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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 831–836

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Bifurcation for a class of singular elliptic problems with quadratic convection term

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Received 23 February 2004; accepted 1 March 2004

Available online 17 April 2004

Presented by Philippe G. Ciarlet

Abstract

We study the bifurcation problem $-\Delta u = g(u) + \lambda|\nabla u|^2 + \mu$ in Ω, $u = 0$ on $\partial\Omega$, where $\lambda, \mu \geqslant 0$ and Ω is a smooth bounded domain in $\mathbb{R}^N$. The singular character of the problem is given by the nonlinearity g which is assumed to be decreasing and unbounded around the origin. In this Note we prove that the above problem has a positive classical solution (which is unique) if and only if $\lambda(a + \mu) < \lambda_1$, where $a = \lim_{t\to+\infty} g(t)$ and λ_1 is the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$. We also describe the decay rate of this solution, as well as a blow-up result around the bifurcation parameter. ***To cite this article: M. Ghergu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Bifurcation pour une classe de problèmes elliptiques singuliers à terme quadratique de convection. On étudie le problème elliptique de bifurcation $-\Delta u = g(u) + \lambda|\nabla u|^2 + \mu$ dans Ω, $u = 0$ sur $\partial\Omega$, où $\lambda, \mu \geqslant 0$ et Ω est un domaine borné régulier de $\mathbb{R}^N$. Le caractère singulier de ce problème est donné par la nonlinéarité g, qui est décroissante et non bornée autour de l'origine. Dans cette Note on montre que le problème ci-dessus admet une solution classique positive (qui, de plus, est unique) si et seulement si $\lambda(a + \mu) < \lambda_1$, où $a = \lim_{t\to+\infty} g(t)$ et λ_1 est la première valeur propre de l'opérateur de Laplace dans $H_0^1(\Omega)$. Nous établissons également le taux de décroissance de cette solution, ainsi qu'un résultat d'explosion autour du paramètre de bifurcation. ***Pour citer cet article : M. Ghergu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit Ω un domaine borné et régulier de $\mathbb{R}^N$. On suppose que $g : (0, \infty) \to (0, \infty)$ est une fonction de Hölder décroissante telle que $\lim_{t\searrow 0} g(t) = +\infty$. Soit $a := \lim_{t\to\infty} g(t) \in [0, \infty)$ et $\lambda, \mu \geqslant 0$. On désigne par λ_1 la première valeur propre de l'opérateur de Laplace $(-\Delta)$ dans $H_0^1(\Omega)$.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.03.020

On cherche des solutions classiques du problème

$$\begin{cases} -\Delta u = g(u) + \lambda|\nabla u|^2 + \mu & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases} \tag{1}$$

Les résultats principaux de cette Note sont contenus dans

Théorème 0.1. *Les propriétés suivantes sont vraies* :

(i) *Le problème* (1) *admet une solution si et seulement si* $\lambda(a+\mu) < \lambda_1$;

(ii) *Soit* $\lambda^* = \lambda^*(\mu) := \lambda_1/(a+\mu)$, *pour chaque* $\mu > 0$. *Alors le problème* (1) *admet une solution unique* u_λ *pour tout* $\lambda < \lambda^*$ *et, de plus, l'application* $(0, \lambda^*) \ni \lambda \mapsto u_\lambda$ *est croissante. Si la fonction* g *vérifie la condition* $\limsup_{t\searrow 0} t^\alpha g(t) < +\infty$, *pour un certain* $\alpha \in (0,1)$, *alors la suite* $(u_\lambda)_\lambda$ *a les propriétés suivantes* :

(ii1) *Pour tout* $0 < \lambda < \lambda^*$ *il existe deux constantes positives* c_1, c_2 *dépendantes de* λ *telles que* $c_1 \operatorname{dist}(x, \partial\Omega) \leqslant u_\lambda \leqslant c_2 \operatorname{dist}(x, \partial\Omega)$ *dans* Ω ;

(ii2) *On a* $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;

(ii3) *La suite* $(u_\lambda)_\lambda$ *vérifie* $u_\lambda \to +\infty$ *si* $\lambda \nearrow \lambda^*$, *uniformément sur les sous-ensembles compacts de* Ω.

La démonstration de ce résultat repose sur le principe du maximum combiné avec des estimations elliptiques et un théorème de Hörmander concernant les fonctions sur-harmoniques.

On remarque aussi que l'hypothèse de décroissance sur g autour de l'origine implique une condition du type Keller–Osserman qui est équivalente à la *propriété du support compact* formulée dans Bénilan, Brezis et Crandall [1].

1. The main result

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We assume that $g:(0,\infty) \to (0,\infty)$ is a Hölder function which is decreasing and satisfying $\lim_{t\searrow 0} g(t) = +\infty$. Set $a := \lim_{t\to\infty} g(t) \in [0,\infty)$. Assume that λ and μ are nonnegative parameters and let λ_1 denote the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

We are concerned in this paper with classical solutions of the boundary value problem

$$\begin{cases} -\Delta u = g(u) + \lambda|\nabla u|^2 + \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Our main result is

Theorem 1.1. *The following properties hold true*:

(i) *Problem* (2) *has a solution if and only if* $\lambda(a+\mu) < \lambda_1$.

(ii) *Denote* $\lambda^* = \lambda^*(\mu) := \lambda_1/(a+\mu)$, *for any* $\mu > 0$. *Then problem* (2) *has a unique solution* u_λ *for all* $\lambda < \lambda^*$ *and the sequence* $(u_\lambda)_{\lambda<\lambda^*}$ *is increasing with respect to* λ. *Moreover, if*

$$\limsup_{t\searrow 0} t^\alpha g(t) < +\infty, \quad \textit{for some } \alpha \in (0,1), \tag{3}$$

then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ *has the following properties.*

(ii1) *For all* $0 < \lambda < \lambda^*$ *there exists two positive constants* c_1, c_2 *depending on* λ *such that* $c_1 \operatorname{dist}(x, \partial\Omega) \leqslant u_\lambda \leqslant c_2 \operatorname{dist}(x, \partial\Omega)$ *in* Ω;

(ii2) $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;
(ii3) $u_\lambda \to +\infty$ *as* $\lambda \nearrow \lambda^*$, *uniformly on compact subsets of* Ω.

Assumption (3) implies $\int_0^1 (\int_0^t g(s)\,\mathrm{d}s)^{-1/2}\,\mathrm{d}t < +\infty$. As proved by Bénilan, Brezis and Crandall in [1], the above Keller–Osserman-type growth condition around the origin is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and $-\Delta u + g(u) = h$ a.e. in $\mathbb{R}^N$.

We split the proof of Theorem 1.1 into several steps.

Step 1. *Existence of solutions.* If $\lambda = 0$ then, by Lemma 2.2 in [2], problem (2) has a solution for any $\mu \geqslant 0$. Next, we suppose that $\lambda > 0$ and fix $\mu \geqslant 0$. Denote $v = \mathrm{e}^{\lambda u} - 1$. Then

$$\begin{cases} -\Delta v = \Phi_\lambda(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Phi_\lambda(s) = \lambda(s+1)g(\lambda^{-1}\ln(s+1)) + \lambda\mu(s+1)$, for all $s \in (0,\infty)$. Then Φ_λ is not monotone, but the mapping $(0,\infty) \ni s \mapsto s^{-1}\Phi_\lambda(s)$ is decreasing for all $\lambda > 0$, $\lim_{s\to+\infty} s^{-1}\Phi_\lambda(s) = \lambda(a+\mu)$ and $\lim_{s\searrow 0} s^{-1}\Phi_\lambda(s) = +\infty$, uniformly for $\lambda > 0$. We first remark that Φ_λ satisfies the hypotheses in Lemma 2.2 in [2] (see also [8]), provided that $\lambda(a+\mu) < \lambda_1$. Hence problem (4) has at least one solution. On the other hand, since $g \geqslant a$ in $(0,\infty)$, we obtain

$$\Phi_\lambda(s) \geqslant \lambda(a+\mu)(s+1), \quad \text{for all } \lambda, s \in (0,\infty). \tag{5}$$

If $\lambda(a+\mu) \geqslant \lambda_1$ and problem (4) has a solution v then, by (5), v is a super-solution of $-\Delta z = \lambda_1(z+1)$ in Ω, $z = 0$ on $\partial\Omega$. Since 0 is sub-solution of this boundary problem we deduce that there exists z that fulfills the above properties. Let $\varphi_1 > 0$ be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$. Hence $\lambda_1 \int_\Omega \varphi_1 z\,\mathrm{d}x = \lambda_1 \int_\Omega \varphi_1(z+1)\,\mathrm{d}x$, a contradiction. This shows that problem (4) has no solutions if $\lambda(a+\mu) \geqslant \lambda_1$.

Step 2. *Uniqueness of the solution.* Fix $\lambda \geqslant 0$. Let u_1 and u_2 be two classical solutions of problem (2) with $\lambda < \lambda^*$. It is enough to show that $u_1 \leqslant u_2$ in Ω. Supposing the contrary, we deduce that $\max_{\overline{\Omega}}\{u_1 - u_2\} > 0$ is achieved in a point $x_0 \in \Omega$. This yields $\nabla(u_1 - u_2)(x_0) = 0$ and $0 \leqslant -\Delta(u_1 - u_2)(x_0) = g(u_1(x_0)) - g(u_2(x_0)) < 0$, a contradiction. We conclude that $u_1 \leqslant u_2$ in Ω. Hence $u_1 = u_2$.

Step 3. *Dependence on* λ. Fix $0 \leqslant \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_1}, u_{λ_2} be the unique solutions of problem (2) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. If $\{x \in \Omega;\ u_{\lambda_1} > u_{\lambda_2}\}$ is nonempty, then $\max_{\overline{\Omega}}\{u_{\lambda_1} - u_{\lambda_2}\} > 0$ is achieved in Ω. At that point, say $\bar{x}$, we have $\nabla(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = 0$ and $0 \leqslant -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = g(u_{\lambda_1}(\bar{x})) - g(u_{\lambda_2}(\bar{x})) + (\lambda_1 - \lambda_2)|\nabla u_{\lambda_1}|^2(\bar{x}) < 0$, which is a contradiction. Hence $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$ and, by the maximum principle, $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

Step 4. *Regularity.* Fix $0 < \lambda < \lambda^*$, $\mu > 0$ and assume that g satisfies the growth condition (3). Taking again $v = \mathrm{e}^{\lambda u} - 1$ it follows that $v_\lambda = \mathrm{e}^{\lambda u_\lambda} - 1$ is the unique solution of problem (4). Since $\lim_{s\searrow 0} s^{-1}(\mathrm{e}^{\lambda s} - 1) = \lambda$, we conclude that (ii1) and (ii2) in Theorem 1.1 are established if we prove the following

(a) $\tilde{c}_1\,\mathrm{dist}(x,\partial\Omega) \leqslant v_\lambda(x) \leqslant \tilde{c}_2\,\mathrm{dist}(x,\partial\Omega)$ in Ω, for some positive constants $\tilde{c}_1, \tilde{c}_2 > 0$;
(b) $v_\lambda \in C^{1,1-\alpha}(\overline{\Omega})$.

Proof of (a). Since g is monotone and $g(s) \leqslant cs^{-\alpha}$ near the origin, there exists positive numbers A, B and C such that

$$\Phi_\lambda(s) \leqslant As + Bs^{-\alpha} + C, \quad \text{for all } 0 < \lambda < \lambda^* \text{ and } s > 0. \tag{6}$$

Fix $m > 0$ such that $m\lambda_1\|\varphi_1\|_\infty < \lambda\mu$. Combining this with (5) we deduce that

$$-\Delta(v_\lambda - m\varphi_1) = \Phi_\lambda(v_\lambda) - m\lambda_1\varphi_1 \geqslant \lambda\mu - m\lambda_1\varphi_1 \geqslant 0 \quad \text{in } \Omega. \tag{7}$$

Since $v_\lambda - m\varphi_1 = 0$ on $\partial\Omega$, we obtain

$$v_\lambda \geqslant m\varphi_1 \quad \text{in } \Omega. \tag{8}$$

The last relation combined with the standard estimate

$$C_1 \operatorname{dist}(x, \partial\Omega) \leqslant \varphi_1(x) \leqslant C_2 \operatorname{dist}(x, \partial\Omega) \quad \text{for any } x \in \Omega$$

imply $v_\lambda(x) \geqslant \tilde{c}_1 \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$, for some positive constant $\tilde{c}_1 > 0$. The first inequality in the statement of (a) is therefore established. For the second one, we apply an idea found in Gui and Lin [4]. Using (8) and the estimate (6), by virtue of Lemma 2.1 in [2] (see also [6]) we deduce that $\Phi_\lambda(v_\lambda) \in L^1(\Omega)$, that is, $\Delta v_\lambda \in L^1(\Omega)$. Using now the smoothness of $\partial\Omega$, there exists $\delta \in (0,1)$ such that for all $x_0 \in \Omega_\delta := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \leqslant \delta\}$, there exists $y \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\operatorname{dist}(y, \partial\Omega) = \delta$ and $\operatorname{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$. Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K-1)\delta$ and let ξ be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta\xi = \Phi_\lambda(\xi) & \text{in } B_K(0) \setminus B_1(0), \\ \xi > 0 & \text{in } B_K(0) \setminus B_1(0), \\ \xi = 0 & \text{on } \partial\big(B_K(0) \setminus B_1(0)\big), \end{cases}$$

where $B_r(0)$ denotes the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, ξ is radially symmetric. Hence $\xi(x) = \tilde{\xi}(|x|)$ and

$$\begin{cases} \tilde{\xi}'' + \dfrac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi}) = 0 & \text{in } (1, K), \\ \tilde{\xi} > 0 & \text{in } (1, K), \\ \tilde{\xi}(1) = \tilde{\xi}(K) = 0. \end{cases}$$

Integrating above we find

$$\tilde{\xi}'(t) = \tilde{\xi}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}\Phi_\lambda\big(\tilde{\xi}(r)\big)\,\mathrm{d}r = \tilde{\xi}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}\Phi_\lambda\big(\tilde{\xi}(r)\big)\,\mathrm{d}r,$$

where $1 < a < t < b < K$. With the same arguments as above we obtain $\Phi_\lambda(\tilde{\xi}) \in L^1(1, K)$ which implies that both $\tilde{\xi}'(1)$ and $\tilde{\xi}'(K)$ are finite. Hence $\tilde{\xi} \in C^2(1, K) \cap C^1[1, K]$. Furthermore,

$$\xi(x) \leqslant \widetilde{C} \min\big\{K - |x|, |x| - 1\big\}, \quad \text{for any } x \in B_K(0) \setminus B_1(0). \tag{9}$$

Fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\operatorname{dist}(y_0, \partial\Omega) = \delta$ and $\operatorname{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$. Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define $\bar{v}(x) = \xi((x - y_0)/\delta)$, for all $x \in \overline{\Omega}$. We show that $\bar{v}$ is a super-solution of problem (4). Indeed, for all $x \in \Omega$ we have

$$\Delta\bar{v} + \Phi_\lambda(\bar{v}) = \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}'\right) + \Phi_\lambda(\tilde{\xi}) \leqslant \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi})\right) = 0,$$

where $r = |x - y_0|/\delta$. We have obtained that $\Delta\bar{v} + \Phi_\lambda(\bar{v}) \leqslant 0 \leqslant \Delta v_\lambda + \Phi_\lambda(v_\lambda)$ in Ω, $\bar{v}, v_\lambda > 0$ in Ω, $\bar{v} = v_\lambda$ on $\partial\Omega$, and $\Delta v_\lambda \in L^1(\Omega)$. By Lemma 2.3 in [2] (see also [8]) we deduce that $v_\lambda \leqslant \bar{v}$ in Ω. Combining this with (9) we obtain

$$v_\lambda(x_0) \leqslant \bar{v}(x_0) \leqslant \widetilde{C} \min\left\{K - \frac{|x_0 - y_0|}{\delta}, \frac{|x_0 - y_0|}{\delta} - 1\right\} \leqslant \frac{\widetilde{C}}{\delta}\operatorname{dist}(x_0, \partial\Omega).$$

Hence $v_\lambda \leqslant \widetilde{C}\,\delta^{-1}\operatorname{dist}(x, \partial\Omega)$ in Ω_δ and the second inequality in the statement of (a) follows.

Proof of (b). Let G be the Green function associated to the Laplace operator in Ω with respect to Dirichlet boundary condition. Then, for all $x \in \Omega$, $v_\lambda(x) = -\int_\Omega G(x, y)\Phi_\lambda(v_\lambda(y))\,\mathrm{d}y$ and $\nabla v_\lambda(x) = -\int_\Omega G_x(x, y)\Phi_\lambda(v_\lambda(y))\,\mathrm{d}y$. If $x_1, x_2 \in \Omega$, using (6) we obtain

$$\begin{aligned}&\left|\nabla v_\lambda(x_1)-\nabla v_\lambda(x_2)\right|\\&\quad\leqslant\int_\Omega\left|G_x(x_1,y)-G_x(x_2,y)\right|\cdot(Av_\lambda+C)\,\mathrm{d}y+B\int_\Omega\left|G_x(x_1,y)-G_x(x_2,y)\right|\cdot v_\lambda^{-\alpha}(y)\,\mathrm{d}y.\end{aligned}$$

Now, taking into account that $v_\lambda\in C(\overline{\Omega})$, by the standard regularity theory (see [3]) we deduce that $\int_\Omega|G_x(x_1,y)-G_x(x_2,y)|\cdot(Av_\lambda+C)\,\mathrm{d}y\leqslant\tilde{c}_1|x_1-x_2|$. On the other hand, with the same arguments as in the proof of Theorem 1 in [4], we deduce that $\int_\Omega|G_x(x_1,y)-G_x(x_2,y)|\cdot v_\lambda^{-\alpha}(y)\leqslant\tilde{c}_2|x_1-x_2|^{1-\alpha}$. The last two inequalities imply $u_\lambda\in C^2(\Omega)\cap C^{1,1-\alpha}(\overline{\Omega})$.

Step 5. *Asymptotic behaviour of the solution.* In order to conclude the asymptotic behaviour for u_λ, it is enough to show that $\lim_{\lambda\nearrow\lambda^*}v_\lambda=+\infty$ on compact subsets of Ω. To this aim, we use some techniques developed in [7]. Due to the special character of our problem, we will be able to show in what follows that, in certain cases, L^2-boundedness implies H_0^1-boundedness! We argue by contradiction. Since $(v_\lambda)_{\lambda<\lambda^*}$ is a sequence of nonnegative super-harmonic functions in Ω then, by a theorem of Hörmander (see [5, Theorem 4.1.9]), we can find a subsequence of $(v_\lambda)_{\lambda<\lambda^*}$ (still denoted by $(v_\lambda)_{\lambda<\lambda^*}$) which converges in $L^1_{\mathrm{loc}}(\Omega)$ to some v^*. The monotony of v_λ yields (up to a subsequence) $v_\lambda\nearrow v^*$ a.e. in Ω.

We first show that $(v_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. Suppose the contrary. Passing eventually at a subsequence, we have $v_\lambda=M(\lambda)w_\lambda$, where

$$M(\lambda)=\|v_\lambda\|_{L^2(\Omega)}\to\infty\quad\text{as }\lambda\nearrow\lambda^*\quad\text{and}\quad w_\lambda\in L^2(\Omega),\quad\|w_\lambda\|_{L^2(\Omega)}=1.\tag{10}$$

Relation (6) yields $(M(\lambda))^{-1}\Phi_\lambda(v_\lambda)\to0$ in $L^1_{\mathrm{loc}}(\Omega)$ as $\lambda\nearrow\lambda^*$, that is,

$$-\Delta w_\lambda\to0\quad\text{in }L^1_{\mathrm{loc}}(\Omega)\text{ as }\lambda\nearrow\lambda^*.\tag{11}$$

By Green's first identity, we have

$$\int_\Omega\nabla w_\lambda\cdot\nabla\phi\,\mathrm{d}x=-\int_\Omega\phi\Delta w_\lambda\,\mathrm{d}x=-\int_{\mathrm{Supp}\,\phi}\phi\Delta w_\lambda\,\mathrm{d}x,\quad\text{for all }\phi\in C_0^\infty(\Omega).\tag{12}$$

Using (11) we obtain

$$\left|\int_{\mathrm{Supp}\,\phi}\phi\Delta w_\lambda\,\mathrm{d}x\right|\leqslant\int_{\mathrm{Supp}\,\phi}|\phi||\Delta w_\lambda|\,\mathrm{d}x\leqslant\|\phi\|_\infty\int_{\mathrm{Supp}\,\phi}|\Delta w_\lambda|\,\mathrm{d}x\to0\quad\text{as }\lambda\nearrow\lambda^*.\tag{13}$$

Relations (12) and (13) yield

$$\int_\Omega\nabla w_\lambda\cdot\nabla\phi\,\mathrm{d}x\to0\quad\text{as }\lambda\nearrow\lambda^*,\ \text{for all }\phi\in C_0^\infty(\Omega).\tag{14}$$

Recall that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using (6) and Hölder's inequality, we have

$$\begin{aligned}\int_\Omega|\nabla w_\lambda|^2&=-\int_\Omega w_\lambda\Delta w_\lambda=-\frac{1}{M(\lambda)}\int_\Omega w_\lambda\Delta u_\lambda=\frac{1}{M(\lambda)}\int_\Omega w_\lambda\Phi_\lambda(v_\lambda)\\&\leqslant\frac{A}{M(\lambda)}\int_\Omega w_\lambda v_\lambda+\frac{B}{M(\lambda)}\int_\Omega w_\lambda v_\lambda^{-\alpha}+\frac{C}{M(\lambda)}\int_\Omega w_\lambda\\&=A\int_\Omega w_\lambda^2+\frac{B}{M(\lambda)^{1+\alpha}}\int_\Omega w_\lambda^{1-\alpha}+\frac{C}{M(\lambda)}\int_\Omega w_\lambda\leqslant A+\frac{B}{M(\lambda)^{1+\alpha}}|\Omega|^{(1+\alpha)/2}+\frac{C}{M(\lambda)}|\Omega|^{1/2}.\end{aligned}$$

The above estimates imply that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Thus, there exists $w \in H_0^1(\Omega)$ such that

$$w_\lambda \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad w_\lambda \to w \quad \text{strongly in } L^2(\Omega) \quad \text{as } \lambda \nearrow \lambda^*. \tag{15}$$

Combining relations (10) and (15), we obtain $\|w\|_{L^2(\Omega)} = 1$. On the other hand, by (14) and (15) we find $\int_\Omega \nabla w \cdot \nabla\phi \, dx = 0$, for all $\phi \in C_0^\infty(\Omega)$. So $w = 0$, which contradicts $\|w\|_{L^2(\Omega)} = 1$. Hence $(v_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. As before for w_λ, we obtain that $(v_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Thus, up to a subsequence,

$$v_\lambda \rightharpoonup v^* \quad \text{weakly in } H_0^1(\Omega), \quad v_\lambda \to v^* \quad \text{strongly in } L^2(\Omega), \quad v_\lambda \to v^* \quad \text{a.e. in } \Omega \quad \text{as } \lambda \nearrow \lambda^*. \tag{16}$$

Now we can proceed to get a contradiction. We first observe that $-\int_\Omega \Delta v_\lambda \varphi_1 \, dx = \int_\Omega \Phi_\lambda(v_\lambda)\varphi_1 \, dx$, for all $\lambda < \lambda^*$. Using now (5) we find

$$\lambda_1 \int_\Omega v_\lambda \varphi_1 \geqslant \lambda(a+\mu) \int_\Omega (v_\lambda + 1)\varphi_1 \, dx, \quad \text{for all } 0 < \lambda < \lambda^*. \tag{17}$$

By (16), we can use Lebesgue's dominated convergence theorem in order to pass to the limit with $\lambda \nearrow \lambda^*$ in (17). We obtain $\lambda_1 \int_\Omega v^* \varphi_1 \, dx \geqslant \lambda_1 \int_\Omega (v^* + 1)\varphi_1 \, dx$, contradiction. This shows that $\lim_{\lambda \nearrow \lambda^*} v_\lambda = +\infty$, uniformly on compact subsets of Ω. Consequently, the sequence $(u_\lambda)_{\lambda<\lambda^*}$ has the same property. This concludes the proof of Theorem 1.1. □

Acknowledgements

M. Ghergu is supported by the MIRA Project through a Ph.D. fellowship *en cotutelle* at the Université de Savoie. He warmly thanks Prof. I. Ionescu for his guidance and for kind support. This paper has been completed while V. Rădulescu was visiting the Université de Picardie Jules Verne at Amiens in February 2004. He is grateful to Prof. O. Goubet for invitation and for useful discussions.

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