i \leq d$ and $n \geq 1$. We observe that $v^n \in (E^{1,2}(\Omega))^d$, $\|v^n\|_{1,d} = 1$ and

$$\begin{cases} -\text{div } (a_i \nabla v_i^n) + \displaystyle\sum_{j=1}^{d} b_{ij} v_j^n = \dfrac{f_i^n}{\|u^n\|_{1,d}}, \quad \text{in } \Omega, \text{ for } i = 1, \cdots, d \\ v_i^n = 0, \quad \text{on } \Gamma_{\mathcal{D}}, \text{ for } i = 1, \cdots, d \\ \dfrac{\partial v_i^n}{\partial \nu} = \dfrac{g_i^n}{\|u^n\|_{1,d}}, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d . \end{cases} \tag{2.6}$$

Lemma 1. *The sequence (v^n) is bounded in the space E.*

Proof of Lemma 1. Taking into account (2.3), (2.4) and the assumption $\|u^n\|_{1,d} \to \infty$ we obtain that the L^2-sequences $r_i^n = \frac{f_i^n}{\|u^n\|_{1,d}}$ and $s_i^n = \frac{g_i^n}{\|u^n\|_{1,d}}$ converge to 0 in $L^1(\Omega)$, respectively in $L^1(\Gamma_{\mathcal{N}})$. Set

$$M = \max_{i,j} \{ \|a_i\|_{L^\infty(\Omega)}, \|b_{ij}\|_{L^\infty(\Omega)} \}.$$

Fix $p > 1$ such that $p < \frac{N}{N-1}$. Set

$$w_i^n = [(1 + |v_i^n|)^{(Np-N-p)/(N-p)} - 1] \operatorname{sgn} v_i^n .$$

By Proposition IX.5 in [4] it follows that $w_i^n \in H_0^1(\Omega)$. Multiplying by w_i^n in (2.6) and integrating by parts we find

$$- \int_{\Gamma_{\mathcal{N}}} a_i s_i^n w_i^n - \frac{N-(N-1)p}{N-p} \int_{\Omega} a_i (1 + |v_i^n|)^{-N(2-p)/(N-p)} |\nabla v_i^n|^2$$

$$+\int_\Omega \sum_{j=1}^d b_{ij} v_j^n w_i^n = \int_\Omega r_i^n w_i^n \,.$$

Thus, by (1.2) and the fact that $|w_i^n| \le 1$ we deduce that

$$\begin{aligned}&\alpha\,\frac{N-(N-1)p}{N-p}\int_\Omega (1+|v_i^n|)^{-N(2-p)/(N-p)}\,|\nabla v_i^n|^2 \\ &\le M\,\|s_i^n\|_{L^1(\Gamma_{\mathcal N})} + \|r_i^n\|_1 + M\,\|v^n\|_{1,d}\,.\end{aligned} \tag{2.7}$$

Therefore,

$$\int_\Omega \frac{|\nabla v_i^n|^2}{(1+|v_i^n|)^{N(2-p)/(N-p)}} \le C_1\,. \tag{2.8}$$

On the other hand, by Sobolev inclusions and Hölder's inequality,

$$\begin{aligned}&||v_i^n||_{p^*}^p \le C\int_\Omega |\nabla v_i^n|^p \\ &\le C\Big(\int_\Omega \frac{|\nabla v_i^n|^2}{(1+|v_i^n|)^{N(2-p)/(N-p)}}\Big)^{p/2}\Big(\int_\Omega (1+|v_i^n|)^{\frac{Np}{N-p}}\Big)^{(2-p)/2},\end{aligned} \tag{2.9}$$

where C depends only on p. Relations (2.8) and (2.9) yield

$$||v_i^n||_{p^*} \le C\,||\nabla v_i^n||_p \le C_2\,\|1+|v_i^n|\|_{p^*}^{\frac{N(2-p)}{2(N-p)}} \le C_3\left(1+||v_i^n||_{p^*}^{\frac{N(2-p)}{2(N-p)}}\right). \tag{2.10}$$

We distinguish two different situations:

Case 1: $N \ge 3$. This implies $1 > \frac{N(2-p)}{2(N-p)}$. Hence, by (2.10), the sequence (v^n) is bounded in $(L^{p^*}(\Omega))^d$, so in $(L^p(\Omega))^d$. Returning now to (2.10) we have

$$\int_\Omega |\nabla v_i^n|^p \le C$$

which shows that (v^n) is bounded in $(E^{1,p}(\Omega))^d$, for any $p < \frac{N}{N-1}$.

Case 2: $N = 2$. This implies $1 = \frac{N(2-p)}{2(N-p)}$, so the above argument does not work. However, it is possible to repeat it, but for a modified sequence v^n. Indeed, we observe that if the constant C_3 appearing in (2.10) is less than 1, then the boundedness of (v^n) in $(E^{1,p}(\Omega))^d$ follows with the same argument. But C_3 depends only on C_1, so on the value of

$$M\,\|s_i^n\|_{L^1(\Gamma_{\mathcal N})} + \|r_i^n\|_1 + M\,\|v^n\|_{1,d}.$$

But (r_i^n) and (s_i^n) converge to 0 in $L^1(\Omega)$, respectively in $L^1(\Gamma_{\mathcal N})$. Thus, in order to get $C_3 < 1$, it is sufficient to define v_i^n by $v_i^n = \varepsilon\,\frac{u_i^n}{||u_i^n||_{1,d}}$, for $\varepsilon > 0$

small enough. This choice is possible due to the linearity of the system (2.5). □

The key fact in the proof of the above result is the boundedness of (v^n) in $(L^1(\Omega))^d$ combined with the linearity of the problem (2.6).

Proof of Proposition 1 continued (case $N \leq 3$). Let $V^n = (V_1^n, \cdots, V_d^n) \in (E^{1,2}(\Omega))^d$ be the unique solution of the coercive problem

$$\begin{cases} -\text{div }(a_i \nabla V_i^n) + \displaystyle\sum_{j=1}^{d} b_{ij} v_j^n = 0, \quad \text{in } \Omega, \text{ for } i = 1, \cdots, d \\ V_i^n = 0, \quad \text{on } \Gamma_{\mathcal{D}}, \text{ for } i = 1, \cdots, d \\ \dfrac{\partial V_i^n}{\partial \nu} = 0, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d\,. \end{cases} \tag{2.11}$$

It follows by Lemma 1 that the sequence $(v^n)_n$ is bounded in $(L^{p^*}(\Omega))^d$, for any $p < \frac{N}{N-1}$. Our hypothesis $N \leq 3$ implies $p^* \geq 2$, provided that $\frac{2N}{N+2} \leq p < \frac{N}{N-1}$. Hence, the sequence $(v^n)_n$ is bounded in $(L^2(\Omega))^d$. After multiplication in (2.11) by V_i^n and integration we find

$$\int_\Omega |\nabla V_i^n|^2 \leq \alpha^{-1} \sum_{j=1}^{d} \int_\Omega |b_{ij} v_j^n V_i^n| \leq \alpha^{-1} M \sum_{j=1}^{d} \|v_j^n\|_2 \cdot \|V_i^n\|_2 \leq C |V^n|_{2,d}. \tag{2.12}$$

It follows that $(V^n)_n$ is bounded in $(E^{1,2}(\Omega))^d$. On the other hand, by (2.6) and (2.11),

$$\begin{cases} -\text{div }(a_i \nabla (v_i^n - V_i^n)) = \dfrac{f_i^n}{\|u^n\|_{1,d}}, \quad \text{in } \Omega, \text{ for } i = 1, \cdots, d \\ v_i^n - V_i^n = 0, \quad \text{on } \Gamma_{\mathcal{D}}, \text{ for } i = 1, \cdots, d \\ \dfrac{\partial (v_i^n - V_i^n)}{\partial \nu} = \dfrac{g_i^n}{\|u^n\|_{1,d}}, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d\,. \end{cases} \tag{2.13}$$

Observing that the sequence $(v_n - V_n)_n$ is bounded in $(L^1(\Omega))^d$ and arguing as in the proof of Lemma 1, we deduce that $(v_n - V_n)_n$ is bounded in $(E^{1,p}(\Omega))^d$, for any $p < \frac{N}{N-1}$. So, up to a subsequence, we can assume that

$$v^n - V^n \rightharpoonup 0 \qquad \text{weakly in } (E^{1,p}(\Omega))^d, \, \forall p < \tfrac{N}{N-1}. \tag{2.14}$$

But, by Lemma 1 and passing again at a subsequence,

$$v^n \rightharpoonup v \qquad \text{weakly in } (E^{1,p}(\Omega))^d, \, \forall p < \tfrac{N}{N-1}. \tag{2.15}$$

Hence, by (2.14) and (2.15),

$$V^n \rightharpoonup v \qquad \text{weakly in } (E^{1,p}(\Omega))^d,\ \forall p < \tfrac{N}{N-1}. \tag{2.16}$$

But $(V^n)_n$ is bounded in $(E^{1,2}(\Omega))^d$, so $v \in (E^{1,2}(\Omega))^d$. Taking into account (2.11) we obtain that the same convergence holds in $(E^{1,2}(\Omega))^d$ and $v \in (E^{1,2}(\Omega))^d$. By (2.15) and (2.16) we deduce that we can pass at the limit in (2.11) and we find

$$\begin{cases} -\text{div}\ (a_i \nabla v_i) + \displaystyle\sum_{j=1}^{d} b_{ij} v_j = 0, \quad \text{in } \Omega,\ \text{for } i = 1, \cdots, d \\ v_i = 0, \quad \text{on } \Gamma_{\mathcal{D}},\ \text{for } i = 1, \cdots, d \\ \dfrac{\partial v_i}{\partial \nu} = 0, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d\,. \end{cases} \tag{2.17}$$

By the uniqueness of the solution in $(E^{1,2}(\Omega))^d$ we conclude that $v = 0$. Consequently, (v^n) converges weakly to 0 in E which implies, by Rellich's theorem that we can assume $v_n \to 0$ strongly in $(L^1(\Omega))^d$ which contradicts $\|v^n\|_{1,d} = 1$. □

Proof of Theorem 1 continued. We are now in position to conclude the proof of Theorem 1 in the case $N \le 3$. This time we argue as in the proof of Lemma 1 but with u^n instead of v^n. Indeed, since $(u^n) \subset (E^{1,2}(\Omega))^d$ is bounded in $(L^1(\Omega))^d$ we may repeat the same arguments as in the proof of Lemma 1 to show that (u^n) is bounded in E. In particular, this implies that, passing eventually at a subsequence, there exists $u \in E$ such that

$$u^n \rightharpoonup u \qquad \text{weakly in } (E^{1,p}(\Omega))^d,\ \forall p < \tfrac{N}{N-1}.$$

Hence, u is solution to the problem (1.1).

In the case $N \ge 4$ we shall employ several times the above arguments. For this aim we define the sequence $V^n_{(k)}$ by $V^n_{(1)} = V^n$ and, for any $k \ge 2$, let $V^n_{(k)} = (V^n_{1,k}, \cdots, V^n_{d,k}) \in\ (E^{1,2}(\Omega))^d$ be the unique solution of the problem

$$\begin{cases} -\text{div}\ \left(a_i \nabla V^n_{i,k}\right) + \displaystyle\sum_{j=1}^{d} b_{ij} V^n_{i,k-1} = 0, \quad \text{in } \Omega,\ \text{for } i = 1, \cdots, d \\ V^n_{i,k} = 0, \quad \text{on } \Gamma_{\mathcal{D}},\ \text{for } i = 1, \cdots, d \\ \dfrac{\partial V^n_{i,k}}{\partial \nu} = 0, \quad \text{on } \Gamma_{\mathcal{N}}, \quad \text{for } i = 1, \cdots, d\,. \end{cases} \tag{2.18}$$

Fix $1 \le p < \frac{N}{N-1}$.

Lemma 2. *The sequence* $(V^n_{(1)})_n$ *is bounded in* $(E^{1,Np/(N-p)}(\Omega))^d$.

Proof of Lemma 2. We repeat the argument applied in the proof of Lemma 1, but for $V^n_{(1)}$ instead of v^n. We already know that (v^n) is bounded in $(L^p(\Omega))^d$. Multiplying in (2.11) by

$$w^n_i = [(1+|V^n_i|)^{(p-1)N/(N-2p)} - 1]\, sgn\, V^n_i \in H^1_0(\Omega)$$

we find

$$\begin{aligned}
&\frac{(p-1)N}{N-2p}\alpha \int_\Omega \frac{|\nabla V^n_i|^2}{(1+|V^n_i|)^{\frac{2N-p(N+2)}{N-2p}}}\\
&\leq \frac{(p-1)N}{N-2p}\int_\Omega a_i \frac{|\nabla V^n_i|^2}{(1+|V^n_i|)^{\frac{2N-p(N+2)}{N-2p}}} = -\int_\Omega (\sum_{j=1}^d b_{ij} v^n_j) w^n_i\\
&\leq M||v^n||_{1,d} + M\,||v^n||_{p,d}\Big(\int_\Omega (1+|V^n_i|)^{\frac{(p-1)N}{N-2p}\cdot\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}.
\end{aligned}$$

Hence,

$$\int_\Omega \frac{|\nabla V^n_i|^2}{(1+|V^n_i|)^{\frac{2N-p(N+2)}{N-2p}}} \leq C_1 + C_2\left(\int_\Omega (1+|V^n_i|)^{\frac{Np}{N-2p}}\right)^{\frac{p-1}{p}}. \tag{2.19}$$

We observe that the hypothesis $N \geq 4$ implies $p < \frac{N}{N-1} \leq \frac{2N}{N+2}$, so $\frac{2(N-p)}{Np} > 1$. Therefore, by Sobolev inclusions and Hölder's inequality, we obtain

$$\int_\Omega |\nabla V^n_i|^{\frac{Np}{N-p}} \leq \Big(\int_\Omega \frac{|\nabla V^n_i|^2}{(1+|V^n_i|)^{\frac{2N-p(N+2)}{N-2p}}}\Big)^{\frac{Np}{2(N-p)}} \Big(\int_\Omega (1+|V^n_i|)^{\frac{Np}{N-2p}}\Big)^{\frac{2N-p(N+2)}{2(N-p)}}. \tag{2.20}$$

By (2.19) and (2.20) we find

$$\begin{aligned}
&\Big(\int_\Omega |\nabla V^n_i|^{\frac{Np}{N-p}}\Big)^{\frac{N-p}{Np}} \qquad (2.21)\\
&\leq \Big[C_1 + C_2\Big(\int_\Omega (1+|V^n_i|)^{\frac{Np}{N-2p}}\Big)^{\frac{p-1}{p}}\Big]^{1/2}\Big(\int_\Omega (1+|V^n_i|)^{\frac{Np}{N-2p}}\Big)^{\frac{2N-p(N+2)}{2Np}}\\
&\leq C_3\Big(\int_\Omega (1+|V^n_i|)^{\frac{Np}{N-2p}}\Big)^{\frac{2N-p(N+2)}{2Np}} + C_4\Big(\int_\Omega (1+|V^n_i|)^{\frac{Np}{N-2p}}\Big)^{\frac{N-2p}{2Np}}.
\end{aligned}$$

Our choice $p < \frac{N}{N-1} < \frac{N}{2}$ implies $\frac{Np}{N-p} < N$. Therefore, by Sobolev inclusions, the space $E^{1,Np/(N-p)}(\Omega)$ is continuously embedded in $L^{Np/(N-2p)}(\Omega)$, namely

$$\Big(\int_\Omega |V_i^n|^{\frac{Np}{N-2p}}\Big)^{\frac{N-2p}{Np}} \le C\Big(\int_\Omega |\nabla V_i^n|)^{\frac{Np}{N-p}}\Big)^{\frac{N-p}{Np}}. \qquad (2.22)$$

Thus, by (2.21) and (2.22), we deduce that

$$\begin{aligned}\Big(\int_\Omega |V_i^n|^{\frac{Np}{N-2p}}\Big)^{\frac{N-2p}{Np}} &\le C\Big(\int_\Omega |\nabla V_i^n|^{\frac{Np}{N-p}}\Big)^{\frac{N-p}{Np}}\\ &\le C_5\Big(\int_\Omega (1+|V_i^n|)^{\frac{Np}{N-2p}}\Big)^{\frac{2N-p(N+2)}{2Np}} + C_6\Big(\int_\Omega (1+|V_i^n|)^{\frac{Np}{N-2p}}\Big)^{\frac{N-2p}{2Np}}\\ &\le C_7 + C_8 ||V_i^n||_{\frac{Np}{N-2p}}^{\frac{2N-p(N+2)}{2(N-2p)}} + C_9\, ||V_i^n||_{\frac{Np}{N-2p}}^{1/2}.\end{aligned}$$

Observing that $\frac{2N-p(N+2)}{2(N-2p)} < 1$, the above relations yield

$$||V_i^n||_{\frac{Np}{N-2p}} \le C\, ||\nabla V_i^n||_{\frac{Np}{N-p}} \le C_{10} + C_{11}\, ||V_i^n||_{\frac{Np}{N-2p}}^{1/2}. \qquad (2.23)$$

This implies that (V_i^n) is bounded in $L^{Np/(N-2p)}(\Omega)$. Then, again by (2.23), the sequence (∇V_i^n) is bounded in $L^{Np/(N-p)}(\Omega)$ which implies the boundedness of (V_i^n) in $E^{1,Np/(N-p)}(\Omega)$. □

Proof of Theorem 1 concluded. It follows by Lemma 2 that the sequence $(V_{(1)}^n)$ is bounded in $(L^{Np/(N-2p)}(\Omega))^d$. If $\frac{Np}{N-2p} \ge 2$, then we get the boundedness of $(V_{(1)}^n)$ in $(L^2(\Omega))^d$ and the proof is concluded with exactly the same arguments as in the case $N \le 3$, but for v^n replaced by $V_{(1)}^n$. The condition $\frac{Np}{N-2p} \ge 2$ holds true if $p \ge \frac{2N}{N+4}$. Taking into account the restriction $p < \frac{N}{N-1}$ we find either $N = 4$ or $N = 5$. If not, we will repeat the arguments done in the proof of Lemma 2. It is sufficient to point out that the proof of Lemma 2 is based on the observation that $(V_{(1)}^n)$ is bounded in $(E^{1,p^*}(\Omega))^d$, provided that (v^n) is bounded in $(L^p(\Omega))^d$. Now, with the same arguments, one can show that the boundedness of $(V_{(1)}^n)$ in $(L^{Np/(N-p)}(\Omega))^d$ implies the boundedness of $(V_{(2)}^n)$ in $(E^{1,Np/(N-2p)}(\Omega))^d$, since $\frac{Np}{N-2p}$ is the Sobolev conjugated exponent of $\frac{Np}{N-p}$. This holds true provided that $\frac{Np}{N-3p} \ge 2$ and $p < \frac{N}{N-1}$, namely for $N = 6$ or $N = 7$. For greater values of N the proof relies on the same principles. □

We remark that the solution obtained by approximation in the above proof is unique. Indeed, let $f^{n,1}, f^{n,2} \in (L^2(\Omega))^d$ and $g^{n,1}, g^{n,2} \in (L^2(\Gamma_{\mathcal{N}}))^d$ be such that conditions (2.1)-(2.4) are fulfilled. Denote by $u^{n,1}$, respectively

$u^{n,2}$ the corresponding (unique) solutions in $(E^{1,2}(\Omega))^d$ of the problem (2.5). Since the sequence $(u^{n,1} - u^{n,2})$ is bounded in $(L^1(\Omega))^d$, it follows with the same arguments as in the above proof that

$$u^{n,1} - u^{n,2} \rightharpoonup 0 \qquad \text{weakly in } (E^{1,p}(\Omega))^d,\ \forall p < \tfrac{N}{N-1}$$

which implies the uniqueness of the solution obtained by approximation.

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