C. R. Acad. Sci. Paris, t. 329, Série I, p. 281-286, 1999
Équations aux dérivées partielles/*Partial Differential Equations*

# Perturbations of nonsmooth symmetric nonlinear eigenvalue problems

**Marco DEGIOVANNI** [a], **Vicenţiu RĂDULESCU** [b]

[a] Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via Trieste 17, 25121 Brescia, Italy
E-mail: m.degiovanni@dmf.bs.unicatt.it

[b] Department of Mathematics, University of Craiova, 1100 Craiova, Romania
E-mail: varadulescu@hotmail.com

(Reçu le 18 mars 1999, accepté le 7 juin 1999)

**Abstract.** We consider a symmetric semilinear boundary value problem having infinitely many solutions. We prove that, if we perturb this problem in a non-symmetric way, then the number of solutions goes to infinity as the perturbation tends to zero. The growth conditions on the nonlinearities do not ensure the smoothness of the associated functional. © 1999 Académie des Sciences/Éditions scientifiques et médicales Elsevier SAS

***Perturbations des problèmes non linéaires aux valeurs propres symétriques non réguliers***

**Résumé.** *On considère un problème semi-linéaire symétrique avec une infinité de solutions. On montre que, si l'on perturbe ce problème d'une manière non symétrique, alors le nombre de solutions devient de plus en plus grand lorsque la perturbation tend vers zéro. Les conditions de croissance sur les non-linéarités ne garantissent pas la régularité de la fonctionnelle associée.* © 1999 Académie des Sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

Soit $\Omega \subset \mathbb{R}^N$ un ouvert borné. Pour $r > 0$ fixé arbitrairement on considère le problème suivant : trouver $(u, \lambda) \in \mathrm{H}_0^1(\Omega) \times \mathbb{R}$ tel que :

$$\begin{cases} f(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega), \\ -\Delta u = \lambda f(x,u) \ \text{ dans } \mathcal{D}'(\Omega), \\ \displaystyle\int_\Omega |\mathrm{D}u|^2 \, \mathrm{d}x = r^2. \end{cases} \tag{1}$$

Note présentée par Haïm BRÉZIS.

On suppose que $f : \Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction de Carathéodory avec les propriétés suivantes :

($f1$) $f(x,-s) = -f(x,s)$, p.p. sur $\Omega$ et pour chaque $s \in \mathbb{R}$ ;

($f2$) il existe $a \in \mathrm{L}^1(\Omega)$, $b \in \mathbb{R}$ et $0 \le p < \frac{2N}{N-2}$ (si $N > 2$) tels que

$$0 < s\, f(x,s) \le a(x) + b\,|s|^p, \quad F(x,s) \le a(x) + b\,|s|^p,$$

p.p. sur $\Omega$ et pour chaque $s \in \mathbb{R} \setminus \{0\}$, où $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$ ;

($f3$) $\sup\limits_{|s| \le t} |f(x,s)| \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, pour chaque $t > 0$.

THÉORÈME 1. – *Supposons que les conditions* ($f1$)–($f3$) *sont satisfaites. Alors le problème* (1) *admet une suite* $(\pm u_n, \lambda_n)$ *de solutions distinctes.*

Ensuite notre objectif est d'analyser le problème perturbé :

$$\begin{cases} f(x,u),\ g(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega), \\ -\Delta u = \lambda\big(f(x,u) + g(x,u)\big) \ \text{ dans } \mathcal{D}'(\Omega), \\ \displaystyle\int_\Omega |\mathrm{D}u|^2\,\mathrm{d}x = r^2, \end{cases} \tag{2}$$

où $g : \Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction de Carathéodory qui n'est pas nécessairement impaire par rapport à la seconde variable. On suppose quand même que $g$ satisfait

($g1$) $0 < s\,g(x,s) \le a(x) + b\,|s|^p$ p.p. sur $\Omega$ et pour chaque $s \in \mathbb{R} \setminus \{0\}$ ;

($g2$) $\sup\limits_{|s| \le t} |g(x,s)| \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, pour chaque $t > 0$ ;

($g3$) $G(x,s) \le C_g(1 + |s|^p)$, p.p. sur $\Omega$ et pour chaque $s \in \mathbb{R}$, avec $C_g > 0$, où $G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t$.

On démontre que le nombre de solutions du problème perturbé (2) devient de plus en plus grand si la perturbation est assez petite, dans un sens précisé ultérieurement. Plus précisement, on a :

THÉORÈME 2. – *Supposons que les conditions* ($f1$)–($f3$) *et* ($g1$)–($g3$) *sont satisfaites. Alors, pour chaque entier* $n \ge 1$, *il existe* $\varepsilon_n > 0$ *tel que le problème* (2) *admet au moins* $n$ *solutions distinctes si* $g$ *est une fonction telle que la condition* ($g3$) *soit satisfaite pour* $C_g = \varepsilon_n$.

La preuve des théorèmes 1 et 2 repose sur un argument variationnel. D'abord on pose

$$S_r = \left\{ u \in \mathrm{H}^1_0(\Omega) \ :\ \int_\Omega |\mathrm{D}u|^2\,\mathrm{d}x = r^2 \right\}$$

et on étudie les points critiques sur $S_r$ de la fonctionnelle continue et paire $I : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ définie par :

$$I(u) = -\int_\Omega F(x,u)\,\mathrm{d}x.$$

*Remarque* 1. – Si ($f2$), ($f3$) sont remplacées par la condition standard $0 < s\,f(x,s) \le a_1(x)\,|s| + b\,|s|^p$ avec $a_1 \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega)$, alors $I$ est de classe $\mathrm{C}^1$ et le théorème 1 se trouve dans [8], Theorem 8.17. Avec nos hypothèses, $f$ peut avoir la forme $f(x,s) = \alpha(x)\,\gamma(s)$ avec $\alpha \in \mathrm{L}^1(\Omega)$, $\alpha \ge 0$, $\gamma \in \mathrm{C_c}(\mathbb{R})$, $\gamma$ impaire et $s\,\gamma(s) \ge 0$ pour chaque $s \in \mathbb{R}$. Dans ce cas là, $I$ est bien sûr continue, mais pas localement Lipschitz.

*Remarque* 2. – Lorsque $f$ et $g$ satisfont la condition standard qu'on vient de mentionner, les résultats du type du théorème 2 sont bien classiques (*voir* par example [7]). Des résultats de perturbation, plutôt différents des nôtres, où le problème perturbé admet encore une infinité de solutions, peuvent être trouvés dans [8], [9]. Dans un cadre non régulier, un résultat dans la ligne du théorème 2 a été démontré dans [4] lorsque $f$ et $g$ satisfont la condition standard, mais la fonction $u$ est contrainte par un obstacle, de sorte que l'équation se transforme dans une inéquation variationnelle.

---

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For some fixed $r > 0$, consider the problem: find $(u, \lambda) \in \mathrm{H}_0^1(\Omega) \times \mathbb{R}$ such that:

$$\begin{cases} f(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega), \\ -\Delta u = \lambda f(x,u) \quad \text{in } \mathcal{D}'(\Omega), \\ \displaystyle\int_\Omega |\mathrm{D}u|^2 \, \mathrm{d}x = r^2, \end{cases} \tag{1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that the following conditions hold:

($f1$) $f(x,-s) = -f(x,s)$, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$;

($f2$) there exist $a \in \mathrm{L}^1(\Omega)$, $b \in \mathbb{R}$ and $0 \le p < \frac{2N}{N-2}$ (if $N > 2$) such that:

$$0 < s\, f(x,s) \le a(x) + b\,|s|^p, \quad F(x,s) \le a(x) + b\,|s|^p,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R} \setminus \{0\}$, where $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$;

($f3$) $\sup\limits_{|s| \le t} |f(x,s)| \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, for every $t > 0$.

We notice that, if $N = 1$, then in condition ($f2$) the term $b\,|s|^p$ can be substituted by any continuous function $\varphi(s)$ of $s$, while, if $N = 2$, the same term can be substituted by $\exp(\varphi(s))$, with $\varphi(s)s^{-2} \to 0$ as $|s| \to \infty$.

THEOREM 1. – *Assume that hypotheses* ($f1$)–($f3$) *hold. Then Problem* (1) *admits a sequence* $(\pm u_n, \lambda_n)$ *of distinct solutions.*

Then we want to study what happens when the energy functional is subjected to a perturbation which destroys the symmetry.

Consider the problem: find $(u, \lambda) \in \mathrm{H}_0^1(\Omega) \times \mathbb{R}$ such that:

$$\begin{cases} f(x,u),\ g(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega), \\ -\Delta u = \lambda\big(f(x,u) + g(x,u)\big) \quad \text{in } \mathcal{D}'(\Omega), \\ \displaystyle\int_\Omega |\mathrm{D}u|^2 \, \mathrm{d}x = r^2, \end{cases} \tag{2}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We make no symmetry assumption on $g$, but we impose only:

($g1$) $0 < s\, g(x,s) \le a(x) + b\,|s|^p$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R} \setminus \{0\}$;

($g2$) $\sup\limits_{|s| \le t} |g(x,s)| \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, for every $t > 0$;

($g3$) $G(x,s) \le C_g\,(1 + |s|^p)$, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, for some $C_g > 0$, where $G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t$.

Our second result shows that the number of solutions of Problem (2) becomes greater and greater, as the perturbation tends to zero. More precisely, we have:

THEOREM 2. – *Assume that hypotheses* ($f1$)–($f3$) *and* ($g1$)–($g3$) *hold. Then, for every positive integer* $n$, *there exists* $\varepsilon_n > 0$ *such that Problem* (2) *admits at least* $n$ *distinct solutions, provided that* ($g3$) *holds for* $C_g = \varepsilon_n$.

We will prove Theorems 1 and 2 by a variational argument. First, we set:

$$S_r = \left\{ u \in \mathrm{H}_0^1(\Omega) \,:\, \int_\Omega |\mathrm{D}u|^2 \,\mathrm{d}x = r^2 \right\}$$

and we study the critical points on $S_r$ of the even continuous functional $I : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ defined by:

$$I(u) = -\int_\Omega F(x,u)\,\mathrm{d}x.$$

*Remark* 1. – If $(f2)$, $(f3)$ are substituted by the more standard condition $0 < s\,f(x,s) \le a_1(x)\,|s| + b\,|s|^p$ with $a_1 \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega)$, then $I$ is of class $\mathrm{C}^1$ and Theorem 1 can be found in [8], Theorem 8.17. Under our assumptions, $f$ could have the form $f(x,s) = \alpha(x)\,\gamma(s)$ with $\alpha \in \mathrm{L}^1(\Omega)$, $\alpha \ge 0$, $\gamma \in \mathrm{C}_c(\mathbb{R})$, $\gamma$ odd and $s\,\gamma(s) \ge 0$ for any $s \in \mathbb{R}$. In such a case, $I$ is clearly continuous, but not locally Lipschitz.

*Remark* 2. – When $f$ and $g$ are subjected to the standard condition we have mentioned, results like Theorem 2 go back to Krasnoselskii [7]. For perturbation results, quite different from ours, where the perturbed problem still has infinitely many solutions, we refer the reader to [8], [9]. In a nonsmooth setting, a result in the line of Theorem 2 has been proved in [4] when $f$ and $g$ satisfy the standard condition, but the function $u$ is subjected to an obstacle, so that the equation becomes a variational inequality.

From $(f2)$ it easily follows that $I(u) < 0$ and that $\sup I_r(u) = 0$, where $I_r = I_{|S_r}$.

Since $I$ is only continuous, we will apply the nonsmooth techniques developed in [1], [3], [4], [5]. In the following, we will adopt the notations of such papers.

LEMMA 1. – *The following facts hold:*

(a) *if* $u \in S_r$ *satisfies* $|\mathrm{d}I_r|(u) < +\infty$, *then* $f(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ *and there exists* $\mu \in \mathbb{R}$ *such that:*

$$\|\mu\,\Delta u + f(x,u)\|_{\mathrm{H}^{-1}} \le |\mathrm{d}I_r|(u)\,;$$

(b) *the functional* $I_r$ *satisfies* $(\mathrm{PS})_c$ *for any* $c < 0$;

(c) *if* $u \in S_r$ *is a critical point of* $I_r$, *then there exists* $\lambda > 0$ *such that* $(u, \lambda)$ *is a solution of Problem* (1).

*Proof.* – (a) Set also

$$I_{r,\mathrm{est}}(w) = \begin{cases} I(w) & \text{if } w \in S_r, \\ +\infty & \text{if } w \in \mathrm{H}_0^1(\Omega) \setminus S_r. \end{cases}$$

Then it is immediately seen that $|\mathrm{d}I_{r,\mathrm{est}}|(u) = |\mathrm{d}I_r|(u)$, where we are using the weak slope introduced in [5] (*see* also [1], Definition 2.1). By [1], Theorem 4.13, there exists $\alpha \in \partial I_{r,\mathrm{est}}(u)$ with $\|\alpha\|_{\mathrm{H}^{-1}} \le |\mathrm{d}I_{r,\mathrm{est}}|(u)$, where $\partial$ stands for the subdifferential introduced in [1], Definition 4.1. Taking into account $(f2)$, we deduce from [6], Theorem 3.3, that:

$$I^0(u\,;0) \le 0, \quad I^0(u\,;2u) \le -2\int_\Omega f(x,u)\,u\,\mathrm{d}x < +\infty.$$

Actually, the same proof shows a stronger fact, namely that

$$\overline{I}^0(u\,;0) \le 0, \quad \overline{I}^0(u\,;2u) \le -2\int_\Omega f(x,u)\,u\,\mathrm{d}x < +\infty.$$

Therefore, we can apply [1], Corollary 5.10, obtaining $\beta \in \partial I(u)$ and $\mu \in \mathbb{R}$ with $\alpha = \beta - \mu\,\Delta u$. From [6], Theorems 3.3 and 2.25, we conclude that $f(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and $\beta = -f(x,u)$. Then (a) easily follows.

(b) Let $c < 0$ and let $(u_n)$ be a $(\mathrm{PS})_c$-sequence for $I_r$. By the previous point, we have $f(x,u_n) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and there exists a sequence $(\mu_n)$ in $\mathbb{R}$ with

$$\left\|\mu_n\,\Delta u_n + f(x,u_n)\right\|_{\mathrm{H}^{-1}} \longrightarrow 0.$$

Up to a subsequence, $(u_n)$ is convergent to some $u$ weakly in $\mathrm{H}^1_0(\Omega)$ and a.e. From $(f2)$ it follows $I(u) = c < 0$, hence $u \neq 0$. Again by $(f2)$ and Lebesgue's theorem, we deduce that

$$0 < \int_\Omega f(x,u)\,u\,\mathrm{d}x = \lim_n \int_\Omega f(x,u_n)\,u_n\,\mathrm{d}x = \lim_n \mu_n \int_\Omega |\mathrm{D}u_n|^2\,\mathrm{d}x.$$

Therefore, up to a further subsequence, $(\mu_n)$ is convergent to some $\mu > 0$ and

$$\left\|\Delta u_n + \frac{1}{\mu} f(x,u_n)\right\|_{\mathrm{H}^{-1}} \longrightarrow 0.$$

From [6], Lemma 4.8, we deduce that $(u_n)$ is precompact in $\mathrm{H}^1_0(\Omega)$ and (b) follows.

(c) Arguing as in (b), we find that $f(x,u) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and that there exists $\mu > 0$ with $\mu\Delta u + f(x,u) = 0$. Then the assertion easily follows. □

LEMMA 2. – *There exists a sequence $(b_n)$ of essential values of $I_r$ strictly increasing to* 0.

*Proof.* – We will adapt some arguments from [4] to our concrete situation. Let $\psi : ]-\infty, 0[ \to \mathbb{R}$ be an increasing diffeomorphism. From Lemma 1 it follows that $\psi \circ I_r$ satisfies $(\mathrm{PS})_c$ for every $c \in \mathbb{R}$. Then by [2], Theorem 1.4.13, we have that $\{u \in S_r \,:\, \psi \circ I_r(u) \le b\}$ has finite genus for every $b \in \mathbb{R}$. If $(c_n)$ is the sequence defined as in [4], Theorem 2.12, with respect to $\psi \circ I_r$, it follows that $c_n \to +\infty$ as $n \to \infty$. Therefore, there exists a sequence $(b'_n)$ of essential values of $\psi \circ I_r$ strictly increasing to $+\infty$. Then $b_n = \psi^{-1}(b'_n)$ has the required properties. □

*Proof of Theorem* 1. – Combining Lemma 1 with [4], Theorem 2.10, we deduce that each $b_n$ is a critical value of $I_r$. Again from Lemma 1 we conclude that there exists a sequence $(\pm u_n, \lambda_n)$ of solutions of Problem (1) with $I(u_n) = b_n$ strictly increasing to 0. □

Now we introduce the continuous functional $J_r : S_r \to \mathbb{R}$ defined by

$$J(u) = I(u) - \int_\Omega G(x,u)\,\mathrm{d}x.$$

LEMMA 3. – *For every $\eta > 0$, there exists $\varepsilon > 0$ such that $\sup\limits_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, provided that $(g3)$ holds for $C_g = \varepsilon$.*

*Proof.* – By Sobolev inclusions, we have

$$0 \le I_r(u) - J_r(u) = \int_\Omega G(x,u)\,\mathrm{d}x \le C_g \int_\Omega (1 + |u|^p)\,\mathrm{d}x < \eta, \qquad \text{for any } u \in S_r,$$

if $g$ is chosen as in the hypothesis. □

*Proof of Theorem* 2. – As in the proof of Theorem 1, let us consider a strictly increasing sequence $(b_n)$ of essential values of $I_r$ such that $b_n \to 0$ as $n \to \infty$. Given $n \ge 1$, take some $\delta > 0$ with

$b_n + \delta < 0$ and $2(b_j - b_{j-1}) < \delta$ for $j = 2, \dots, n$. We apply [4], Theorem 2.6, to $I_r$ and $J_r$. So, for any $j = 1, \dots, n$, there exists $\eta_j > 0$ such that $\sup_{u \in S_r} |I_r(u) - J_r(u)| < \eta_j$ implies the existence of an essential value $c_j \in ]b_j - \delta, b_j + \delta[$ of $J_r$. We now apply Lemma 3 for $\eta = \min\{\eta_1, \dots, \eta_n\}$. Thus we obtain $\varepsilon_n > 0$ such that $\sup_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, if $(g3)$ holds with $C_g = \varepsilon_n$. It follows that $J_r$ has at least $n$ distinct essential values $c_1, \dots, c_n$ in the interval $] - \infty, 0[$.

Now Lemma 1 can be clearly adapted to the functional $J_r$. Then we find $u_1, \dots, u_n \in S_r$ and $\lambda_1, \dots, \lambda_n > 0$ such that each $(u_j, \lambda_j)$ is a solution of Problem (2) with $J_r(u_j) = c_j$. □

## References


[1] Campa I., Degiovanni M., Subdifferential calculus and nonsmooth critical point theory, Quaderni Sem. Mat. Brescia, 15/97, Brescia, 1997.

[2] Canino A., Degiovanni M., Nonsmooth critical point theory and quasilinear elliptic equations, in: Topological Methods in Differential Equations and Inclusions (Montréal, PQ, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 472, Kluwer Acad. Publ., Dordrecht, 1995, pp. 1–50.

[3] Corvellec J.-N., Degiovanni M., Marzocchi M., Deformation properties for continuous functionals and critical point theory, Topol. Methods Nonlin. Anal. 1 (1993) 151–171.

[4] Degiovanni M., Lancelotti S., Perturbations of even nonsmooth functionals, Differ. Integ. Eq. 8 (1995) 981–992.

[5] Degiovanni M., Marzocchi M., A critical point theory for nonsmooth functionals, Ann. Mat. Pura Appl. 167 (4) (1994) 73–100.

[6] Degiovanni M., Zani S., Multiple solutions of semilinear elliptic equations with one-sided growth conditions, in: Advanced Topics in Nonlinear Operator Theory, Math. Comput. Modelling, (in press).

[7] Krasnoselskii M.A., Topological methods in the theory of nonlinear integral equations, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow 1956, The Macmillan Co., New York, 1964.

[8] Rabinowitz P.H., Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conf. Series Math. 65, Amer. Math. Soc., Providence, 1986.

[9] Struwe M., Variational methods, Springer-Verlag, Berlin, 1990.