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GROUND STATE SOLUTIONS OF MAGNETIC SCHRÖDINGER EQUATIONS WITH EXPONENTIAL GROWTH

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Abstract. In this paper, we investigate the following nonlinear magnetic Schrödinger equation with exponential growth:

$$(-i\nabla + A(x))^2u + V(x)u = f(x, |u|^2)u \text{ in } \mathbb{R}^2,$$

where V is the electric potential and A is the magnetic potential. We prove the existence of ground state solutions both in the indefinite case with subcritical exponential growth and in the definite case with critical exponential growth. In order to overcome the difficulty brings from the presence of magnetic field, by using subtle estimates and establishing a new energy estimate inequality in complex field, we weaken the Ambrosetti-Rabinowitz type condition and the strict monotonicity condition, which are commonly used in the indefinite case. Furthermore, in the definite case, we introduce a Moser type function involving magnetic potential and some new analytical techniques, which can also be applied to related magnetic elliptic equations. Our results extend and complement the present ones in the literature.

1. **Introduction and main results.**

This paper is concerned with the following magnetic Schrödinger equation

$$\begin{cases} (-i\nabla + A(x))^2u + V(x)u = f(x, |u|^2)u, \\ u \in H^1(\mathbb{R}^2, \mathbb{C}), \end{cases} \tag{1}$$

where i is the imaginary unit and $f \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ is the reaction. We denote by $A : \mathbb{R}^2 \to \mathbb{R}^2$ the magnetic potential, and $V : \mathbb{R}^2 \to \mathbb{R}$ is the electric potential. These potentials are the sources of the electromagnetic field $(E, B) = (\nabla V, \nabla \times A)$.

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Problem (1) is the nonlinear stationary version of the following linear Schrödinger equation, which governs the nanoscopic world:

$$(-i\hbar\nabla + A(x))^2 u + V(x)u = i\hbar\partial_t u. \tag{2}$$

Several physical motivations lead to study problems (1) and (2), including superconductivity. This phenomenon occurs for certain materials at very low temperatures; they begin to enjoy an "infinite conductivity" and to hate magnetic fields. What is remarkable is that the only explanations of this observable phenomenon on a human scale are of a quantum order. At the heart of these phenomena, a model proposed by de Gennes makes it possible to understand surface superconductivity, see Fournais and Helffer [18].

We assume the following basic hypotheses:

(V1) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$, $V(x)$ is 1-periodic in x_1 and x_2, and

$$\sup[\sigma(-\Delta_A + V) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\Delta_A + V) \cap (0, \infty)],$$

where $-\Delta_A := (-i\nabla + A)^2$;

(A1) $A \in L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ with $p > 2$, and $B = \text{curl } A$ is 1-periodic in x_1, x_2 ;

(F1) $f \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f(x,t)$ is 1-periodic in x_1, x_2, and

$$\lim_{|t|\to\infty} \frac{f(x,t)t^{1/2}}{e^{\alpha t}} = 0, \text{uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > 0;$$

(F2) $\lim_{t\to 0^+} f(x,t) = 0$, uniformly on $x \in \mathbb{R}^2$.

Throughout this paper, we restrict the range of the variable t to the positive half line $t \geq 0$, since we will only consider the nonlinearity $f(x,t) = f(x, |u|^2)$. Hypothesis (V1) is associated with the *tunnel effect*, which appears when the electric potential V has some symmetries, see Helffer and Sjöstrand [20].

Equation (1) arises from looking for the standing waves solution $\psi(x,t) := e^{iEt/\hbar}u(x)(E \in \mathbb{R})$ of the nonlinear evolution equation:

$$i\hbar\frac{\partial\psi}{\partial t} = [-i\nabla - A(x)]^2 \psi + U(x)\psi - f(|\psi|^2)\psi, \qquad \text{in } \mathbb{R}^N \times \mathbb{R},$$

where $\hbar$ is Planck constant, $U(x)$ is a real electric potential and the nonlinear term and f is a superlinear function. The magnetic potential $A = (A_1, A_2, ..., A_N)$ is a source for the magnetic field $B := \text{curl}A$, where curl is the usual curl operator if $N = 3$ and $B = (B_{j,k})$, $1 \leq j, k \leq N$ with $B_{jk} = \partial_j A_k - \partial_k A_j$ for general N. Schrödinger equation is a fundamental assumption in quantum mechanics, which combines the concept of matter wave with the wave equation. The standing wave solution of the Schrödinger equation can be used to describe optical soliton in light, the motion of superconductors in magnetic fields, and physical phenomena such as the Bose-Einstein condensates, which plays a crucial role in theories of nonlinear optics, electromagnetism, superconductivity and so on. When sending Planck constant $\hbar$ to zero, it performs formally the transition from quantum mechanics to classical mechanics. Therefore, it is of great significance to investigate standing wave solution of Schrödinger equation due to its profound physical meaning and application value.

In the past decades, large quantities of excellent results were obtained which concern the existence, multiplicity and dynamical behavior for the Schrödinger equation without magnetic fields, namely $A \equiv 0$, see [14–16, 19, 24, 27, 29, 30, 33, 35, 37, 38, 40]. Meanwhile, the nonlinear magnetic Schrödinger equation aroused great interests to researchers recently for its relevance in semiconductor theory, condensed matter physics and plasma physics. To our knowledge, the first result concerning the magnetic Schrödinger equation was obtained by Esteban-Lions [17], where they proved the existence of stationary solutions when $N = 2, 3$ by using concentration-compactness lemma and solving appropriate minimization problems for the corresponding energy functional. Arioli-Szulkin [4] considered a semilinear stationary Schrödinger equation in a magnetic field:

$$(-i\nabla + A)^2 u + V(x)u = g(x, |u|)u, \qquad x \in \mathbb{R}^N. \tag{3}$$

By using constrained minimization and concentration-compactness technique when g is of critical growth, and employing a minimax argument when g is of subcritical growth, they obtained the existence of nontrivial solutions for (3) in both cases with the spectrum $\sigma(-\Delta_A + V) \subset (0, +\infty)$. For the semiclassical magnetic Schrödinger equation

$$\left(\frac{\varepsilon}{i}\nabla - A(z)\right)^2 u + V(z)u = f(|u|^2)u, \qquad z \in \mathbb{R}^N, \tag{4}$$

Alves-Figueiredo-Furtado [1] established the relationship between the number of solutions and the topology of the set where the potential attains its minimum value for (4) by combining variational methods, penalization techniques and Ljusternik-Schnirelmann theory, when the nonlinearity $f \in \mathcal{C}^1$ is of polynomial growth and the potential V satisfies

(V0) $V_0 := \inf_{z\in\mathbb{R}^N} V(z) > 0$ and there exists an open bounded set $\Omega \subset \mathbb{R}^N$ such that

$$V_0 < \min_{z\in\partial\Omega} V(z)$$

and $M := \{z \in \Omega : V(z) = V_0\} \neq \emptyset$.

While f is only continuous, the methods used in [1] become invalid, so Ji-Rădulescu [21, 23] developed new analytical techniques to obtain the existence and concentration of solutions for problem (4). We notice that the methods used in [1, 17, 21] require that $\inf_{x\in\mathbb{R}^N} V(x) > 0$ or $\sigma(-\Delta_A + V) \subset (0, \infty)$, which leads to that the quadratic part of the corresponding energy functional for (1) can be defined as a norm, with which they can derive easily the mountain-pass geometry. While $V(x)$ is sign-changing and the spectrum of the operator $-\Delta_A + V$ has a negative part, that is V satisfies (V1), the fundamental mountain-pass geometry of the corresponding energy functional cannot be derived, so the methods developed in [1, 4, 17, 21, 41] failed to deal with problem (1). In such situation, the energy functional associated with (1) is strongly indefinite near the origin (the indefinite case), which is more difficult and seldom investigated on the magnetic Schrödinger equation. It is natural to consider whether there exist nontrivial solutions for magnetic Schrödinger equation in the strongly indefinite case, and we will give an affirmative answer to this question in the present paper.

On the other hand, the real-valued indefinite Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x, u), \qquad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{5}$$

with (V1) when $A = 0$ is widely studied in the literature. An effective tool dealing with the strongly indefinite problem is generalized linking theorem, which is proposed by Kryszewski-Szulkin [24] and improved by Li-Szulkin [26] and Ding [14] later. Here we mention several papers which achieved outstanding results in such field and inspired our research on the problem (1). To obtain the ground state solutions for problem (5), Szulkin-Weth [35] developed a new approach, which is based on a direct and simple reduction of the indefinite variational problem to a definite one and derived a new minimax characterization of the corresponding critical value. They started their argument by showing that for each $u \in E\backslash E^-$, the set $\mathcal{N}$ intersects $\hat{E}(u)$ in exactly one point which is a unique global maximum point on $\Phi|_{\hat{E}(u)}$ where Φ is the corresponding energy functional of problem (5). Here $E := H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum; $\mathcal{N}$ is Nehari-Pankov manifold firstly introduced by Pankov [30] which is defined by

$$\mathcal{N} := \{u \in E\backslash E^- : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^-\},$$

and $\hat{E}(u) := E^- \oplus \mathbb{R}^+u$. In order to obtain this conclusion, under the strict monotonicity condition

(S1) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,

they proved firstly the following key inequality:

$$g(s) := f(x,u)\left[s\left(\frac{s}{2}+1\right)u + (1+s)v\right] + F(x,u) - F(x,z) < 0, \tag{6}$$

where $s \geq -1$, $u, v \in \mathbb{R}$, $z(s) := (1+s)u + v$, $F(x,t) = \int_0^t f(x,s)ds$ and $x \in \mathbb{R}^N$ is fixed. Later, by using non-Nehari manifold method and developing a more direct analytical technique, Tang [36] showed the existence of Nehari-Pankov type ground state solutions for (5) under a weaker condition than (S1). Through proving the following inequality,

$$\left(\frac{1-t^2}{2}\tau - t\sigma\right) f(x,\tau) \geq \int_{t\tau+\sigma}^{\tau} f(x,s)ds,\ x \in \mathbb{R}^N,\ \tau \neq 0,\ t \geq 0 \text{ and } \sigma \in \mathbb{R}, \tag{7}$$

he derives a key energy estimates inequality connecting $\Phi(u)$, $\langle\Phi'(u), u\rangle$ and $\langle\Phi'(u), w\rangle$ where $w \in E^-$, which is crucial in his proof process. For more results concerning real-valued indefinite Schrödinger equation, we refer the readers to [30, 37] and the references therein. Nevertheless, due to the existence of magnetic potential A, which makes (1) a complex-valued problem, the method dealing with real-valued indefinite Schrödinger equation cannot be applied to problem (1). More precisely, since complex numbers cannot be compared, we cannot derive the inequalities like (6) and (7) in [35, 36], which leads to a failure when dealing with problem (1) by using the methods in [35, 36]. Therefore, new tricks and techniques are required to investigate the complex-valued indefinite Schrödinger equation (1).

In addition, unlike most of the previous papers on the magnetic Schrödinger equations which focus on the polynomial growth of nonlinearities, in this paper, we consider the case when the nonlinearity is of subcritical exponential growth, that is f satisfies (F1). As is known to all, the Sobolev embedding yields $H^1(\mathbb{R}^2) \subset L^s(\mathbb{R}^2)$ for all $s \in [2, \infty)$ but $H^1(\mathbb{R}^2) \not\subseteq L^\infty(\mathbb{R}^2)$. Instead of Sobolev inequality, in dimension 2, Trudinger-Moser inequality established by Cao [7] is used to preserve the variational structure when the nonlinearity is of exponential growth

Lemma 1.1. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2, \mathbb{R})$, then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) < \infty.$$

Moreover, if $0 < \alpha < 4\pi$ *and* $u \in H^1(\mathbb{R}^2, \mathbb{R})$ *satisfies* $\|\nabla u\|_2^2 \le 1$ *and* $\|u\|_2 \le M < \infty$*, then there exists a constant* $C = C(M, \alpha)$ *such that*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx \le C(M, \alpha).$$

To the best of our knowledge, there are only a few papers focusing on the magnetic Schrödinger equation with exponential growth. By adopting an argument of penalization method and qualitative analysis, Ji-Rădulescu [22] showed the existence and multiplicity of multi-bump solutions for the nonlinear magnetic Schrödinger equation with critical exponential growth

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + Z(x))u = f(|u|^2)u, \qquad \text{in } \mathbb{R}^2.$$

More recently, d'Avenia-Ji [5] studied the following magnetic Schrödinger equation with critical exponential growth in $\mathbb{R}^2$,

$$(\frac{\varepsilon}{i}\nabla - A(x))^2 u + V(x)u = f(|u|^2)u, \qquad \text{in } \mathbb{R}^2.$$

By using penalization technique and Ljusternik-Schnirelmann theory, they proved multiplicity and concentration of solutions for ε small. It should be pointed out that the existing papers concerning the magnetic Schrödinger equation with exponential growth considered only the case where $V(x)$ is a positive potential bounded away from zero which is the so-called definite case. However, dealing with the indefinite case (1) is more complicated since it's difficult to show the boundedness and non-vanishing of (PS) sequence and the weakly sequential continuity of $\Psi(u) = \int_{\mathbb{R}^2} F(x, |u|^2)dx$ with the exponential growth nonlinearity when applying the generalized linking theorem. So far we have not find a paper investigating the indefinite magnetic Schrödinger equation with exponential growth.

Motivated by the works mentioned above, in the first part of present paper, we dedicate to study the existence of nontrivial solutions and Nehari-Pankov type ground state solutions for (1). There are several main obstacles we must overcome in the process. To begin with, compared with the definite case, the sign-changing and strongly indefinite potential $V(x)$ destroy the mountain-pass geometry of the corresponding energy functional, which makes it more difficult to show the existence and compactness of (PS) sequence. In addition, due to the presence of magnetic potential $A(x)$ which makes (1) a complex-valued problem, the methods dealing with real-valued indefinite Schrödinger equations developed in [35, 36] is invalid for (1). Furthermore, the nonlinearity with exponential growth causes several new difficulties in applying methods in dealing with our problem. Consequently, our problem is more complicated than the pattern investigated in [5, 21, 35]. Before stating our results, we present several assumptions on f:

(F3) $\lim_{t\to\infty} \frac{F(x,t)}{t} = \infty$ as $t \to \infty$ uniformly in $x \in \mathbb{R}^2$;

(F4) $f(x,t)t \ge F(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^2 \times (0, \infty)$, and there exist $T_0 > 0$ and $\gamma > 1$ such that

$$t \ge T_0 \Rightarrow f(x,t)t \ge \gamma F(x,t) > 0 \quad \text{and} \quad f(x,t) \ge \frac{1}{4\beta_2^2} \Rightarrow f(x,t)t - F(x,t) > 0,$$

where $\beta_2 > 0$ is defined in (2.1).

(F5) $f(x,t)$ is non-decreasing in t on $(0, \infty)$.

In the case of indefinite magnetic Schrödinger equations with subcritical exponential growth, our main results are the following.

Theorem 1.2. *Suppose that* (V1)*,* (A1)*,* (F1)-(F2) *and* (F4) *are satisfied. Then problem* (1) *has a nontrivial solution.*

Theorem 1.3. *Suppose that* (V1)*,* (A1)*,* (F1)-(F3) *and* (F5) *are satisfied. Then problem* (1) *has a ground state solution of Nehari-Pankov type.*

Remark 1.4. In our theorem, F is assumed to be superlinear growth at the infinity, which is weaker than the Ambrosetti-Rabinowitz condition commonly used in related literature such as in [5, 21, 22],

(AR) $0 < \mu F(x,t) \le 2f(x,t)t$ for some $\mu > 2$ and all $t \in \mathbb{R}\backslash\{0\}$, $x \in \mathbb{R}^N$,

which implies $F(x,t) \ge c|t|^{\mu/2} > 0$ for $|t| \ge 1$ and $x \in \mathbb{R}^N$. Furthermore, by subtle estimates, we establish a new inequality related to $I(u)$, $I(tu+v)$, $\langle I'(u), u\rangle$ and $\langle I'(u), v\rangle$ for $t \ge 0$, $u \in X$ and $v \in X^-$ (see Section 3), with which we can derive easily the boundedness of (PS) sequence and weaken the strict monotonicity condition to (F5).

In the next part, we consider the case that V and f are asymptotically periodic, i.e. V and f satisfy

(V1') $V(x) = V_0(x) + V_1(x)$, where V_0 and V_1 satisfy

i)$V_0 \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$, V_0 is 1-periodic in x_1, x_2, and

$$\sup[\sigma(-\Delta_A + V_0) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(-\Delta_A + V_0) \cap (0, \infty)];$$

ii) $V_1 \le 0$, $V_1 \in \Theta$, and $\|V_1\|_\infty < \frac{\Lambda}{2}$.

(F1') $f(x,t) = f_0(x,t) + f_1(x,t)$ where f_0 and f_1 satisfy

i) $f_0 \in \mathcal{C}^(\mathbb{R}^2 \times \mathbb{R})$, f_0 is 1-periodic in x_1, x_2, $f_0(x,t) = o(t)$ as $|t| \to 0$ uniformly in $x \in \mathbb{R}^2$, and $f_0(x,t)$ is nondecreasing in t on $[0, \infty)$.

ii) $f_1 \in \mathcal{C}(\mathbb{R}^2, [0, \infty))$ and there exists $q > 2$, $a \in \Theta$ such that $f_1(x,t) \le a(x)\left[1 + |t|^{q-2}\left(e^{\alpha t} - 1\right)\right]$, for all $\alpha > 0$. Moreover, $-V_1(x)t + F_1(x,t) > 0$ for $|x| < 1 + \sqrt{2}$ and $t \ne 0$.

Here, Θ denotes the class of functions $g \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}) \cap L^\infty(\mathbb{R}^2, \mathbb{R})$ such that for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^2 : |g(x)| \ge \varepsilon\}$ has finite Lebesgue measure. Compared with the case $V(x)$ is periodic, dealing with the asymptotically periodic magnetic Schrödinger equation is more difficult, since the usual method to recover the compactness of (PS) sequence of the corresponding energy functional of (1) is combining concentration compactness lemma with a new "translation" Υ (see Section 2), with which it can be shown that the energy functional is invariant with respect to the action of $\mathbb{Z}^N$, which is valid only for the periodic case. Furthermore, due to the different structure on $V(x)$ compared with (V1), a new working space should be defined in this case, which causes several extra difficulties. As far as we know, there are few results on the existence of ground state solutions to (1) when $V(x)$ is asymptotically periodic. Therefore, extra efforts are needed to overcome the difficulties caused by the dropping of periodicity of $V(x)$.

When the potential $V(x)$ is asymptotically periodic and nonlinearity is of subcritical exponential growth, we have the following result.

Theorem 1.5. *Suppose that* (V1')*,* (A1)*,* (F1')*,* (F1)-(F3) *and* (F5) *are satisfied. Then problem* (1) *has a ground state solution of Nehari-Pankov type.*

In the last part of present paper, we consider the case where $V(x)$ is a positive potential and the nonlinearity f is of critical exponential growth, i.e. V and f satisfy

(V2) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}^+)$, $V(x)$ is 1-periodic in x_1, x_2, and $\sigma(-\Delta_A + V) \in (0, +\infty)$;

(F1") $f \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f(x,t)$ is 1-periodic in x_1, x_2, and there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{f(x,t)t^{1/2}}{e^{\alpha t}} = 0 \ (+\infty), \text{uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > \alpha_0 \ (\alpha < \alpha_0);$$

respectively. By virtue of Trudinger-Moser inequality, the existence of nontrivial solutions for real-valued Schrödinger equation with critical exponential growth has been studied widely, see [2, 3, 12, 13] and references therein. The main difficulties caused by the nonlinearity of critical exponential growth are showing that the Cerami sequence or the minimizing sequence $\{u_n\}$ is non-vanishing and proving that the weak limit $\tilde{u}$ of $\{u_n\}$ is a solution of the original equation. To address these obstacles, the Moser type function $w_n(x)$ supported in $B(0,\rho)$ is defined in [13] as follows

$$w_n(x) = \begin{cases} \frac{\sqrt{\log n}}{\sqrt{2\pi}}, & 0 \le |x| \le \rho/n, \\ \frac{\log(\rho/|x|)}{\sqrt{2\pi \log n}}, & \rho/n \le |x| \le \rho, \\ 0, & |x| \ge \rho, \end{cases}$$

with which one can show that the Cerami sequence does not vanish when the minimax-level is less than the threshold. This method is then applied widely in related literatures such as [8, 10, 31] and so on. Nevertheless, unlike in the real-valued Schrödinger equation case where one can easily deduce that $w_n \in H^1(\mathbb{R}^2)$ and $\|\nabla w_n\|_2 = 1$ by a direct computation, which is significant in the proof process, it is difficult to show that Moser's function belongs to the usual working space involving the magnetic potential for problem (1)

$$H^1_A(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\},$$

where $\nabla_A u := (\nabla + iA)u$. This leads to a failure when applying directly the method developed in [13] to problem (1). Moreover, for the magnetic Schrödinger equation with critical exponential growth, in [6], by exploiting the ideas dealing with related scalar problem and applying Lagrange multipliers theorem, Barile-Figueiredo showed the existence of a complex solution to (1) with $f(x,|u|^2) = f(|u|^2)$ under the conditions (F1"), (F2) and

(S2) $f(s)s - F(s) \ge 0$ for every $s \in \mathbb{R}$ with $s \ge 0$ where $F(s) = \int_0^s f(t)t$.

(S3) there exist $\lambda > 0$ and $q \in (2,\infty)$ such that $f(s) \ge \lambda s^{(q-2)/2}$ for every $s \in \mathbb{R}$ with $s \ge 0$.

It should be emphasized that the assumption (S3) is essential in their arguments, with which the minimax-level for the corresponding energy functional can be chosen small enough such that the main difficulty caused by the critical exponential growth can be overcome. However, their methods and results do not reveal the essential characteristic of the critical exponential growth since there is no relationship between their assumptions and the exponential growth velocity α_0 in (F1"). In the present paper, we intend to adopt a direct way to obtain the existence of ground state solution for (1) under the assumption

(F6) $\liminf_{t\to\infty} \frac{tF(x,t)}{e^{\alpha_0 t}} \ge \nu > \frac{4}{\alpha_0 \rho^2}$ uniformly on $x \in \mathbb{R}^2$, where ρ is a positive constant satisfying $\rho < \sqrt{\|V\|_\infty + 4\zeta^2}$ and $\zeta > 0$ is a positive constant defined in (A2), which takes account of the exponential growth velocity α_0. Besides (F6), we assume that A and f satisfy :

(A2) $A \in L^\infty(B(0,\rho),\mathbb{R}^2)$ with $\zeta := \operatorname{ess\,sup}_{x\in B(0,\rho)} |A(x)|$.

(F7) there exists $\mu > 2$ such that

$$2tf(x,t) \geq \mu F(x,t) > 0, \qquad \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+;$$

(F8) there exists $M_0 > 0$ and $T_1 > 0$ such that

$$F(x,t) \leq M_0|f(x,t)|, \qquad \forall\ |t| \geq T_1,\ x \in \mathbb{R}^2.$$

In the process of proving that the (PS) sequence of the corresponding energy functional for (1) is nonvanishing, we introduce a Moser type function involving the magnetic potential, which can be also applied to related magnetic elliptic equations. We are now in a position to state our last result of the present paper.

Theorem 1.6. *Suppose that* (V2), (A1)-(A2), (F1"), (F2) *and* (F5)-(F8) *are satisfied. Then problem* (1) *has a ground state solution of Nehari type.*

This paper is organized as follows. In Section 2, we give the variational setting and preliminaries and the proof of Theorem 1.2. Theorem 1.3 and Theorems 1.5-1.6 are proved respectively in Sections 3-5.

Notation

- $\|\cdot\|_p$ denotes the usual norm of $L^p(\mathbb{R}^N,\mathbb{C})$ for $p \in [1,\infty]$;
- $B(x,R)$ denotes the ball centered at x with the radius R;
- C_i $(i = 1,2,...)$ denote positive constants which may be different in different places.

2. Variational setting and preliminaries.

2. Variational setting and preliminaries. To establish the variational structure of problem (1), we set

$$H^1_A(\mathbb{R}^2,\mathbb{C}) := \left\{u \in L^2(\mathbb{R}^2,\mathbb{C}) :\ \nabla_A u \in L^2(\mathbb{R}^2,\mathbb{C})\right\},$$

where $\nabla_A u := (\nabla + iA)u$. $H^1_A(\mathbb{R}^2,\mathbb{C})$ is a Hilbert space endowed with inner product

$$\langle u,v\rangle_A := Re\left\{\int_{\mathbb{R}^3} (\nabla_A u\overline{\nabla_A v} + u\bar{v})dx\right\},$$

where $Re(w)$ denotes the real part of $w \in \mathbb{C}$ and $\bar{w}$ denotes its conjugate, and $\mathcal{C}_0^\infty(\mathbb{R}^2,\mathbb{C})$ is dense in $H^1_A(\mathbb{R}^2,\mathbb{C})$ with respect to the norm $\|u\|_A = \langle u,u\rangle_A^{1/2}$ (see [17]). Noting that $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2})$ for any $z_1,z_2 \in \mathbb{C}$, we have

$$\int_{\mathbb{R}^2} |\nabla_A u|^2 dx = \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |A(x)|^2|u|^2 dx - 2Re\int_{\mathbb{R}^2} \nabla u \cdot iA(x)\bar{u}dx,$$

which implies that $H^1_A(\mathbb{R}^2,\mathbb{C})$ and $H^1(\mathbb{R}^2,\mathbb{C})$ are incomparable, that is in general $H^1(\mathbb{R}^2,\mathbb{C}) \not\subseteq H^1_A(\mathbb{R}^2,\mathbb{C})$ and $H^1_A(\mathbb{R}^2,\mathbb{C}) \not\subseteq H^1(\mathbb{R}^2,\mathbb{C})$. The following diamagnetic inequality established by Lieb-Loss [28] is a crucial tool to deal with magnetic elliptic problems

$$|\nabla|u|(x)| = \left|Re\left(\nabla u\frac{\bar{u}}{|u|}\right)\right| = \left|Re\left((\nabla u + iAu)\frac{\bar{u}}{|u|}\right)\right| \leq |\nabla_A u(x)|, \text{ for a.e. } x \in \mathbb{R}^2,$$

by which $|u| \in H^1(\mathbb{R}^2,\mathbb{R})$ if $u \in H^1_A(\mathbb{R}^2,\mathbb{C})$ and the embedding $H^1_A(\mathbb{R}^2,\mathbb{C}) \hookrightarrow L^q(\mathbb{R}^2,\mathbb{C})$ is continuous for $2 \leq q < \infty$ and locally compact for $1 \leq q < \infty$.

Let $\mathcal{L} := -\Delta_A + V$. Since $V \in \mathcal{C}(\mathbb{R}^2,\mathbb{R}) \cap L^\infty(\mathbb{R}^2,\mathbb{R})$ due to (V1), $\mathcal{L}$ is a self-adjoint operation with domain $\mathcal{D}(\mathcal{L}) = H^2(\mathbb{R}^2,\mathbb{C})$ (see [16]). Let $\{E(\lambda) : -\infty \leq$

$\lambda \le +\infty\}$ and $|\mathcal{L}|$ be the spectral family and the absolute value of $\mathcal{L}$ respectively and denote the square root of $|\mathcal{L}|$ by $|\mathcal{L}|^{1/2}$. Set $\mathcal{L} = U|\mathcal{L}|$ the polar decomposition of $\mathcal{L}$, where $U = id - E(0) - E(0^-)$ commuting with $\mathcal{L}$, $|\mathcal{L}|$, and $|\mathcal{L}|^{1/2}$. Let $X = \mathcal{D}(|\mathcal{L}|^{1/2})$ and define

$$X^- = E(0^-)X, \qquad X^0 = [E(0) - E(0^-)]X \qquad \text{and} \qquad X^+ = [id - E(0)]X.$$

Then for any $u \in X$, we have $u = u^- + u^0 + u^+$, where

$$u^- = E(0^-)u \in X^-, \;\; u^0 = [E(0) - E(0^-)]u \in X^0 \text{ and } u^+ = [id - E(0)]u \in X^+.$$

Under the assumption (V1), $X^0 = \{0\} = ker\{\mathcal{L}\}$, then $X = X^- \oplus X^+$ is a Hilbert space with the inner product and the corresponding norm defined by

$$\langle u, v\rangle = \langle |\mathcal{L}|^{1/2}u, |\mathcal{L}|^{1/2}v\rangle_2, \;\; u, v \in X$$

and

$$\|u\| = \langle u, u\rangle^{1/2}, \;\; u \in X$$

respectively. For any $u \in X$, it is easy to verify that $u = u^- + u^+$ and

$$\mathcal{L}u = \pm|\mathcal{L}|u, \qquad \forall\, u \in X^\pm \cap \mathcal{D}(\mathcal{L}).$$

In such situation, $X = H^1_A(\mathbb{R}^2, \mathbb{C})$ with the equivalent norm in $H^1_A(\mathbb{R}^2, \mathbb{C})$ (see [4, 34]), which means there exist $\mathcal{C}_1$, $\mathcal{C}_2$ and $\beta_q > 0$ such that

$$\mathcal{C}_1\|u\| \le \|u\|_A \le \mathcal{C}_2\|u\| \qquad \text{and} \qquad \|u\|_q \le \beta_q\|u\|, \qquad \forall\, u \in X, \; q \in [2, \infty). \tag{8}$$

Under assumptions (V1), (F1) and (F2), the corresponding energy functional of problem (1) can be defined as:

$$I(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |u|^2), \qquad \forall\, u = u^+ + u^- \in X.$$

Using (F1) and (F2), for any $\varepsilon > 0$ and $\alpha > 0$, there exists $C_\varepsilon > 0$ such that for any $(x, t) \in (\mathbb{R}^2, \mathbb{R}^+)$,

$$|f(x, t)| \le \varepsilon + C_\varepsilon t^{-1/2}(e^{\alpha t} - 1) \;\text{ and }\; |F(x, t)| \le \varepsilon t + C_\varepsilon t^{1/2}(e^{\alpha t} - 1). \tag{9}$$

In view of Lemma 1.1 and (9), $I \in \mathcal{C}^1(X, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \langle u^+, v^+\rangle - \langle u^-, v^-\rangle - Re\int_{\mathbb{R}^2} f(x, |u|^2)u\bar{v}dx, \qquad \forall\, u, v \in X.$$

When $\sigma(\mathcal{L}) \subset (0, +\infty)$ and the nonlinearity f is of polynomial growth, it's standard to achieve the mountain-pass geometry of the functional I. If instead, $\sigma \cap (-\infty, 0) \neq \emptyset$, we shall use the following lemma to obtain the infinite dimensional linking geometry.

Lemma 2.1 ([4, 24]). *Let E be a Hilbert space and suppose that $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the following hypotheses:*
(i) $I(u) = \frac{1}{2}\langle Lu, u\rangle - \psi(u)$, where L is a bounded self-adjoint linear operator, ψ is bounded below, weakly sequentially lower semicontinuous and $\nabla\psi$ is weakly sequentially continuous;
(ii) there exists a closed separable L-invariant subspace Y such that the quadratic form $u \mapsto \langle Lu, u\rangle$ is negative definite on Y and positive semidefinite on $Y^\perp$;
(iii) there are constants $b, \rho > 0$ such that $I|S_\rho \cap Y^\perp \ge b$, where $S_\rho := \{u \in E : \|u\| = \rho\}$;
(iv) there is $z_0 \in S_1 \cap Y^\perp$ and $R > \rho$ such that $I|\partial M \le 0$, where $M := \{u =$

$y + \lambda z_0 : y \in Y, \|u\| < R, \lambda > 0\}$.
Then there exists a sequence $\{u_n\}$ *such that*

$$\nabla I(u_n) \to 0 \qquad \text{and} \qquad I(u_n) \to c, \tag{10}$$

for some $c \in [b, sup_{\bar{M}} I]$.

Lemma 2.2. *If* (V1), (F1) *and* (F2) *are satisfied, then* $\psi(u)$ *is bounded below, weakly lower semicontinuous and* $\nabla\psi$ *is weakly sequentially continuous, where*

$$\psi(u) := \frac{1}{2}\int_{\mathbb{R}^2} F(x, |u|^2)dx.$$

Proof. We only prove that $\nabla\psi$ is weakly sequentially continuous here since the proof of other parts is standard. Note that

$$(\nabla\psi(u), v) = Re\int_{\mathbb{R}^2} f(x, |u|^2)u\bar{v}dx, \qquad \forall\ u, v \in X.$$

Let $u_n \rightharpoonup u$ in X, then $\|u_n\| \le C_1$ for some constants $C_1 > 0$. For any $\varepsilon > 0$, the decay of integral implies that there exists $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^2\setminus B(0,R_\varepsilon)} |v|^2dx < \varepsilon^2.$$

Let $\alpha \in (0, 2\pi/C_1^2\mathcal{C}_2^2)$, then it follows from Lemma 1.1, (9) and Hölder inequality that

$$\begin{aligned}
&\left|Re\int_{\mathbb{R}^2\setminus B(0,R_\varepsilon)} f(x, |u_n|^2)u_n\bar{v}dx\right| \\
&\le \int_{\mathbb{R}^2\setminus B(0,R_\varepsilon)} |f(x, |u_n|^2)||u_n||v|dx \\
&\le \left\{\left(\int_{\mathbb{R}^2} |u_n|^2dx\right)^{\frac{1}{2}} + C_2\left[\int_{\mathbb{R}^2}\left(e^{\alpha|u_n|^2} - 1\right)^2 dx\right]^{\frac{1}{2}}\right\}\left[\int_{\mathbb{R}^2\setminus B(0,R_\varepsilon)} |v|^2dx\right]^{\frac{1}{2}} \\
&\le \left\{\|u_n\|_2^2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\mathcal{C}_2^2\|u_n\|^2(|u_n|/\mathcal{C}_2\|u_n\|)^2} - 1\right)dx\right]^{\frac{1}{2}}\right\}\left[\int_{\mathbb{R}^2\setminus B(0,R_\varepsilon)} |v|^2dx\right]^{\frac{1}{2}} \\
&\le C_3\varepsilon,
\end{aligned} \tag{11}$$

where we use the fact $(e^m - 1)^n \le e^{mn} - 1$ for $m \ge 0$ and $n > 1$. Similarly, we can derive

$$\left|Re\int_{\mathbb{R}^2\setminus B(0,R_\varepsilon)} f(x, |u|^2)u\bar{v}dx\right| \le C_3\varepsilon.$$

Moreover, the absolute continuity of integrals implies that there exists $\delta > 0$ such that

$$\int_A |v|^2dx < \varepsilon^2$$

for any $A \subset B(0, R_\varepsilon)$ with $meas(A) < \delta$. Since $\{u_n\}$ is bounded in X, there exists $M_\varepsilon > 0$ such that

$$meas(\Omega_n[M_\varepsilon, \infty)) \le \delta \text{ and } meas(\Omega[M_\varepsilon, \infty)) \le \delta,$$

where $\Omega_n[M_\varepsilon,\infty) := \{x \in B(0,R_\varepsilon) : |u_n(x)| \geq M_\varepsilon\}$ and $\Omega[M_\varepsilon,\infty) := \{x \in B(0,R_\varepsilon) : |u(x)| \geq M_\varepsilon\}$. Set $\Omega(M_\varepsilon) := \{x \in B(0,R_\varepsilon) : |u(x)| = M_\varepsilon\}$, then through a similar argument of (11), one has

$$\begin{aligned}&\left|Re\int_{\Omega_n[M_\varepsilon,\infty)\cup\Omega(M_\varepsilon)} f(x,|u_n|^2)u_n\bar{v}dx\right|\\ &\leq \left\{\left(\int_{\mathbb{R}^2}|u_n|^2dx\right)^{\frac{1}{2}} + C_2\left[\int_{\mathbb{R}^2}\left(e^{\alpha|u_n|^2}-1\right)^2dx\right]^{\frac{1}{2}}\right\}\left[\int_{\Omega_n[M_\varepsilon,\infty)\cup\Omega(M_\varepsilon)}|v|^2dx\right]^{\frac{1}{2}} \leq C_4\varepsilon,\end{aligned}$$

and

$$\left|Re\int_{\Omega(M_\varepsilon)} f(x,|u_n|^2)u_n\bar{v}dx\right| \leq C_4\varepsilon.$$

Since $u_n \to u$ in $L^s_{loc}(\mathbb{R}^2)$ for $2 \leq s < \infty$ and $u_n \to u$ a.e. on $\mathbb{R}^2$, then

$$f(x,|u_n|^2)u_n\bar{v}\chi_{|u_n|\leq M_\varepsilon} \to f(x,|u|^2)u\bar{v}\chi_{|u|\leq M_\varepsilon}, \text{ a.e. in } B(0,R_\varepsilon)\backslash\Omega(M_\varepsilon).$$

Moreover,

$$\left|f(x,|u_n|^2)u_n\bar{v}\chi_{|u_n|\leq M_\varepsilon}\right| \leq |v| \max_{x\in B(0,R_\varepsilon),|t|\leq M_\varepsilon} f(x,t^2)t, \qquad \forall x \in B(0,R_\varepsilon).$$

Therefore, the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\int_{B(0,R_\varepsilon)\backslash(\Omega_n[M_\varepsilon,\infty)\cup\Omega(M_\varepsilon))} f(x,|u_n|^2)u_n\bar{v} = \lim_{n\to\infty}\int_{B(0,R_\varepsilon)\backslash\Omega[M_\varepsilon,\infty)} f(x,|u|^2)u\bar{v}dx.$$

Noting that $|Re(z_1-z_2)| \leq |z_1-z_2|$ for any z_1, $z_2 \in \mathbb{C}$, one has

$$\begin{aligned}&\lim_{n\to\infty} Re\int_{B(0,R_\varepsilon)\backslash(\Omega_n[M_\varepsilon,\infty)\cup\Omega(M_\varepsilon))} f(x,|u_n|^2)u_n\bar{v}\\ &= \lim_{n\to\infty} Re\int_{B(0,R_\varepsilon)\backslash\Omega[M_\varepsilon,\infty)} f(x,|u|^2)u\bar{v}dx.\end{aligned}$$

Taking account of all the estimates above, we can derive

$$\begin{aligned}&\lim_{n\to\infty}\left|Re\left(\int_{\mathbb{R}^2} f(x,|u_n|^2)u_n\bar{v}dx - \int_{\mathbb{R}^2} f(x,|u|^2)u\bar{v}dx\right)\right|\\ &\leq \lim_{n\to\infty}\left|Re\int_{\mathbb{R}^2\backslash B(0,R_\varepsilon)} f(x,|u_n|^2)u_n\bar{v}dx - Re\int_{\mathbb{R}^2\backslash B(0,R_\varepsilon)} f(x,|u|^2)u\bar{v}dx\right|\\ &\quad + \lim_{n\to\infty}\left|Re\int_{B(0,R_\varepsilon)\backslash(\Omega_n[M_\varepsilon,\infty)\cup\Omega(M_\varepsilon))} f(x,|u_n|^2)u_n\bar{v}dx\right.\\ &\qquad\qquad \left. - Re\int_{B(0,R_\varepsilon)\backslash\Omega[M_\varepsilon,\infty)} f(x,|u|^2)u\bar{v}dx\right|\\ &\quad + \lim_{n\to\infty}\left|Re\int_{\Omega_n[M_\varepsilon,\infty)\cup\Omega(M_\varepsilon)} f(x,|u_n|^2)u_n\bar{v}dx - Re\int_{\Omega[M_\varepsilon,\infty)} f(x,|u|^2)u\bar{v}dx\right|\\ &\leq C_5\varepsilon.\end{aligned}$$

Due to the arbitrariness of ε, we have

$$\lim_{n\to\infty}\left|Re\left(\int_{\mathbb{R}^2} f(x,|u_n|^2)u_n\bar{v}dx - \int_{\mathbb{R}^2} f(x,|u|^2)u\bar{v}dx\right)\right| = 0,$$

which implies

$$\lim_{n\to\infty}(\nabla\psi(u_n),v) = (\nabla\psi(u),v).$$

Hence $\nabla\psi$ is weakly sequentially continuous. □

Lemma 2.3. *If* (V1)*,* (F1) *and* (F2) *are satisfied, then there exist constants $b,\rho>0$ such that $I|S_\rho\cap X^+\geq b$.*

Proof. By (F1) and (F2), fixing $q>2$, for any $\alpha,\varepsilon>0$, there exists constant $C_6>0$ such that

$$|F(x,t)|\leq \varepsilon|t|+C_6|t|^{q/2}(e^{\alpha t}-1),\qquad \forall\ t\geq 0. \tag{12}$$

It follows from Lemma 1.1, (12) and Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^2}|F(x,|u|^2)|dx&\leq \frac{1}{2\beta_2^2}\|u\|_2^2+C_7\left(\int_{\mathbb{R}^2}|u|^{2q}dx\right)^{\frac12}\left[\int_{\mathbb{R}^2}\left(e^{2\alpha|u|^2}-1\right)dx\right]^{1/2}\\&\leq \frac12\|u\|^2+C_7\|u\|_{2q}^q\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\mathcal{C}_2^2\|u\|^2(|u|/\mathcal{C}_2\|u\|)^2}-1\right)dx\right]^{1/2}\\&\leq\frac12\|u\|^2+C_8\|u\|^q,\qquad \forall\ u\in X^+,\ \|u\|<\frac{\sqrt{2\pi/\alpha}}{\mathcal{C}_2},\end{aligned}$$

which implies

$$I(u)=\frac12\|u\|^2-\frac12\int_{\mathbb{R}^2}F(x,|u|^2)dx\geq\frac14\|u\|^2-\frac{C_8}{2}\|u\|^q,\ \forall\ u\in X^+,\|u\|<\frac{\sqrt{2\pi/\alpha}}{\mathcal{C}_2}.$$

Therefore, there exists $b>0$ and $0<\rho<\frac{\sqrt{2\pi/\alpha}}{\mathcal{C}_2}$ such that $I|S_\rho\cap X^+\geq b$. □

Lemma 2.4. *If* (V1)*,* (F1)*-*(F2) *and* (F4) *are satisfied, there exists $z_0\in S_1\cap X^+$ and $R_1>\rho$ such that $I|\partial M\leq 0$, where*

$$M:=\{u=y+\lambda z_0: y\in X^-,\|u\|<R_1,\lambda>0\}.$$

Proof. Arguing by contradiction, for any $z_0\in S_1\cap X^+$, we assume that there exists $y_n\in X^-$ and $\lambda_n>0$ such that $\|y_n+\lambda_n z_0\|\to\infty$ with $I(y_n+\lambda_n z_0)>0$. Let $u_n=y_n+\lambda_n z_0$ and $v_n=\frac{u_n}{\|u_n\|}:=v_n^-+t_nz_0$, then $\|v_n\|=1$ and there exists $v\in X$ and $t>0$ such that up to a subsequence, $v_n\rightharpoonup v$ in X, $v_n\to v$ in $L^s_{loc}(\mathbb{R}^2,\mathbb{C})$ for $2\leq s<\infty$ and $v_n\to v$ a.e. on $\mathbb{R}^2$ with $t_n\to t$, which follows that

$$\begin{aligned}0\leq\frac{I(u_n)}{\|u_n\|^2}&=\frac{\lambda_n^2\|z_0\|^2}{2\|u_n\|^2}-\frac{\|y_n\|^2}{2\|u_n\|^2}-\frac{1}{2\|u_n\|^2}\int_{\mathbb{R}^2}F(x,|u_n|^2)dx\\&=\frac{t_n^2}{2}\|z_0\|^2-\frac12\|v_n^-\|^2-\frac12\int_{\mathbb{R}^2}\frac{F(x,|u_n|^2)}{\|u_n\|^2}dx.\end{aligned}$$

Next, we discuss in two cases: (1) $t=0$; (2) $t\neq 0$.

Case (1). $t=0$, by (F4), one has

$$0\leq\frac12\|v_n^-\|^2+\frac12\int_{\mathbb{R}^2}\frac{F(x,|u_n|^2)}{\|u_n\|^2}dx\leq\frac{t_n^2}{2}\|z_0\|^2\to 0.$$

This implies $\|v_n^-\|^2\to 0$ and $1=\|v_n^-+t_nz_0\|^2\to 0$, which is a contradiction.

Case (2). $t\neq 0$, then for a.e. $x\in\{y\in\mathbb{R}^2,v(y)\neq 0\}$, one has $\lim_{n\to\infty}|u_n(x)|=\infty$, which, together with (F4), implies that

$$\begin{aligned}0&\leq\overline{\lim_{n\to\infty}}\left[\frac{t_n^2}{2}\|z_0\|^2-\frac12\|v_n^-\|^2-\frac12\int_{\mathbb{R}^2}\frac{F(x,|u_n|^2)}{|u_n|^2}|v_n|^2dx\right]\\&\leq\frac{t^2}{2}\|z_0\|^2-\frac12\int_{\mathbb{R}^2}\underline{\lim_{n\to\infty}}\frac{F(x,|u_n|^2)}{|u_n|^2}|v_n|^2dx=-\infty,\end{aligned}$$

also a contradiction. The proof is now complete. $\square$

Lemma 2.5. *If* (V1), (F1)-(F2) and (F4) *are satisfied, then any sequence* $\{u_n\} \subset X$ *satisfying* (10) *is bounded.*

Proof. Let $\{u_n\} \subset X$ satisfying (10), then

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^2}[f(x,|u_n|^2)|u_n|^2 - F(x,|u_n|^2)]dx := \frac{1}{2}\int_{\mathbb{R}^2}\tilde{F}(x,|u_n|^2)dx. \end{aligned} \tag{13}$$

Arguing by contradiction, we assume that $\|u_n\| \to \infty$. (F1) and (F4) show that there exist positive constants $\mathcal{C}_3$, α and R such that

$$|f(x,t)t^{1/2}| \le \mathcal{C}_3 e^{\alpha t}, \quad \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \tag{14}$$

and

$$|f(x,t)t^{1/2}| \ge \mathcal{C}_3 e^{1/4}, \quad \forall\ x \in \mathbb{R}^2, t \ge R. \tag{15}$$

Let $R' = \max\{T_0, R\}$ and define collections Ω_1 and Ω_2 as follows:

$$\Omega_1 := \left\{(x,t) \in \mathbb{R}^2 \times [-R', R'] :\ f(x,t) < \frac{1}{4\beta_2^2}\right\}$$

and

$$\Omega_2 := \left\{(x,t) \in \mathbb{R}^2 \times [-R', R'] :\ f(x,t) \ge \frac{1}{4\beta_2^2}\right\}.$$

By virtue of (F1) and (F4), there exists a constant $\mathcal{C}_4$ such that

$$|f(x,t)t^{\frac{1}{2}}|^2 \le \mathcal{C}_4 \tilde{F}(x,t), \qquad \forall\ (x,t) \in \Omega_2. \tag{16}$$

Using (13) and (F4), one has

$$\begin{aligned} c + o(1) =& \frac{1}{2}\int_{\mathbb{R}^2}[f(x,|u_n|^2)|u_n|^2 - F(x,|u_n|^2)]dx \\ \ge& \frac{1}{2}\int_{|u_n| \le R'}\tilde{F}(x,|u_n|^2) + \frac{\gamma-1}{2\gamma}\int_{|u_n|>R'} f(x,|u_n|^2)|u_n|^2. \end{aligned} \tag{17}$$

As proved in [11, Lemma 2.4], for all $t \ge 0$ and $s \ge e^{1/4}$, there holds

$$st \le (e^{t^2} - 1) + s(\log s)^{1/2}. \tag{18}$$

Now, let $t = |u_n^+|/(\mathcal{C}_2\|u_n\|)$ and $s = |f(x,|u_n|^2)||u_n|/\mathcal{C}_3$ in (18), then combining (14), (17) and (18) , one has

$$\begin{aligned} &\frac{1}{\|u_n\|^2}\int_{|u_n|>R'}|f(x,|u_n|^2)u_n u_n^+|dx \\ \le& \frac{1}{\|u_n\|}\left\{\mathcal{C}_2\mathcal{C}_3\int_{|u_n|>R'}\left(e^{(|u_n^+|/\mathcal{C}_2\|u_n\|)^2} - 1\right)dx \right. \\ &\left. + \mathcal{C}_2\int_{u_n|>R'}|f(x,|u_n|^2)||u_n|\left[\log\left(\frac{1}{\mathcal{C}_3}|f(x,|u_n|^2)||u_n|\right)\right]^{1/2}\right\} \\ \le& \frac{1}{\|u_n\|}\left(C_9 + \sqrt{\alpha}\mathcal{C}_2\int_{|u_n|>R'} f(x,|u_n|^2)|u_n|^2 dx\right) \end{aligned}$$

$$\leq \frac{C_9 + 2\sqrt{\alpha}\mathcal{C}_2\gamma c/(\gamma-1) + o(1)}{\|u_n\|} = o(1).$$

Similarly, we have

$$\frac{1}{\|u_n\|^2}\int_{|u_n|>R'} |f(x,|u_n|^2)u_n u_n^-|dx = o(1).$$

Furthermore, it follows from (9), (13) and (16) that

$$\frac{1}{\|u_n\|^2}\int_{\Omega_1} |f(x,|u_n|^2)u_n u_n^+|dx \leq \frac{1}{4\beta_2^2\|u_n\|^2}\int_{\Omega_1} |u_n||u_n^+|dx \leq \frac{\|u_n\|_2\|u_n^+\|_2}{4\beta_2^2\|u_n\|^2} \leq \frac{1}{4},$$

and

$$\begin{aligned}\frac{1}{\|u_n\|^2}\int_{\Omega_2} |f(x,|u_n|^2)||u_n||u_n^+|dx \leq & \frac{\sqrt{\mathcal{C}_4}}{\|u_n\|^2}\left(\int_{\Omega_2}\tilde{F}(x,|u_n|^2)\right)^{1/2}\|u_n^+\|_2 \\ \leq & \frac{\sqrt{2c\mathcal{C}_4}\beta_2 + o(1)}{\|u_n\|} = o(1).\end{aligned}$$

Through the same argument as above, we can obtain

$$\frac{1}{\|u_n\|^2}\int_{\Omega_1} |f(x,|u_n|^2)u_n u_n^-|dx \leq \frac{1}{4} \quad \text{and} \quad \frac{1}{\|u_n\|^2}\int_{\Omega_2} |f(x,|u_n|^2)u_n u_n^-|dx \leq o(1).$$

Collecting the estimates above, we can derive

$$\begin{aligned}1 + o(1) = & \frac{\|u_n\|^2 - \langle I'(u_n), u_n^+ - u_n^-\rangle}{\|u_n\|^2} \\ \leq & \frac{1}{\|u_n\|^2}\left[\int_{\mathbb{R}^2} |f(x,|u_n|^2)||u_n||u_n^-|dx + \int_{\mathbb{R}^2} |f(x,|u_n|^2)||u_n||u_n^+|dx\right] \\ = & \frac{1}{\|u_n\|^2}\left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{|u_n|>R}\right)\left[|f(x,|u_n|^2)||u_n||u_n^+| + |f(x,|u_n|^2)||u_n||u_n^-|\right]dx \\ \leq & \frac{1}{2} + o(1),\end{aligned}$$

which is a contradiction, so $\{u_n\}$ is bounded. □

Proof of Theorem 1.2. In view of Lemmas 2.1-2.5, there exists a sequence $\{u_n\} \in X$ satisfying (10) and a constant $C_{10} > 0$ such that $\|u_n\| \leq C_{10}$. Assume

$$\delta := \overline{\lim_{n\to\infty}} \sup_{y\in\mathbb{R}^2} \int_{B(y,1)} |u_n|^2 dx = 0.$$

Then Lions' concentration compactness lemma implies $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Let $\alpha \in (0, 4\pi/\mathcal{C}_2^2 C_{10}^2)$, using (F1) and (F2), we have that for fixed $q > 2$, there exists $C_{11} > 0$ such that

$$f(x,t) \leq \frac{c}{C_{10}^2\beta_2^2} + C_{11}|t|^{(q-2)/2}(e^{\alpha t} - 1),$$

which, together with Lemma 1.1, implies

$$\begin{aligned}&\int_{\mathbb{R}^2} f(x,|u_n|^2)|u_n|^2 dx \\ \leq & \frac{c}{C_{10}^2\beta_2^2}\int_{\mathbb{R}^2} |u_n|^2 dx + C_{11}\int_{\mathbb{R}^2} |u|^q(e^{\alpha|u_n|^2} - 1)dx\end{aligned} \tag{19}$$

$$\leq \frac{c}{C_{10}^2\beta_2^2}\|u\|_2^2 + C_{11}\|u\|_{2q}^q\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\mathcal{C}_2^2\|u_n\|^2(|u_n|/\mathcal{C}_2\|u_n\|)^2}-1\right)dx\right]^{\frac{1}{2}} \leq c+o(1). \tag{20}$$

It follows from (F4), (10) and (19) that

$$c+o(1) = I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = \frac{1}{2}\int_{\mathbb{R}^2}[f(x,|u_n|^2)|u_n|^2dx - F(x,|u_n|^2)]dx \leq \frac{c}{2}+o(1).$$

This contradiction implies $\delta > 0$. Hence there exists $\{y_n\} \subset \mathbb{R}^2$ such that

$$\int_{B(y_n,1)}|u_n|^2dx > \frac{\delta}{2}. \tag{21}$$

In view of (A1), for all $z \in \mathbb{Z}^2$, there holds

$$B(x+z) - B(x) = \text{curl } A(x+z) - \text{curl } A(x) = 0,$$

which means $A(x+z)-A(x) = \nabla\phi_z(x)$, for some $\phi_z \in H^1_{loc}(\mathbb{R}^2,\mathbb{R})$ (see [25, Lemma 1.1]). Define translation $\Upsilon: X\times\mathbb{Z}^2 \to X$ by letting $\Upsilon_z u(x) = u(x+z)e^{i\phi_z(x)}$. Though a direct computation, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}|\nabla_A[\Upsilon_z u(x)]|\,dx\\
=&\int_{\mathbb{R}^2}\left|\nabla\left[u(x+z)e^{i\phi_z(x)}\right]+iA(x)u(x+z)e^{i\phi_z(x)}\right|dx\\
=&\int_{\mathbb{R}^2}\left|e^{i\phi_z(x)}\nabla u(x+z)+iu(x+z)\nabla\phi_z(x)e^{i\phi_z(x)}+iA(x)u(x+z)e^{i\phi_z(x)}\right|dx\\
=&\int_{\mathbb{R}^2}\left|[\nabla u(x+z)+iA(x+z)u(x+z)]e^{i\phi_z(x)}\right|dx = \int_{\mathbb{R}^2}|\nabla_A u(x)|dx.
\end{aligned}$$

and

$$\int_{\mathbb{R}^2}|\Upsilon_z u(x)|^q dx = \int_{\mathbb{R}^2}\left|u(x+z)e^{i\phi_z(x)}\right|^q dx = \int_{\mathbb{R}^2}|u(x)|^q dx, \qquad \forall\ q\ \in[2,\infty).$$

where we note that $|e^{i\phi_z(x)}| = 1$ for any $x \in \mathbb{R}^2$ due to $\phi_z \in H^1_{loc}(\mathbb{R}^2,\mathbb{R})$. Consequently, for each $z \in \mathbb{Z}^2, \Upsilon_z$ is well defined and isometry. Furthermore, let $v_n = \Upsilon_{[y_n]}u_n$, where $[x]$ denotes the largest integer not exceeding x, then (21) implies

$$\int_{B(0,1+\sqrt{2})}|v_n|^2dx > \frac{\delta}{2}.$$

The periodicity of V and f yields

$$I(v_n)\to c \qquad\text{and}\qquad \nabla I(v_n)\to 0.$$

Then the boundedness of $\{v_n\}$ can be deduced from Lemma 2.5. Going if necessary to a subsequence, there exists $v \in X$ such that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^s_{loc}(\mathbb{R}^2,\mathbb{C})$ for $s\in[1,\infty)$, which leads to

$$\int_{B(0,1+\sqrt{2})}|v_n|^2dx > \frac{\delta}{2}.$$

and then $v \neq 0$. Recalling Lemma 2.1, the weakly sequentially continuity of $\psi(u)$ implies, for any $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^2,\mathbb{C})$,

$$\langle I'(v),\xi\rangle = \langle v^+,\xi^+\rangle - \langle v^-,\xi^-\rangle - Re\int_{\mathbb{R}^2}f(x,|v|^2)v\bar{\xi}dx$$

$$= \lim_{n\to\infty}\left[\langle v_n^+,\xi^+\rangle - \langle v_n^-,\xi^-\rangle - Re\int_{\mathbb{R}^2} f(x,|v_n|^2)v_n\bar{\xi}dx\right] = 0.$$

Hence $I'(v) = 0$ and v is a nontrivial solution of problem (1). □

3. **Ground state solutions.** In this section, we prove the existence of ground state solution of Nehari-Pankov type for problem (1), i.e. we show that there exists $u \in X$ such that $I(u) = \inf_{\mathcal{N}} I$, where

$$\mathcal{N} := \{u \in X\backslash X^- : \langle I'(u), u\rangle = 0 \text{ and } \langle I'(u), v\rangle = 0 \text{ for all } v \in X^-\}.$$

Lemma 3.1. *Under the assumptions* (F1)-(F3) *and* (F5)*, for any $z_1, z_2 \in \mathbb{C}$ and $t \geq 0$, there holds*

$$\frac{1-t^2}{2}f(x,|z_1|^2)|z_1|^2 - tf(x,|z_1|^2)Re(z_1\bar{z}_2) + \frac{1}{2}F(x,|tz_1+z_2|^2) - \frac{1}{2}F(x,|z_1|^2) \geq 0. \tag{22}$$

Proof. Note that by (F5), we can get

$$f(x,t)t \geq F(x,t) \geq 0, \qquad \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+.$$

Define function $g(t) : \mathbb{R} \to \mathbb{R}$ as follows

$$g(t) = \frac{1-t^2}{2}f(x,|z_1|^2)|z_1|^2 - tf(x,|z_1|^2)Re(z_1\bar{z}_2) + \frac{1}{2}F(x,|tz_1+z_2|^2) - \frac{1}{2}F(x,|z_1|^2).$$

If $z_1 = 0$, then $g(t) = \frac{1}{2}F(x,|z_2|^2) \geq 0$. So in the following part, we assume $z_1 \neq 0$. By virtue of (F3) and (F5), it is easy to verify that

$$g(0) = \frac{1}{2}[f(x,|z_1|^2)|z_1|^2 - F(x,|z_1|^2)] + \frac{1}{2}F(x,|z_2|^2) \geq 0 \text{ and } g(t) \to +\infty, \text{as } t \to \infty.$$

Assume that $g(t)$ achieves its minimum at $t_0 \in [0,\infty)$, then we have $g'(t_0) = 0$, i.e.

$$t_0 f(x,|z_1|^2)|z_1|^2 + f(x,|z_1|^2)Re(z_1\bar{z}_2) - f(x,|tz_1+z_2|^2)Re(z_1\overline{t_0z_1+z_2}) = 0. \tag{23}$$

Set $z = t_0z_1 + z_2$. We discuss in two cases: 1) $Re(z_1\bar{z}) = 0$; 2) $Re(z_1\bar{z}) \neq 0$.

Case 1. $Re(z_1\bar{z}) = 0$, which leads to

$$\begin{aligned}
g(t_0) &= \frac{1-t_0^2}{2}f(x,|z_1|^2)|z_1|^2 - t_0f(x,|z_1|^2)Re(z_1\overline{z-t_0z_1}) + \frac{1}{2}F(x,|z|^2) - \frac{1}{2}F(x,|z_1|^2)\\
&= \frac{1-t_0^2}{2}f(x,|z_1|^2)|z_1|^2 + t_0^2f(x,|z_1|^2)|z_1|^2 + \frac{1}{2}F(x,|z|^2) - \frac{1}{2}F(x,|z_1|^2)\\
&= \frac{1}{2}\left[f(x,|z_1|^2)|z_1|^2 - F(x,|z_1|^2)\right] + \frac{t_0^2}{2}f(x,|z_1|^2)|z_1|^2 + \frac{1}{2}F(x,|z|^2) \geq 0.
\end{aligned}$$

Hence we have (22) holds.

Case 2. $Re(z_1\bar{z}) \neq 0$, which means $z_1 \neq 0$ and $z \neq 0$. We consider the two possibilities: (i) $|z_1| = |z|$ (ii) $|z_1| \neq |z|$. When $|z_1| = |z|$, let $z_1 = x_1 + iy_1$ and $z = x_2 + iy_2$ where x_i, y_i $(i = 1,2)$ denote the real part and imaginary part of z_1 and z, respectively, then

$$Re(z_1\bar{z}) = x_1x_2 + y_1y_2 \leq \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} = |z_1||z| = |z_1|^2,$$

which implies

$$\begin{aligned}
g(t_0) &= \frac{1-t_0^2}{2}f(x,|z_1|^2)|z_1|^2 - t_0f(x,|z_1|^2)Re(z_1\overline{z-t_0z_1}) + \frac{1}{2}F(x,|z|^2) - \frac{1}{2}F(x,|z_1|^2)\\
&= \frac{1-t_0^2}{2}f(x,|z_1|^2)|z_1|^2 - t_0f(x,|z_1|^2)Re(z_1\bar{z}) + t_0^2f(x,|z_1|^2)|z_1|^2
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1-t_0^2}{2} f(x,|z_1|^2)|z_1|^2 - t_0 f(x,|z_1|^2)|z_1|^2 + t_0^2 f(x,|z_1|^2)|z_1|^2 \\
&= \frac{(1-t_0)^2}{2} f(x,|z_1|^2)|z_1|^2 \geq 0.
\end{aligned}
$$

When $|z_1| \neq |z|$, without loss of generality, we assume that $|z_1| < |z|$. Using (23), one has $f(x,|z_1|^2) = f(x,|z|^2)$, which, together with (F5), implies $f(x,t) = const$ for $t \in [|z_1|^2,\ |z|^2]$, and then

$$
F(x,|z|^2) - F(x,|z_1|^2) = \int_{|z_1|^2}^{|z|^2} f(x,t)dt = f(x,|z_1|^2)(|z|^2 - |z_1|^2). \tag{24}
$$

Note that

$$
|z|^2 = (t_0 z_1 + z_2)\overline{t_0 z_1 + z_2} = t_0^2 |z_1|^2 + 2t_0 Re(z_1 \bar{z}_2) + |z_2|^2. \tag{25}
$$

It follows from (F5), (24) and (25) that

$$
\begin{aligned}
g(t_0) &= \frac{1-t_0^2}{2} f(x,|z_1|^2)|z_1|^2 - t_0 f(x,|z_1|^2) Re(z_1\bar{z}_2) + \frac{1}{2}[F(x,|z|^2) - F(x,|z_1|^2)] \\
&= \frac{1-t_0^2}{2} f(x,|z_1|^2)|z_1|^2 - t_0 f(x,|z_1|^2) Re(z_1\bar{z}_2) + \frac{1}{2} f(x,|z_1|^2)(|z|^2 - |z_1|^2) \\
&= \frac{1}{2} f(x,|z_1|^2)|z_2|^2 \geq 0.
\end{aligned}
$$

Here we also have (22) holds. The proof is now complete. □

Lemma 3.2. *If* (V1), (F1)-(F3) *and* (F5) *hold, then for any* $u \in X$, $v \in X^-$ *and* $t \geq 0$, *there holds*

$$
I(u) \geq I(tu+v) + \frac{1}{2}\|v\|^2 + \frac{1-t^2}{2}\langle I'(u), u\rangle - t\langle I'(u), v\rangle. \tag{26}
$$

Proof. According to the definition of I, one has

$$
\begin{aligned}
&I(u) - I(tu+v) \\
=&\frac{1-t^2}{2}\|u^+\|^2 - \frac{1-t^2}{2}\|u^-\|^2 + t\langle u^-, v\rangle + \frac{1}{2}\|v\|^2 + \frac{1}{2}\int_{\mathbb{R}^2}[F(x,|tu+v|^2) - F(x,|u|^2)]dx \\
=&\frac{1-t^2}{2}\langle I'(u), u\rangle - t\langle I'(u), v\rangle + \frac{1}{2}\|v\|^2 + Re\int_{\mathbb{R}^2}\left[\frac{1-t^2}{2} f(x,|u|^2)|u|^2 - t f(x,|u|^2)u\bar{v}\right. \\
&\left. + \frac{1}{2}F(x,|tu+v|^2) - \frac{1}{2}F(x,|u|^2)\right] dx,
\end{aligned} \tag{27}
$$

where we use the fact $\langle u^+, v\rangle = 0$ in the process, due to $v \in X^-$. Combining (22) with (27), we have that (26) holds. □

From Lemma 3.2, we can derive the following corollaries directly.

Corollary 3.3. *If* (V1), (F1)-(F3) *and* (F5) *are satisfied, then for any* $u \in X$ *and* $t \geq 0$, *there holds*

$$
I(u) \geq I(tu^+) + \frac{t^2}{2}\|u^-\|^2 + \frac{1-t^2}{2}\langle I'(u), u\rangle + t^2\langle I'(u), u^-\rangle. \tag{28}
$$

Corollary 3.4. *If* (V1), (F1)-(F3) *and* (F5) *hold, then for any* $u \in \mathcal{N}$, $v \in X^-$ *and* $t \geq 0$, *there holds*

$$
I(u) \geq I(tu+v), \tag{29}
$$

Lemma 3.5. *If* (V1), (F1)-(F3) *and* (F5) *hold, then there exist* $\rho > 0$, $c_* \in [\kappa_1, m]$ *and* $\{u_n\} \in X$ *such that*

$$I(u_n) \to c_*, \qquad \nabla I(u_n) \to 0, \tag{30}$$

where

$$\kappa_1 = \inf\{I(u) : u \in X^+, \|u\| = \rho\} \qquad \text{and} \qquad m = \inf_{\mathcal{N}} I.$$

Proof. For any $u \in X^+$, using (12) and Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x,|u|^2)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4\beta_2^2}\|u\|_2^2 + C_1\left(\int_{\mathbb{R}^2}|u|^{2q}dx\right)^{\frac{1}{2}}\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\mathcal{C}_2^2\|u\|^2(|u|/\mathcal{C}_2\|u\|)^2}-1\right)dx\right]^{\frac{1}{2}} \\ &\geq \frac{1}{4}\|u\|^2 - C_2\|u\|^q, \qquad \forall\ \|u\| < \frac{\sqrt{2\pi/\alpha}}{\mathcal{C}_2}, \end{aligned}$$

which shows that there exists $\rho > 0$ such that $\kappa_1 > 0$. Furthermore, for any $u \in \mathcal{N}$, there exists $s > 0$ such that $su^+ \in (X^- \oplus \mathbb{R}^+ u) \cap S_\rho$, then combined with (29) we can deduce $m \geq \kappa_1 > 0$. Choose $\{v_k\} \subset \mathcal{N}$ such that

$$m \leq I(v_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}.$$

By virtue of (F5) and the definition of m, one has

$$m \leq I(v_k) = \frac{1}{2}(\|v_k^+\|^2 - \|v_k^-\|^2) - \frac{1}{2}\int_{\mathbb{R}^2} F(x,|v_k|^2)dx \leq \frac{1}{2}\|v_k^+\|^2,$$

which implies $\|v_k^+\| \geq \sqrt{2m} > 0$. Let $e_k = \frac{v_k^+}{\|v_k^+\|}$, then $e_k^+ \in X^+$ and $\|v_k^+\|$=1. In view of Lemma (2.4), there exists $R_k > \max\{\rho, \|v_k\|\}$ such that $\sup I(\partial M_k) \leq 0$, where

$$M_k := \{u = y + \lambda e_k :\ y \in X^-,\ \|u\| \leq R_k,\ \lambda > 0\}, \qquad k \in \mathbb{N}.$$

Consequently, Lemma 2.1 yields that there exist constants $c_k \in [\kappa_1, \sup_{\overline{M_k}} I]$ and sequences $\{u_{k,n}\} \subset X$ such that

$$I(u_{k,n}) \to c_k \quad \text{and} \quad \nabla I(u_{k,n}) \to 0,\ k \in \mathbb{N}.$$

Since $v_k \in M_k$, using Corollary 3.4, one has

$$I(v_k) \geq I(tv_k + w), \qquad \forall\ t \geq 0,\ w \in X^-.$$

It follows that $I(v_k) = \sup_{u\in M_k} I(u)$, which implies

$$I(u_{k,n}) \to c_k < m + \frac{1}{k} \qquad \text{and} \qquad \nabla I(u_{k,n}) \to 0,\ k \in \mathbb{N}.$$

Choose $\{n_k\} \subset \mathbb{N}$ such that

$$I(u_{k,n_k}) < m + \frac{1}{k} \qquad \text{and} \qquad \|\nabla I(u_{k,n_k})\| < \frac{1}{k},\ k \in \mathbb{N}.$$

Now, let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Going if necessary to a subsequence, we have

$$I(u_k) \to c_* \in [\kappa_1, m] \qquad \text{and} \qquad \nabla I(u_k) \to 0.$$

□

Lemma 3.6. *If* (V1), (F1)-(F3) *and* (F5) *hold, then any sequence* $\{u_n\} \subset X$ *satisfying* (30) *is bounded.*

Proof. Let $\{u_n\} \subset X$ satisfying (30). Arguing indirectly, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, then there exists $w \in X$ such that up to a subsequence, $w_n \rightharpoonup w$ in X, $w_n \to w$ in $L^s_{loc}(\mathbb{R}^2, \mathbb{C})$ for $2 \le s < \infty$ and $w_n \to w$ a. e. on $\mathbb{R}^2$. We claim that

$$\delta := \overline{\lim_{n\to\infty}} \sup_{y\in\mathbb{R}^2} \int_{B(y,1)} |w_n^+|^2 dx > 0. \tag{31}$$

Indeed, if it is not true, then the Lions' concentration compactness lemma implies $w_n^+ \to 0$ for $2 < q < \infty$. For fixed $s > \sqrt{2(c_* + 1)}$, set $\varepsilon = \frac{1}{2s^2\beta_2^2}$ and $\alpha \in (0, \frac{4\pi}{s^2 C_2^2})$ in (12), then one has

$$\begin{aligned}
\int_{\mathbb{R}^2} F(x, |sw_n^+|^2)dx &\le \frac{1}{2\beta_2^2}\|w_n^+\|_2^2 + C_3 \int_{\mathbb{R}^2} |sw_n^+|^q \left(e^{\alpha s^2 |w_n^+|^2} - 1\right) dx \\
&\le \frac{1}{2} + C_3 s^q \|w_n^+\|_{2q}^q \left[\int_{\mathbb{R}^2} \left(e^{2\alpha s^2 C_2^2 \|w_n^+\|^2 (|w_n^+|/C_2\|w_n^+\|)^2} - 1\right) dx\right]^{\frac{1}{2}} \\
&= \frac{1}{2} + o(1).
\end{aligned}$$

Let $t_n = \frac{s}{\|u_n\|}$. Combining (28) with (30), we have

$$\begin{aligned}
c_* + o(1) = I(u_n) &\ge I(t_n u_n^+) + \frac{t_n^2}{2}\|u_n^-\|^2 + \frac{1 - t_n^2}{2}\langle I'(u_n), u_n\rangle + t_n^2 \langle I'(u_n), u_n^-\rangle \\
&= \frac{t_n^2}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |t_n u_n^+|^2)dx \\
&= \frac{s^2}{2}\|w_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |sw_n^+|^2)dx > c_* + \frac{3}{4} + o(1).
\end{aligned}$$

This contradiction implies that our claim is true. Through a similar argument of the last part of the proof of Theorem 1.2, there exists $v_n := \Upsilon_{[y_n]} w_n$ such that $\|v_n\| = \|w_n\|$ and

$$\int_{B(0,1+\sqrt{2})} |v_n|^2 dx > \frac{\delta}{2}. \tag{32}$$

Going if necessary to a subsequence, $v_n \rightharpoonup v$ in X, $v_n \to v$ in $L^s_{loc}(\mathbb{R}^2, \mathbb{C})$ $(2 \le s < \infty)$ and $v_n \to v$ a.e. on $\mathbb{R}^2$. Define $\tilde{u}_n = \Upsilon_{[y_n]} u_n$, then $\frac{\tilde{u}_n}{\|u_n\|} = v_n \to v \ne 0$ a.e. on $\mathbb{R}^2$. For a.e. $x \in \{y \in \mathbb{R}^2 : v(y) \ne 0\}$, we have $|\tilde{u}_n| = |v_n|\|u_n\| \to \infty$ as $n \to \infty$. It follows from (30) and (F3) that

$$\begin{aligned}
0 = \lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} &= \frac{1}{2}\lim_{n\to\infty}\left[\|w_n^+\|^2 - \|w_n^-\|^2 - \int_{\mathbb{R}^2} \frac{F(x, |u_n|^2)}{\|u_n\|^2} dx\right] \\
&\le \frac{1}{2} - \frac{1}{2}\int_{\mathbb{R}^2} \underline{\lim_{n\to\infty}} \frac{F(x + [y_n], |\tilde{u}_n|^2)}{|\tilde{u}_n|^2} |v_n|^2 dx = -\infty.
\end{aligned}$$

This contradiction implies that $\{u_n\}$ is bounded in X. □

Proof of Theorem 1.3. In view of Lemmas 3.2-3.6, through a similar argument of the proof of Theorem 1.2, there exists a bounded sequence $u_n \rightharpoonup u \in X \backslash \{0\}$ such that

$$\lim_{n\to\infty} I(u_n) = c_* \qquad \text{and} \qquad I'(u) = 0, \tag{33}$$

which shows $I(u) \geq m$. As is proved in [39, Lemma 6.15], the functional I is weakly upper semicontinuous, which implies $u \notin X^-$. Otherwise, using (F5) and (33) that

$$0 \geq -\frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x,|u|^2) = I(u) \geq \varlimsup_{n\to\infty} I(u_n) = c_* > 0,$$

a contradiction. Hence $u \in \mathcal{N}$ and $I(u) \geq m$. It follows from (F5) and Fatou's Lemma that

$$\begin{aligned} m \geq c_* &= \lim_{n\to\infty}\left[I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right] \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} \varliminf_{n\to\infty}\left[f(x,|u_n|^2)|u_n|^2 - F(x,|u_n|^2)\right]dx \\ &= \frac{1}{2}\int_{\mathbb{R}^2}\left[f(x,|u|^2)|u|^2 - F(x,|u|^2)\right]dx = I(u) - \frac{1}{2}\langle I'(u), u\rangle = I(u). \end{aligned}$$

Therefore, $I(u) \leq m$ and so $I(u) = m = \inf_{\mathcal{N}} I > 0$. □

4. **The asymptotic case.** In this section, we consider the case where V is asymptotically periodic and f is of subcritical exponential growth. In such situation, we define a new operator $\mathcal{L}_0 := -\Delta_A + V_0$ with the spectral family $\{\mathcal{E}(\lambda) : -\infty \leq \lambda \leq +\infty\}$. Through a similar argument of Section 2, we can define a new functional space $X_0 = \mathcal{D}(|\mathcal{L}_0|^{\frac{1}{2}})$ with the inner product and norm defined by

$$\langle u, v\rangle_0 = \langle |\mathcal{L}_0|^{\frac{1}{2}} u, |\mathcal{L}_0|^{\frac{1}{2}} v\rangle_2, \ \forall\, u, v \in X_0,$$

and $\|u\|_0 = \langle u, u\rangle_0^{1/2}$, respectively. Furthermore, for any $u \in X_0$, $u = u^{\mathcal{E}-} + u^{\mathcal{E}+}$ where

$$u^{\mathcal{E}-} := \mathcal{E}(0^-)u \in X_0^{\mathcal{E}-} := \mathcal{E}(0^-)X_0, \ u^{\mathcal{E}+} := (id - \mathcal{E}(0))u \in X_0^{\mathcal{E}+} := (id - \mathcal{E}(0))X_0,$$

and $X_0 = H_A^1(\mathbb{R}^2, \mathbb{C})$ with the equivalent norm in $H_A^1(\mathbb{R}^2, \mathbb{C})$, which means there exist constants still donated by $\mathcal{C}_1$, $\mathcal{C}_2$ and β_q such that

$$\mathcal{C}_1\|u\|_0 \leq \|u\|_A \leq \mathcal{C}_2\|u\|_0 \qquad \text{and} \qquad \|u\|_q \leq \beta_q\|u\|_0, \qquad \forall\, u \in X_0, \ q \in [2, \infty).$$

Define the limit function of I_0 as follows

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^2}[|\nabla_A u|^2 + V_0|u|^2]dx - \frac{1}{2}\int_{\mathbb{R}^2} F_0(x, |u_n|^2)dx, \ \forall\, u \in X_0.$$

It follows from (V1'), (F1), (F1') and (F2) that $I_0 \in \mathcal{C}^1(X_0, \mathbb{R})$ and

$$\langle I_0'(u), v\rangle = \langle u^{\mathcal{E}+}, v^{\mathcal{E}+}\rangle - \langle u^{\mathcal{E}-}, v^{\mathcal{E}-}\rangle - \int_{\mathbb{R}^2} f_0(x, |u|^2)|u|^2 dx, \ \forall\, u, v \in X_0.$$

The corresponding Nehari-Pankov manifold is defined by

$$\mathcal{N}_0 := \{u \in X_0 \backslash X_0^{\mathcal{E}-} : \langle I'(u), u\rangle = 0 \text{ and } \langle I'(u), v\rangle = 0 \text{ for all } v \in X_0^{\mathcal{E}-}\}.$$

Lemma 4.1. *If* (V1'), (F1), (F1') *and* (F2) *are satisfied, and* $u_n \rightharpoonup 0$ *in* X_0, *there hold*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F_1(x, |u_n|^2)dx = 0, \qquad \lim_{n\to\infty} Re\int_{\mathbb{R}^2} f_1(x, |u_n|^2)u_n\bar{v} = 0, \qquad \forall\, v \in X_0, \tag{34}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} V_1(x)|u_n|^2 dx = 0, \qquad \lim_{n\to\infty} Re\int_{\mathbb{R}^2} V_1(x)u_n\bar{v} = 0, \qquad \forall\, v \in X_0. \tag{35}$$

Proof. Let $u_n \rightharpoonup 0$ in X_0, then $\|u_n\|_0 \le C_1$ for some $C_1 > 0$. For any $\varepsilon > 0$, by (F1') ii), there exists $R_1 > 0$ such that $meas[U_\varepsilon(R_1)] < \varepsilon$, where $U_\varepsilon(R_\varepsilon) = \{x \in \mathbb{R}^2, |a(x)| \ge \varepsilon, |x| \ge R_1\}$. Recall that for any $\alpha > 0$ and fixed $q > 2$, there holds

$$f_1(x,t) \le a(x)\left[1 + t^{(q-2)/2}\left(e^{at} - 1\right)\right]. \tag{36}$$

Then (36) and Hölder inequality yield

$$\begin{aligned}
&Re \int_{U_\varepsilon(R_1)} f_1(x,|u_n|^2)u_n\bar{v}dx \qquad (37)\\
\le& \int_{U_\varepsilon(R_1)} f_1(x,|u_n|^2)|u_n||\bar{v}|dx\\
\le& \int_{U_\varepsilon(R_1)} a(x)|u_n||v|dx + \int_{U_\varepsilon(R_1)} \left[a(x)|u_n|^{q-1}\left(e^{\alpha|u_n|^2} - 1]\right)|v|\right]dx\\
\le& C\|a\|_\infty |U_\varepsilon(R_1)|^{1/3q}\|u_n\|_q^{q-1}\|v\|_{3q}\left[\int_{\mathbb{R}^2}\left(e^{3q\alpha\mathcal{C}_2^2\|u_n\|_0^2(|u_n|/\mathcal{C}_2\|u_n\|_0)^2} - 1\right)dx\right]^{3/q}\\
&+ \|a\|_\infty|U_\varepsilon(R_1)|^{\frac{1}{3}}\|u_n\|_3\|v\|_3 \le C_3\varepsilon, \qquad (38)
\end{aligned}$$

where we let $\alpha \in (0, 4\pi/(3qC_1^2\mathcal{C}_2^2))$ in (36). In addition, Since $u_n \to 0$ in $L^s_{loc}(\mathbb{R}^2,\mathbb{C})$ for $s \in [2,\infty)$, then for n large enough,

$$\int_{B(0,R_1)} |u_n|^s dx < \varepsilon,$$

which leads to

$$\begin{aligned}
&Re \int_{B(0,R_1)} f_1(x,|u_n|^2)u_n\bar{v}dx\\
\le& \|a\|_\infty\|v\|_{2q}\left(\int_{B(0,R_1)} |u_n|^q dx\right)^{\frac{q-1}{q}}\left[\int_{\mathbb{R}^2}\left(e^{2q\alpha\mathcal{C}_2^2\|u_n\|_0^2(|u_n|/\mathcal{C}_2\|u_n\|_0)^2} - 1\right)dx\right]^{\frac{2}{q}}\\
&+ \|a\|_\infty\|v\|_2\int_{B(0,R_1)} |u_n|^2dx \le C_4\varepsilon. \qquad (39)
\end{aligned}$$

where we choose $\alpha \in (0, 2\pi/(qC_1^2\mathcal{C}_2^2))$ in (39). Similarly, we have

$$\begin{aligned}
&Re \int_{\mathbb{R}^2\setminus[U_\varepsilon(R_1)\cup B(0,R_1)]} f_1(x,|u_n|^2)u_n\bar{v}dx\\
\le& \varepsilon\|u_n\|_q^{q-1}\|v\|_{2q}\left[\int_{\mathbb{R}^2}\left(e^{2q\alpha\mathcal{C}_2^2\|u_n\|_0^2(|u_n|/\mathcal{C}_2\|u_n\|_0)^2} - 1\right)dx\right]^{\frac{2}{q}}\\
&+ \varepsilon\|v\|_2\int_{B(0,R_1)} |u_n|^2dx \le C_5\varepsilon. \qquad (40)
\end{aligned}$$

Combining (37), (39) with (40), we derive

$$\lim_{n\to\infty} Re \int_{\mathbb{R}^2} f_1(x,|u_n|^2)u_n\bar{v}dx = 0.$$

Through a similar process above, we have that (34) and (35) hold. □

Lemma 4.2. *If* (V1'), (F1'), (F1)-(F3) *and* (F5) *are satisfied, then for any* $t \ge 0$, $u \in X_0$, $v \in X_0^{\mathcal{E}-}$, *there holds*

$$I(u) \ge I(tu+v) + \frac{1}{2}\|v\|_0^2 - \frac{1}{2}\int_{\mathbb{R}^2} V_1(x)|v|^2dx + \frac{1-t^2}{2}\langle I'(u),u\rangle - t\langle I'(u),v\rangle. \tag{41}$$

Proof. The proof is similar to Lemma 3.2, so we omit here. □

Corollary 4.3. *If* (V1'), (F1'), (F1)-(F3) *and* (F5) *hold, then for any* $t \geq 0$, $u \in X_0$, *there holds*

$$I(u) \geq I(tu^{\mathcal{E}+}) + \frac{t^2}{2}\|u^{\mathcal{E}-}\|_0^2 - \frac{t^2}{2}\int_{\mathbb{R}^2} V_1(x)|u^{\mathcal{E}-}|^2 dx + \frac{1-t^2}{2}\langle I'(u), u\rangle + t^2\langle I'(u), u^{\mathcal{E}-}\rangle. \tag{42}$$

Lemma 4.4. *If* (V1'), (F1'), (F1)-(F3) *and* (F5) *hold, then for any* $t \geq 0$, $u \in X_0$, $v \in X_0^{\mathcal{E}-}$, *there exist constants* $\rho > 0$, $\tilde{c} \in [\kappa_2, \tilde{m}]$ *and a sequence* $\{u_n\} \subset X_0$ *such that*

$$I(u_n) \to \tilde{c}, \qquad \nabla I(u_n) \to 0. \tag{43}$$

where

$$\kappa_2 = \inf\{I(u) : u \in X_0^{\mathcal{E}+}, \|u\| = \rho\} \text{ and } \tilde{m} = \inf_{u \in \mathcal{N}_0} I(u).$$

Proof. For any $u \in X^{\mathcal{E}+}$, using (12), (V1') and Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_0^2 + \frac{1}{2}\int_{\mathbb{R}^2} V_1(x)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |u|^2) dx \\ &\geq \frac{1}{2}\|u\|_0^2 - \frac{\|V_1\|_\infty}{2}\|u\|_2^2 - \frac{1}{4\beta_2^2}\|u\|_2^2 \\ &\quad + C_6\left(\int_{\mathbb{R}^2} |u|^{2q} dx\right)^{\frac{1}{2}}\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\mathcal{C}_2^2\|u\|_0^2(|u|/\mathcal{C}_2\|u\|_0)^2} - 1\right) dx\right]^{\frac{1}{2}} \\ &\geq \left(\frac{1}{4} - \frac{\|V_1\|_\infty}{2\Lambda}\right)\|u\|_0^2 - C_7\|u\|_0^q, \qquad \forall \ \|u\|_0 < \frac{\sqrt{2\pi/\alpha}}{\mathcal{C}_2}, \end{aligned}$$

which shows that there exists small $\rho > 0$ such that $\kappa_2 > 0$. The rest part is similar to Lemma (3.5), so we omit here. □

Lemma 4.5. *If* (V1'), (F1'), (F1)-(F3) *and* (F5) *hold, then any sequence satisfying* (43) *is bounded.*

Proof. Arguing indirectly, we assume that there exists $\{u_n\} \subset X_0$ satisfying (42) and $\|u_n\|_0 \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|_0}$, then $\|w_n\|_0 = 1$. Going if necessary to a subsequence, $w_n \rightharpoonup \tilde{w}$ in X_0. When $\tilde{w} \neq 0$, by a standard argument, we can derive a contradiction. Therefore, we only consider the case $\tilde{w} = 0$, which follows that $w_n^{\mathcal{E}+} \rightharpoonup 0$ and $w_n^{\mathcal{E}-} \rightharpoonup 0$ in X_0. If

$$\delta := \overline{\lim_{n\to\infty}} \sup_{y \in \mathbb{R}^2} \int_{B(y,1)} |w_n^{\mathcal{E}+}|^2 dx > 0, \tag{44}$$

then the Lions' concentration compactness lemma implies $w_n^{\mathcal{E}+} \to 0$ for $2 < q < \infty$. For fixed $s > \sqrt{2(\tilde{c}+1)}$, set $\varepsilon = \frac{1}{2s^2\beta_2^2}$ and $\alpha \in (0, \frac{2\pi}{s^2\mathcal{C}_2^2})$ in (12), then one has

$$\begin{aligned} &\int_{\mathbb{R}^2} F(x, |sw_n^{\mathcal{E}+}|^2) dx \\ \leq &\frac{1}{2\beta_2^2}\|w_n^{\mathcal{E}+}\|_2^2 + C_8\int_{\mathbb{R}^2} |sw_n^{\mathcal{E}+}|^q\left(e^{\alpha s^2|w_n^{\mathcal{E}+}|^2} - 1\right) dx \\ \leq &\frac{1}{2} + C_9 s^q\|w_n^{\mathcal{E}+}\|_{2q}^q\left[\int_{\mathbb{R}^2}\left(e^{2\alpha s^2\mathcal{C}_2^2\|w_n^{\mathcal{E}+}\|_0^2(|w_n^{\mathcal{E}+}|/\mathcal{C}_2\|w_n^{\mathcal{E}+}\|_0)^2} - 1\right) dx\right]^{1/2} = \frac{1}{2} + o(1). \end{aligned}$$

Let $t_n = \frac{s}{\|u_n\|_0}$. Combining (35) with (42), we have

$$\begin{aligned}\tilde{c} + o(1) = I(u_n) &\geq I(t_n u_n^{\mathcal{E}+}) + \frac{t_n}{2}\|u_n^{\mathcal{E}-}\|_0^2 - \frac{t_n^2}{2}\int_{\mathbb{R}^2} V_1(x)|u_n^{\mathcal{E}-}|^2 dx \\ &\quad + \frac{1-t_n^2}{2}\langle I'(u_n), u_n\rangle + t_n^2\langle I'(u_n), u_n^{\mathcal{E}-}\rangle \\ &\geq \frac{t_n^2}{2}\|u_n\|_0^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |t_n u_n^{\mathcal{E}+}|^2)dx \\ &= \frac{s^2}{2}\|w_n\|_0^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |s w_n^{\mathcal{E}+}|^2)dx > \tilde{c} + \frac{3}{4} + o(1).\end{aligned}$$

This contradiction implies that our claim is true. Through a similar argument of last part of the proof of Theorem 1.2, there exists $v_n := \Upsilon_{[y_n]} w_n$ such that $\|v_n\|_0 = \|w_n\|_0$ and

$$\int_{B(0,1+\sqrt{2})} |v_n|^2 dx > \frac{\delta}{2}. \tag{45}$$

Going if necessary to a subsequence, $v_n \rightharpoonup v$ in X, $v_n \to v$ in $L^s_{loc}(\mathbb{R}^2, \mathbb{C})$ $(2 \leq s < \infty)$ and $v_n \to v$ a. e. on $\mathbb{R}^2$. Define $\tilde{u}_n = \Upsilon_{[y_n]} u_n$, then $\frac{\tilde{u}_n}{\|u_n\|_0} = v_n \to v \neq 0$ a. e. on $\mathbb{R}^2$. For a. e. $x \in \{y \in \mathbb{R}^2 : v(y) \neq 0\}$, we have $|\tilde{u}_n| = |v_n|\|u_n\|_0 \to \infty$. It follows from (43) and (F3) that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|_0^2} \\ &= \frac{1}{2}\lim_{n\to\infty}\left[\|w_n^{\mathcal{E}+}\|_0^2 - \|w_n^{\mathcal{E}-}\|_0^2 + \int_{\mathbb{R}^2} V_1(x)|w_n|^2 dx - \int_{\mathbb{R}^2} \frac{F(x, |u_n|^2)}{\|u_n\|_0^2} dx\right] \\ &\leq \frac{1}{2} - \frac{1}{2}\int_{\mathbb{R}^2} \lim_{n\to\infty} \frac{F(x + [y_n], |\tilde{u}_n|^2)}{|\tilde{u}_n|^2}|v_n|^2 dx = -\infty.\end{aligned}$$

This contradiction implies that $\{u_n\}$ is bounded in X_0. □

Lemma 4.6. *If* (V1'), (F1'), (F1)-(F3) *and* (F5) *hold, for any $u \in X_0 \backslash X_0^{\mathcal{E}-}$, there exists $t > 0$ and $w \in X_0^{\mathcal{E}-}$ such that $tu + w \in \mathcal{N}_0$.*

Proof. Observe that $X_0^{\mathcal{E}-} \oplus \mathbb{R}u = X_0^{\mathcal{E}-} \oplus \mathbb{R}^+ u$. Without loss of generality, we can assume that $u \in X_0^{\mathcal{E}+}$. Through similar arguments of Lemmas 2.3 and 2.4, we can deduce that $I(tu) > 0$ for small $t > 0$ and $I(u) \leq 0$ for $u \in (X_0^{\mathcal{E}-} \oplus \mathbb{R}^+ u)\backslash B_R(0)$ for some $R > 0$, which implies $0 < \sup_{u \in X_0^{\mathcal{E}-}\oplus\mathbb{R}^+ u} I(u) < \infty$. Taking advantage of the weakly upper semi-continuity of I on $X_0^{\mathcal{E}-} \oplus \mathbb{R}^+ u$, there exists $u_0 \in X_0^{\mathcal{E}-} \oplus \mathbb{R}^+ u$ such that $I(u_0) = \sup_{u \in X_0^{\mathcal{E}-}\oplus\mathbb{R}^+ u} I(u)$. Then u_0 is a critical point of $I|_{X_0^{\mathcal{E}-}\oplus\mathbb{R}^+ u}$, so $\langle I'(u_0), u_0\rangle = \langle I'(u_0), v\rangle = 0$ for all $v \in X_0^{\mathcal{E}-} \oplus \mathbb{R}^+ u$. Consequently, $u_0 \in \mathcal{N}_0 \cap (X_0^{\mathcal{E}-} \oplus \mathbb{R}^+ u)$. □

Proof of Theorem 1.5. In view of Lemmas 4.4 and 4.5, there exists a bounded sequence $\{u_n\} \subset X_0$ satisfying (43). Going if necessary to a subsequence, $u_n \rightharpoonup u$ in X_0. Suppose that $u = 0$, then $u_n \to 0$ in $L^s_{loc}(\mathbb{R}^2, \mathbb{C})$ for $s \in [2, \infty)$. Note that

$$I_0(u) = I(u) - \frac{1}{2}\int_{\mathbb{R}^2} V_1(x)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} F_1(x, |u|^2)dx, \qquad \forall\, u \in X_0,$$

and

$$\langle I_0'(u), v\rangle = \langle I'(u), v\rangle - Re\int_{\mathbb{R}^2} V_1(x)u\bar{v}dx - Re\int_{\mathbb{R}^2} f_1(x, |u|^2)u\bar{v}dx, \qquad \forall\, u, v \in X_0,$$

which, together with (43) and Lemma 4.1, implies that

$$I_0(u_n) \to \tilde{c}, \qquad \nabla I_0(u_n) \to 0.$$

Through a similar argument of the proof of Theorem 1.2, there exists $\{y_n\} \in \mathbb{R}^2$ such that

$$\int_{B([y_n],1+\sqrt{2})} |u_n|^2 dx > \frac{\delta}{2}.$$

for some $\delta > 0$. Let $v_n(x) = \Upsilon_{[y_n]} u_n$, then $\|v_n\|_0 = \|u_n\|_0$ and

$$\int_{B(0,1+\sqrt{2})} |v_n|^2 dx > \frac{\delta}{2}. \tag{46}$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in X_0, $v_n \to v$ in $L^s_{loc}(\mathbb{R}^2, \mathbb{C})$ for $s \in [2, \infty)$, and $v_n \to v$ a. e. on $\mathbb{R}^2$. (46) implies $v \neq 0$. Due to the periodicity of $V_0(x)$ and $f_0(x,t)$ in x, one has

$$I_0(v_n) \to \tilde{c}, \qquad \nabla I_0(v_n) \to 0.$$

Analogous to the last part of Theorem 1.3, we can show that $I_0'(v) = 0$ and $I_0(v) \leq \tilde{c}$. Therefore, $v^{\mathcal{E}+} \neq 0$. By virtue of Lemma 4.6, there exists $t_0 > 0$ and $w_0 \in X_0^{\mathcal{E}-}$ such that $t_0 v + w_0 \in \mathcal{N}_0$, which leads to $I(t_0 v + w_0) \geq \tilde{m}$. Since $f_0(x,t)$ is nondecreasing in t on $(0,\infty)$, similar to (22), we can derive

$$\frac{1-t^2}{2} f_0(x,|z_1|^2)|z_1|^2 - t f_0(x,|z_1|^2) Re(z_1\bar{z}_2) + \frac{1}{2} F_0(x,|tz_1+z_2|^2) + \frac{1}{2} F_0(x,|z_1|^2) \geq 0. \tag{47}$$

It follows from (V1') and (47) that

$$\begin{aligned}
\tilde{m} &\geq \tilde{c} \\
&\geq I_0(v) \\
&= I_0(t_0 v + w_0) + \frac{1}{2}\|w_0\|_0^2 + \frac{1-t_0^2}{2}\langle I_0'(v), v\rangle - t_0 \langle I_0'(v), v\rangle + \int_{\mathbb{R}^2} \left[\frac{1-t^2}{2} f_0(x,|z_1|^2)|z_1|^2 \right. \\
&\quad \left. - t f_0(x,|z_1|^2) Re(z_1\bar{z}_2) + \frac{1}{2} F_0(x,|tz_1+z_2|^2) + \frac{1}{2} F_0(x,|z_1|^2)\right] dx \\
&\geq I_0(t_0 v + w_0) + \frac{1}{2}\|w_0\|_0^2 \\
&= \frac{1}{2}\|w_0\|_0^2 + I(t_0 v + w_0) - \frac{1}{2}\int_{\mathbb{R}^2} V_1(x)|t_0 v + w_0|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} F_1(x, |t_0 v + w_0|^2) dx \\
&> I(t_0 v + w_0) \geq \tilde{m},
\end{aligned}$$

This contradiction implies $\tilde{u} \neq 0$. Through a same argument of the last part of the proof of Theorem 1.3, we can prove that $I'(\tilde{u}) = 0$ and $I(\tilde{u}) = \tilde{m} = \inf_{\mathcal{N}_0} I$, hence $\tilde{u} \in X_0$ is a ground state solution to (1). □

5. **The critical case.** In this section, we consider the case when the spectrum of the operator $\mathcal{L}$ has a positive infimum and f is of critical exponential growth (that is f satisfies assumption (F1") and V satisfies (V2), which leads to $X^- = \{0\}$). Then for any $u \in X$,

$$\|u\| = \left\{\int_{\mathbb{R}^2} (|\nabla_A u|^2 + V|u|^2) dx\right\}^{\frac{1}{2}},$$

and

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} F(x, |u_n|^2) dx.$$

Using (F1") and (F2), for any $\varepsilon > 0$ and $\alpha > \alpha_0$, there exists $C_\varepsilon > 0$ such that for any $(x,t) \in (\mathbb{R}^2, \mathbb{R}^+)$,

$$|f(x,t)| \leq \varepsilon + C_\varepsilon t^{-1/2}(e^{\alpha t} - 1) \text{ and } |F(x,t)| \leq \varepsilon t + C_\varepsilon t^{1/2}(e^{\alpha t} - 1),$$

which, together with Lemma 1, implies $I \in \mathcal{C}^1(X, \mathbb{R})$ and

$$\langle I'(u), v\rangle = Re\left(\frac{1}{2}\int_{\mathbb{R}^2}[\nabla_A u\overline{\nabla_A v} + V(x)u\bar{v}]dx - \frac{1}{2}\int_{\mathbb{R}^2} f(x,|u|^2)u\bar{v}dx\right).$$

The corresponding Nehari manifold is defined by

$$\mathcal{M} := \{u \in X\backslash\{0\} : \langle I'(u), u\rangle = 0\}.$$

Lemma 5.1. *If* (V2), (F1"), (F2) *and* (F5) *hold, there holds*

$$I(u) \geq I(tu) + \frac{1-t^2}{2}\langle I'(u), u\rangle, \qquad \forall\ u \in X,\ t \geq 0. \tag{48}$$

Proof. It follows from (F2) and (F5) that

$$I(u) - I(tu) - \frac{1-t^2}{2}\langle I'(u), u\rangle = \frac{1}{2}\int_{\mathbb{R}^2}[(1-t^2)f(x,|u|^2)|u|^2 + F(x,|tu|^2) - F(x,|u^2|)]dx \geq 0.$$

Here we use the fact that for any $z \in \mathbb{C}$ and $t \geq 0$,

$$(1-t^2)f(x,|z|^2)|z|^2 - F(x,|tz|^2) - F(x,|z|^2) = \int_1^t 2s|z|^2\left[f(x,s^2|z|^2) - f(x,|z|^2)\right]ds \geq 0.$$

□

In view of Lemma 5.1, through a similar arguments in Section 2, we can derive the following lemmas and corollaries.

Corollary 5.2. *If* (V2), (F1"), (F2) *and* (F5) *hold, then*

$$I(u) = \max_{t\geq 0} I(tu), \qquad \forall u \in X,\ t \geq 0.$$

Lemma 5.3. *If* (V2), (F1"), (F2) *and* (F5) *hold, there exist a constant* $c^* \in (0, m^*)$ *where* $m^* = \inf_{u\in\mathcal{M}} I(u)$ *and a sequence* $\{u_n\} \in X$ *such that*

$$I(u_n) \to c^*, \qquad \|I'(u_n)\|(1 + \|u_n\|) \to 0. \tag{49}$$

Lemma 5.4. *If* (V2), (F1"), (F2), (F5) *and* (F7) *are satisfied, any sequence* $\{u_n\}$ *satisfying* (49) *is bounded.*

Lemma 5.5. *If* (V2), (F1"), (F2) *and* (F5) *hold, for any* $u \in X\backslash\{0\}$, *there exists* $t_u > 0$ *such that* $t_u u \in \mathcal{M}$.

Proof. For fixed $u \in X\backslash\{0\}$, define $g(t) := I(tu)$ on $[0, \infty)$, which follows that

$$g'(t) = 0 \Leftrightarrow t^2\|u\|^2 - \int_{\mathbb{R}^2} f(x,|tu_n|^2)|tu_n|^2dx \Leftrightarrow \langle I'(tu), tu\rangle = 0 \Leftrightarrow tu \in \mathcal{N}.$$

By virtue of (F1") and (F2), analogous to the proof of Lemma 2.3, we have $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for t large. Therefore, there exists $t_u > 0$ such that $g(t_u u) = \max_{t\in(0,\infty)} g(t)$ and then $t_u u \in \mathcal{M}$. □

In spired by [6, 13], we define a Moser type function involving the magnetic potential $w_n(x) = e^{i\phi(x)}u_n(x)$, where $\phi(x) = A(0)\cdot x = \sum_{j=1}^2 A_j(0)x_j$ and

$$u_n(x) = \begin{cases} \frac{\sqrt{\log n}}{\sqrt{2\pi}}, & 0 \leq |x| \leq \rho/n, \\ \frac{\log(\rho/|x|)}{\sqrt{2\pi}\log n}, & \rho/n \leq |x| \leq \rho, \\ 0, & |x| \geq \rho. \end{cases}$$

where $\rho < \sqrt{\|V\|_\infty + 4\zeta^2}$. Firstly, we claim that $w_n(x) \in X$. It suffices to show that $\nabla_A(e^{i\phi(x)}u) \in L^2_{loc}(\mathbb{R}^2, \mathbb{C})$ for $u \in H^1(\mathbb{R}^2, \mathbb{R})$ and *supp* $u \subset B(0, \rho)$. Recall that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2})$ for any $z_1, z_2 \in \mathbb{C}$, then we have

$$\int_{\mathbb{R}^2} \left|\nabla_A(e^{i\phi(x)}u)\right|^2 dx = \int_{\mathbb{R}^2} \left|\nabla(e^{i\phi(x)}u) + iA(x)e^{i\phi(x)}u\right|^2 dx$$

$$= \int_{\mathbb{R}^2} \left|\nabla(e^{i\phi(x)}u)\right|^2 dx + \int_{\mathbb{R}^2} \left|A(x)e^{i\phi(x)}u\right|^2 dx - 2Re \int_{\mathbb{R}^2} \nabla(e^{i\phi(x)}u) \cdot iA(x)\overline{e^{i\phi(x)}u}dx.$$

Note that

$$\nabla e^{i\phi(x)} = ie^{i\phi(x)}\nabla\phi(x) = ie^{i\phi(x)}A(0).$$

Through a direct computation, we get

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|\nabla(e^{i\phi(x)}u)\right|^2 dx &= \int_{\mathbb{R}^2} \nabla(e^{i\phi(x)}u) \cdot \overline{\nabla(e^{i\phi(x)}u)}dx \\
&= \int_{\mathbb{R}^2} \left(u\nabla e^{i\phi(x)} + e^{i\phi(x)}\nabla u\right) \cdot \left(\overline{u\nabla e^{i\phi(x)}} + \overline{e^{i\phi(x)}\nabla u}\right) dx \\
&= \int_{\mathbb{R}^2} \Big(\left|\nabla e^{i\phi(x)}\right|^2 |u|^2 + ie^{i\phi(x)}u\nabla\phi(x) \cdot \nabla u e^{-i\phi(x)} \\
&\qquad + e^{i\phi(x)}\nabla u \cdot \nabla\phi(x)(-ie^{-i\phi(x)}u) + |e^{i\phi(x)}|^2|\nabla u|^2\Big) dx \\
&= \int_{\mathbb{R}^2} (|A(0)|^2u^2 + |\nabla u|^2)dx.
\end{aligned}$$

and

$$\begin{aligned}
&Re \int_{\mathbb{R}^2} \nabla(e^{i\phi(x)}u) \cdot iA(x)\overline{e^{i\phi(x)}u}\ dx \\
=&Re \int_{\mathbb{R}^2} \left(ie^{i\phi(x)}u\nabla\phi(x) + e^{i\phi(x)}\nabla u\right) \cdot iA(x)e^{-i\phi(x)}u\ dx \\
=& - \int_{\mathbb{R}^2} A(x) \cdot \nabla\phi(x)u^2\ dx = - \int_{\mathbb{R}^2} A(x) \cdot A(0)u^2dx.
\end{aligned}$$

It follows from (A2) that $|A(x)| \le \zeta$ a. e. on $B(0, \rho)$. On the basis of the equalities above, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|\nabla_A(e^{i\phi(x)}u)\right|^2 dx &= \int_{\mathbb{R}^2} \left[|A(0)|^2u^2 + |\nabla u|^2 + |A(x)|^2u^2 + 2A(x) \cdot A(0)u^2\right] dx \\
&\le \int_{B(0,\rho)} |\nabla u|^2dx + 4\zeta^2 \int_{B(0,\rho)} u^2dx < \infty,
\end{aligned}$$

from which we can deduce that our claim is true. Moreover, via an elementary computation, one has

$$\begin{aligned}
\int_{B(0,\rho)} |w_n|^2dx &= \left(\int_{B(0,\rho/n)} + \int_{B(0,\rho)\setminus B(\rho/n)}\right) u_n^2dx \\
&= \frac{\rho^2 \log n}{2n^2} + 2\pi \int_{p/n}^{\rho} \frac{r\log^2(\rho/r)}{2\pi\log n}dr \\
&= \frac{\rho^2 \log n}{2n^2} + \frac{\rho^2}{\log n}\left[\frac{1}{4} - \frac{1}{4n^2} - \frac{\log n}{2n^2} - \frac{\log^2 n}{2n^2}\right] \\
&= \rho^2\left(\frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2}\right) := \rho^2\varrho_n^2 > 0, \text{for } n \ge 2.
\end{aligned}$$

Hence we have

$$\begin{aligned}\|w_n\|^2 &= \int_{\mathbb{R}^2}\left[\left|\nabla_A\left(e^{i\phi(x)}u_n\right)\right|^2 + V(x)\left|e^{i\phi(x)}u_n\right|^2\right]dx\\ &\le \int_{B(0,\rho)}|\nabla u_n|^2dx + 4\zeta^2\int_{B(0,\rho)}u_n^2dx + \int_{B(0,\rho)}V(x)u_n^2dx = 1 + (\|V\|_\infty + 4\zeta^2)\rho^2\varrho_n^2.\end{aligned}$$

Lemma 5.6. *If* (V2), (F1”), (F2), (F5)-(F8) *are satisfied, then there exists* $k\in\mathbb{N}$ *such that*

$$\max_{t\ge 0} I(tw_k) < \frac{2\pi}{\alpha_0}.$$

Proof. By virtue of (F6) and (F8), we can obtain that there exists $\varepsilon>0$ and $t_\varepsilon>0$ such that

$$tF(x,t) \ge (\nu-\varepsilon)e^{\alpha_0 t}, \qquad \forall x \in\mathbb{R}^2,\ |t|\ge t_\varepsilon, \tag{50}$$

and

$$\frac{\nu-\varepsilon}{1+\varepsilon} > \frac{4}{\alpha_0^2\rho^2}, \tag{51}$$

respectively. In the following part, we discuss in three cases, where the inequalities hold for $n\in\mathbb{N}$ large enough.

Case i. $t\in\left[0,\sqrt{2\pi/\alpha_0}\right]$. It follows from (F7) that

$$\begin{aligned}I(tw_n) &= \frac{t^2}{2}\|w_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}F(x,|tw_n|^2)x\\ &\le \frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t^2 \le \frac{\pi}{\alpha_0} + \frac{\pi\rho^2(\|V\|_\infty+4\zeta^2)}{2\alpha_0\log n} < \frac{2\pi}{\alpha_0}.\end{aligned}$$

Case ii. $t\in\left[\sqrt{2\pi/\alpha_0},\sqrt{4\pi(1+\varepsilon)/\alpha_0}\right]$. Since $|tw_n|\ge\sqrt{t_\varepsilon}$ for $x\in B(0,\rho/n)$, by virtue of (51), one has

$$\begin{aligned}I(tw_n) &= \frac{t^2}{2}\|w_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}F(x,|tw_n|^2)dx\\ &\le \frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t^2 - \frac{\pi^2\rho^2(\nu-\varepsilon)}{t^2n^2\log n}e^{\frac{\alpha_0 t^2\log n}{2\pi}}\\ &\le \frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t^2 - \frac{\alpha_0\pi\rho^2(\nu-\varepsilon)}{4(1+\varepsilon)n^2\log n}e^{\frac{\alpha_0 t^2\log n}{2\pi}}\\ &:= \varphi_n(t) := a_1t^2 - a_2e^{a_3t^2},\end{aligned}$$

where

$$a_1 = \frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2},\ a_2 = \frac{\alpha_0\pi\rho^2(\nu-\varepsilon)}{4(1+\varepsilon)n^2\log n},\ a_3 = \frac{\alpha_0\log n}{2\pi}.$$

Let $t_n\in[0,\infty)$ such that $\varphi'(t_n)=0$, which means

$$\begin{aligned}t_n &= \left(\frac{1}{a_3}\log\frac{a_1}{a_2a_3}\right)^{\frac{1}{2}}\\ &= \left\{\frac{2\pi}{\alpha_0\log n}\log\left[\frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}\frac{4(1+\varepsilon)n^2\log n}{\alpha_0\pi\rho^2(\nu-\varepsilon)}\frac{2\pi}{\alpha_0\log n}\right]\right\}^{\frac{1}{2}}\\ &= \left\{\frac{4\pi}{\alpha_0}\left[1+\frac{\log[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2]+\log[4(1+\varepsilon)]-\log[(\nu-\varepsilon)\alpha_0^2\rho^2]}{2\log n}\right]\right\}^{\frac{1}{2}}.\end{aligned}$$

Hence $\varphi_n(t)$ achieves its maximum at t_n, that is for each $n\in\mathbb{N}$,

$$\begin{aligned}\varphi_n(t)\le\varphi_n(t_n)=&a_1t_n^2-\frac{a_1}{a_3}\\ =&\frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t_n^2-\frac{\pi[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2]}{\alpha_0\log n},\ \forall\, t>0.\end{aligned}\tag{52}$$

Using (51), we have

$$\begin{aligned}&\frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t_n^2\\ =&\frac{2\pi}{\alpha_0}\left[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2\right]\\ &\times\left[1+\frac{\log[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2]+\log[4(1+\varepsilon)]-\log[(\nu-\varepsilon)\alpha_0^2\rho^2]}{2\log n}\right]\\ \le&\frac{2\pi}{\alpha_0}\left[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2+\frac{\log[4(1+\varepsilon)]-\log[(\nu-\varepsilon)\alpha_0^2\rho^2]}{2\log n}\right]+O\left(\frac{1}{\log^2 n}\right)\\ \le&\frac{2\pi}{\alpha_0}\left(1+\frac{(\|V\|_\infty+4\zeta^2)\rho^2}{4\log n}\right)+O\left(\frac{1}{\log^2 n}\right).\end{aligned}\tag{53}$$

Combining (52) with (53), we derive

$$\begin{aligned}I(tw_n)\le&\varphi_n(t_n)\\ \le&\frac{2\pi}{\alpha_0}\left(1+\frac{(\|V\|_\infty+4\zeta^2)\rho^2}{4\log n}\right)-\frac{\pi[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2]}{\alpha_0\log n}+O\left(\frac{1}{\log^2 n}\right)\\ \le&\frac{2\pi}{\alpha_0}\left[1-\frac{2-(\|V\|_\infty+4\zeta^2)\rho^2}{4\log n}\right]+O\left(\frac{1}{\log^2 n}\right)<\frac{2\pi}{\alpha_0}.\end{aligned}$$

Case iii. $t\in[\sqrt{4\pi(1+\varepsilon)/\alpha_0},\infty]$. Similarly, we have $|tw_n|\ge\sqrt{t_\varepsilon}$ for $x\in B(0,\rho/n)$ and

$$\begin{aligned}I(tw_n)=&\frac{t^2}{2}\|w_n\|^2-\frac12\int_{\mathbb{R}^2}F(x,|tw_n|^2)dx\\ \le&\frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t^2-\frac{\pi^2\rho^2(\nu-\varepsilon)}{t^2\log n}e^{\left(\frac{\alpha_0}{2\pi}t^2-2\right)\log n}\\ \le&\frac{2\pi(1+\varepsilon)[1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2]}{\alpha_0}-\frac{\alpha_0\pi\rho^2(\nu-\varepsilon)n^{2\varepsilon}}{4(1+\varepsilon)\log n}<\frac{2\pi}{\alpha_0},\end{aligned}$$

where at the second inequality, we use the fact that the function

$$\psi_n(t):=\frac{1+(\|V\|_\infty+4\zeta^2)\rho^2\varrho_n^2}{2}t^2-\frac{\pi^2\rho^2(\nu-\varepsilon)}{t^2\log n}e^{\left(\frac{\alpha_0}{2\pi}t^2-2\right)\log n}$$

is decreasing on $[\sqrt{4\pi(1+\varepsilon)/\alpha_0},\infty]$ since its stationary point of $\psi_n(t)$ tends to $\sqrt{4\pi/\alpha_0}$ as $n\to\infty$ and the last inequality can be deduced due to $\frac{n^{2\varepsilon}}{\log n}\to\infty$ as $n\to\infty$. Indeed, set

$$a_4=(\|v\|_\infty+4\zeta^2)\rho^2\quad\text{and}\quad a_5=\pi^2\rho^2(\nu-\varepsilon).$$

By a direct computation, one has

$$\psi_n'(t)=t\left[1+2a_4\varrho_n^2-\frac{a_5(\alpha_0\pi^{-1}t\log n-2)}{t^4\log n}e^{(\frac{\alpha_0}{2\pi}t^2-2)\log n}\right].$$

Since $\psi_n(t) > 0$ as $t \to 0$ and $\psi_n'(t) < 0$ as $n \to \infty$, there exists $t_n > 0$ such that $\psi_n'(t_n) = 0$, which implies

$$1 + 2a_4\varrho_n^2 = \frac{a_5(\alpha_0\pi^{-1}t_n\log n - 2)}{t_n^4\log n}e^{(\frac{\alpha_0}{2\pi}t_n^2-2)\log n}. \tag{54}$$

It is easy to see that $\{t_n\}$ is bounded due to (54). Passing to a subsequence, we have that $t_n \to t_0 > 0$. Note that

$$\frac{a_5(\alpha_0\pi^{-1}t_n\log n - 2)}{t_n^4\log n}e^{(\frac{\alpha_0}{2\pi}t_n^2-2)\log n} \to \infty, \quad \text{if } t_0 > \sqrt{\frac{4\pi}{\alpha_0}}, \text{ as } n \to \infty,$$

and

$$\frac{a_5(\alpha_0\pi^{-1}t_n\log n - 2)}{t_n^4\log n}e^{(\frac{\alpha_0}{2\pi}t_n^2-2)\log n} \to 0, \quad \text{if } t_0 < \sqrt{\frac{4\pi}{\alpha_0}}, \text{ as } n \to \infty,$$

while

$$1 + 2a_4\varrho_n^2 \to 1, \text{ as } n \to \infty.$$

Therefore, the stationary point of $\psi_n(t)$ tends to $\sqrt{\frac{4\pi}{\alpha_0}}$ as $n \to \infty$. The proof is now complete. $\square$

Applying Lemmas 5.5 and 5.6, we can derive the following lemma.

Lemma 5.7. *If* (V2), (F1”), (F2) *and* (F5)-(F8) *hold, then* $m^* = \inf_{\mathcal{M}} I < 2\pi/\alpha_0$.

Through a similar argument of [13, Lemma 2.1], we can deduce the following lemma.

Lemma 5.8. *If* (F1”), (F2) *and* (F6) *are satisfied,* $u_n \rightharpoonup u$ *in* X *and there exists constant* $C_1 > 0$ *such that*

$$\int_{\mathbb{R}^2} f(x, |u_n|^2)|u_n|^2 dx \le C_1.$$

Then for every $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{C})$*, there holds*

$$\lim_{n\to\infty} Re \int_{\mathbb{R}^2} f(x, |u_n|^2)u_n\bar{\xi} = \lim_{n\to\infty} Re \int_{\mathbb{R}^2} f(x, |u|^2)u\bar{\xi}.$$

Proof of Theorem 1.6. In view of Lemmas 5.3 and 5.4, there exists a sequence $\{u_n\} \subset X$ satisfying (49) and $\|u_n\| \le C_2$ for some $C_2 > 0$, hence $\|u_n\|_2 \le C_3$ for some $C_3 > 0$. By (F6) and (49), one has

$$c^* + o(1) = I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \ge \frac{\mu - 2}{2\mu}\int_{\mathbb{R}^2} f(x, |u_n|^2)|u_n|^2 dx,$$

which shows

$$\int_{\mathbb{R}^2} f(x, |u_n|^2)|u_n|^2 dx \le C_4, \tag{55}$$

for some $C_4 > 0$. Assume

$$\delta := \overline{\lim_{n\to\infty}} \sup_{y\in\mathbb{R}^2} \int_{B(y,1)} |u_n|^2 dx = 0.$$

Then Lions' concentration compactness lemma implies $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2, \infty)$. Define

$$\Omega_n(a, b) := \{x \in \mathbb{R}^2, a \le |u_n(x)| < b\}.$$

For any $\varepsilon \in (0, T_1/C_4M_0)$, choose $R_\varepsilon > \sqrt{C_4M_0/\varepsilon}$, by virtue of (F8) and (55), one has

$$\begin{aligned}\int_{\Omega_n(R_\varepsilon,\infty)} F(x,|u_n|^2)dx \leq & M_0\int_{\Omega_n(R_\varepsilon,\infty)} f(x,|u_n|^2)dx \\ \leq & \frac{M_0}{R_\varepsilon^2}\int_{\Omega_n(R_\varepsilon,\infty)} f(x,|u_n|^2)|u_n|^2 < \varepsilon. \end{aligned} \tag{56}$$

It follows from (F2) and (F6) that there exists $r_\varepsilon \in (0,1)$ such that

$$\int_{\Omega_n(0,r_\varepsilon)} F(x,|u_n|^2)dx \leq \int_{\Omega_n(0,r_\varepsilon)} f(x,|u_n|^2)|u_n|^2dx \leq \frac{\varepsilon}{C_3^2}\|u_n\|_2^2 \leq \varepsilon. \tag{57}$$

Due to the continuity of f, there holds

$$\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} F(x,|u_n|^2)dx \leq C_5\|u_n\|_3^3 = o(1) \tag{58}$$

and

$$\int_{\Omega_n(r_\varepsilon,1)} f(x,|u_n|^2)|u_n|^2dx \leq C_5\|u_n\|_3^3 = o(1). \tag{59}$$

Combining (56), (57), (58) and (59) with the arbitrariness of ε, we can obtain

$$\int_{\mathbb{R}^2} F(x,|u_n|^2)dx = o(1). \tag{60}$$

Applying Lemma 5.7 and (60), there exists $\tilde{\varepsilon} > 0$ such that

$$\|u_n\|^2 \leq 2m^* + o(1) = \frac{4\pi}{\alpha_0}(1-3\tilde{\varepsilon}) + o(1). \tag{61}$$

Choose $q \in (1,2)$ such that

$$\frac{(1+\tilde{\varepsilon})(1-3\tilde{\varepsilon})q}{1-\tilde{\varepsilon}} < 1. \tag{62}$$

It follows from Lemma 1, (F1”), (61) and (62) that

$$\begin{aligned}\int_{\Omega_n(1,\infty)} |f(x,|u_n|^2)|^q dx \leq & C_6\int_{\Omega_n(1,\infty)} \left[e^{\alpha_0(1+\tilde{\varepsilon})q|u_n|^2} - 1\right]dx \\ \leq & \int_{\mathbb{R}^2} \left[e^{\alpha_0(1+\tilde{\varepsilon})q\|u_n\|^2(|u_n|/\|u_n\|)^2} - 1\right]dx \leq C_7, \end{aligned}$$

which, together with Hölder inequality, implies

$$\int_{\Omega_n(1,\infty)} f(x,|u_n|^2)|u_n|^2dx \leq \left[\int_{\Omega_n(1,\infty)} |f(x,|u_n|^2)|^q dx\right]^{\frac{1}{q}} \|u\|^2_{\frac{2q}{q-1}} = o(1). \tag{63}$$

Taking account of (57), (58), (60) and (63), we have

$$c^* + o(1) = I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = \frac{1}{2}\int_{\mathbb{R}^2}[f(x,|u_n|^2)|u_n|^2 - F(x,u_n)]dx < \frac{\varepsilon}{2} + o(1).$$

This contradiction implies $\delta > 0$. Then there exists a sequence $\{y_n\} \subset \mathbb{R}^2$ such that

$$\int_{B(y_n,1)} |u_n|^2dx > \frac{\delta}{2}.$$

Similar to proof of Theorem 1.2, we let $v_n = \Upsilon_{[y_n]}u_n$, then

$$\int_{B(0,1+\sqrt{2})} |v_n|^2dx > \frac{\delta}{2}, \tag{64}$$

and

$$I(v_n) \to c^*, \qquad \|I(v_n)\|(1+\|v_n\|) \to 0.$$

Lemma 5.4 shows that $\{v_n\}$ is bounded. Going if necessary to a subsequence, there exists $\tilde{v} \in X$ such that $v_n \rightharpoonup \tilde{v}$ in X, $u_n \to \tilde{v}$ in $L^s_{loc}(\mathbb{R}^2, \mathbb{C})$ for $s \in [2, \infty)$ and $v_n \to \tilde{v}$ a. e. on $\mathbb{R}^2$. Then $\tilde{v} \neq 0$ due to (64). For any $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{C})$, by applying Lemma 5.8, we have

$$\lim_{n\to\infty} Re \int_{R^2} f(x, |v_n|^2) v_n \bar{\xi} = Re \int_{\mathbb{R}^2} f(x, |\tilde{v}|^2 \tilde{v} \bar{\xi}.$$

Therefore

$$\langle I'(\tilde{v}), \xi \rangle = \lim_{n\to\infty} \langle I'(v_n), \xi \rangle = 0,$$

which shows $I'(\tilde{v}) = 0$ and $I(\tilde{v}) \geq m^*$. Through the same process of the proof of Theorem 1.3, we can deduce that $I(\tilde{v}) = m^* = \inf_{\mathcal{M}} I(u)$. The proof is now complete. □

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